DEVELOPMENT OF POINT-SET TOPOLOGY: SELECTED ASPECTS

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Declaration

I hereby declare that this thesis, entitled "Development of Point Set Topology : Selected Aspects", submitted to the University of Dhaka for the degree of Master of Philosophy in Mathematics is my own work (except where indicated and acknowledged otherwise in the text), and has not at any time been previously submitted to this university or any other university / institution for award of any degree / diploma at any level.

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Certificate

This is to certify that the thesis entitled " *Development of Point-Set Topology : Selected Aspects*", submitted by *Rahima Khatun* to the *University of Dhaka* for the degree of *Master of Philosophy* in *Mathematics*, is a record of bonafide research work carried out by her under my supervision and guidance in the Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh. She has completed the assignment to my satisfaction and is permitted to submit her thesis for examination by a duly constituted Board of Examiners.

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Abstract

This thesis presents an overview of some of the major topics in point-set topology from a contemporary standpoint with special reference to their historical development.

Chapter 1 gives a broad outline of development of point-set topology upto 1940. Although the genesis of point-set topology can be traced, if one wishes, to a distant past, the definition of derived sets by G. Cantor (1874) is arguably a prime source of point-set topology.

Chapter 2 deals with various equivalent ways of defining a topological space and some of the basic concepts needed.

Chapter 3 deals with the so-called separation axioms and the related properties of first and second axioms of countability.

Chapter 4 deals with convergence in topology. The fact that convergent sequences are inadequate to determine a given topology prompted mathematicians to generalize the notion of sequence in a suitable manner. The two mutually equivalent resulting theories of net convergence and filter convergence are described in this chapter.

Chapter 5 deals with compactness. Four types of compactness are defined and studied in some detail. We discuss the question; when do the compact sets of a topological space coincide with its closed sets? We also

discuss two properties which lie strictly between T_1 and T_2 separation axioms.

Chapter 6 treats the theory of uniform spaces, which are needed to carry over the concept of uniform continuity from metric spaces to topological spaces.

Chapter 7 is the main thrust of the present work. Much of the groundwork for point-set topology was done by Fréchet around the turn of the preceding century. The emergence of the point-set topology as a distinct discipline is credited to the appearance of Hausdorff's *Grundzüge* (1914). In Chapter 7 we compare and contrast, in considerable detail, the contribution of Fréchet and Hausdorff to the founding of point-set topology. We also provide an account and analysis of Hausdorff's personality and motivation. Additional historical notes and remarks scattered throughout the text shed light on the historical development of particular topics.

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Explanation

This work owes a great deal to many of the books cited in the list of major references.

Chapter 1 leans heavily on Bushaw [2].

Chapter 2 is modeled after Willard [20].

Chapter 3 draws on most of the books cited in our list.

Chapter 4 is based on Willard [20] and Wilansky [18].

Chapter 5 draws on many of the books cited in our list and on the papers mentioned in the body of the text.

Chapter 6 leans on Willard.

Chapter 7 is the raison d'être (justification.) of this work. Our discussion of Fréchet work was greatly facilitated by Taylor's exhaustive study [18]. Our description of Hausdorff's work and personality owe a good deal to the references cited in the text. The critical analysis and comparison of the contribution of Fréchet and Hausdorff to point-set topology is our contribution.

The List of Major References include only items which were directly available to us and were frequently consulted during the preparation of this work. Additional references are given in the text and in footnotes as and when necessary. Secondary references (whose originals were not available to us) are marked by asterisks. The footnotes are an integral part of the text.

Table of Contents

Chapter 1

Introduction

This thesis aims at a critical analysis of some selected aspects of point–set topology (also known as general topology). General topology has also been described as generalized analysis or abstract analysis.

The notion of a general or abstract topological space presupposes the general concept of "set". It is therefore quite understandable that the general theory of topological space was developed after the theory of sets. What is interesting is the fact that one of the prime motivations which led G. Cantor (1845-1918) to the general theory of sets also served as the prime motivation for the ultimate development of the notion of a topological space. This is the notion of ***derived set*** of a set of real numbers, which was introduced by Cantor in 1873 in course of his researches on the theory of convergence of Fourier series.

The time period from the early eighteen eighties to the appearance of F. Hausdorff's "Grundzüge der Mengenlehre" [6] in 1914 may be considered as the gestation period for general topology. The forces shaping the edifice of general topology during this period are discussed in considerable detail in Manheim's book "The Genesis of Point-Set Topology" [12]. This book is based on a dissertation at Columbia University. Its conclusion is succinctly expressed in one brief sentence [12, p. 142]. "Looking backwards, point set topology developed from arithmetization [of analysis] and set theory". Manheim therefore begins his book with a discussion of "The difficulties associated with the appearance of calculus" and of "Early attempts to make analysis rigorous" (headings of Chapters I and II). The subsequent four chapters bear the headings "Fourier series and the beginning of researches into point sets", "Arithmetization of analysis", "Development of point set theory" and "The emergence of point set topology". The final Chapter VIII "From Newton to Hausdorff" gives a "Summary of investigations important to the development of point set topology", which is followed by a brief Epilogue. Manheim's unequivocal statement "The appearance in 1914 of Hausdorff's *Grundzüge der Mengelehre* marks the emergence of point-set topology as a separate discipline", echoes the opinion expressed in N. Bourbaki [3, Historical Note to Chapter 1, p. 166].

The notion of a topological space was, so to say " in the air" for quite sometime, at least since the publication of M. Fréchet's doctoral dissertation ~ *Sur Quelques Points du Calcul Fonctionnel*, Rendiconti di Circolo Mathematico di Palermo 22 (1906), 1-74. (Fréchet's work is discussed in more detail in Chapter 7). But it was Hausdorff who gave a clear-cut definition of a topological space in terms of neighbourhood systems. Hausdorff's motivation was the familiar properties of (spherical) neighbourhoods in the Euclidean space $\mathbf{R}^n$. Hausdorff also recast much of Fréchet's work in a more neat and coherent form; for example he introduced the term " metric space" and developed the theory of point–sets in metric spaces in a systematic manner. This also helped him to formulate the notion of a general topological space in a precise manner.

However, the roots of general topology lie deeper than just in neighbourhoods. Manheim [12] traces the roots of general topology to early nineteenth century in the rigorous treatment of real variable theory. Bushaw begins his book "Elements of General Topology" [2] with a "Historical Introduction" (pp. 1– 8), where he says [p. 2] "An enthusiast, by stretching a point here and there, might extend the history of general topology back to a surprisingly remote date, but there is probably little to be gained by going back beyond the researches of Georg Cantor. Starting about 1870 with some moderately straightforward problems in the theory of Fourier series, Cantor was led to investigate properties of subsets of the real number system and of the n-dimensional Euclidean space. These sets were not necessarily intervals, curves, surfaces, polyhedra, and so on, of the familiar kinds but could be very bizarre indeed. To carry out these investigations he not only had to come to grips with the previously evaded concept of the infinite, developing a whole new arithmetic of transfinite numbers, but also introduced and made extensive use of certain new purely distance-related concepts; for example, that of the derivative (*Ableitung*) of a set. He defined the derivative of a set X of points in n-dimensional Euclidean space to be the set of all points x with the property that infinitely many points of X could be found within any arbitrarily small distance of x."

The purpose of the thesis is to trace the development of point-set topology from the appearance of Hausdorff's seminal work [6] to until about 1940. But at several places some later developments have been considered. We shall concentrate on some selected aspects, rather than attempt a complete coverage.

At just about the time when Cantor's own research was tapering off, partly because of vigorous opposition from tradition-bound contemporaries, other mathematicians began to elaborate on and extend the scope of his ideas. Perhaps the most important of his immediate followers in the line that led to general topology were members of an Italian school that included G. Ascoli, C. Arzela, S. Pincherle and V. Volterra; in the eighteen eighties, they began to apply Cantor's distance-related concepts to "spaces" that were not spaces in the conventional sense at all—for example, "spaces" in which the typical "point" might be a curve or a function. Precise references to these pioneering papers are given in [12, p. 113].

The next major step was taken by Maurice Fréchet (1878-1956), who made an observation that seems perfectly natural now but was radical for its time. In his previously cited doctoral thesis, published in 1906, he suggested that much of the work that had been done and was being done on distance-related concepts in a number of specific "spaces" might be done more economically by considering a single abstract, but appropriately restricted, concept of "distance" defined for pairs of equally abstract "points" and developing its properties once and for all. He suggested and explored several alternative ways of doing this, but his most influential proposal was the concept of what is now called a metric space [in Fréchet's original terminology, Class (E)]. A metric space may be described as a nonempty set S together with a function that assigns to any pair of elements x and y of S a real number $d(x, y)$, which satisfies certain natural conditions. The number $d(x, y)$ is interpreted as the "distance" between the "points" x and y.

The most familiar spaces of nineteenth-century mathematics and many other mathematical systems can be regarded as specific metric spaces, and naturally any conclusion obtained for metric spaces in general can be applied at once to any of these particular spaces. The restrictions on the "distance function" d were shrewdly chosen; they were liberal enough to cover a great variety of important mathematical systems, yet tight enough to provide the basis for a theory within which most of the important distance-related concepts (which included, as it turned out, convergence, continuity, and so on) can be defined in a natural way, and many of the important theorems about them can be proved.

This process of generalization did not stop with the introduction of metric spaces; within a few years of the publication of Fréchet's thesis, several mathematicians (especially F. Riesz, F. Hausdorff, and Fréchet himself) observed that the distance function was not really needed for most of the purposes served by the concept of a metric space, and that almost everything one wanted to do could be done using some such subsidiary concept as that of a "neighbourhood", itself a generalization of an older idea: In a metric space, a (spherical) neighbourhood of a point x is the set of all points y satisfying $d(x, y) < \varepsilon$, where ε is some positive real number called the radius of the neighbourhood. The typical spherical neighbourhood of a point x in a metric space is accordingly the set of all elements "within a certain distance of" x. Now we may let Hausdorff speak for himself [6, p. 213].

"These spherical neighbourhoods, as we will call them, have a series of properties of which only a very few are needed at first. As indicated above, we now change our standpoint by disregarding the distances by means of which we defined neighbourhoods, and put those properties at the head as axioms.

By a *topological space* we mean a set E in which certain subsets U_x, which we call neighbourhoods, are assigned to the elements (points), and in fact according to the following:

Neighbourhood Axioms

(A) To each point x there corresponds at least one neighbourhood; every neighbourhood U_x has x as an element.

(B) If U_x and V_x are two neighbourhoods of the same point x, there exists a neighbourhood W_x of x which is a subset of both.

(C) If y belongs to U_x, there exists a neighbourhood U_y which is a subset of U_x.

(D) For any two distinct points x, y there are two neighbourhoods U_x, U_y without any common point" .

It is not hard to show that the spherical neighbourhoods in a metric space do satisfy Hausdorff's "neighbourhood axioms"; so every metric space may be regarded as a "topological space"; but the latter concept is more general. Despite this greater generality, however, much of the theory of metric spaces (and therefore of distance-related aspects of many specific spaces) can be moved almost intact into the new setting. A brief return to the idea of continuity will illustrate the point. The familiar definition of continuity for functions from one metric space into another may be readily paraphrased to read: f: X→Y is continuous at x if, for every neighbourhood U_y of $y = f(x)$ there exists a neighbourhood U_x of x such that $f(u) \in U_y$ for all $u \in U_x$. This statement, continues to be meaningful if the spaces involved are merely topological spaces in Hausdorff's sense, and may therefore be taken as a definition of continuity for functions from one topological space into another. Here again, the definition and the "neighbourhood axioms" are of such a nature that natural counterparts of many classical theorems about continuity can be proved at this more general level.

The advantages of the topological space concept over the metric space concept go far beyond than a process of generality for the sake of generalization. The first successes of the program of isolating certain originally distance-related mathematical concepts in one theory made it natural to hope and expect that This theory might ultimately lie at a very fundamental level in the structure of mathematics as a whole and presuppose only such still more fundamental theories as set theory and its logical prerequisites. A distance function, however, is a function into the real number system, and some of the most essential parts of metric space theory – the proof of the fact that neighborhoods in a metric space satisfy Hausdorff's axioms (A) – (D), for example depend heavily on properties of the real number system. Thus any attempt to put metric space theory ahead of the theory of real numbers on the scale of increasing logical priority would necessarily lead to circularity. The transition to Hausdorff's concept of a topological space, which did not depend on the theory of real numbers or on any other more special mathematical constructs, removed this difficulty and made it possible for the young theory to be put, uncontaminated, in its proper place.

The introduction of topological spaces was therefore a most satisfactory step forward. At one stroke it simplified the theory, widened its scope, and made it as mathematically self-contained as could be expected. This makes it easy to understand the rapid shift in interest, in the years immediately following 1914, from metric space theory to general topology.

There was nevertheless a certain price to be paid for this progress. The successive generalizations preserved many of the features of the original systems, but not all; and useful things were sometimes among the casualties. For instance, it is a routine matter to define the important concept of uniform continuity for functions from one metric space into another, but there is no adequate way of carrying this concept over into the theory of topological spaces in general. A. Weil [17] in 1937 developed the concept of uniform topological space, which allowed the notion of uniform continuity to be carried over to topological spaces endowed with a "uniformity".

Not all improvements or modifications in Hausdorff's definitions were in the direction of greater generality. For instance, no one today defines a topological space in terms of neighbourhoods, as done by Hausdorff. Many substitutes for the Hausdorff's definitions have been proposed, and several do have their respective groups of followers, but whichever definition is adopted in the upshot it is equivalent to what Hausdorff's definition would be without axiom (D), which has proved to be a little too restrictive. Developments in the theory of topological spaces since 1914 have not been confined to skirmishes about definitions, however, and much has been done to enrich the theory itself. Some of the major milestones are mentioned below.

First in historical order was the establishment, in 1920, of the periodical *Fundamenta Mathematicae,* published in Warsaw. The publication of this journal was part of a program for the rejuvenation of Polish mathematics outlined by Z. Janiszewski (Janiszewski himself worked in combinatorial topology), who had suggested that the growth of a strong mathematical tradition in Poland (which reemerged as an independent and sovereign state in 1919 after more than a century of foreign occupation) might best be furthered by choosing one or two fields in which to concentrate at first. One field chosen for this role was the infant theory of topological spaces, and *Fundamenta Mathematicae* was especially hospitable to contributions to this theory. The program was strikingly successful, leading to the rise of a strong school of Polish topologists that included W. Sierpinski, C. Kuratowski, and many others; and it provided topologists all over the world with what was to a large extent their own journal. This golden period of polish mathematics is described in detail by one of its prominent participants, K. Kuratowski in the book : A Half Century of Polish Mathematics, Pergamon Press, 1980.

Polish mathematicians were especially instrumental in the development of functional analysis, which is largely the theory of topological vector spaces. It is legitimate to say that the confluence of general topology with algebra has culminated in modern functional analysis, the subject toward which Pincherle, Fréchet, Riesz and others had been moving at the turn of the century.

The years 1935-1940 were notably fruitful for general topology. It was in this period the concept of convergence in topological spaces began to be clarified by use of the new, and more or less equivalent, theories of nets and filters; the theory of compactifications (that is, extensions of a given topological space that have the important property called "compactness") began to take something like definitive shape; and the important concept of a uniform space, less general than that of a topological space but more general than that of a metric space, was introduced.

The Russian mathematician P.S. Alexandroff and the German mathematician Heinz Hopf (not to be confused with Eberhard Hopf (no relation) of the "Wiener-Hopf Technique" fame). are two great names in topology. While Alexandroff primarily occupied himself with general topology, Hopf's main concern was algebric or combinatorial topology. Beginning in 1924, Alexandroff regularly visited German universities, specially Góittingen and Berlin, where he came in contact with Hopf. In the winter of 1927, both of them visited Princeton University (in USA) where J.W. Alexander, S. Lefschetz and O.Veblen formed a strong group in topology, interested mainly in its algebraic and combinatorial aspects. During the early nineteen thirties they planned to write a book on topology at R. Courant's suggestion, based on lectures and seminars they had given and conducted, sometimes jointly, sometimes separately, at the University of Góttingen (under Courant's able and enlightened leadership the Mathematical

Institute of the University of Göttingen was the undisputed Mecca of mathematicians during those days). They envisaged a three-volume major work covering the entire gamut of algebric, combinatorial and set-theoretic topology. The first part of the book [1] appeared in late 1935. Alexandroff and Hopf's close cooperation was threatened by the rising tense political situation in Europe and was completely broken off by the outbreak of the Second World War in 1939. No further part of this work was ever published but the first part alone is a substantial work which encompasses both general, algebraic and combinatorial topology and remains, to this day, a valuable reference. The book is still available in reprint (Chelsea Publishing Company). In a lengthy introduction [pp. 1-22], the authors give a masterly account of the genesis of combinatorial and algebric topology beginning with Euler's formula for polyhedra.

It was also in these years that a group of young French mathematicians writing under the captivating collective pseudoname of N. Bourbaki began issuing installments of their compendious *Éléments des mathématiques*, with general topology as the subject of Book III of this treatise. Besides giving an account of general topology that will probably be accepted as more or less definitive for some time to come, Bourbaki, by putting the section on topology so early in the treatise – preceded only by Book I on Set Theory and Book II on Algebra – forcibly reminded the mathematical world that when contemporary mathematics is organized in a logically (not necessarily pedagogically or historically) suitable order general topology belongs to a very basic level indeed.

Since 1940 research in general topology has continued to be abundant, and although many of the primary problems have been satisfactorily settled, it is still a living and evolving branch of mathematics.

Chapter 2
Topological Spaces

Topologies, Metrics, Equivalent Metrics

There are several equivalent ways of defining what is meant by a topology on a non-empty set X .The definition universally adopted nowadays is in terms of open sets. We begin by reviewing this definition and discuss the equivalent definitions one by one.

Given a non-empty set X, a non-empty collection $\mathscr{T}$ of subsets of X is called a ***topology*** iff $\mathscr{T}$ is closed under formation of finite intersections and arbitrary unions. Since the union of the empty subfamily of $\mathscr{T}$ is the empty set ∅ and its intersection is X, it follows that every topology $\mathscr{T}$ contains ∅ and X amongst its members. The collection consisting of ∅ and X is clearly a topology on X; it is called the ***indiscrete*** (or, ***trivial***) topology on X. On the other hand the collection consisting of all subsets of X (that is the power set of X) is likewise a topology on X; it is called the ***discrete*** topology. Every other topology on X lies in between these two extreme topologies. Observe that these two topologies coincide if and only if X is a singleton set.

The subsets of X constituting a given topology are called its ***open*** sets. It should be carefully observed that the notion of an open set is tied to a given topology. Thus, for example, in the indiscrete topology, no subset of X other than ∅ and X is open, whereas in the discrete topology every subset of X is open.

A special and very important class of topologies are derived from metrics. A ***metric*** d on a non-empty set is a mapping from X × X into the set of real numbers which is positive definite and symmetric and which satisfies the triangle inequality. Given a metric d on X and a positive real number ε, the set of all points x in X whose "distance" (measured in the metric d) from a given point a in X is less than ε is called the ε-neighbourhood of a. We denote it by $U_\varepsilon(a)$ or $U(a, \varepsilon)$. A subset A of the metric space (X, d) is called ***open*** iff every point $a \in A$ contains a sufficiently small ε-neighbourhood of a. An ε-neighbourhood is itself an open set according to this definition . The family of all open sets of X (thus defined) constitutes a topology on X. It is called the ***metric topology*** induced by d.

Example 1.1 The discrete metric $d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$

induces the discrete topology.

Different metrics on X may induce the same topology on X.

Definition 1.1 Two metrics on X are called ***equivalent*** if and only if they induce the same topology on X.

Example 1.2 Let $d(x, y) = \sqrt{\{(x_1 - y_1)^2 + (x_2 - y_2)^2\}}$,

$$\mu(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

$$\sigma(x, y) = |x_1 - y_1| + |x_2 - y_2|;$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$;

then d, μ, σ are all metrics on $\mathbf{R}^2$ and any two of these are equivalent.

Theorem 1.1 (*Criterion for equivalence of metrics*). Two metrics d, ρ are equivalent if and only if for every point $x \in X$ and every positive ε, there exist two positive numbers δ, η such that

$$U_\rho(x, \delta) \subseteq U_d(x, \varepsilon) \text{ and } U_d(x, \eta) \subseteq U_\rho(x, \varepsilon)$$

Example 1.3 Given a metric d on X,

$$d_1(x, y) = \sqrt{d(x, y)}$$

defines an equivalent metric on X.

Setting $d_{n+1}(x, y) = \sqrt{d_n(x, y)}$, for $n = 1, 2, \ldots,$

we get a sequence of equivalent metrics on X whose limit is the discrete metric on X, because,

$$\lim_{n \to \infty} d_n(x, y) = \lim_{n \to \infty} \sqrt[2^n]{\{d(x, y)\}} = \begin{cases} 0, \text{ for } x = y \\ 1, \text{ for } x \neq y \end{cases}$$

Example 1.4. Given a metric ρ on X,

$$\rho_1(x, y) = \rho(x, y) / \{1 + \rho(x, y)\}$$

defines an equivalent metric on X, in which the distance between any two points is less than 1.

Setting $\rho_{n+1}(x, y) = \rho_n(x, y) / \{1 + \rho_n(x, y)\}$, for $n = 1, 2, \ldots,$ we get a sequence of equivalent metrics on X whose limit, however, is not a metric, because

$$\lim_{n \to \infty} \rho_n(x, y) = 0, \quad \text{for all } x, y \in X.$$

Equivalent Definitions of Topology

Definition 1.2 If X is a topological space and $E \subseteq X$, we say E is closed iff $X \setminus E$ is open.

Theorem 1.2 If $\mathcal{F}$ is the collection of closed sets in a topological space X then

(1) The intersection of any family of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

(2) The union of any finite family of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

In particular, $\varnothing$ and X belong to $\mathcal{F}$.

Conversely, given a non-empty set X and any family $\mathcal{F}$ of subsets of X satisfying (1), (2), the collection of complements of members of $\mathcal{F}$ is a topology on X in which the family of closed sets is precisely $\mathcal{F}$.

Definition 1.3 If X is a topological space and $E \subseteq X$, the ***closure*** of E, denoted by $\bar{E} = cl(E)$ in X is the intersection of all closed subsets of X which contain E. It is the smallest closed set in X which contains E.

Theorem 1.3 Given a topological space X, the mapping $A \to \bar{A}$, called a closure operation of $\mathscr{P}(X)$ into itself, has the following properties:

(1) $E \subseteq \bar{E}$;

(2) $\overline{(\bar{E})} = \bar{E}$;

(3) $\overline{A \cup B} = \bar{A} \cup \bar{B}$;

(4) $\bar{\varnothing} = \varnothing$;

(5) E is closed in X iff $\bar{E} = E$.

Conversely, given a non-empty set X and a mapping $A \to \bar{A}$ of $\mathscr{P}(X)$ having the properties (1) – (4), if we define closed sets in X using (5), then the result is a topology on X whose closure operation is the given mapping $A \to \bar{A}$.

Remark 1.1 A mapping $A \to \bar{A}$ of $\mathscr{P}(X)$ into itself which has the properties (1) – (4), is called a ***Kuratowski closure operation***.

Remark 1.2 The "closed disk" (or closed ε-neighbourhood)

$$\bar{U}(x, \in) = \{y \in X : \rho(x, y) \leq \in\}$$

in a metric space (X, ρ) is a closed set but it need not be the closure of the " open disk" (open ε-neighbourhood)

$$U(x, \in) = \{y \in X : \rho(x, y) < \in\}.$$

Example 1.5 Let X be a set of more than one element and d be the discrete metric on X, then $U(x, 1) = \{x\}$, whereas $U(x, \bar{1}) = X$.

Definition 1.4 If X is a topological space and $E \subseteq X$, ***the interior of E***, denoted by $E^\circ = \text{Int}(E)$ is the union of all open sets contained in E. It is the largest open set contained in E.

The notion of interior and closure are dual to each other, in much the same way as " open" and " closed" arc. Thc formal nature of this duality is brought to light by the relations

$$X \setminus E^\circ = \text{cl}(X \setminus E), \quad X \setminus \bar{E} = (X \setminus E)^\circ$$

Thus any theorem about closures in a topological space can be translated into a theorem about interiors, and vice–versa.

Theorem 1.4 Given a topological space X, the mapping $A \to A^\circ$ of $\mathscr{P}(X)$ into itself, called an interior operation, has the following properties:

(1) $A^\circ \subseteq A$;

(2) $(A^\circ)^\circ = A^\circ$;

(3) $(A \cap B) = A^\circ \cap B^\circ$;

(4) $X^\circ = X$;

(5) A is open in X if and only if $A^\circ = A$.

Conversely, given a non-empty set X and a mapping $A \to A^\circ$ of $\mathscr{P}(X)$ into itself which has the properties (1) – (4), if we define open sets using (5), then the result is a topology on X whose interior operation is the given mapping $A \rightarrow A^\circ$.

Remark 1.3 A mapping $A \rightarrow A^\circ$ of $\mathscr{P}$ (X) into itself which has the properties (1) – (4), is called a ***Tietze interior operation***.

Definition 1.5 If X is a topological space and $E \subseteq X$, the ***frontier of E*** is the set $Fr(E) = cl(E) \cap cl(X \setminus E)$. The frontier of any set is a closed set.

J. Albuquerque : ~ *La notion de ' Frontière' en topologie*, Portugalae Math, **2** (1941), 280-289 [Math.Reviews, **4**, p. 87], has shown that also the notion of a frontier of a set can be used to characterize a given topology.

Here we mention some relations between the frontier, closure and interior operations :

Theorem 1.5 For any subset E of a topological space X,

(1) $\bar{E} = E \cup Fr(E)$;

(2) $E^\circ = E \setminus Fr(E)$;

(3) $X = E^\circ \cup Fr(E) \cup (X \setminus E)^\circ$ is a disjoint union.

Definition 1.6. A ***neighbourhood*** of x in a topological space X is any set U which contains an open set V containing x. In other words, U is a neighbourhood of x iff $x \in U^\circ$. The collection $\mathscr{U}_x$ of all neighbourhoods of x is called the ***neighbourhood system*** at x.

Theorem 1.6 The neighbourhood system $\mathscr{U}_x$ at x in a topological space X has the following properties :

(1) If U $\in \mathscr{U}_x$, then $x \in$ U ;

(2) If U, V $\in \mathscr{U}_x$,then U $\cap$ V $\in \mathscr{U}_x$;

(3) If U $\in \mathscr{U}_x$, then there is a V $\in \mathscr{U}_x$, such that U $\in \mathscr{U}_y$ for each $y \in$ V ;

(4) If U $\in \mathscr{U}_x$ and U $\subseteq$ V, then V $\in \mathscr{U}_x$;

Moreover,

(5) G $\subseteq$ X is open iff G contains a neighbourhood of each of its points.

Conversely, given a non-empty set X and a collection $\mathscr{U}_x$ of subsets of X assigned to each $x \in$ X, having the properties (1) – (4), if we define "open set" using (5), then the result is a topology on X in which the neighbourhood system at each $x \in$ X is precisely $\mathscr{U}_x$.

Remark 1.4 Originally Hausdorff defined a topological space in terms of neighbourhood systems. His motivation was the theory of metric spaces .

Neighbourhood Base, Base and Subbase

Definition 1.7 A ***neighbourhood base*** at x in the topological space X is a subcollection $\mathscr{B}_x$ taken from the neighbourhood system $\mathscr{U}_x$, having the property that each $U \in \mathcal{U}_x$ contains some $V \in \mathcal{B}_x$. That is, $\mathcal{U}_x$ must be determined by $\mathcal{B}_x$ as follows :

$$\mathcal{U}_x = \{U \subseteq X : U \text{ contains for some } V \in \mathcal{B}_x\} \ .$$

Once a neighbourhood base at x has been chosen its elements are called ***basic neighbourhoods*** .

Definition 1.8 Let $(X, \mathcal{T})$ be a topological space, a ***base*** for $\mathcal{T}$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that every $\mathcal{T}$ open set is expressible as the union of a subfamily of $\mathcal{B}$.

That is, $\mathcal{T}$ can be recovered from $\mathcal{B}$ by taking all possible unions of subcollections of $\mathcal{B}$. Moreover, $\mathcal{B}$ is a base for X iff whenever G is an open set in X and $x \in G$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq G$.

Example 1.6 (1) In **R**, the collection $\mathcal{B}$ of all open intervals is a base for the usual topology. More generally, in any metric space X, the collection of all open disks about points of X is a base for the metric topology on X.

(2) The collection of all singleton subsets of X is a base for the discrete topology on X.

Definition 1.9 Suppose (X, $\mathscr{T}$) is a topological space, a ***subbase*** for $\mathscr{T}$ is a collection $\mathscr{S} \subseteq \mathscr{T}$ such that the collection of all finite intersections of elements from $\mathscr{S}$ forms a base for $\mathscr{T}$.

Theorem 1.7 Any collection of subsets of a set X is a subbase for some topology on X.

Subspaces

Definition 1.10 A be a non-empty subset of a topological space (X,$\mathscr{T}$). The family $\mathscr{T}_A$ of all intersections of A with $\mathscr{T}$- open subsets of X is a topology on A; it is called the ***relative topology*** on A or ***the relativization*** of $\mathscr{T}$ to A, and the topological space (A, $\mathscr{T}_A$) is called a subspace of (X, $\mathscr{T}$).

In other words; a subset H of A is a $\mathscr{T}_A$ - open set, if and only if there exists a $\mathscr{T}$ -open subset $G \subseteq X$ such that $H = G \cap A$.

Example 1.7 Given the topology

$\mathscr{T} = \{\varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$ on $X = \{a, b, c, d, e\}$

and the subset $A = \{a, d, e\}$ of X, the relativization of $\mathscr{T}$ to A is

$$\mathscr{T}_A = \{\varnothing, \{a\}, \{d\}, \{a, d\}, \{d, e\}, A\}.$$

Remark 1.5 A subspace of a subspace is a subspace. That is, if $B \subseteq A \subseteq X$, then the relative topology induced on B by the relative topology of A in X is just the relativization of $\mathscr{T}$ to B.

Theorem 1.8 If A is a subspace of a topological space X, then :

(a) $F \subseteq A$ is closed in A iff $F = K \cap A$ where K is closed in X;

(b) If $E \subseteq A$, then $cl_A(E) = A \cap cl_X(E)$;

(c) If $x \in A$, then V is a neighbourhood of x in A iff

$V = U \cap A$, where U is a neighbourhood of x in X ;

(d) If $x \in A$, and if $\mathscr{B}_x$ is a neighbourhood base at x in X, then

$\{B \cap A : B \in \mathscr{B}_x\}$ is a neighbourhood base at x for $\mathscr{T}_A$;

(e) If $\mathscr{B}$ is a neighbourhood base for X, then

$\{B \cap A : B \in \mathscr{B}\}$ is a neighbourhood base for $\mathscr{T}_A$.

Product Spaces

For each $\alpha \in A$, let X_α be a set. The Cartesian product of the family of sets $(X_\alpha)_{\alpha\in A}$ is the set

$$\prod_{\alpha\in A} X_\alpha = \{ x : A \rightarrow \bigcup_{\alpha\in A} X_\alpha : x(\alpha) \in X_\alpha \text{ for each } \alpha \in A \},$$

which we often denote simply by $\prod X_\alpha$. Thus $\prod X_\alpha$ is a set of functions defined on the index set A. The value of $x \in \prod X_\alpha$ at α is denoted by x_α, and x_α is referred to as the α-th coordinate of x. The space X_α is the α-th factor (or, component) space. The map $\pi_\beta : \Pi X_\alpha \rightarrow X_\beta$, defined by $\pi_\beta(x) = x_\beta$, is called the projection map of ΠX_α on X_β. (or, the β th projection map).

When each X_α is a topological space one would wish to have a "suitable" topology on $\prod X_\alpha$ which is both natural and useful. An example of the first requirement is that the product topology on $\mathbf{R} \times \mathbf{R}$ should coincide with the usual topology on $\mathbf{R}^2$. An example of the second requirement is that $\prod X_\alpha$ should have a property P if every component space has the property P. The requirement of the naturality is satisfied, for example, by the ***box topology*** on X_α ; this topology is generated by so-called boxes (with open sides). A box is a subset of $\prod X_\alpha$ of the form $\prod U_\alpha$, where for each $\alpha \in A$, U_α is a non-empty open subset of X_α. The box topology is certainly natural but when the number of component spaces is infinite, this topology contains far too many open sets. It was the Russian mathematician Tychonoff (Tyhonov, in modern transliteration) who found the just the right topology which yield useful results in this situation also.

Definition 1.11 The Tychonoff topology (or, product topology) on ΠX_α is obtained by taking as a *base* for the open sets, sets of the form ΠU_α, where

(1) U_α is open in X_α, for each $\alpha \in A$,

(2) For all but finitely many components, $U_\alpha = X_\alpha$.

Remark 1.6 (1) could have been replaced by

(a) $U_\alpha \in \mathscr{B}_\alpha$, where for each α, $\mathscr{B}_\alpha$ is a fixed base for the topology of X_α.

The set ΠU_α, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$, can be written as

$$\Pi U_\alpha = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}).$$

Thus the product topology is precisely that topology which has for a *subbase* the collection

$$\{ \pi_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \text{ is open in } X_\alpha \}$$

The sets U_α can be restricted to come from fixed base in X_α.

In case only a finite number of spaces $X_1, X_2, \ldots, X_n$ is involved, the product topology on $\prod_{k=1}^{n} X_k$ coincides with the box topology, so in these cases, the product topology will always seem "natural".

Examples 1.8 (1) For each $\alpha \in A$, let X_α be a discrete space. Then ΠX_α will be a discrete space if and only if A is finite.

(2) If $Y_\alpha \subseteq X_\alpha$ for each $\alpha \in A$, then the product topology on ΠY_α coincides with its topology as a subspace of ΠX_α .

Remark 1.7 The Tychonoff topology is the weakest topology on ΠX_α induced by the family of $\{ \pi_\alpha : \alpha \in A \}$ of all projection maps.

Theorem 1.9 A map $f: X \to \Pi X_\alpha$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.

Definition 1.12 Let X be a set and X_α is a topological space with $f_\alpha : X \to X_\alpha$, for each $\alpha \in A$. The ***weak topology*** induced on X by the collection $\{ f_\alpha : \alpha \in A \}$ of functions is the coarsest topology on X which makes each f_α continuous. It is that topology on X for which the sets $f_\alpha^{-1}(U_\alpha)$, $\alpha \in A$ and U is open in X, form a subbase.

Quotient Spaces

Dual to the notion of the weak topology induced on X by a collection of maps $f_\alpha : X \to X_\alpha$, we have the notion of the *strong topology* induced on Y by a collection of maps $g_\alpha : Y_\alpha \to Y$, which is the finest topology on Y making all these maps continuous. In the particular case, when there is only one map $g : X \to Y$, the resulting strong topology on Y is called the ***quotient topology*** induced on Y by g.

Definition 1.13 If X is a topological space, Y is a set and $g : X \to Y$ is an onto mapping, then the collection $\mathscr{T}_g$ of subsets of Y defined by

$$\mathscr{T}_g = \{G \subseteq Y : g^{-1}(G) \text{ is open in } X\}$$

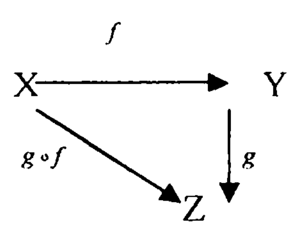
is a topology on Y, called the ***quotient topology*** induced on Y by g. When Y is given some such quotient topology, it is called a quotient space of X, and the inducing map f is called a ***quotient map.***

The quotient topology induced on Y by g is the finest topology on Y making g continuous. The quotient topology can be completely described thus : $F \subseteq Y$ is closed in the quotient topology induced by g iff : $g^{-1}(F)$ is closed in X.

Theorem 1.11 If X and Y are topological spaces and f: X $\rightarrow$ Y is continuous and either open nor closed, then the topology $\mathscr{T}$ on Y is the quotient topology induced by f.

Example 1.9. Consider $X = [0, 2\pi]$ and $Y = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ with their usual topologies, define f: X $\rightarrow$ Y by $f(x) = (\cos x, \sin y)$. Then f is continuous and closed; so the unit circle with its usual topology is a quotient space of the closed interval $[0, 2\pi]$ with its usual topology.

Theorem 1.12 Let Y have the quotient topology induced by a map f of X onto Y . Then an arbitrary map g : Y $\rightarrow$ Z is continuous iff $g \circ f$: X $\rightarrow$ Z is continuous.



Historical Note

The definition of a topological space in terms of open sets originated with H. Tietze : ~ Beiträge zur allgemeinen Topologie I, Math. Annalen. **88** (1923), 290-312. and P. Alexandroff : Zur Begründung der n-dimensionalen mengentheoretischen Topologie, Math. Annalen **94** (1925), 296-308.

The definition of a topological space via closed sets is merely a mater of set-theoretic duality as manifested in De Morgan's laws.

The definition of a topological space in terms of the closure operation is due to C. Kuratowski : Sur l'opération $\bar{A}$ de l'Analysis Situs, Fund. Math. **3** (1922), 182-199.

The use of the word "topology" to denote the collection of open sets of a topological space, now widespread, is a surprisingly recent development. According to Thron [16], it was Kelley [8] who first used the word "Topology" in this sense.

The box topology appears in Tietze : Über Analysis Situs, Abhandlugen der math. Seminar der Univ. Hamburg, **2** (1923), 27-70.

The Tychonoff topology was introduced by A. Tychonoff : Über die topologische Erweiterung von Räumen, Math. Annalen, **102** (1930), 544-561.

The quotient topology is due to R. L. Moore : Concerning Upper Semi-Continuous Collections of Continua, Trans. Amer. Math. Soc. **27** (1925), 416-428. and P. Alexandroff : Über stetige Abbildung kompakter Räume, Math. Annalen, **96** (1926), 555-571.

Chapter 3

Separation Axioms

Most topological spaces possess special properties which go beyond the basic requirements of a topology. For example, every metric topology has the property that, if x and y are two distinct points then there exist disjoint open sets A, B such that $x \in A$ and $y \in B$. Indeed, this motivated Hausdorff to demand this property as a requirement for his definition of a topology in terms of neighbourhoods. A topological space possessing this property is called a ***Hausdorff space*** and the property is an instance of a "separation axiom". A separation axiom is a property demanded of a topology that assures the possibility of "separating" certain types of subsets of a topological space.

The nomenclature " separation axiom" (*Trennungsaxiom* in German) is due to Tietze (1923). Alexandroff and Hopf (1935) in their book "Topologie"[1] used the letter T with subscripts to denote the various separation axioms. In this chapter, we give an overview of the various separation axioms currently in use; later we shall discuss some lesser known separation axioms in connection with our study of compactness.

Chatterjee-Ganguly-Adhikari [4, p.117] formulate the separation axioms using the notion of weakly and strongly separated sets.

T_o - Space

We call a topological space a ***T_0 - space*** if, for each pair of distinct points x, y there is a neighbourhood of x not containing y, or a neighbourhood of y not containing x.

The T_o- axiom is equivalent to the requirement that two distinct points always have distinct closures.

Example 3.1 Let X be a set of more than one element with the indiscrete topology. Then X is not a T_o - space.

Example 3.2 Let X = { a, b } be a set of two elements with the topology $\mathscr{T} = \{ \varnothing, \{ a \}, X \}$. Then X is a T_o - space.

Theorem 3.1 Subspaces and products of T_o - spaces are T_o - spaces.

Comments on the formulation of the Separation Axiom T_o

A topological space X is a T_o - space iff given any two distinct points of X, there is a neighbourhood of *at least* one point which does not contain the other.

The formulation of T_o - axiom as found in some books is open to misinterpretation. It is of course permitted that in a T_0 – space, *each* of two distinct points may have a neighbourhood containing one point but not the other. So the formulation in [18, p. 46] or in [16, p. 91] using the phrase "either or" is misleading. Even in the carefully written text of Willard [20, p. 85] the formulation "whenever x and y are distinct points in X, there is an open set containing one and not the other", is subject to misinterpretation, in as much as it might be construed as demanding the existence of such neighbourhood for each of the two points. Authors who enunciate the T_0 - axiom without ambiguity include Alexandroff & Hopf [1, p.58], Hocking and Young [7, p. 37], Gemignani [5, p. 83] and Chatterjee-Ganguly-Adhikari [4,p. 118].

Remark 3.1 The T_o - separation axiom is sometimes named after ***Kolmogoroff*** [1, p. 58].

T_1 – Space

We call a topological space a ***T_1 – space*** if, for each pair of distinct points x, y in X, there is a neighbourhood of x not containing y and a neighbourhood of y not containing x (briefly: for every pair of distinct points each point has a neighbourhood not containing other).

Example 3.3 The cofinite topology on X makes it a T_1 - space, for if $x \neq y$, then $\{ y \}^c$ is an open set containing x but not y and $\{ x \}^c$ is an open set containing y but not x.

Remark 3.2 Every T_1- space is a T_o- space but the converse is not true in general.

Example 3.4 X = { *a*, *b* } with the topology $\mathscr{T}$ = { ∅, { *a* }, X } is a T_o - space but not a T_1 – Space.

Theorem 3.2 Subspaces and products of T_1 – spaces are T_1 – spaces.

The T_1 – axiom is equivalent to the requirement that every singleton subset of X coincides with its closure; hence to the requirement that every singleton subset is closed [9, p. 90]. We further observe that the only T_1 – topology on a finite set X is the discrete topology.

The T_1 – separation axiom is often named after Fréchet. A T_1 – space is sometimes called a Riesz space. Čech (1938) showed that the theory of general topological spaces can be reduced to "the theory of Kolmogoroff spaces", that is T_0 – spaces. This is done by identifying points having coincident closures. See also Thron [16, p. 91], who attributes the result to M. H. Stone.

Separation axioms between T_0 and T_1

Bhattacherjee-Ganguly-Adhikari [4, p. 106] call a topological space a T_D-space if the derived set of every singleton set is closed. This axiom lies strictly between T_0 and T_1 as shown by examples give there (p. 157). They also prove that a topological space is a T_D- space if and only if every derived set is closed in X. Such spaces were studied as early as 1911 by E. R. He-drick : On Properties of a domain for which any derived set is closed, Trans. Amer. Math. Asoc. **12** (1911), 285-294. It is therefore justified to call them He-drick spaces.

C. E. Aull and W. J. Thron : ~ Separations axioms between T_0 and T_1, Indag. Math, **24** (1963), 26-37; Math Reviews, **25** (1963), 304-305,(# 1529) have given as many as seven separation properties which lie between T_0 and T_1; these properties however arranged in order of increasing strength.

T_2 - space (Hausdorff space)

We call a topological space X a T_2 - space (or, Hausdorff space), iff whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

Theorem 3.3 Every metric space is Hausdorff.

Theorem 3.4 Every subspace of a T_2 - space, and every product of T_2 - spaces are T_2 - spaces.

Remark 3.3 A Hausdorff space is a T_1- space but a T_1- space need not be Hausdorff.

Separation axioms between T_1 and T_2

In chapter 5, we discuss two separation axioms which lie between T_1 and T_2, because one of them involves the notion of compactness and implies the other.

Regular and T_3 – spaces

We call a topological space X is a regular space iff whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$.

In other words, a topological space is regular iff the set of closed neighbourhoods of any point is a local base at that point.

The "good" topological spaces are those which are regular, in the sense that many useful results hold for regular spaces. The topology on a regular space X may no longer reflect the set-theoretic character of X. For example, a trivial (indiscrete) space is regular; thus a regular space need not be Hausdorff or even T_1; nor a Hausdorff space needs to be regular [20, p. 92]. To remedy this unpleasant situation, we define a T_3-space to be a regular T_1 space (see Example below). Since in a T_1-space, every singleton set is closed it follows that, every T_3-space is T_2.

Example 3.5 A topological space is a T_3-space if it is a regular T_1- space.

Example 3.6 A Hausdorff space need not be regular. Let X be the real line with neighbourhoods of any non-zero point being as in the usual topology, while neighbourhoods of 0 will be sets of the form U\A, where U is a neighbourhood of 0 in the usual topology and A = { $1/n$: n = 1, 2, ...}. Then X is Hausdorff since this topology is finer than the usual topology, which is Hausdorff. But A is closed in X and cannot be separated from 0 by disjoint open sets, so X is not regular.

Remark 3.4 Chatterjee-Ganguly-Adhikari [4] treat regular space and T_3 - space as synonymous.

Alexandroff and Hopf formulate the third separation axiom as follows and name it after L. Vietoris : every pair of disjoint closed sets , of which one is a singleton set, possess disjoint neighbourhoods . Since a point of a topological space need not be a closed set, the axiom is not a sharpening of the first separation axiom. They define a T_3 - space on a regular space as a T_1 - space, which satisfies this third separation axiom.

Gamignani calls the separation property defining a regular space T_3 and calls a topological space regular if it is both T_3 and T_1 and adds following parenthesis:

1. A space X is said to be T_3 if given any closed subset F of X and any point x of X which is not in F, there are open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.
2. A space X is said to be regular if X is both T_3 and T_1.

Remark 3.6 A normal space need not be regular [20, p. 100], but the addition of the T_1-axiom to normality ensures that every T_4-space is T_3, hence in particular regular.

Theorem 3.6 Every metric space is normal.

An overriding property of a normal space is contained in a famous and very important theorem known as Urysohn's Lemma.

Theorem 3.7 (***Urysohn's Lemma***) Given disjoint closed sets A, B in a normal space X, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. The converse of this theorem is also true but of little importance. Urysohn's lemma, on the other hand, ranks among the greatest theorems in topology, since it provides, starting from scratch, a bare-hands construction of a continuous function where none was assumed to exist.

Remark 3.7 Alexandroff & Hopf calls the separation property defining a normal space the fourth separation axiom and names if after Tietze.

Separation axioms between T_3 and T_4
(Completely regular space and Tychonoff space or $T_{3\frac{1}{2}}$-space)

We call a topological space X is a ***completely regular*** iff whenever A is a closed set in X and $x \in A$, there is a continuous function $f: X \to I = [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

Remark 3.8 By Urysohn's lemma, every T_4-space is completely regular.

A completely regular T_1 - space is called a ***Tychonoff space*** or a $T_{3\frac{1}{2}}$***-space.***

Every completely regular space is regular. For suppose A is closed, $x \notin A$, and $f : X \to I$ is a continuous function with $f(x) = 0$ and $f(A) = \{1\}$. Then, $f^{-1}([0, \frac{1}{2}])$ and $f^{-1}([\frac{1}{2}, 1])$ are disjoint open sets in X containing x and A, respectively. But a completely regular space need not be Hausdorff , as any indiscrete space of more than one point illustrates; and this is the reason Tychonoff spaces enjoy a separate identity.

Theorem 3.8 Every subspace of a completely regular (or Tychonoff) space is completely regular (respectively, Tychonoff).

Theorem 3.9 A product space is completely regular iff each factor space is completely regular.

Completely normal space or T_5 – space

Willard [20, p. 105] calls a space is completely normal iff every subspace is normal. Chatterjee-Ganguly-Adhikari [4] defines the concept in terms of weak and strong separation properties.

Definition 3.1 (Weak and strong separation) Let $(X, \mathscr{T})$ be a topological space. Two non-empty subsets A and B are said to be

(i) ***weakly separated*** in $(X, \mathscr{T})$ if there exist two open sets U and V, such that $A \subseteq U$, $B \subseteq V$, $A \cap V = \varnothing$, and $B \cap U = \varnothing$;

(ii) ***strongly separated*** in $(X, \mathscr{T})$, if there exist two open sets G and H, such that $A \subseteq G$, $B \subseteq H$, and $G \cap H = \varnothing$.

Definition 3.2 X is called a completely normal space iff every pair of weakly separated sets is strongly separated.

These two definitions are equivalent [4, p. 145].

A ***T_5-space*** is a completely normal T_1-space.
We have the implications

$$T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

In general none of these implications is reversible.

As already indicated, the terminology in the literature with respect to the separation axioms – beyond Hausdorff's – is more than a little confused. Some writers interchange our usage, using T_3, Tychonoff and T_4 for those spaces, which need not to T_1 (and regular, completely regular and normal for those that are). Others use T_3 and regular to mean the same thing, which sometimes means it includes T_1 and sometimes not (and likewise for Tychonoff and completely regular, and T_4 and normal).

Countability and separability

Countability and separability are two properties which do not, strictly speaking, fall under separation axioms. But they are closely related and this is the natural place to discuss these properties.

A topological space in which every point has a countable neighbourhood base is said to satisfy the first axiom of countable or to be first countable.

(X, $\mathscr{T}$) is second countable (or satisfies the second axiom of countability) iff its topology has a countable base.

Remark 3.9 Every second countable space is first countable.

Theorem 3.10 An uncountable discrete space is first countable but not second countable.

Theorem 3.11 The continuous open image of a second countable space is second countable.

Remark 3.10 One should note carefully that "X is first or (second) countable" means something different from "X is countable".

Definition 3.3 Two topologies $\mathscr{T}$ and $\mathscr{T}'$ on the same set X are called sequentially equivalent iff they determine the same family of convergent sequences. That is,

$$x_n \to x \text{ in } \mathscr{T} \iff x_n \to x \text{ in } \mathscr{T}^{\prime}.$$

The topology of a first countable space is uniquely determined by the family of its convergent subsequences. In other words, we have the

Theorem 3.12 Let $\mathscr{T}$, $\mathscr{T}^{\prime}$ be first countable topologies on X. Then $\mathscr{T} = \mathscr{T}^{\prime}$ if and only if $\mathscr{T}$ and $\mathscr{T}^{\prime}$ are sequentially equivalent.

Definition 3.4 A subset D of a topological space X is called ***dense*** (in X) iff $cl_X(A) = X$.

Definition 3.5 A topological space X is ***separable*** iff X has a countable dense subset.

Example 3.8 The real line with the usual topology is separable, since the set of rational numbers is a dense subset.

Example 3.9 A discrete space is separable iff it is countable.

Theorem 3.13 A continuous image of a separable space is separable.

Remark 3.11 Subspaces of separable spaces need not be separable. However, an open subspace of a separable space is separable.

Theorem 3.14 A metric space is separable if and only if it is second countable [4, p. 255].

Historical Notes

The T_1-axiom is sometimes named after F. Riesz : ~ Stetigkeitsbegriff und abstrakte Mengenlehre, Atti IV Congr. Internat. Mat. Roma **2** (1908), 18-24.

Regular spaces were first introduced by L. Vietoris : ~ Stetige Mengen, Monatsh. Math. **31** (1921), 173-204. Their importance was stablished by Tychonoff (1929) who proved that every Tychonoff space is homeomorphic to some subspace of a cube (any product of a family of closed and bounded intervals). The name Tychonoff space was suggested by Tukey : ~ Convergence and Uniformity in Topology, Annals of Math, **2** (1940), p.84 [Math. Review 2, p. 67].

The T_4-axiom was introduced by H. Tietze : ~ Beiträge zur allgemeinen Topologie I, Math. Annalen **88** (1923), 290-312.

Urysohn's lemma was indeed, proved by Urysohn : ~ Über die Mächtigkeit der zusammenhängenden Mengen, Math. Annalen **94** (1925), 262-295.

Completely normal was introduced by Tietze (in the paper mentioned above).

The first and second axioms of countability were introduced by Hausdorff in his *Grundzüge* [6].

The notion of separability is due to Fréchet (in the paper mentioned on p.2).

Chapter 4
Convergence in Topology

Sequence

A sequence (x_n) in a topological space X is said to converge to $x \in X$, and we write $x_n \to x$, iff for each neighbourhood U of x, there is some positive integer n_0 such that $n \geq n_0$ implies $x_n \in U$. Briefly, $x_n \to x$ iff (x_n) is eventually in every neighbourhood U of x.

It is clear that we can replace "neighbourhood" with "basic neighbourhood" in this definition without altering its impact.

Example 4.1 Let ρ be a metric on X. Then $x_n \to x$ in the topology induced by ρ iff $\lim_{n \to \infty} \rho(x_n, x) = 0$.

For metric spaces, the following results are well-known.

Theorem 4.1

1. $x \in X$ is a cluster point of $A \subseteq X$ if and only if there is a sequence (x_n) in A which converges to x.
2. $A \subseteq X$ is open in X if and only if whenever (x_n) is a sequence converging to a limit in A, then x_n is eventually in A.
3. $A \subseteq X$ is closed if and only if whenever (x_n) is a sequence converging to a limit $x \in X$, then $x \in A$.

4. $f : X \to Y$ is continuous at x_0 if and only if whenever $x_n \to x_0$ in X, then $f(x_n) \to f(x_0)$ in Y.

In view of this theorem we can say that the topology of a metric space is completely determined by its family of convergent sequences

We now give an example to show that two different topologies on the same set X may determine the same family of convergent sequences.

Let X be any uncountable set, $\mathcal{T}_1$ be the discrete topology on X and $\mathcal{T}_2$ be the cocountable topology on X. A sequence (x_n) in X is convergent to $x_0 \in X$ in either topology if and only if there exists n_0 such that $x_n = x_0$ holds for all $n \geq n_0$ [18, p. 27, Example 3].

Remark 4.1 For a first countable space Theorem 4.1 holds verbatim [20, p. 71]. In other words, the topology of a first countable space is completely determined by sequential convergence.

The fact that sequences are in general inadequate to determine a given topology, led mathematicians to more general theories of convergence suited to general topological spaces; one such theory is based on a notion of a net, which generalizes the concept of a sequence. The other theory is based on the notion of a filter, a concept that is defined from distilling the set-theoretic properties of the family of neighbourhoods of a point.

The two approaches to a generalized theory of convergence in a topological space appear at first to be distinct, but ultimately they turn out to be equivalent. Nets resemble sequences strongly, and are handier to use in discussion of continuity of functions, and algebraic operations, while filters are preferable in dealing with compactness and completeness. It is therefore important to know both approaches and make use of nets or filters according to expediency.

Nets

Definition 4.1 A set A is a directed set iff there is a relation $\leq$ on A such that

(i) $\alpha \in \alpha$, for each $\alpha \in A$;

(ii) $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_3$ imply $\alpha_1 \leq \alpha_3$;

(iii) Given $\alpha_1, \alpha_2 \in A$ then there is some $\alpha_3 \in A$ with $\alpha_1 \leq \alpha_3$, $\alpha_2 \leq \alpha_3$.

In other words, each pair of elements of A has a common upper bound in A.

Definition 4.2 A ***net*** in a set X is a function $P : A \to X$ on any directed set A with values in X. The point $P(\alpha)$ is usually denoted x_α, and we speak of the ***net*** $(x_\alpha)_{\alpha\in A}$ or simply the net (x_α).

Definition 4.3 A ***subnet*** of a net $P: A \to X$ is the composition $P \circ \varphi$, where $\varphi : M \to A$ is an increasing cofinal function from a directed set M to A. That is,

(a) $\varphi(\mu_1) \leq \varphi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (φ is increasing);

(b) for each $\alpha \in A$, there is some $\mu \in M$ such that $\alpha \leq \varphi(\mu)$ (φ is cofinal in A).

For $\mu \in M$, the point $P \circ \varphi(\mu)$ is often written x_α, and we usually speak of " the ***subnet*** (x_α) of (x_α)".

If (x_α) is a net in X, a set of the form $\{x_\alpha : \alpha \geq \alpha_0\}$, for $\alpha_0 \in A$, is called a ***tail*** of (x_α).

Remark 4.2 The set **N** of positive integers is a directed set when given its usual order. Thus, every sequence (x_n) is a net. We also note that every subsequence of a sequence (x_n) is a subnet of the net (x_n). But the converse is not true in general; there is no guarantee that a subnet of (x_n) is a subsequence, because it may not be a sequence at all, for a subnet can have a much richer index set then the original sequence.

Definition 4.3 Let (x_α) be a net in a topological space X. Then (x_α) ***converges to*** $x \in X$ written $x_\alpha \to x$, iff for each neighbourhood U of x, there is some $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$. Thus $x_\alpha \to x$ iff each neighbourhood of x contains a tail of (x_α).
This is sometimes expressed by saying : (x_α) converges to x iff it is eventually in every neighbourhood of x.

Definition 4.4 x is a ***limit point*** of (x_α) iff for each neighbourhood U of x and for each $\alpha_0 \in A$ there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. This is sometimes expressed by saying : (x_α) has x as a limit point iff (x_α) is frequently in each neighbourhood of x.

Remark 4.3 Some authors (e.g. Willard [20]) use the term cluster point for a limit point; but we prefer to use the term cluster point only in relation to subsets of a topological space.

Example 4.2 (The neighbourhood net determined by a point) A fixed neighbourhood base $\mathscr{B}_x$ of $x \in X$, becomes a directed set under the relation $U \leq V$ iff $U \supseteq V$. For each $U \in \mathscr{B}_x$ we choose a point $x_U \in U$; then (x_U) is a net in X called ***the neighbourhood net*** at x.

Theorem 4.2 If a net (x_λ) converges to x, then every subnet of (x_λ) converges to x.

Theorem 4.3 A net (x_n) has x as a limit point if and only if it has a subnet which converges to x.

Corollary 4.1 If a subnet of (x_α) has a x as a limit point., so does (x_α).

The analogue of Theorem 4.1 holds if the word "sequence" is replaced by "net".

Therefore, the topology (of any topological space) is uniquely determined by the family of its convergent nets. It is to be carefully noted that the domain of such nets can be, and must allowed to be, any directed set of any cardinality.

Definition 4.5 A net (x_α) in a set X is an ***ultranet (universal net)*** iff each subset E of X, (x_α) is either eventually in E or eventually in X \ E [20, p .76].

It follows from the definition that if an ultranet is frequently in E then it is eventually in E, in particular, an ultranet in a topological space must converge to each of its limit points.

Theorem 4.5 If (x_λ) is an ultranet in X and f: X → Y, then $(f(x_\lambda))$ is an ultranet in Y.

Filter

Definition 4.6 A non-empty collection $\mathcal{F}$ of subsets of a set X is called a *filter* in X if

(i) $\varnothing \in \mathcal{F}$;

(ii) $A \in \mathcal{F}$ and $B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) $A \in \mathcal{F}$ and $B \supseteq A$ imply $B \in \mathcal{F}$.

Observe that (iii) implies $X \in \mathcal{F}$.

Example 4.3 Let X be a topological space and $x \in X$. Let $\mathscr{U}_x$ be the collection of all neighbourhoods of x. Thus $\mathscr{U}_x$ is a filter, called the ***neighbourhood filter*** at x.

Definition 4.7 Given a filter $\mathscr{F}$ in a topological space X, we say that $\mathscr{F}$ ***converges*** to x, in symbols, $\mathscr{F} \to x$, iff $\mathscr{F}$ induces the neighbourhood filter at x.

The neighbourhood filter at x obviously converges to x, for any $x \in X$.

Definition 4.8 A non-empty collection of $\mathscr{B}$ of non-empty subsets of an abstract set X is called ***filter base*** in X iff the intersection of two sets in $\mathscr{B}$ contains a set in $\mathscr{B}$.

Thus a filter is itself a filter base. Conversely, if $\mathscr{B}$ is a filter base in X, then the collection of all subsets of X which contains a set in $\mathscr{B}$ is seen to be a filter called the filter generated by $\mathscr{B}$.

Example 4.4 Any neighbourhood base at $x \in X$ is a filter base for the neighbourhood filter at x.

Example 4.5 Let X be any set, $A \subseteq X$. Then $\{F \subseteq X : A \subseteq F\}$ is a filter on X, for which the singleton family $\{A\}$ is a filter base.

Definition 4.9 (i) A filter $\mathscr{F}$ in a topological space has $x \in X$ as a cluster point iff every neighbourhood of x meets every member of $\mathscr{F}_x$ (that is, $U \cap A \neq \varnothing$ holds for every $U \in \mathscr{U}_x$ and $A \in \mathscr{F}$).

(ii) The filter $\mathscr{F}$ is said to converges to x iff $\mathscr{U}_x \subseteq \mathscr{F}$.

Theorem 4.6 $\mathscr{F}$ has x as a cluster point iff there is a filter finer than $\mathscr{F}$ (i.e. a filter which contains $\mathscr{F}$) which converges to x.

The analogue of theorem 4.1 holds verbatim for filter convergence.

Theorem 4.7 A filter $\mathscr{F}$ converges to x_o in ΠX_α iff $\pi_\alpha(\mathscr{F}) \to \pi_\alpha(x_o)$ in X_α, for each α.

Definition 4.10 A filter $\mathscr{F}$ is an ***ultra filter*** iff there is no strictly finer filter than $\mathscr{F}$.

Theorem 4.8 A filter $\mathscr{F}$ on X is an ultra filter iff for each $E \subseteq X$, either $E \in \mathscr{F}$ or $X \setminus E \in \mathscr{F}$

Theorem 4.9. Every filter $\mathscr{F}$ is contained in some ultra filter.

Theorem 4.10 ff f maps X onto Y and $\mathscr{F}$ is an ultra filter on X, then $f(\mathscr{F})$ is an ultra filter on Y.

Definition 4.11 Given is a filter $\mathscr{F}$ on X, we choose a point $x \in F$ for every $F \in \mathscr{F}$. Let for any $A = \{(x, F)\}$; A besomes a directed set under the relation : $(x_1, F_1) \leq (x_2, F_2)$ iff $F_2 \subseteq F_1$; so the map $P : A \to X$ defined by $P(x, F) = x$ is a net in X. It is called the ***net based*** on $\mathscr{F}$

Definition 4.12 Given a net $(x_\alpha)_{\alpha\in A}$ in X. Let $\mathscr{B}$ be the set of all subsets of X of the form $\mathscr{B}_\gamma = \{x_\alpha : \gamma \leq \alpha\}$ for each $\gamma \in A$, then $\mathscr{B}_\gamma$ is a filter base in X; it generates the filter

$$\mathscr{F} = \{F \subseteq X : F \supseteq \mathscr{B}_\gamma \text{ for some } \gamma \in A\},$$

It is called the ***filter generated*** by the net (x_α).

Theorem 4.11 (1) The filter generated by a net in X converges to x if and only if the net in X converges to x.

(2) Every net based on a filter $\mathscr{F}$ in X converges to $x \in X$ if and only if the filter converges to x.

In view of this theorem, we can say that net convergence and filter convergence are equivalent Theories.

Historical note

E. H. Moore and H. L. Smith developed a "General Theory of Limits" in their paper of the same title in American Journal of Mathematics **44** (1922), 102-121. Garrett Birkhoff (not to be confused with his father, the eminent mathematician G. D. Birkhoff (1882-1944)), recast the theory in its present form in his paper "Moore-Smith Convergence in General Topology". Ann. Math. **38** (1937), 39-56.

The theory of filters is due to H. Cartan (not to be confused with his father, the eminent mathematician E. Cartan (1874-1956)), "Théorie des Filtres" and "Filtres et Ultrafiltres" C. R. Acad. Sci. Paris **205** (1937), 595-598 and 777-779. Our account of the theory is based on Willard [20].

Chapter 5

Compactness

Historically the concept of compactness arose from considerations of accumulation points of infinite sets, and from the possibility of selecting convergent subsequences of sequences. Here we have in mind the Theorem of Bolzano–Weierstrass (every infinite bounded set of real numbers has an accumulation, or cluster, point) and the Theorem of Arzela-Ascoli (every equi continuous and bounded family of functions on a closed and bounded interval contains a uniformly convergent subsequence).

The immediate motivation for the present-day notion of compactness is to be found in Borel's covering theorem (1895), to the effect that every infinite open cover of a closed and bounded interval has a finite subcover, Heine (1872) had, following Weierstrass, introduced the notion of uniform continuity and had proved that every pointwise continuous function on a closed and bounded interval is uniformly continuous. This theorem is an immediate consequence of Borel's theorem. For this reason Borel's covering theorem is more commonly known as the Heine-Borel theorem. The Heine–Borel theorem generalizes to $\mathbf{R}^n$.

Among infinite sets encountered in analysis, compact sets are definitely the most well-behaved. All finite sets are compact, and all infinite compact sets retain many of the properties of finite sets, particularly if some suitable hypothesis of an analytical nature is added. This point is well illustrated by Edwin Hewitt in his instructive essay " The Role of Compactness in Analysis", American Math. Monthly 67 (1960), 499-596.

The term compact was coined by Fréchet (1906). He applied it to abstract spaces in which every infinite sequence of points contains a convergent subsequence; or, equivalently in which every infinite set has an accumulation, or a cluster, point. Applied to general topological spaces, this condition defines the sequentially compact spaces and the countably compact spaces, respectively. Hausdorff was the first to notice that the present-day definition in terms of existence of finite subcovers of open covers, is equivalent to the above definitions in the context of metric spaces.

Hausdorff thus adopted the conclusion of the Heine-Borel on closed and bounded subsets of the Euclidean space $\mathbf{R}^n$ as the definition of compactness in metric spaces. It was left to Alexandroff and Urysohn to apply this definition to general topological spaces; they called such spaces ***bicompact***. Bicompactness won over countable compactness and sequential compactness when Tychonoff (1935) proved that this property was preserved in the passage to arbitrary topological products of compact spaces. This conclusion fails in general for sequential compactness as well as for countable compactness. The term bicompactness remained in use throughout the nineteen thirties. Bourbaki took the lead in replacing the term bicompact by compact; he additionally demanded that a compact topological space should necessarily be Hausdorff. Nowadays Bourbaki's terminology is the accepted one, except that the requirement of being Hausdorff is dropped by most authors.

Four types of compactness

Definition 5.1 A topological space $(X, \mathscr{T})$ is called ***compact*** if every open cover has a finite subcover. A subset $S \subseteq X$ is called compact if and only if the subspace $(S, \mathscr{T}_S)$ is compact (in the relative topology).

Since every $\mathscr{T}_S$-open set has the form $A \cap S$, where $A \subseteq X$ is $\mathscr{T}$- open and $\bigcup_{i \in I} (A_i \cap S) = (\bigcup_{i \in I} A_i) \cap S$, it follows that S is compact if and only if every covering of S by $\mathscr{T}$- open sets has a finite subcover. Thus, in dealing with compactness of subsets, the relative topology does not come into the picture directly.

Theorem 5.1 X is compact if and only if every family of closed sets having the finite intersection property has a non-empty intersection. (That means, if $(A_i)_{i \in I}$ is a family of closed sets such that every finite subfamily has non-empty intersection, then $\bigcap_{i \in I} A_i \neq \varnothing$).

This result is useful in proving many theorems; for example;

Theorem 5.2 Every closed subspace of a compact space is compact.

In the definition of compactness the index set I of an open cover can be any arbitrary infinite set; in particular it may be uncountable – indeed its cardinality can exceed that of the continuum. If we restrict the index set I to be countable, we get the class of countably compact topological spaces. Two other types of compactness are obtained by taking as definitions the conclusions of the Bolzano-Weierstrass theorem for sets and sequences of real numbers.

Definition 5.2 A topological space (X, $\mathscr{T}$) is said to be

(i) ***countably compact***, if every countable open cover of X has a finite sub cover.

(ii) ***sequentlally compact***, if every infinite sequence in X contains a convergent subsequence (or equivalently has a limit point in X).

(iii) ***Fréchet compact*** (or ***Bolzano-Weierstrass compact***), if every infinite subset of X has an accumulation (or cluster) point.

As before, these definition apply to a subset of X with respect to the relative topology. As such the analogue of Theorem 5.2 (on closed subsets) continues to hold for these three types of compactness.

Of the four type of compactness mentioned above, sequential compactness and Fréchet compactness are considered to be less important. Also, properties of a countably compact space are almost similar to the corresponding properties of a compact space, with the limitation of countability. In order to distinguish the first defined notion of compactness from this and other types of compactness, we shall sometimes refer to it as Borel compactness.

Relations among the four types of compactness

Some relations between the four types of compactness are mentioned below. There are only a few implications that hold in general. But if we impose addition condition(s) on X (e.g. X is first, or second countable), then additional implications hold. We mention below some of these implications [4, pp. 219-221].

Theorem 5.3 (1) Every compact, or sequentially compact, space is countably compact.

(2) Every countably compact space is Fréchet compact.

Theorem 5.4 (1) Every countably compact, or Fréchet compact, first countable space is sequentially compact.

(2) Every Fréchet compact T_1 space is countably compact.

Theorem 5.5 For a second countable T_1-space, the four types of compactness are mutually equivalent.

Theorem 5.6 For a metric space, Borel compactness, Fréchet compactness and sequential compactness are mutually equivalent.

Remark 5.1 Lipschutz [10, p. 157] calls X countably compact if X is Fréchet compact in our terminology.

Remark 5.2 A metric space need not be second countable (or even separable) [10, p. 133, Examples 3,4]. So for a metric space in general Borel compactness and countable compactness need not be equivalent. However, for a second countable metric space they are equivalent.

Lindelöf Space

A property closely related to compactness is known as the Lindelöf property.

Definition 5.3 X is called a ***Lindelöf space*** if every open cover of X has a countable subcover.

Remark 5.3 The spelling "Lindeloff" used in Chatterjee-Ganguly-Adhikari [4] is unwarranted.

Theorem 5.7 (1) Every second countable space is a Lindelöf space. In particular $\mathbf{R}^n$ (with the usual topology) is a Lindelöf space.

(2) Every countably compact Lindelöf space is compact.

Remark 5.4 1. **R** with the cofinite topology is Lindelöf but not first countable.

2. A second countable (hence Lindelöf) Fréchet compact space need not be countably compact [4, p. 221].

The importance and usefulness of net and filter convergence is perhaps most convincingly demonstrated by the following result.

Theorem 5.8 For a topological space X, the following statements are equivalent :

(a) X is compact.

(b) each filter in X has a cluster point.

(c) each net in X has a limit point.

(d) each ultranet in X converges.

(e) each ultrafilter in X converges.

A very important property of compactness is the following :

Theorem 5.9 A continuous image of a compact space is compact.

Remark 5.5 An analogous result holds for countable compactness and sequential compactness, but not for Fréchet compactness.

Compactness in Hausdorff space

A compact subset of a topological space need not be closed [4, p. 211, Example 2]. However, this is true if X is Hausdorff. Indeed, Hausdorff spaces have very pleasant properties relating to compactness. That is why Bourbaki makes the separation axioms T_2 a prerequisite for compactness.

Theorem 5.10 (1) Every compact subset of a Hausdorff space is closed.

(2) Disjoint compact subsets of Hausdorff can be separated by disjoint open sets.

(3) Every compact Hausdorff space is normal (hence a T_4 space).

Tychonoff's Theorem

Suppose $(X_\alpha, \mathscr{T}_\alpha)_{\alpha \in A}$ any non-empty family of topological spaces if the Cartesian product $X = \prod_{\alpha \in A} X_\alpha$, endowed with the product (Tychonoff) topology, is compact then every component space X_α must be compact, because each projection map $\pi_\alpha : X \to X_\alpha$ is continuous. Of utmost importance is the fact that the converse holds.

Theorem 5.11 (Tychonoff) If $(X_\alpha, \mathscr{T}_\alpha)_{\alpha \in A}$ is any non-empty family of topological spaces the the product space $X_\alpha = \prod X_\alpha$ is compact in the product topology.

Remark 5.6 Tychonoff's theorem has been designated as "the most important theorem in general topology" [20, p. 120]. Now, the very definition of the Cartesian product of an uncountable infinite family of sets presupposes the "notorious" but indispensable axiom of choice first formulated explicitly by Zerlomc . Tychonoff's theorem is thus an indirect consequence of the axiom of choice. J. L. Kelley proved conversely that " ~ The Tychonoff Product Theorem Implies the Axiom of Choice", Fundamenta. Math. **37** (1950), 75-76 [Math Review **12**, p. 626]. From a logical standpoint, the Tychonoff's theorem is thus equivalent to the axiom of choice.

Compactness in linearly ordered spaces

Suppose (X, ≤) is a linearly (or, totally) ordered set, which has no largest or smallest element. Then all "open intervals", that is sets of the form

$$(a, b) = \{ x \in X : a < x < b \},$$

together with the empty set, constitute a base for a topology on X. This is called the interval topology (or order topology) on X.

For linearly ordered spaces we have the following analogue of the Heine-Borel theorem .

Theorem 5.12 (Haar-König theorem) A linearly ordered set, with its interval topology, is compact if and only if it is order-complete (i.e. iff every non-empty subset has a least upper bound and a greatest lower bound [4, p. 212].

Corollary 5.13 A well-ordered set, with its interval topology, is compact if and only if it contains a maximal element.

Remark 5.7 For X = **R**, the interval topology coincides with the usual topology. Therefore, the Heine-Borel theorem for **R** can be viewed as a special case of the Haar-König theorem.

Compactness in metric spaces

Defination 5.4 Given a metric space (X, d), a finite subsets A $\subseteq$ X is called an ***ε-net*** for X if for every $x \in$ X there exists a point $a \in$ A such that $d(x, a) < \varepsilon$. X is called ***totally bounded*** if it has an ε-net for every $\varepsilon > 0$.

Definition 5.5 A sequence (x_n) is called a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ whenever $m > N$.

Remark 5.8 Every convergent sequence is a Cauchy sequence, but the converse is not true in general.

Definition 5.6 A metric space is called ***complete*** iff every Cauchy sequence in it is convergent (to a limit belonging to X).

Theorem 5.14 A metric space is compact iff it is complete and totally bounded [4, p. 261].

Remark 5.9 This theorem is to be considered as the proper analogue for metric spaces of the Heine-Borel for $\mathbf{R}^n$: "A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded". Since every closed subset of a complete metric space is complete and every bounded subset of $\mathbf{R}^n$ is totally bounded, the above theorem yields the Heine-Borel theorem as a corollary.

When are compact and closed equivalent?

It is well-known that in a compact topological space, every closed set is compact; but the converse is not true in general. It is equally well-known that in a Hausdorff space, every compact set is closed, but – again – the converse is not true in general. There are, however, spaces in which the compact sets coincide with the closed sets – compact Hausdorff spaces, for example. Levine [11] investigated the question posed in the heading above and obtained a nice answer.

Definition 5.7 A topological space (X, $\mathscr{T}$) will be called ***closed-compact*** iff the closed sets in X coincide with the compact sets in X, that is, if and only if the two-way implication "A $\subseteq$ X is closed iff A $\subseteq$ X is compact" holds.

Definition 5.8 A topological space (X, $\mathscr{T}$) is called ***maximal compact*** iff (i) (X, $\mathscr{T}$) is compact, and (ii) whenever $\mathscr{T}^*$ is a strictly finer topology on X than $\mathscr{T}$, then (X, $\mathscr{T}^*$) is not compact.

Remark 5.10 Every closed-compact space X is compact and T_1, because X is closed (therefore compact) and singleton sets are compact (therefore closed).

Example 5.1 Let (R, $\mathscr{T}$) be the space of rationals with the relative topology and let (X, $\mathscr{T}^*$) be the one point compactification of (R, $\mathscr{T}$). Since (R, $\mathscr{T}$)

is not locally compact it follows that (X, $\mathscr{T}^*$) is not Hausdorff, but (X, $\mathscr{T}^*$) is closed-compact.

The principal result of Levine's is the following .

Theorem 5.15 (X, $\mathscr{T}$) is closed-compact if and only if (X, $\mathscr{T}$) is maximal compact.

Since every compact Hausdorff space is maximal compact, it is also closed-compact. Levine proved that for frist countable spaces the reverse implication holds :

Theorem 5.16 (X, $\mathscr{T}$) be a first countable topological space. Then (X, $\mathscr{T}$) is closed-compact if and only if it is compact and Hausdorff.

Between T_1 and T_2

Wilansky [19] has investigated two properties of topological spaces which lie (strictly) between T_1 and T_2–separation axiom; these are the compact-closed and unique limit spaces.

Definition 5.9 A topological space (X, $\mathscr{T}$) is called a ***compact-closed space*** iff every compact set in X is closed.

Definition 5.10 A topological space (X, $\mathscr{T}$) is called a ***unique-limit space*** iff every convergent sequence in X has a unique limit in X to which it converges.

The reader should carefully note the distinction between closed-compact and compact-closed spaces. Every closed-compact space is of course compact-closed, but the converse is not true in general.

Theorem 5.17 The following implications hold

$$T_2 \Rightarrow \text{Compact-Closed} \Rightarrow \text{Unique Limit} \Rightarrow T_1.$$

Example 5.2 Let X be a countably infinite set and if $\mathscr{T}$ be the cofinite topology (in which a proper subset is closed if and only if it is finite). It makes X a compact T_1 - space but not a unique limit space, since every sequence of distinct members of X converges to every member of X.

Example 5.3 Let X be an uncountable set and $\mathscr{T}$ be the cocountable topology on X. Then X is a compact-closed space, but not a Hausdorff space.

Theorem 5.18 A first countable space X is a unique limit space $\Leftrightarrow$ X is compact-closed space $\Leftrightarrow$ X is a T_2 – space.

The topology of a first countable space is completely determined by its sequences. Therefore, a first countable unique limit space is a Hausdorff space.

Wilansky also investigated the relation between a topological space X and its one-point compactification X^+ in considerable detail, vis-a vis the properties under consideration here. Some of these results are mentioned below.

Theorem 5.19 If X is compact-closed then X^+ is a unique limit space.

Remark 5.11 Wilansky gives an example [19, p. 265] such that X is a unique limit space but X^+ is not.

Definition 5.11 A topological space X is called a ***Kuratowski space*** if it has the following property : If $S \cap K$ is closed for all closed-compact subsets K of S, then S itself is closed.

In the literature such spaces are simply called *k*-spaces, Kelley [8, p.231] proves that every Hausdorff space which is either first countable, or is locally compact, is a Kuratowski space.

Theorem 5.20 Suppose X is a compact-closed space; then X^+ is a compact-closed space if and only if X is a Kuratowski space.

Corollary 5.21 Suppose X is Hausdorff; then

(1) X^+ is a unique limit space.

(2) X^+ is a compact-closed space if and only if X is a Kuratowski space.

(3) X^+ is Hausdorff if and only if X is locally compact.

Theorem 5.22 Every first countable unique limit space is compact-closed and every first countable compact-closed space is Hausdorff.

Chapter 6
Uniform space

If X is any set, we denote by Δ (x), or simply Δ, the ***diagonal*** $\{(x, x) : x \in X\}$ in $X \times X$.

Definition 6.1 A diagonal uniformity on a set X is a collection $\mathscr{D}(X)$, or, simply $\mathscr{D}$, of subsets of $X \times X$, called ***surroundings***, which satisfy:

(a) $D \in \mathscr{D} \Rightarrow \Delta \subseteq D$, in any $\Delta \in D$;

(b) $D_1, D_2 \in \mathscr{D} \Rightarrow D_1 \cap D_2 \in \mathscr{D}$;

(c) $D \in \mathscr{D} \Rightarrow E \circ E \subseteq D$ for some $E \in \mathscr{D}$;

(d) $D \in \mathscr{D} \Rightarrow E^{-1} \subseteq D$ for some $E \in \mathscr{D}$;

(e) $D \in \mathscr{D}, D \subseteq E \Rightarrow E \in \mathscr{D}$.

When X has such a structure, we call X a ***uniform space***. The uniformity $\mathscr{D}$ is called separating and X is said to be separated iff $\cap\{D : D \in \mathscr{D}\} = \Delta$.

A base for the uniformity $\mathscr{D}$ (also called a ***base*** for the ***surroundings*** on X) is any subcollection $\mathscr{E}$ of $\mathscr{D}$.

Definition 6.2 $\mathscr{E}$ is a ***base*** for $\mathscr{D}$ iff $\mathscr{E} \subset \mathscr{D}$ and each $D \in \mathscr{D}$ contains some $E \in \mathscr{E}$.

Clearly, a collection $\mathscr{E}$ of subsets of X × X is a ***base for some uniformity*** iff its sets satisfy (a), (c), (d) and the following modified form of (b):

(b_1) $D_1, D_2 \in \mathscr{E} \Rightarrow D_3 \subseteq D_1 \cap D_2$ for some $D_3 \in \mathscr{E}$.

A ***subbase*** for $\mathscr{D}$ is a subcollection $\mathscr{E}$ of $\mathscr{D}$ such that all finite intersections of elements of $\mathscr{E}$ form a base for $\mathscr{D}$.

Example 6.1. Given any set X, the collection $\mathscr{D}$ consisting of the single set X × X is a uniformity on X, called the ***trivial*** uniformity

Example 6.2 Given any set X, the collection $\mathscr{D}$ of all subsets of X × X which contain Δ is a uniformity on X, called the ***discrete uniformity***. It has for a base the singleton collection { Δ}.

Example 6.3 Any metric ρ on a set X generates a metric uniformity $\mathscr{D}_\rho$ on X, namely the uniformity having for a base the sets $D_{\rho,\epsilon}$, $\epsilon > 0$, where $D_{\rho,\epsilon} = \{(x, y) \in X \times X : \rho(x, y) < \epsilon\}$

Uniformities which can be generated in this way from metrics are called ***metrizable.***

If ρ is any metric on X, the uniformities generated by ρ and 2ρ coincide, so that different metrics may give rise to the same uniformity. Thus a uniformity represents less structure on a set than a metric.

Definition 6.3 The topology associated with a diagonal uniformity $\mathscr{D}$ will be called the uniform topology $\mathscr{T}$ generated by $\mathscr{D}$. Whenever the topology on a topological space X can be obtained in this way from a uniformity, X is called a ***uniformizable topological space***.

Example 6.4 (i) The trivial uniformity on a set X generates the trivial (indiscrete) topology.

(ii) The discrete uniformity on a set X generates the discrete topology.

(iii) The topology generated by the metric uniformity on **R** is the usual topology.

(iii) The uniform topology generated by a metric coincide with the metric topology.

Theorem 6.1 A toplogical space is uniformizable iff it is completely regular [20,p. 256].

Definition 6.4 Let X have a diagonal uniformity $\mathscr{D}$. A net (x_α) in X is ***Cauchy net*** iff for each D $\subseteq \mathscr{D}$, there is some $\alpha_o \in$ A such that $(x_{\alpha_1}, x_{\alpha_2}) \in$ D whenever $\alpha_1, \alpha_2 \geq \alpha_o$.

Theorem 6.2 Every convergent net is a Cauchy net.

Definition 6.5 If every Cauchy net in a uniform space X converges, then X is called a complete uniform space.

Theorem 6.3 A space X with a uniformity $\mathscr{D}$ generated by the metric ρ is complete iff (X, ρ) is a complete metric space.

Theorem 6.4

(1) Every closed subset A of a complete uniform space X is complete.

(2) Every complete subspace A of a Hausdorff uniform space X is closed.

Theorem 6.5 A non-empty product of uniform space is complete iff each component space is complete.

Definition 6.6 A uniformity $\mathscr{D}$ on X is ***totally bounded*** iff for each D ∈ $\mathscr{D}$, there is a finite cover $\{ U_1, U_2, \ldots, U_n \}$ of X such that $U_k \times U_k \subseteq D$, for each k.

Theorem 6.6 A uniform space X is totally bounded iff each net in X has a Cauchy subnet [20, p. 262].

Finally, we have the analogue of theorem 5.14 for compact metric space.

Theorem 6.7 A uniform space X is compact iff it is complete and totally bounded .

Uniform Continuity

It is true that the theory of topological spaces arise as a generalization of metric spaces; but the development of the theory of topological space has not stripped metric spaces of their importance or usefulness. Metric spaces continue to provide the setting for the formulation and analysis of a number of general ideas that are not quite within the reach of general topology: the concept of uniform continuity, or that of a Cauchy sequence, both depend on the possibility of comparing degrees of closeness throughout a space. As, Fréchet [2, p. 102] remarks : "the introduction to distance is absolutely necessary if one wants to extend the notion of uniform continuity [from the classical setting to a more abstract one]".

Definition 6.7 Given metric spaces (X, d) and (Y, ρ), a function $f : X \to Y$ is said to be ***uniformly continuous*** on X, if for every positive real number ε, there exists a positive real number δ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$; that is, such that $f(U(x, \delta)) \subseteq U(f(x), \varepsilon)$ for every point x of X.

Informally, the idea is that if two points are sufficiently close together in X, their images will be desirably close together in Y.

Definition 6.8 A mapping f from a uniform space X into uniform space Y is called uniformly continuous on X iff for each surrounding V of Y there is a suounding U of X such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$.

Theorem 6.8 Let $\mathscr{U}, \mathscr{V}$ be uniformities on X. Then $\mathscr{U} \supseteq \mathscr{V}$ if and only if the identity map $i : (X, \mathscr{U}) \rightarrow (X, \mathscr{V})$ is uniformly continuous.

We have the following generalization of Heine's theorem on uniform continuity.

Theorem 6.9 Every continuous function f from a compact uniform space $(X, \mathscr{U})$ into a uniform space $(Y, \mathscr{V})$ is uniformly continuous .

Corollary 6.10 The uniformity of a compact space is unique; that is, a compact topological space has at most one uniformity which induces its topology.

Historical Note

The theory of uniform spaces was developed by A. Weil whose monograph [17] contains many of the basic result on uniform spaces.

Chapter 7
Fréchet, Hausdorff and the emergence of point set topology : A Critical Analysis

Fréchet and Point-set topology

It is generally agreed that the emergence of point-set topology as a distinct discipline dates from the appearance of Hausdorff's *Grundzüge der Mengenlehre*. However, every subject has a prehistory and general topology is no exception. Towards the end of the nineteenth century mathematicians began to study properties of sets of functions etc. A noteworthy result of this time is the theorem of Arzela – Ascoli, which served has a motivating example for the study of "abstract spaces"; these are sets about the nature of whose elements no assumptions are made except that there spaces may be required to satisfy such axioms as are imposed on them. The study of abstract spaces in this very broad sense was first undertaken by Fréchet in his thesis. He studied three particular classes of abstract spaces, which he designated as V, H, E classes [15].

The first attempts to abstract what is common to properties of sets of points and sets of functions are due to Fréchet (in the paper cited on page 2) and F. Riesz (in the paper cited on page 43). The former started from the notion of sequential limit and did not succeed in constructing a convenient and fruitful system of axioms, but at least he recognized the relationship between the principle of Bolzano-Weierstrass and the Heine-Borel theorem; in this connection he introduced the word "compact". As to F. Riesz, who took as his starting point the concept of point of accumulation (or rather of "derived set" which amounts to the same thing), his theory was incomplete and appeared only in outline form.

Fréchet's extensive work on abstract spaces played a prominent role in the emergence of general topology. However, while Hausdorff formulated a precise set of axioms for a topological space. Fréchet – even after the appearance of Hausdorff's *Grundzüge* – preferred to stick to his won version of abstract spaces (some of which were not topological spaces in the accepted sense).

A.E. Taylor has published a major study [15] on the work of M. Fréchet which appeared in two parts and comprises over 160 pages < 1 >. The two parts, taken together, give a comprehensive account of Fréchet's work on point-set theory and general topology. Prior to 1912, Fréchet had considered three methods of developing an axiomatic point-set theory :
(1) the method of L-classes , (2) the method of V-classes, and (3) the method of E-classes . These are discussed at considerable length by Taylor. In all three methods an element p is a limit element of a set S if there exists a sequence $\{p_n\}$ of distinct elements $p_1, p_2, p_3 \ldots$, such that the sequence converges to (or has the limit) p. The collection of limit elements (if any) of the set S is called the ***derived set*** of S and is denoted by S′. It may be empty. For an L – class the notion of a convergent sequence with its limit is a primitive notion satisfying certain axioms. For a V – class or an E – class the notion of a convergent sequence is defined with the aid of a real – valued binary function (a function of two elements). In the case of a V – class a value of this binary function is called by Fréchet a voisinage (which translates as "neighborhood", but which is not a set of elements, as in

standard modern terminology today, but a non – negative real number). In the case of an E – class, Fréchet speaks of an écart instead of a voisinage. An E – class is in fact a metric space and the écart of two elements is their distance apart. The name “metric space” for an E – class was introduced by Felix Hausdorff (German name *Metrischer Raum*) on page 211 of the *Grundzüge*.

Fréchet’s work on abstract point-set theory had a particularly significant impact on several American mathematicians of whom we may mention E. R. Hedrick, E. W. Chittenden and T. H. Hildebrandt, some of whom continued to work on Fréchet’s line even after the appearance of Hausdorff’s *Grundzuge* <2>. Some of their work is discussed by Taylor [15]. Another American mathematician who worked on and developed a form of “General Analysis” was E. H. Moore.

Among works of American mathematicians directly inspired by Fréchet’s work, we mention the following <3>:

E. R. Hedrick : On properties of a domain for which any derived set is closed, Trans. Amer. Math. Soc. **12** (1911) 285 – 294.

T. H. Hildebrandt : A contribution to the foundations of Fréchet’s calcul fonctionnel, Amer. J. Math. **34** (1912) 237 – 290.

E. W. Chittenden : Converse of the Heine – Borel theorem in Riesz domain, Bull. Amer. Math. Soc. **21** (1914 / 1915) 179 – 183.

E. W. Chittenden : On the equivalence of ecart and voisinage, Trans. Amer. Math. Soc. **18** (1917) 161 – 166.

C. E. Aull's paper : E. W. Chittenden and the early history of general topology , Topology and its Applications **12** (1981) 115 – 125, discusses more instances of Fréchet's influence on American mathematicians.

E. H. Moore's doctoral student R. E. Root's work is briefly discussed by Taylor [15] because he recognized the importance of the notion of neighbourhood in the context of "Iterated Limits in General Analysis", American Journals of Mathematics **13** (1914), 79 – 133. "The paper has its origin in the thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of 'neighbourhood' or 'vicinity' of an element is fundamental. In the domain of general analysis various way of determining a neighbourhood have been employed, notably the notion of voisinage used by M. Fréchet and the relations K_1 and K_2 used by E. H. Moore . . ." says Root. It was not the perpose of Root's work to develop a systematic theory of abstract general topology. But he did formulate some axioms of neighbourhood independently and ahead of Hausdorff, which are reproduced in Taylor [15, pp. 298 – 299].

A great deal of work was published during the period 1900 – 1930 on Heine-Borel theorem, is modifications and generalizations and the precise extent of their validity, Of which, Chittenden's paper cited above is an example. Hildebrandt's paper : The Borel theorem and its generalizations, Bulletin Amer. Math. Soc. **22** (1926), 423 – 474, gives an exposition of many of these results in light of the historical development. Borel originally (1895) proved his theorem for denumerable (countably infinite) open coverings. Lebesgue (1898) extended it to uncountable coverings; for this reason many authors (e. g. Taylor) speak of "Borel property" and "Borel-Lebesgue property" to distinguish between denumerable and uncountable coverings. The appellation "Heine-Borel theorem" seems to have been first used by Schoenflies (1900), a nomenclature which is still widely used. W. H. Young wrote extensively on point-sets in Euclidean spaces during the turn of the century, he proved various generalizations and modifications of the Heine–Borel theorem, beginning with the following theorem on overlapping intervals (1902).

"Given any set of intervals, overlapping in any manner, on a straight line, a countable set from among them may be found having the same internal points".

This theorem yields the Heine-Borel theorem as an immediate corollary. Young also proved the following extension of the Heine-Borel theorem:

"Given any closed set of points on a straight line and a set of intervals so that every point of the closed set of points is an internal point of at least one of the intervals, then there exist a finite number of the given intervals having the same property".

The justification for attaching E. Heine's name the Borel's theorem lies in the fact that the classic theorem on uniform continuity "A function continuous at every point of a closed and bounded interval is uniformly continuous on that interval", can be very easily proved using Borel's theorem. Now this Theorem on Uniform Continuity is the culmination of Heine's paper "Elemente der Functionenlehere", Journal für die reine und angewandte Mathematik, **74** (1872), 172 – 188, which – as Heine explicitely acknowledges – was primarily an exposition of material from Weierstrass's lectures at the University of Berlin. Whether the mere fact that Borel's covering theorem affords a proof of the theorem on uniform continuity (which is definitely not due to Heine <4>) is sufficient ground to attach Heine's name to Borel's theorem, appears debatable. But the appellation "Heine-Borel theorem" is so well-entrenched that any attempt to dislodge Heine's name may cause discomfort, or even consternation.

According to Fréchet's own testimony he did not read Hausdorff's "interesting book" until after the First World War. But even then Fréchet continued to stick to his multifarious approach to abstract spaces, and never adopted Hausdorff's approach to the theory of topological spaces.

In the years 1924 – 1926 Alexandroff (and initially also Urysohn < 5 >) visited Fréchet and also maintained close contact with him through correspondence. These contacts are described in much detail by Taylor. Even these contacts with Alexandorff who had thoroughly studied Hausdorff's *Grundzüge* and influenced by it, could not win over Fréchet's reluctance to adopt the approach laid out by Hausdorff; Fréchet considered Hausdorff's approach as too restrictive.

In 1928 Fréchet published his version of the theory of abstract spaces ~ *Les espaces abstraaits et leur théorie considérée comme introduction à, l'analyse générale* (The theory of abstract spaces considered as an introduction to general analysis) < 6 >.

Fréchet took considerable pains to describe the origin, the plan and the method of exposition of the book. Fréchet consciously omitted many proofs, giving the book the nature of a major survey work of mostly his own researches.

Fréchet's book was most certainly not appropriately designed as an instrument for aiding a student who wished to learn systematically the most important things about the state of topology in the second half of the nineteen twenties. For such a student an effective instrument would have been one that selected a certain line of development to reach the fundamental ideas and results without much distraction with side issues, and displayed enough of the arguments and proofs needed along the way to enable the student to understand the subject and become proficient in demonstrating the theorems and making investigations independently. Fréchet's decision to omit proofs and merely to describe a great assortment of ideas and results, with not much selective emphasis, made the book merely a compendium of definitions, facts, and relationships, with a guide to the periodical literature as the only help. This deprived the book of the appeal of a well-planned textbook, which would instruct, inspire, and stimulate young scholars.

Fréchet's book was also too late on the scene to have any hope of displacing the influence of Hausdorff's *Grundzüge*. Moreover, it was not constructed in a manner to capture the minds of young French mathematicians who might readily have preferred a French book to a German book on Topology.

Fréchet's monograph on abstract general spaces had an important precursor "Esquisse d'une théorie des ensembles abstraits" (1922). This lengthy essay was contributed upon invitation of the University of Calcutta for inclusion in a Festschrift for Sir Asutosh Mookerjee. The Festschrift was occasioned by Sir Asutosh's Silver Jubilee and published in two volumes. The "Esquisse" appeared in Volume II (Science), pp. 333 – 394. The contents of the "Esquisse" are discussed in some detail by Taylor [15] <7> because it had a noticeable influence on Alexandroff and Urysohn at the start of their researches in general topology; they published a number of papers directly inspired by the "Esquisse". Indeed, it seems that "Esquisse" had decidedly more impact than Fréchet's later monograph of 1928.

In the same year (1928) there also appeared (in two parts) a book on set theory (in Polish) by the young Polish mathematician Waclaw Sierpinski; the second part was on topology and it included a good and systematic account of the theory of Fréchet's *H*-classes. The book became widely known through its English translation, which appeared in 1934 [13]. "This book was of great value as a text book. I know of no other place where the theory of *H*-classes is developed as clearly, systematically, and thoroughly. It is ironic that Sierpinski's book does a better job of putting *H*-classes in a favourable light than is done in any of Fréchet's own writings" – says Taylor.

A. Appert was another mathematician who, like Sierpinski, made a conscious effort to synthesize Fréchet's multifarious approach to the theory of abstract spaces with Hausdorff's clear cut approach to topological spaces in a short monograph which appeared in 1938 in two parts (totalling 108 pages) in the series "Actualités scientiiques et industrielles" <8>.

Appert's monograph was entitled "*Propriétés des espaces abstraits les plus généraux*" and represented the last attempt to popularize Fréchet's approach; in contrast to Fréchet's monograph it included complete proofs and many examples. One example of a topological space due to Appert is of particular interest; some of its pertinent properties are described in [14, pp. 117 – 118]. Its underlying set is **N** and the assigned topology has a number-theoretic connection. For any subset E of **N** and any $n \in \mathbf{N}$, let

$$N(n, E) = \{ m \in E;\ m \leq n \}. \text{ Then } \lim_{n \to \infty} N(n, E) / n$$

is called the asymptotic density of E, provided this limit exists. The Appert space is the set **N** endowed with the following topology $\mathscr{T}$:

$$A \in \mathscr{T} \Leftrightarrow \text{ either (i) } 1 \notin A,$$

$$\text{or (ii) } 1 \in A \text{ and } \lim_{n \to \infty} N(n, A) / n = 1.$$

We include a proof that $\mathscr{T}$ is indeed a topology on $\mathbf{N}$.

(i) Let $A_i \in \mathscr{T}$ for every $i \in I$; to show $\bigcup_{i \in I} A_i \in \mathscr{T}$.

Case 1. If $1 \notin A_i$ for every $i \in I$, then $1 \notin \bigcup_{i \in I} A_i$. So $\bigcup_{i \in I} A_i \in \mathscr{T}$.

Case 2. If $1 \in A_i$ for at least one $i_0 \in I$; then $\lim_{n \to \infty} N(n, A_i) / n = 1$.

Since $N(n, E \cup F) \geq N(n, E)$, We have $N(n, \bigcup_{i \in I} A_i) \geq N(n, A_i)$

So $\quad N(n, \bigcup_{i \in I} A_i) / n \quad \geq \quad N(n, A_i) / n$

Given $\varepsilon > 0$, there exists n_0 such that $N(n, A_i) / n > 1 - \varepsilon$ for all $n > n_0$.

Therefore, $\quad 1 \geq N(n, \bigcup_{i \in I} A_i) / n > 1 - \varepsilon$ for all $n > n_0$.

Therefore, $\lim_{n \to \infty} N(n, \bigcup A_i) / n = 1$. So, $\bigcup_{i \in I} A_i \in \mathscr{T}$.

(ii) Let $A_1, A_2 \in \mathscr{T}$.

Case 1. $1 \notin A_1$ or $1 \notin A_2$. Then $1 \notin A_1 \cap A_2$. So $A_1 \cap A_2 \in \mathscr{T}$.

Case 2. $1 \in A_1$ and $1 \in A_2$. Then

$$\lim_{n\to\infty} N(n, A_1)/n = 1 \quad \text{and} \quad \lim_{n\to\infty} N(n, A_2)/n = 1.$$

To show $\lim_{n\to\infty} N(n, A_1 \cap A_2)/n = 1$;

or equivalently $\lim_{n\to\infty} N(n, A_1^c \cup A_2^c)/n = 0$;

because $(E \cap F)^c = E^c \cup F^c$ and $N(n, E^c) = n - N(n, E)$; where E^c denotes the complement of E in **N**.

Since $\lim_{n\to\infty} N(n, A_1^c)/n = 0$ and $\lim_{n\to\infty} N(n, A_2^c)/n = 0$,

given $\varepsilon > 0$ there exists n_0 such that

$N(n, A_1^c)/n \leq \varepsilon/2$ and $N(n, A_2^c)/n \leq \varepsilon/2$, whenever $n > n_0$.

Clearly, $N(n, E \cup F) \leq N(n, E) + N(n, F)$. So,

$$0 \leq N(n, A_1^c \cup A_2^c)/N \leq N(n, A_1^c)/n + N(n, A_2^c)/n \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

whenever $n > n_0$.

Therefore $\lim_{n\to\infty} N(n, A_1^c \cup A_2^c)/n = 0$, Hence, $A_1 \cap A_2 \in \mathscr{T}$

Taylor sums up Fréchet's work in topology in these words. "When considering evaluating the total body of Fréchet's work on topology in the period 1907 – 1928, I think it must be said that it was diffuse, too general to fit well with the needs and tastes of the times, and not accompanied by the development of a methodology to attack with significant success problems whose conquest might have helped to give his work prestige. Fréchet did, in fact, pose problems, but usually he left them unsolved or only partially solved". Taylor felt that Fréchet lacked the disposition, and perhaps the talent, for the sort of work that involves the development of technique or new ideas for attacking specific hard problems successfully. "When Fréchet was established in Paris, in late 1928, after leaving Strasbourg, he was almost exactly fifty years old. A very full and long life still lay ahead of him. But his important work in topology was over. Activity in topology was flourishing in Europe and America and the direction of work in topology had passed him by" concludes Taylor. To this we may add the following remark : The glory of being considered the founding father of point-set topology could have and perhaps, should have, belonged to Fréchet <9>. It is an irony of fate that Fréchet's seminal work was too diffuse to earn him this glory.

Hausdorff and his "*Grundzüge*"

Hausdorff's *Grundzüge der Mengenlehre* ranks among the most influential mathematical books of all times. Its impact was immediate and decisive. The title, however, is not fully indicative of its contents. Hausdorff not only presents in the first sixth chapters a through and elegant account of Cantor's set theory (the book is dedicated to "Georg Cantor, the creator of set theory, in thankful reverence") but goes on in the seventh chapter to apply set theoretic concepts and methods to point-sets in general spaces. Hausdorff begins with an illuminating discussion of various approaches to the study of point-sets in general spaces, pointing out that any one of the three concepts *distance, neighbourhood, limis*, can be taken as a starting point for building up a general theory. He points out that with the help of distance one can define neighbourhoods and limits; with the help of neighbourhoods one can define limits but not in general distance; with the help of limits one can define in general neither neighbourhoods nor distances. "For various reasons, we prefer to build up the fundamental considerations of this chapter on neighbourhoods and bring up the other two concepts only later. However, in order to give the reader at once the feeling of a complete picture we begin with the special type of neighbourhoods defined by distance" [6, p. 211] . Saying this, Hausdorff defines a metric space in the now familiar fashion (except that, he denotes the distance between two points x and y by $\overline{xy}$). We have elsewhere (page 6) already quoted Hausdorff's neighbourhood axioms for a topological space, which were directly motivated by properties of neighbourhoods in metric spaces. Henry Blumberg published a lengthy review of Hausdorff's *Grundzüge* in the

Bulletin of the American Mathematical Society in 1920 (Volume. **27**, pp. 116-120). In addition to giving a quite a detailed account of the contents of the book, Blumberg's review accurately points outs its overriding qualities and the originality of the author. This review is still worth reading; we confine ourselves to quoting the first two paragraphs and the last of Blumberg's review.

"If there are still mathematicians who hold the theory of aggregates under general suspicion, and are reluctant to grant it full recognition as a rigorous mathematical discipline, they will find it hard to retain their doubts under fire of the logic of Hausdorff's treatise. It would be difficult to name a volume in any field of mathematics, even in the unclouded domain of number theory, that surpasses the *Grundzüge* in clearness and precision.

But it is only in a subsidiary role that the *Grundzüge* is an answer to the skeptics. Its most striking feature is that it is a work of art of a master. No one thoroughly acquainted with its contents could fail to withhold admiration for the happy choice and arrangement of subject matter, the careful diction, the smooth, vigorous and concise literary style, and the adaptable notation; above all things, however, for the highly pleasing unifications and generalizations and the harmonious weaving of numerous original results into the texture of the whole."

"One may quarrel with the author for his abstract style, for his Euclidean manner of grading the proofs, so that no difficulties remain and none but mild climaxes are reached, for his finish that may excite admiration but hardly activity on the reader's part. One may crave for a book that is built like a drama around a single idea – a more sketchy book, leaving more to the reader's imagination, a book with a less diversified and more emphatic message. But such remonstrance would be like quarrelling with Beethoven for having written symphonies instead of operas. There is no such thing as *the* book. Hausdorff's *Grundzüge* is a treatise, and as a treatise it necessarily falls short of the summum bonum. But as a treatise it is of the first rank."

Allen Shields published in the "Years Ago" column of The Mathematical Intelligencer, Volume **11**, Number **1**(1989), (pp. 6 - 9) a note, "Felix Hausdorff : *Grundzüge der Mangenlehre*", which contains a brief biographical account of Hausdorff and describes some of the contents of the book from a contemporary standpoint. Shields points out that "after defining metric spaces in Chapter Seven, the author defines what are today called Hausdorff spaces. This seems to be the place where abstract topological spaces were first defined". Allen rightly says "This is the book from which succeeding generations of mathematicians learned the elements of set theory and point set topology", but he makes little or no reference to the measure of Hausdorff's accomplishment.

The astonishing fact remains that prior to the appearance of the *Grundzüge*, Hausdorff had not published anything at all on what must be regarded as the most original and influential part of the work – the theory of topological and metric spaces (Chapters 7, 8, 9). As Katetov says [article on Hausdorff, ~ *Dictionary of Scientific Biography*, Volume **6** (1981), p. 176], "The *Grundzüge* is a very rare case in mathematical literature; the foundations of a new discipline are laid without the support of any previously published comprehensive work".

Interestingly enough, the preface of the *Grundzüge* reveals Hausdorff's own views on his labours. Expressing the hope that while the book might profitably be read by a student of the middle semesters (i.e., an advanced undergraduate), he declares that it would have failed its purpose if it did not offer the professional colleagues something new, at least in methodological and formal aspects. Saying that one should logically and systematically organize scattered facts, remove unnecessarily special or complicating assumptions from previous results, that one should strive for attaining simplicity and generality – surely the minimum that can be demanded of an author dealing with material already treated by others – Hausdorff expresses his belief of having reasonably fulfilled this demand and of having opened some new lines of inquiry. He points out, especially, that by axiomatizing point-set theory, many theorems on point-sets on the real line have been so transformed, generalized, decomposed, and tied up in another context that a mere reference to the existing literature would give no correct picture. Thus, despite these rather modest and unassuming words, there is no question that Hausdorff was aware of what he had accomplished with this book. So, while he was no doubt pleased with the impact created by the *Grundzüge* on the subsequent development, he was hardly surprised.

The definition of a topological space by means of a set of axioms on neighbourhoods [6, p. 213] is, beyond question, Hausdorff's greatest contribution. The concept of a neighbourhood as such was, of course, nothing new; Hausdorff's great credit was to make it the point of departure for an axiomatic development – abstract in form, but as Bourbaki has observed [3, Historical Note to Chapter 1, p. 166], adapted in advance to applications – and to have found the right set of axioms. That, in analogy with metric spaces, he incorporated the separation axiom since named after him, detracts nothing from this credit. The *Grundzüge* was the source to which the spectacular rise of point-set topology in the nineteen twenties and thirties is due. Many of the basic notions and concepts of the subject that are to be found here – even the name metric space – have come down to us unaltered; one has to envy Hausdorff for his singular acumen and extraordinary foresight.

As an instance of the novelties in the *Grundzüge*, we mention Hausdorff's definition of an ordered pair. It is well – known that the intuitive notion of an ordered pair can be made precise in terms of sets. The approach usually adopted is that of N. Wiener who defined:

$$(a, b) = \{ \{a\}, \{a, b\} \}.$$

The crucial property of ordered pairs;

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

follows from this definition, but requires some thinking.

Hausdorff's approach is much simpler. He takes two symbols, 1 and 2, which are different from all other elements under consideration and defines:

$$(a, b) = \{ \{a, 1\}, \{b, 2\} \}.$$

With this definition the above property is immediate.

Noteworthy also is the following result now known as Hausdorff's Maximal Principle: "Every partially ordered set contains a maximal linearly (totally) ordered subset". Hausdorff proved it using the well – ordering theorem of Zerlemo, "Every set can be well – ordered." J. L. Kelley [8] emphasizes the Hausdorff Minimal Principle and uses it throughout his book.

Felix Hausdorff was born in Breslau (now Wroclaw in Poland) in a prosperous Jewish merchant family, which moved to Leipzig during his childhood. Hausdorff entered the University of Leipzig to study mathematics and astronomy <10>. Following the German tradition he visited the universities of Berlin and Freiburg each for one semester. Until well into the twentieth century the doctorate was the only academic degree in German universities <11>. Hausdorff's dissertation was on the theory of astronomical refraction (1891). Hausdorff continued scientific researches related to his dissertation which resulted in a habilitation essay <12> "On the absorption of light in the atmosphere"(1895). During his formative years Hausdorff devoted at least as much time to music, literature and fine arts as he did to mathematics. Under the pseudonym "Paul Mongré" he wrote some poems and a good many articles of a philosophical nature (in these writings the influence of the philosopher Nietzsche is especially evident). Writing under the same pseudonym he published a satirical and chrnocritical drama "Der Arzt seiner Ehre" (The physician of his honour); in 1898 there appeared a substantial work of a decided his philosophical nature "Das Chaos in kosmischer Auslese" (Chaos in cosmic disposition). Hausdorff's literary works are discussed in some detail in Herbert Mehrtens' monograph "Felix Hausdorff – Ein Mathematiker seiner Zeit" (40 pp, 1980), published on the occasion of unveiling of a plaque in memory of Hausdorff at the Mathematical Institute of the University of Bonn (for the wording this plaque, please see p.94); other valuable sources on the life and work of Hausdorff include the essay "Felix Hausdorff, 1868 – 1942", by W. Krull in

Bonner Gelehrte, University of Bonn (1970). Janusz Czyż of the Mathematical Institute of the Polish Academy of Sciences has undertaken a major study of Hausdorff's works (preprint 1993) which includes a biographical sketch (Chapter 0) dealing in particular with the poems and philosophical works of Hausdorff < 13 >.

Hausdorff was appointed an Extraordinarius (associate professor) in 1902 (in Leipzig University) and shortly thereafter he received a similar appointment at Göttingen, which he turned down. In those days Berlin and Göttingin were the two most prominent research centres in mathematics in Germany; rejecting a call to either of these places was unusual, to say the least. Hausdorff obviously did not wish to leave his familiar multifaceted circle at Leipzig for the small town of Göttingen. According to Wolfgang Krull, the eminent algebraist who was Hausdorff's colleague at Bonn from 1928, the refusal definitely harmed his academic career < 14 >.

Eight years later Hausdorff did accept a similar call to the University of Bonn. Three years later, he went to the University of Greifswald (one of the smaller German universities on the Baltic coast) as an Ordinarius (full professor). In 1921 he returned to the University of Bonn as professor and director of the Mathematical Institute and remained there till his retirement in 1935.

In contrast to four books of a literary and philosophical nature, published under the pseudonym Paul Mongré, Hausdorff published only two books on mathematics including the *Grundzüge*. The second book "*Mengenlehre*" (1927) had much in common with the *Grundzüge*. Besides

42 published mathematical papers, Hausdorff left a considerable amount of unpublished material. Some of these posthumous papers have appeared in a volume edited by G. Bergmann (Felix Hausdorff : Nachgelassene Schriften, Band 1, Stuttgart, 1969). In Volume 69 of the Jahresbericht of the *Deutsche Mathematiker Vereinguing* ,G. Bergmann published a preliminary report on the scientific "Nachlass" of Felix Hausdorff (pp. 62-75). These include full texts of many of the course lectures delivered by Hausdorff from 1897-1935 (in most cases with precise dates). From Bergmann's list of Hausdorff's publications and lectures, we can pinpoint the time when Hausdorff turned his attention to set theory. Hausdorff's early interest also included mathematical statistics and insurance mathematics as evidenced by lectures given during the period 1897-1900, and publication of a substantial paper on probability theory "*Das Risiko bei Zufallsspielen*". In 1899, he lectured for the first time on a topic in pure mathematics, "Selected Topics in Higher Geometry", and in the same year, he published a major paper an non-Euclidean geometry. With the turn of the century he turned his attention to the then "hot" and controversial subject of set theory as evidenced by his lecture on this subject in Summer 1901 and in the same year he published a paper on ordered sets. Hausdorff did not again lecture on set theory until 1910 in Bonn but, his continued interest and occupation with set theory is evidenced by the publication of at least five more substantial papers on set theory during the period 1904-1908.

As already stated, Hausdorff only other published book in mathematics was also on set theory. This book was in reality a revised edition of the *Grundzüge*. According Bergmann (letter to M. R. Chowdhury), to cut costs. Hausdorff was asked by the publisher to reduce the size of the book by at least one-third. Hausdorff did this by dropping the chapters on metric and topological spaces all together. In view of the fact that these chapters were among the most influential of the entire book, the decision to scrap these chapters appears mystifying. He subjected the remaining material to a thorough revision, new edition appeared under the litle "*Mengenlehre*"; it was well received and reprinted several times. But the *Grundzüge* remains the pillar on which Hausdorff's reputation rests.

Magda Dierkesmann (in Volume 69 of the Jahresberict, pp51-59)has given a brief but moving description of Hausdorff as a teacher and as a person ; she was one of the students whom Hausdorff examined (1932) for their teacher's certificate examination in mathematics. Taking leave of him after the oral examinations, she expressed the wish she would visit him on his 70th birthday, which he would no doubt celebrate with his admirers and students; Hausdorff said softly with a melancholy smile "By then everything will be different". Although Hitler had not yet seized power at that time, the growing menace of Nazism was very much in evidence. Despite the clear threat, Hausdorff made no serious attempt to leave Germany while it was still possible. He continued to live quietly in Bonn after his retirement. Fearing impending deportation to a concentration camp, Hausdorff and his wife ended their lives of their own accord on 26th January 1942.

A memorial plaque to Hausdorff was unveiled at the entrance of the Mathematical Institute of the University of Bonn (Wegelerstrase 10) on 25th January 1980. It bears the inscription: (in English translation).

AT THIS UNIVERSITY WORKED, 1921 – 1935
THE MATHEMATICIAN FELIX HAUSDORFF:
8.11.1868 – 26.1.1942.
THE NATIONAL – SOCIALISTS DROVE HIM
TO DEATH BEACAUSE HE WAS A JEW.
WITH HIM WE HONOUR ALL VICTIMS OF THE TYRANNY.
NEVER AGAIN DICTATORSHIP AND WAR!

Taylor in his study [15] of Maurice Fréchet had naturally to deal with Hausdorff also; pages 300 – 304 are particularly relevant to our analysis. Of Hausdorff, Taylor says "He was not primarily a topologist, but his book [*Grundzüge*] established him as a major figure in the development of abstract general topology during a formative period. More precisely, it was Chapters 7 and 8 in the book, and Chapter 7 especially, that exerted strong influence on general topologies.

Hausdorff appended brief notes on the various chapters at the end of his book Concerning the genesis of the definition of topological spaces by means of neighbourhoods, Hausdorff made the following remark [6,p. 456]." I have put forward the outlines of the theory of neighbourhoods developed here in a course of lectures on set theory at the University of Bonn in the Summer Semester 1912". Through the courtesy of G. Bergmann, Taylor obtained a photocopy of the relevant portion of that course from Hausdorff's Nachlass.

The following is a translation of the quotation from that course as it appears on page 301 of [15].

"§6 Neighbourhoods:

Point sets on a straight line in the plane and in space in general in an n – dimensional space $E = E_n$ is given by a system of n – real numbers $(x_1, x_2, \ldots, x_n)$, which we interpret as rectangular (Cartesian) co-ordinates as distance of two points we define

$$xy = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \ldots + (x_n-y_n)^2} \geq 0$$

By a neighbourhood U_x of a point x, we mean the totality of all points y for which $xy < \rho$ holds (ρ is a positive number); interior of a "sphere" of radius ρ. These neighbourhoods have following properties:

(α) Every U_x contains x and is contained in E.

(β) For two neighbourhoods of the same point $U_x \subseteq U_x'$ or $U_x' \subseteq U_x$ holds.

(γ) If y lies in U_x then there is a neighbourhood U_y which is contained in U_x.

(δ) If $x \neq y$ then there are two neighbourhoods U_x, U_y without common point.

The following considerations depend initially only on these property. Therefore, they hold generally when E is a point–set to whose points x corresponds sets U_x having these four properties."

It is clear from this quotation that in developing the theory of point - sets in Euclidean spaces, Hausdorff would use these four properties without explicit reference to the nature of the points or of the neighbourhoods beyond what can be derived by use of these properties alone. In other words, to Hausdorff these four properties were going to play the role of axioms. There is no indication when Hausdorff made the crucial transition to formulating his neighbourhoods axioms for a topological space in the light of these properties. This transition was by no means obvious or automatic because the second property (β) had to be modified suitably. Had Hausdorff blindly imitated these four properties in formulating the neighbourhood axioms for a topological space the result would have been disappointing; for – we know – the neighbourhood systems in many non-metrizable topological spaces do not satisfy (β), but satisfy the weaker requirement (B) as formulated by Hausdorff.

Hausdorff did not include in his book a list of references as it is customary today. He did include an Appendix entitled "Postscripts and Remarks" [6,pp. 449-473] which contains copious references to published papers relating to specific topics or questionts. As general references he mentors Cantor's series of papers "*Über unendliche lineare Punktmannigfaltigkeiten* I – VI" which appeared in Mathematishe Annlen over the years 1872 – 1884. Hausdorff also refers to the then existing only book on set theory. "The theory of sets of points" by W. H Young and G. C.Young, Cambridge 1906, and to A. Schoenflies's now forgotten Report on Set Theory <15> which appeared in two parts. Hausdorff also refers to a substantially revised and improved version of parts of Schoenflies's Report which appeared in 1913 <16> when Hausdorff's manuscript was already in the hands of the printer and could therefore be taken into consideration, as Hausdorff mentions in a footnote, only in the Appendix. Concerning Schoenflies's original report Hausdorff made the following remark "Considering the intended readership of the present book it is to be pointed out that Schoenflies's Report is not suitable for the beginner; but the critical reader can derive much inspiration from it and evaluate for himself the success of the admirably comprehensive representation of such an extended field". This remark is interesting because Schoenflies's Report has been criticized for its shortcomings < 17 >. With the appearance of the *Grundzüge* even its revised edition became obsolete, if not irrelevant. A comparison with Youngs' book and Schoenflies's Report gives a clear picture of Hausdorff's accomplishment (as commented on by Blumberg in his review and analyzed by us in our study).

Hausdorff's remark on Schoenflies's Report (quoted above) leads us to the conclusion that he not only had made a thorough study of the Report but had derived much inspiration from it, despite its somewhat unsatisfactory nature. But his primary sources were, no doubt, Cantor's original papers. The question how or why Hausdorff turned his interest to set theory has not been addressed by any of the writers Krull, Mehrtens, Czyż. Mehrtens (p. 13) reports that in those days mathematicians of the nearby universities of Leipzig and Halle used to meet regularly over coffee and cakes to exchange ideas. Cantor was the senior professor at Halle and we may assume that he did attend at least some of these get-togethers. This is, very likely, how Hausdorff came to develop his interest in set theory. Our conclusion is indirectly supported by Hausdorff's dedication of his book to Cantor "in thankful reverence". This phrase is strongly suggestive of personal acquaintance.

Hausdorff had intended the *Grundzüge* not only as a useful handbook for the "professional colleague" but also as a textbook for a "student of the middle semesters" <18>. As explained by Hausdorff in his preface he knowingly and purposely refrained from citing too many references because in many situations "a mere reference to the existing literature would give no correct picture". For example, he makes no reference to Fréchet when defining a metric space <19>. But, when discussing the possible approaches (via. distances, neighbourhoods, limits) to point – set theory in general spaces. Hausdorff does point out that (p. 456) "M. Fréchet's theory is based on the concept of limits and gives precise reference to *Sur quelques points du calcul fonctionnel*, (1906)".

Taylor [15, p. 303] while addressing the question of who or what might have influenced Hausdorff to develop point-set topology on the basis of the neighbourhood concept, thinks that "Hausdorff probably was influenced by Hilbert and F. Riesz. Careful and industrious scholar that he was, Hausdorff would surely have seen Hilbert's work on the Foundations of Geometry <20>, and would, likewise, have seen the paper that was read at the International Congress of Mathematicians in Rome in 1908". Riesz's paper was entitled "The concept of continuity and abstract set theory" and appeared on pp. 18 – 24 of the Proceeding of the Congress. In it Riesz stressed his view that one should get away from distance and use the notion of neighbourhood instead. However, Riesz himself does not appear to have done any further work along this line.

It is interesting to note what Hausdorff says in his preface about the limited number of references to original sources in his Appendix. "There was also an inner difficulty in reconstructing afterwards germs of ideas and inspirations which were sometimes substantially rearranged during the course of the representation". We can therefore exonerate Hausdorff from any charge of intentionally suppressing sources of his ideas and inspirations, especially because, Hausdorff expressively refers to the [6, p. 449] reader to Schoenflies Report and to Youngs' book for supplementary references to original sources.

It is not easy to describe Hausdorff's personality. About his young days in Leipzig when he was a mathematics lecturer at the university and at the same time writing literary pieces and works, it has been said that he led a double life. Indeed, in those days he spent more time in the company of artists and writers then with mathematicians. The choice of his pseudonym is suggestive of his personality in his young days. The word Mongré is derived from the French phrase "a mon gré" which means "according to my fancy". Thus Hausdorff's literary excursions may be looked upon as an outcome of his inner urge for fanciful writing, something that has no place in mathematics. Hausdorff's literary involvement seems to have ended by 1905 when he turned his attention seriously to mathematical research, whose main thrust was set theory but was no means confined to it.

All writers have described Hausdorff as modest and unassuming. Many have described him as shy, a description which certainly does not apply to his young days when he frequented with artists and writers. It is true, in later years Hausdorff became rather withdrawn; the last mathematical conference he attended was in 1920. Hausdorff has also been described as a critical person with whom few mathematicians would want to open a correspondence. We cannot be sure about the basis of this remark. On the contrary, when Alexandroff and Urysohn wanted to visit him in Bonn, he welcomed them. A part of this correspondence has been quoted by G. Bergmann in his lengthy introduction pp. v - xvi on "Fleix Hausdorff:

Nachgelassene Schriften". It is interesting to remember that Alexandroff and Urysohn also visited M. Fréchet; they recognized and approached Fréchet's pioneering (though diffuse) work in the development of point-set topology. Finally, Hausdorff remains a house hold name to every student of topology and it is no exertion to call him "the creator of point-set topology".

Footnote :

< 1 > Part I was not available to us.

< 2 > Taylor reports [15,p. 303] Fréchet's sensitivity concerning the influence of Hausdorff's *Grundzüge*. Fréchet's daughter told Taylor of her father's annoyed reaction to the credit given to Hausdorff in Bourbaki's History of Mathematics (in French).

< 3 > Fréchet's personal contact with several American mathematicians (as described by Taylor), certainly contributed to furthering his line of research in America.

<4> Hildebrandt (in paper cited on page76) reports (p. 424) that Dirichlet in his lectures on "The Theory of Definite Integrals" of 1854 at the University of Berlin had stated and proved the theorem on uniform continuity (as evidenced by notes of these lectures published only in 1904.

< 5 > Urysohn tragically drowned off the coast of Bretagne (France) while swimming (17 August, 1924).

< 6 > A lengthy review appeared in *Jahrbuch über die Fortschritte der Mathematik* - (1928).

<7> Taylor obviously had no access to the Festschrift itself, for he says of Sir Asutosh Mookerjee "Just who he was and what scientific contact, if any, existed between him and Fréchet are unknown to me".

<8> Also A. Weil's later monograph [17] on uniform spaces appeared in this series.

<9> Indeed, had Hausdorff not burst into the scene with his *Grundzüge* in 1914, this accolade would no doubt belong to Fréchet.

<10> Hausdorff had originally wanted to study music and become a composer. To this his businessman father strongly objected. Hausdorff retained his interest in music throughout his life. Indeed, at several places in the *Grundzüge* (especially p. 62), we find clear evidence of the author's musical bent of mind.

<11> Until well into the twentieth century, the doctorate was the only academic degree at German university. There was (and still is) a state examination for intending high school teachers, but this Teachers Diploma is not, strictly speaking, a university degree.

<12> A pre-requisite for appointment as a university docent (lecturer) after acceptance of a habilitation essay. The candidate is required give a probationary lecture before the faculty. Perhaps, the most famous

probationary lecture was that of Riemann (1854) on the hypotheses which lie at the foundations of geometry.

< 13 > Literary works of Hausdorff published under the pseudonym Paul Mongré are:

1. Sant' Ilario. Gedanken aus der Landschaft Zarathustras, pp. 378, Leipzig, 1897.
2. Das Chaos in kosmischer Auslese, pp. 213, Leipzig, 1898.
3. Ekstasen (Verschiedenes) pp. 216, Leipzig, 1900.
4. Der Arzt seiner Ehre, Berlin, 1912.

< 14 > According to Krull this is borne out by the fact he had to wait eight more years for a similar call (to Bonn), although on account of his published papers, he had long been a thoroughly deserving candidate for a full professorship.

< 15 > Reports on subjects on contemporary interest were an important feature of the *Jahresbericht der Deutschen Mathematiker – Vereinigung* in its early days. The most famous of these reports was Hilbert's "*Theorie der algebraischen Zahlkörpern*" (1897), popularly known as the *Zahlbericht.*

< 16 > "These reports were totally eclipsed by Hausdorff's Grundzüge der Mengenlehre (1914)", writes H. Freudeuthal in the article on Schoenflies in the Dictionary of Scientific Biography.

< 17 > On this point see M. R. Chowdhury, "The Schoenflies – Young Controversy" Jahangirnagar Review A, Volume 5 (1981), 13 – 21. This paper deals with an acrimonious controversy between Young and

Schoenflies; an alternative proof of Young's generalization of the Heine-Borel theorem (cited on page 77) given by Young in 1906, was labeled as incorrect by Schoenflies in the second part of his Report on Set Theory (1908). The ensuring controversy lasted until 1913.

<18 > Indeed, the first sentence of the preface reads "The present work intends to be a Textbook and no Report". Hausdorff obviously wanted to make the point that his *Grundzüge* was not to be seen merely as an improved or up to date version of Schoenflies's Report on Set Theory.

<19>. And for good reason. Though it is a commonplace to say that metric spaces were introduced by Fréchet in his doctoral dissertation whose published version is 'Sur Quelques Points du Calcul Fonctionnel (1906)',

Fréchet never defined a metric in precisely the same form as done later by Hausdorff, Fréchet always spoke of écart and his earliest publication on this subject is 'La notion d'écart et le Calcul fonctionnel, C. R. Acad. Sci. **140** (1905) 772 – 774. We let Fréchet speak for himself. "We consider elements of any nature (points, curves, functions, etc.) such that to each pair A, B of elements there corresponds a well-defined number which is positive or zero, called écart of A and B and denoted by (A, B), satisfying the following properties:

1^0) The écart of A and B is zero if A and B are not distinct and only in this case.

2^0) If A, B, C are any three elements and if the écarts (A, C) and (B, C) are infinitely small, then the same is true for the écart (A, B).

< 20 > It is interesting to observe that Hausdorff and Hilbert both worked on geometry at about the same time. Hilbert's researches culminated in his famous book *Grundlagen der Geometrie* (1901), which exerted a strong influence on the development of abstract algebra.

List of Major References

1. P. S. Alexandroff & H. Hopf : Topologie, Erster Band, Chelsea Publishing Company, 1972. (First published 1935).
2. D.Bushaw : Elements of General Topology, John Wiley & Sons, 1963.
3. N.Bourbaki : General Topology, Part I (translation from French), Addison-Wesley, 1966.
4. B.C.Chatterjee, S.Ganguly & M.R.Adhikari : A Text Book of Topology, Asian Books, 2002.
5. M. C.Gemignani : Elementary Topology, Addison-Wesley, 1967.
6. F. Hausdorff : Grundzüge der Mengenlehre, Chelsea Publishing Company, 1949. (First published 1914).
7. J. G. Hocking & G. S. Young : Topology, Addison – Wesley, 1961.
8. J. L. Kelley : General Topology, D.Van Nostrand Company, 1955.
9. K. Kuratowski : A Half Century of Polish Mathematics, Pergamon Press, 1980.
10. S. Lipschutz, Theory and Problems of General Topology, Schaum's Outline Series, McGraw-Hill, 1965.
11. N. Levine : When are compact and closed equivalent?, Amer. Math. Monthly 72 (1965), 41-44.
12. J. H.Manheim : The Genesis of Point Set Topology, Pergamon, 1964.

13. W. Sierpinski : Introduction to General Topology,University of Toronto Press 1934.

14. L. A. Steen & J. A. Seeback,Jr., Counterexamples in Topology, Holt Rinehart and Winston, 1970.

15. A. E. Taylor : A Study of Maurice Fréchet , II Mainly about his Work on General Topology, 1909 – 1928. Archive for History Exact Sciences, 34 (1985), pp. 279-380.

16. W. J. Thron : Topological Structures, Holt, Rinehart and Winston, 1966.

17. A. Weil : Sur les espaces a structure uniform et sur la topologie generale, Actualités scientifiques et industrielles 551, Herman, 1937.

18. A. Wilansky : Topology for Analysis, Ginn A Xerox Company, 1970.

19. A. Wilansky : Between T_1 and T_2, Amer. Math. Monthly 72 (1965), 261-266.

20. S. Willard : General topology, Addison-Wesley Publishing Company, 1970.