Modified Inferential Methods on Restricted Parameters in Multivariate Regression Analysis: Applications in Socio-demographic Research

A dissertation submitted to the Institute of Statistical Research and Training, University of Dhaka in fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Statistics

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DEDICATION

*This dissertation is dedicated to my beloved sons, **Sheikh Abdur Rahman Bin Sayem** and **Sheikh Abdur An-Naafi'** for his wholehearted affection and providing continuous support of my study.*

CERTIFICATE

This is to certify that Sheikh Mohammad Sayem, PhD Researcher, Institute of Statistical Research and Training (ISRT), University of Dhaka worked on **"Modified Inferential Methods on Restricted Parameters in Multivariate Regression Analysis: Applications in Socio-demographic Research"** for the PhD degree of University of Dhaka under my supervision and guidance. To the best of my knowledge and belief, his work is original and not any part of the subject matter of any degree previously awarded to anybody elsewhere. He is permitted to submit his thesis for the award of the Doctor of Philosophy in Applied Statistics under the University of Dhaka. I wish him every success.

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DECLARATION

I do hereby solemnly declare that the dissertation entitled -**"Modified Inferential Methods on Restricted Parameters in Multivariate Regression Analysis: Applications in Socio-demographic Research"** is the outcome of my own endeavor and research. Neither any part of the research outcome has been borrowed from any previous research nor has been submitted for any other degree in any University or Institute.

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Abstract

Efficient and significant empirical estimate of the multivariate regression parameters will be helpful for the policymaker to make the right decisions about sophisticated interrelated issues in the dynamic world. Since the end of the twentieth century, statisticians are going forward to develop unique working methodology for estimating and testing restricted parameters. This study reviews existing methods and proposed modified maximum likelihood estimator (MMLE), modified multivariate t statistic and modified joint confidence regions to get efficient estimates and test statistic for exact linear restricted parameters of multivariate regression with continuous responses. The proposed estimator is unbiased, consistent and relatively efficient than classical maximum likelihood estimator. Likelihood ratio test, modified Akaike information criterion are applied to select the related predictors of multivariate responses. We also proposed a modified maximum likelihood estimator for restricted parameters of multivariate regression with mixed responses and evaluate the performance of the proposed estimation method based on relative efficiency criterion. A Monte Carlo experiment is conducted to examine relative performance of the modified methods.

We also proposed a modified two parameter weighted estimator (MTPWE) to estimate the stochastic linear restricted parameters in multivariate regression analysis. The study has revealed theoretically and numerically that the proposed MTPWE is consistent based on mean square error criterion and relatively efficient than conventional multivariate least square (MLSE) and weighted mixed estimator (MMWME) in multivariate extension. Moreover, A Monte Carlo

simulation experiment has done to ensure a comparison of the MTPWE to the MLSE and MMWME for different restricted parameters of the various levels of correlation and sample size.

The proposed inferential approach has been also applied to detect the numerical nexus among socio-demographic determinants, food expenditure and total monthly expenditure in “Haor” areas of Bangladesh by using Household Income Expenditure Survey (HIES) dataset 2016. The study reveals that logarithm form of total monthly expenditure and food expenditure as multivariate continuous responses are significantly related to total operating land, logarithm form of family size and total monthly income ($p < 0.01$) considering a restriction on the parameters at 5% level of significance. Based on the simulation study and empirical application, the performance of the modified inferential approach is deemed more realistic than the existing methodology.

Acronyms and Abbreviations

AIC	Akaike Information Criterion
BLUE	Best Linear Unbiased Estimator
CCL	Composite Conditional Likelihood
CSI	Constraint Statistical Inference
FS	Logarithm of Family Size
GORIC	Generalized Order Restricted Information Criterion
HIES	Household Income Expenditure Survey
JCI	Joint Confidence Interval
LM	Lagrange Multiplier Test
LRT	Likelihood Ratio Test
MANOVA	Multivariate Analysis of Variance
MFC	Logarithm of Monthly Food Consumption
MLE	Maximum Likelihood Estimator
MLSE	Multivariate Least Squares Estimate
MMLE	Modified Maximum Likelihood Estimate
MMWME	Modified Multivariate Weighted Mixed Estimate
MND	Multivariate Normal Distribution
MRA	Multivariate Regression Analysis
MRACR	Multivariate Regression Analysis for Continuous Responses
MRLE	Modified Restricted Least Squares Estimate
MTPWME	Modified Two Parameters Weighted Mixed Estimator
MVUE	Minimum Variance Unbiased Estimate

MRLE	Modified Restricted Least Squares Estimate
MRMR	Multivariate Regression with Mixed Response
MSRMLE	Multivariate Restricted Maximum Likelihood Estimate
MVL	Multivariate Logit
OLS	Ordinary Least Square
ORIC	Order Restricted Information Criterion
RB	Relative Bias
RE	Relative Efficiency
REML	Restricted Maximum Likelihood Estimator
TME	Logarithm of Total Monthly Expenditure
TMI	Logarithm of Total Monthly Income as a Worker
TOL	Total Operating Land
W	Wald Test

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PART I

Introduction

Chapter 1

Introduction

1.1 Prelude

Inferential statistics is a wide-ranging discipline based on mathematics, empirical science, and philosophy used to draw a conclusion about a particular parameter of population using sample data. Efficient and significant empirical estimate of population parameters will be helpful for the policymaker to take the right decisions about sophisticated issues in the dynamic world. Sometimes in real-life, parameter originates from the complex domain and covered by different sorts of restriction. The nature of restrictions can be exact linear, stochastic or inequality constraints. Since the end of the twentieth century, statisticians are going forward to develop a unique working methodology for estimating and testing the restricted parameters mathematically as well as statistically in the contemplated decision-making process. Constraint statistical inference (CSI) has grown out for cause and effect analysis of survey or experimental data in various interdisciplinary fields especially, where restrictions are attached with parameters of a statistical model.

Multivariate regression analysis (MRA) a methodology of the statistical modeling used to assess the effect of the predictor variables on the interrelated responses. It can also be used for predicting response variables from a collection of predictor variables. MRA has scope to fortify its applications in almost all walks of life and science. However, in the dynamic world, multivariate regression models are difficult due to a complex network of restriction either in the unknown parameter space in a parametric way or even in the sample space of the random observations. Hence, every tire of the statistical modeling says model specification, parameter estimation, the test of significance of the parameters, criterion of appropriate model selection fall in the new challenge. Now, it became necessary to modify the inferential methods for compatible developments in statistical decision theory and thereby need to regulate the appropriate methodology to meet the demand of the vast interdisciplinary fields of applications.

1.2 Background of the Study

Multivariate techniques have emerged as a powerful inferential tool to analyze relationship among multiple variables at a time. It plays a vital role in developing multivariate estimator for estimating the parameter vector and to evaluate the estimated parameter mostly by testing significance of the parameters. However, statisticians have faced crux when the prior knowledge about the nature of parameters is predefined and multicollinearity problem is raised among the predictors in cause and effect analysis. Ordinary least square or maximum likelihood methods are not efficient to estimate the parameters in such situations.

When the researchers are going to test the parameter jointly in multivariate regression or in multivariate analysis of variance (MANOVA), the test statistics available in existing literature are Likelihood Ratio, Wilks' lambda, Pillai's trace, Hotelling-Lawley trace and Roy's greatest root (Johnson and Wichern, 2013).The power of general Likelihood Ratio or Wilks' lambda test to test the parameters jointly in multivariate regression is also weak (Silvapulle,2006). Generally, the distribution of the error term of multivariate multiple regression models is multivariate normal (Anderson, 2003). But most of the time, the demographic variables may be qualitative or quantitative. One example of the response vectors may be that it consists of desired family size, contraceptive use, marital status which comprises both qualitative and quantitative variables, may not be multivariate normal. Hence, multivariate regression analysis faces a labyrinth to estimate regression parameters or testing it.

1.3 Statement of the Problem

Multivariate regression analysis is getting increasing attention among the socio-demographers due to the complex nature of the interdependence of the response variables. It is more efficient than regression analysis for each dependent variable separately when the correlation structure among the dependent variables is present (Hartung and Knapp, 2014). MRA allows a researcher to answer several questions, such as whether there is a significant relationship between the criterion and predictor variables, whether a given subset of predictors is really important to the relationship or not, and whether the predictors are able to explain a significant amount of the total variation among the criterion variable (Kshirsagar and Ravindra, 2014). The multivariate linear regression has been discussed in both theoretical and applied statistics (Anderson, 1951; Johnson and Wichern, 2013; Bilodeau and Brenner; 1999, Rencher, 2002). This empirically, statistically and mathematically mature method is needed to deal with big data challenge in the digital world. In the complex domain, the multivariate regressions are generally not so simple due to a complex network of constraints in the space of the unknown parameters in a parametric way. There is therefore a growing need for statistical inference to cope with such constrained environments. However, very little research has addressed the problem of i) estimating multivariate regression parameters considering exact or stochastic linear restriction and ii) testing the restricted parameters individually. Detection of the critical value(s) for the test statistic related to the restricted hypothesis is not smooth due to the complex nature of the exact null distribution. Though several information criterions are used to investigate the best model after selecting the appropriate number of variables, the need to develop a consistent model selection criterion considering parameter restriction remains unanswered.

Bangladesh has experienced promising improvements in its overall economic, social and health conditions but the progress is not up to the mark in Haor areas. The socio-economic condition of day laborers and other workers in Haor areas are volatile where predictors may follow some restrictions (linear or stochastic) leading to one-sided hypothesis testing in modeling exercise. This study has tried to apply multivariate regression with restricted parameter for detecting major predictors, like family size, access to safety-net program, income and operating land, of monthly food consumption and overall expenditure.

However, the concept of multivariate regression faces hurdles when response matrix is categorical or mix of both categorical and numerical variables (and both continuous and ordered discrete variables), which is a common scenario in socio-economic and demographic analysis. The way of estimating the joint distribution of mixed responses in multivariate regression is complex and unknown in most of the cases. Consequently, parametric methods to estimate the parameters have still not been established and testing procedure to test the significance of the regression coefficients are needed to be modified.

So, a big challenge for social statisticians is to develop appropriate model by addressing these issues and seek out efficient estimation and powerful hypothesis testing procedure to fit model correctly and predict about future phenomenon.

1.4 Objectives of the Study

The main objective of the study is to develop modified inferential approach on restricted parameters in multivariate regression analysis and its applications in socio-demographic research.

The specific objectives of the study are to-

i. develop appropriate estimation strategy for exact restricted parameters of multivariate regression with continuous responses;

ii. develop suitable testing procedure for multivariate regression parameters of continuous responses under exact restricted alternatives;

iii. construct modified estimators of multivariate regression parameters with continuous responses under stochastic restrictions;

iv. construct modified estimators of multivariate regression parameters with mixed responses;

v. conduct Monte-Carlo simulation to check the adequacy and appropriateness of the suggested methods; and

vi. apply the modified approach to socio-demographic real life data.

1.5 Functional Definition and Notation of Important Terminologies

Constraint Statistical Inference: Statistical models in real-life interest as well as in interdisciplinary research are generally complex in their designs, sampling methodology, associated probability laws, which in turn are often constrained by exact or stochastic restrictions, order, functional, shape or other restraints. In such situations, constrained statistical inference is a branch of decision theory; used for parameter estimation and hypothesis testing considering the parameter restrictions, in the process.

***Linear Regression*:** Linear regression is a statistical technique which is useful for predicting one set of response variables from another set of predictor variables. The response variables will be discrete, continuous or mixed. The model of linear regression is given as

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \tag{1.1}$$

where, $\boldsymbol{Y}$ is the $n \times p$ matrix of response variables and $\boldsymbol{X}$ is the $n \times (k+1)$ predictors. $\boldsymbol{\beta}$ is a $(k+1) \times p$ matrix of unknown parameters and $\boldsymbol{\varepsilon}$ is $n \times p$ matrix of random disturbances.

***Restricted Parameter Space*:** The parameter space $\mathcal{B}$ is the space of possible parameter values that reflects the feasible states of nature relative to the unknown parameters matrix $\boldsymbol{\beta}$ in linear regression. If the prior or non-sample information that reflects knowledge other than that derived from the statistical investigation is amalgamated with unknown parameters, parameter space is defined as restricted parameter space.

***Exact Linear Restriction*:** If the exact information on the particular parameter or linear combination of the parameters in a linear regression model is available by the investigator, the restriction is called exact linear equality restrictions. The form of exact linear equality restrictions:

$$\boldsymbol{R\beta} = \boldsymbol{\xi} \tag{1.2}$$

where $\boldsymbol{R}$ is a $q \times (k+1)\ and\ q < k$ known prior information design matrix that expresses the structure of the information on the individual parameters or some linear combination of the elements of known elements $\boldsymbol{\beta}$ matrix and $\boldsymbol{\xi}$ is a matrix of known elements of order $q \times p$.

***Stochastic Linear Restriction*:** The restrictions may be called stochastic linear restrictions, if uncertainty exists about the prior or non-sample information specification in equation (1.2). The form of stochastic linear restriction can be expressed as

$$\boldsymbol{R\beta} + \boldsymbol{\nu} = \boldsymbol{\xi} \qquad (1.3)$$

$\boldsymbol{\nu}$ is a $q \times p$ unobservable normally distributed random vector with mean vector δ and covariance $\boldsymbol{\Psi}$.

Inequality Restriction: In applied research, there exists in many cases prior information concerning the non-negativity or non-positivity of a regression parameter or linear combination of parameters, or that a parameter or a linear combination of parameters lies between certain upper and lower bounds or that functions are monotonic, convex or quasi-convex. When information of this form is available, it can be presented by the inequality restrictions. The form of inequality restrictions is given by

$$\boldsymbol{R\beta} \geq \boldsymbol{\xi} \qquad (1.3)$$

Uniformly Minimum Variance Unbiased Estimator: Uniformly minimum variance unbiased estimator (UMVUE) is an unbiased estimator that has minimum variance than any other unbiased estimator for all possible observable values of the parameters.

The modified estimator of regression parameter $\boldsymbol{\beta}$ is called UMVUE if and only if 1) modified estimator is unbiased and 2) the variance of modified estimator is minimum compared to that of any other unbiased estimator.

***Relative Efficiency*:** The relative efficiency of modified estimators is the ratio of the variance of modified estimator and the variance of other estimator.

$$\text{Relative Effeciency} = \frac{\text{Variance of modified estimator of } \boldsymbol{\beta}}{\text{Variance of existing estimator of } \boldsymbol{\beta}}$$

The value of relative efficiency is greater than 1 indicates that modified estimator is preferable.

***Coverage Probability*:** The coverage probability of a technique for calculating confidence interval is the proportion of time that the interval contains the true value of interest. Confidence interval with shortest length and high coverage probability is better than others.

Restricted Hypothesis: If C and $\boldsymbol{\mathcal{M}}$ are subsets of an Euclidian space, then restricted hypotheses are

Type A: $\mathcal{H}_0$: $\boldsymbol{\beta} \in \boldsymbol{\mathcal{M}}$ against $\mathcal{H}_1$: $\boldsymbol{\beta} \in \mathsf{C}$ and $\boldsymbol{\beta} \notin \boldsymbol{\mathcal{M}}$

Type B: $\mathcal{H}_1$: $\boldsymbol{\beta} \in \mathsf{C}$ against $\mathcal{H}_2$: $\boldsymbol{\beta} \notin \mathsf{C}$.

1.6 Conceptual Framework of the Study

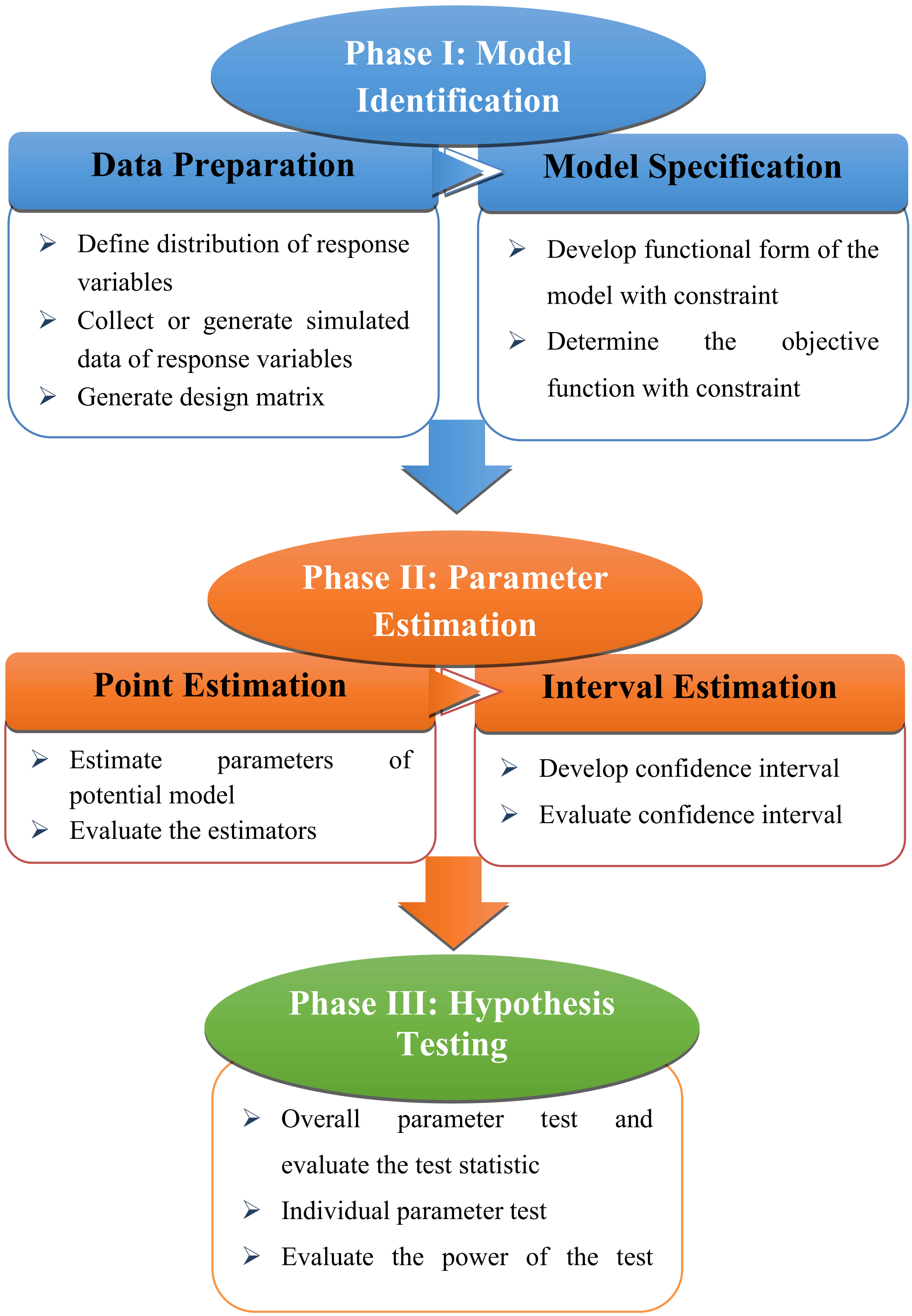


Figure 1.1: Conceptual Framework for Developing Modified Inferential Methods for MRA

1.7 Source of Data

The multivariate data have been simulated from multivariate normal, multivariate Bernoulli distribution and mixed distribution to achieve specific objectives I to VI. The Household Income Expenditure Survey (HIES) data 2016 have been collected and partially used for achieving specific objective VII.

1.8 Statistical Computation

Numerical computation is a challenging task for modifying statistical methodology. Complex statistical modeling faces computational obstacles about Monte Carlo methods, random number generation, big data management, numerical optimization in statistical inference for estimating and testing parameters. Programming language R (graphical user interface, RStudio), statistical software (IBM SPSS Statistics, Stata), Spreadsheet (Microsoft Excel) have been used for the purpose of statistical computation.

1.9 Synoptic View of the Study

This chapter introduces the background of the study, statement of the problem upon which the study is based. The study objectives are defined and relevant concepts are delineated. Contributions of the study are discussed. Operational terminologies and concepts, statistical computation of this study are defined. Chapter II reviews the relevant literature to multivariate regression with continuous, categorical and mixed responses. Estimation and testing procedures related to multivariate regression are discussed. The theoretical background and previous conceptual and empirical research findings are discussed. Chapter III (for objectives I, II and VI) discusses modified inferential approach for multivariate regression with continuous responses considering parameter restriction. Monte Carlo simulation and real life applications of the methods are also presented in this chapter. Chapter IV (for objectives III, V and VI) presents a detailed discussion on multivariate regression with continuous responses, model specification for stochastic restricted parameters, modified estimation and testing methods. Monte Carlo experiments are conducted to evaluate the performance of modified techniques. Chapter V (for objectives IV and V) documented the methodology of estimating and testing the parameter of multivariate regression with mixed responses. Again, Monte Carlo study is performed to evaluate the proposed estimating and testing methods. Chapter VI (for objective VII) presents a real life application of constraint statistical inference**.** Chapter VII conclusions and the implications of the research are delineated and future research directions are presented.

PART II

Multivariate Continuous Responses with Restricted Parameters

Chapter 2

Literature Review of Multivariate Continuous Responses

2.1 Introduction

Linear regression is a statistical methodology used to predict one set of response variables from another set of predictor variables. Both univariate and multivariate regression plays a pivotal role to find significant predictors in decision-making problems. Multivariate linear regression attempts to investigate any significant relationship among responses and predictors; to diagnosis which predictors have dominant effect on the relationship and finally find the answer to the question whether the significant amount of total variation among the response variables can be explained through the predictors (Kshirsagar and Ravindra, 2014).

The nature of the response variables in classical multivariate regression model must be continuous. The multivariate linear regression has been discussed in both theoretical and applied statistics (Anderson, 1951; Johnson and Wichern, 2013; Bilodeau and Brenner, 1999; and Rencher, 2002).

In this chapter, we review the different aspects of multivariate regression for continuous responses considering different parameter constraints. Section 2.1.1 defines the concept and notation of multivariate regression for normal variates. In Section 2.2, we describe the estimation technique of the restricted parameters for univariate regression. Section 2.3 reveals the overall testing procedure of the restricted parameters. Sections2.4 and 2.5 have mentioned the individual testing procedure and joint confidence interval of the restricted parameters respectively. In section 2.6, we conclude the chapter by outlining what we will investigate in the next chapters for continuous responses.

2.1.1 Classical Multivariate Regression with Continuous Responses

Classical univariate regression analyses can assess the relationship among univariate response variable and a set of explanatory variables. But, multivariate regression is used to explain the relationship between more than one quantitative response variables and a set of quantitative explanatory variables. When a correlation structure among the response variables is present, a multivariate regression model is considered as more efficient than regressions analysis for each response variable separately. So, MRA is employed to make inference about correlated response variables based on parsimonious number of predictors when both predictors and correlated response variables are quantitative in nature (Hartung and Knapp, 2014).

Let $\boldsymbol{Y}$ be an $n \times p$ observation matrix of p continuous multivariate response variable and $\boldsymbol{X}$ be a design matrix of $n \times (k+1)$ nonstochastic predictor variables with rank $k \leq n$ where n is the sample size. A multivariate regression model is given as

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}) + \boldsymbol{\varepsilon} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

Here, the distribution of random disturbances is multivariate normal, $\boldsymbol{\varepsilon} \sim MND(0_{n\times p}, \Sigma_{p\times p})$. $\boldsymbol{\beta}$, $\boldsymbol{\Sigma}$ are the unknown parameters where $\boldsymbol{\beta}$ is a $(k+1) \times p$ matrix of regression co-efficient (Johnson and Wichern, 2013).

Estimating unknown parameters of the MRA is one of the challenging issues. The first need of MRA is to fit model from the observed data considering number of assumptions through estimating unknown parameters using maximum likelihood estimator when the distribution of the responses is multivariate normal or by using least-squares approach when no distributional assumptions are made. But, the multivariate regressions are generally not so simple due to a complex network of constraints in the space of the unknown parameters in a parametric way. Hence, there is a growing need for statistical inference to cope with these sorts of constrained environments. Ordinary least square or maximum likelihood methods are not efficient to estimate the restricted parameters. Different studies were conducted separately for estimating the restricted parameters and testing the overall significance of restricted parameters for classical multiple linear regression (Atiqullah, 1969; Nancy, 2014; Speed, 2014; Silvapulle, 2006).

2.2 Estimating Technique of the Restricted Parameters

Statistical inference problem in which stochastic model faces several types of restriction especially, exact linear restriction, stochastic linear restriction, inequality restriction. Ordinary least square or maximum likelihood methods do not take challenges to find standard results of the restricted parameter of the regression model where the distribution of the random error is normal (Silvapulle, 2006). Atiqullah (1969) described a restricted least square estimator for general linear model considering exact linear parameter restriction.

2.2.1 Restricted Least Square Estimator

Let $\boldsymbol{Y}$ be a $n \times 1$vector of observations and $\boldsymbol{X}$ be a full rank design matrix of $n \times k$ nonstochastic predictor variables where n is the sample size. Consider a general linear model as

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}, \text{ with prior restriction } \boldsymbol{R\beta} = \boldsymbol{\xi}$$

The method of Sweep-Out is used to obtain computing formulas for calculating the least squares estimator and its variance matrix in the linear models not necessary of full rank, in which certain restrictions may hold on the actual parameters (Atiqullah, 1969).

$$\widetilde{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}} - \boldsymbol{C}^{-1}\boldsymbol{R}^t(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}^t)^{-1}(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \xi) \; where \; \boldsymbol{X}^t\boldsymbol{X} = \boldsymbol{C}$$

$$\boldsymbol{E}(\widetilde{\boldsymbol{\beta}}) = \boldsymbol{\beta}$$

$$\boldsymbol{V}(\widetilde{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{R}^t(\boldsymbol{R}\boldsymbol{C}^{-1}\boldsymbol{R}^{\boldsymbol{t}})^{-1}\boldsymbol{R}\boldsymbol{C}^{-1})$$

2.2.2 Restricted Maximum Likelihood Estimation

Restricted maximum likelihood estimation (REML) is an approach to estimation that maximizes likelihood over a restricted parameter space (Nancy, 2014). Parameters in dispersion matrices can be estimated by using restricted maximum likelihood estimation (REML) method which can be taken as a substitute to its main contender profile maximum likelihood because, use of profile likelihood may lead to badly biased estimators of parameters of interest when there are a large number of nuisance parameters whereas REML considers degrees of freedom lost in estimating parameters in a model for expected values and gives estimators of the remaining parameters with less bias and better consistency properties (Speed, 2014).

In a general linear model with normal error distribution, REML is an unbiased estimator when the distribution of the response variable is normal. The method has been applied to situations where the parameters satisfy order restrictions (Nancy, 2014). In the multiple linear regression analysis, the linear model is given below

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{b} + \boldsymbol{\varepsilon}$$

where$\boldsymbol{Y}$ is an $n \times 1$ observed data vector, $\boldsymbol{\beta}$ is a $(k+1) \times 1$ vector of fixed effect parameters with $\boldsymbol{X}$ be a design matrix of $n \times (k+1)$, b is a $l \times 1$ vector of random effects, **Z** is a design matrix with dimension $n \times l$and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of error terms which are independent and distributed as $N(0, R)$. The variance of $\boldsymbol{Y}$ is $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{D}\boldsymbol{Z}^{\boldsymbol{t}} + \boldsymbol{R}$. The elements of $\boldsymbol{D}$ and $\boldsymbol{R}$ may be taken to be functions of an unobservable parameter vector. The REML estimators of $\boldsymbol{\beta}$ and its variance are

$$\boldsymbol{\beta}_R = \left(\boldsymbol{X}^t \boldsymbol{V_R}^{-1} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^t \boldsymbol{V_R}^{-1} \boldsymbol{Y}$$

and

$\boldsymbol{V}(\boldsymbol{\beta_R}) = \left(\boldsymbol{X}^t \boldsymbol{V_R}^{-1} \boldsymbol{X}^{-1}\right)$ where $\boldsymbol{V_R} = \boldsymbol{Z}\boldsymbol{D_R}\boldsymbol{Z}^t + \boldsymbol{R_R}$

2.3 Overall Testing Procedure of Restricted Parameters

After estimating the restricted parameters of the regression model, the researcher is usually inquisitive in testing one or more linear hypotheses about individual restricted parameters or linear combination thereof. The Lagrange multiplier test (LM), Likelihood Ratio test (LR) and Wald test (W) are frequently used statistic for testing parametric restrictions in the linear regression model (Wolak, 1989). The first document on this issue seems to be due to Gourieroux et al. (1982), succeeded by the celebrated paper by Self and Liang (1987). Based on the first two documents, Mukerjee and Tu (1995) published a paper related to the exact small sample LRT and also discussed its properties in the case of a classical simple linear regression model with the non-negativity restriction. Multivariate linear regressions are broadly used statistical technique in many applications to model the associations between multiple related responses and a set of predictors. Likelihood ratio test (LRT) is again one of the most popular methods to test the structure of regression coefficient in such cases (Fujikoshi, 1974). Let a classical multivariate linear regression model is

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \text{ where } \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_{(\mathbf{1})} \\ \cdots\cdots \\ \boldsymbol{\beta}_{(\mathbf{2})} \end{bmatrix}$$

and the functional form of likelihood ratio test statistic will be

$$\Lambda = \frac{max_{(\boldsymbol{\beta}_{(\mathbf{1})},\boldsymbol{\Sigma})} L(\widehat{\boldsymbol{\beta}}_{(1)}, \widehat{\boldsymbol{\Sigma}})}{max_{(\boldsymbol{\beta},\boldsymbol{\Sigma})} L(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Sigma}})}$$

So, the likelihood ratio test is based upon the difference between the maximum of the likelihood under the null and under the alternative hypothesis. The restricted likelihood ratio test is uniformly more powerful than the global version for the entire restricted parameter space in many cases (Tsai, 1992).

The exact knowledge of the null distribution of the hypothesis test statistic is essential for the reasonable use of the test statistic. The distribution of two-sided LR tests follows asymptotically central Chi-square (Johnson, 2013) under the null hypothesis. However, in a socio-demographic study, the hypothesis is not always exactly two-sided. A relevant theory related to inequality constraints is scattered in the prior literature under different names especially order restricted inference or one-sided testing. A mixture distribution was applied to multivariate LR tests by different researches where the asymptotic null distribution of the tests was explained to be a mixture of different chi-squared type distribution with binomial mixing probabilities (Silvapulle, 2006).

Silvapulle (2006) addressed and reviewed a substantive number of research related to LRT for testing constrained parameters in different situations. But, the complexity of the procedure has been raised for different reasons, i) most of the times the dispersion matrix of the error is unknown, ii) matching the null distribution of the test statistic for complex hypothesis is difficult. Fonseca *et al.* (2015) has given a path for testing linear inequality constraint on the regression coefficient of univariate model and developed the LRT for unknown error variance which is in the same line as in Mukerjee and Tu (1995).

2.4 Individual Testing Procedure of Restricted Parameters

The classical theory of the statistical inference is primarily flourished on the assumptions that the distributions of target variables are normal. But, most of the times several estimators of socio-economic and business data exhibit fat-tailed distributions. Recently many authors have investigated as to how inferences are affected if the distributional pattern of the statistic departs from normality (Kibria and Joarder, 2006). The suitability of the application of t statistic in real life was assessed by Blettberg and Conedes (1974). The multivariate t statistic is a natural generalization of the univariate Student t statistic which is a more viable alternative to test the significance of the parameter vector. Kelker (1970), Cambanis, Huang and Simons (1981), Fang and Anderson (1990), Kotz and Nadarajah (2004) also discussed the significance of this argument. A p dimensional random vector $\boldsymbol{T} = \left(T_1, T_2, \ldots, T_p\right)^t$ is said to have the p variate t distribution with degree of freedom v, mean vector $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$ and correlation matrix$\boldsymbol{\rho}$, if the statistic can be expressed as

$$\boldsymbol{T} - \boldsymbol{\mu} = \frac{\boldsymbol{Y}}{\sqrt{\frac{\boldsymbol{u}}{v}}} \ ,$$

Where $\boldsymbol{Y}$ is a p variate normal random vector with mean zero and covariance matrix $\boldsymbol{\Sigma},$ and if $\boldsymbol{u}$ is a chi-square variate with degree of freedom v. The functional form of probability density function of $\mathbf{T}$ with parameters $\boldsymbol{\Sigma}, \boldsymbol{\mu}$ and v is given by

$$f(\mathbf{T}) = \frac{\Gamma\left[\frac{(v+p)}{2}\right]}{\Gamma(\frac{v}{2}) v^{\left(\frac{p}{2}\right)} \Pi^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} \left[\mathbf{1} + \frac{1}{v}(\mathbf{T}-\boldsymbol{\mu})^{t} \boldsymbol{\Sigma}^{-1} (\mathbf{T}-\boldsymbol{\mu})\right]^{-\frac{(v+p)}{2}}$$

where $\frac{v}{v-2}\boldsymbol{\Sigma}$ is the covariance matrix if $v \geq 2$ (Kotz and Nadarajah, 2004).

The ultimate target of decision-makers is to improve the power of the test statistic in different circumstances. But, sometimes researchers faced the problem of power loss when testing the hypothesis of linear equality against the hypothesis of linear inequality in linear regression models. The study of one tail alternatives hypothesis testing was originally addressed by Bartholomew (1959) for independent linear models considering normality assumption and extended by Kudo (1963) for multivariate linear models. Nuesch (1966) also treated this study for the classical linear model while Perlman (1969) expanded the outcomes for a more general class of multivariate normal models.

Gourieroux et al. (1982) discussed the asymptotic null distribution of the one-sided test statistics in multivariate normal models when the variance-covariance matrix may depend on a finite number of unknown parameters. Wolak (1987) proposed exact one-tail tests for classical multivariate linear models and Wolak (1989) extended the results from Gourieroux et al. (1982) for restricted hypotheses. Kodde and Palm (1986) presented a Wald-type test that may be used for testing equality and inequality restrictions in general multivariate regression models and Silvapulle (2006) described a score-type test for assessing one-tail alternatives in general regression models that may include correlated observations also. The score test and Wald test statistics which are asymptotically distributed as a mixture of chi-square distributions where the weights may depend on the correlations but not depend on the null parameters. Application of multivariate t statistics is a very promising approach in multivariate analysis. If the distribution of Y follows multivariate normal with a certain mean vector, then the critical region of the test statistic follow non-central multivariate t distribution. Kshirsagar's (1961), Siotani's (1976), Arellano-Valle and Bolfarine's (1995), Fang et al. (2002), Gupta (2000) and Jones' (2002) proposed and modified functional approach of non-central multivariate t in different circumstances.

2.5 Joint Confidence Region of Restricted Parameters

Joint confidence intervals (JCI) construct a joint confidence region (JCR) for a vector of parameters, comprising individual intervals for the separate components, with a coverage confidence level of the simultaneous correctness of all the statements involved (Chen and Hoppe, 2017). Joint confidence regions are numerically analyzed by comparison for regression, the mean value of multiple responses, regression coefficients and individual observation (Belov, 2018). Confidence intervals are used in socio-economic research to indicate the degree of uncertainty in estimates due to random error. It is reasonably well-known that one can get a statistical significance test by constructing confidence interval around an obtained statistic and seeing whether or not the corresponding hypothesized parameter is "captured" by the interval (Knapp, 2017). Since the distributions of the regression parameter of classical multivariate regression with continuous responses are multivariate normal. This study has been derived joint confidence regions for the regression vector with a specified coverage probability.

Many methods are used to computing JCI for fitted values and linear combination of regression coefficients. The commonly used procedures are the Scheffe´, Duncan, Tukey and the Bonferroni method. In the multivariate setting, it is difficult to find the exact joint confidence regions in balanced or unbalanced models. In order to respond to the problem, different approximation procedures are used to obtain good approximate joint confidence regions. A quantitative way of obtaining the critical value that determines the joint confidence region of a given level has been applied to overcome this sort of problem (Belov, 2018). The effective method to construct joint confidence regions with prescribed coverage probability for the regression parameters evaluated at different settings of the predictor variables, which are narrower than bounds obtained without using the predictor constraints.

2.6 Model Selection Technique

Model selection is an important task of any statistical analysis for selecting a well fitted stochastic model from a candidate set of models that fits the input data. Kullback and Leibler (1951) derived information measure known as the Kullback-Leibler (K-L) distance. Later, the K-L distance defined as a directed distance between two models which is the most fundamental of all information measures and it is the logical basis for model selection. In general statistical models, it is called model selection (Linhart and Zucchini, 1986) and especially, in the regression model, it is well-known as a variable selection (Miller, 1990).

Akaike information criterion (AIC), which was proposed by Akaike (1973) and is an estimator of risk based on the Kullback–Leibler (K–L) information between the true model and the candidate model, is being used universally in selecting variables. Akaike (1973) evaluated AIC under the assumption that the distribution of candidate models corresponds to the true distribution. Hence a correction term for the bias of risk in AIC is fixed for any distribution, even if its risk is changed by the true distribution. Akaike Information Criterion is defined by

$$\text{AIC} = -2\{\text{maximum log likelihood} - \text{no. of unknown parameter}\}$$

$$= nln\left|\hat{\Sigma}\right| + np(ln2\pi + 1) + 2\{kp + \frac{1}{2}p(p+1)\}$$

The AIC is considered as an approximately unbiased estimator for R(x), but in the case of over specified candidate model and for small sample size, AIC drastically underestimate the R(x) in multivariate normal regression model (Hurvich and Tsai, 1989). AIC was modified by Satoh (1997) using Corrected AIC (CAIC) and the modified AIC became unbiased estimator for both under specified and over specified models. Modified AIC (MAIC) is written below:-

$$\text{MAIC} = \text{CAIC} + 2ktr(\hat{\Lambda} - I_p) - \{tr(\hat{\Lambda} - I_p)\}^2 - tr\{(\hat{\Lambda} - I_p)^2\},$$

where, CAIC $= n\ln\left|\hat{\Sigma}\right| + np\ln 2\pi + \frac{(n+k)np}{n-k-p-1}$

and $\widehat{\Lambda} = \frac{\mathrm{n-k}}{\mathrm{n-k_F}}\hat{\Sigma}_{\mathrm{F}}\Sigma^{\Lambda-1}$.

Inequality constraints or order restrictions is not considered in AIC to select model. Traditionally AIC does not incorporate order restriction to select appropriate model, but simple order restriction is included by Anraku (1999) in Order Restricted Information Criterion (ORIC), $-2\{l(\widetilde{\theta},\ \tilde{\sigma}) - inf_{\theta,\sigma}\, B(\theta,\sigma)\}$. The simple closed form of penalty term (optimum bias)$\inf_{\theta,\sigma}B(\theta,\sigma)$ is $\{1 + \sum_{i=1}^{p} iw_i\,(p, W, C)\}$ for the restriction $R\theta > 0$, which is applicable for simple order and tree ordered restriction (Silvapulle and Sen, 2005).

Generalized Order Restricted Information Criterion (GORIC) proposed by (Kuiper, Hoijtink and Silvapulle, 2011) is GORIC $= -2\{l\,(\widetilde{\theta},\ \tilde{\sigma})\ - 1 - \sum_{i=1}^{p} iw_i\,(p, W, C)\}$,

where, $l(\widetilde{\theta},\ \widetilde{\sigma})$ is the maximum log likelihood and penalty term is the nonnegative constants; known as chi-bar square weights which arises naturally in constrained statistical inference and defined in the chi-bar square distribution, $pr(\tilde{X}^T W^{-1}\tilde{X}\ \le c)\ =\ \sum_{i=0}^{p} w_i\,(p, W, C)pr(\chi_i^2\ \le c)$ (Silvapulle and Sen, 2005) and (Silvapulle, 1996).

For simple order restriction GORIC reduces to Anraku (1999) model and when no inequality constraints imposed on θ, then for w_p (p,W,C)=1, w_i(p, W, C) = 0; i < p the GORIC reduces to AIC. Probability of choosing correct model using GORIC approaches is unity when sample size tends to infinity(∞).

Chapter 3

Modified Inferential Approach for Multivariate Continuous Responses with Exact Restrictions

3.1 Multivariate Regression Analysis for Continuous Responses (MRACR)

Classical multivariate regression analysis emphasizes the use of sample information in making inferences about the unknown parameters (matrix of regression coefficients) ignoring any non-sample information that may exist about the individual parameters or relationships among the unknown parameters. But, nexus among non-sample information and unknown parameters in the model can develop such statistical decision which accountable for the creation of inefficient improvement policy. Appropriate model specification, efficient estimation technique and powerful hypothesis testing method are needed in these circumstances. Recognizing different constraint of unknown parameters, we consider a modified approach for MRA with continuous responses where can be considered both sample and non-sample information.

In this chapter, different aspects of multivariate regression with exact linear restriction have been described (Sayem and Hossain, 2022). Section 3.1; define the specification of the multivariate regression model with the description of exact linear restriction in subsection 3.2.1. In section 3.3 and subsection 3.3.1 have been explained the modified maximum likelihood estimator and its properties. Section 3.4 reviewed the likelihood ratio test for overall regression parameters whereas section 3.5 described the modified t test for individual regression parameters with exact linear restriction. A modified joint confidence interval for individual regression parameters with exact linear restriction is explained in section 3.6. In section 3.7, Monte Carlo experiment has been conducted to evaluate the performance of modified methods and finally in section 3.8 has been recorded the overall conclusion.

3.2 Model Specification for MRACR

Let $\boldsymbol{Y}$ be an $n \times p$ observation matrix of p continuous multivariate response variable and $\boldsymbol{X}$ be a design matrix of $n \times (k+1)$ nonstochastic predictor variables with rank $k \leq n$ where n is the sample size. Consider the following multivariate regression model

$\boldsymbol{Y} = f(\boldsymbol{X}) + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a$n \times p$ matrix of random error.

Multivariate regression model sometimes faces the challenge of exact linear restriction, stochastic linear restriction and inequality restriction on the parameters.

3.2.1 Exact Linear Restriction in Multivariate Regression with Continuous Responses

There may be precedents in applied research when the researcher has exact information on a particular parameter or linear combination of the parameters. If exact information is available, a multivariate regression model with exact linear restriction is given as

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}, \text{ with prior restriction } \boldsymbol{R\beta} = \boldsymbol{\xi} \text{ or } \boldsymbol{R\beta} \geq \boldsymbol{\xi} \tag{3.1}$$

Assumptions:

a) The values of predictor variables of the design matrix are fixed.

b) The distribution of random disturbances is multivariate normal i.e.$\boldsymbol{\varepsilon} \sim MND(\mathbf{O}, \boldsymbol{\Sigma})$.

c) $\boldsymbol{\beta}, \boldsymbol{\Sigma}$ are the unknown parameters where $\boldsymbol{\beta}$ is a $(k+1) \times p$ matrix of regression coefficient.

d) The exact linear restriction exists among the subset of the parameters which is believed to be true for each response.

3.3 Modified Maximum Likelihood Estimator (MMLE)

On the parameter space of the regression coefficient, consider the following restriction

$$\boldsymbol{R\beta} = \boldsymbol{\xi}, \tag{3.2}$$

Where $\boldsymbol{R}$ is a matrix of order $q \times (k+1)$and $q < k$of known elements and $\boldsymbol{\xi}$ is a matrix of known elements of order $p \times p$. In case of no restrictions, the probability density function of multivariate response becomes

$f(y|\boldsymbol{\beta}, \boldsymbol{\Sigma}) = (2\Pi)^{-\frac{np}{2}} |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}[(\boldsymbol{Y} - \boldsymbol{X\beta})^t (\boldsymbol{Y} - \boldsymbol{X\beta})\boldsymbol{\Sigma}^{-1}]\right\}$, t is a symbol of transpose.

In order to estimate the regression coefficient, the following log-likelihood function is needed to be maximized

$$\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma}) = \sum_{i=1}^{l} \left(-\frac{1}{2}\right) \{\ln|\boldsymbol{\Sigma}| + (\boldsymbol{Y}_i - \boldsymbol{X\beta})^t \boldsymbol{\Sigma}^{-1} (\boldsymbol{Y}_i - \boldsymbol{X\beta})\}. \tag{3.3}$$

Using equation (3.3), the maximum likelihood estimator (MLE) of $\boldsymbol{\beta}$ is $\boldsymbol{\beta}_{mle} = (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{y}$.

To take the restriction "equation 3.2" into account of the inferential procedure, a modified maximum likelihood estimator (MMLE) for multivariate regression is proposed in a similar manner as used in univariate regression by imposing restrictions on the log-likelihood function (3.3), the objective functions to be maximized, thus becomes

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \ell(\boldsymbol{\beta}) + \lambda^t(\boldsymbol{R\beta} - \boldsymbol{\xi}) \tag{3.4}$$

where λ is a vector of Lagrangian multipliers and t is the symbol of matrix transpose. Method of Lagrange multipliers is a strategy for finding the maximum and minimum of the function subjected to the constraints (Hoffmann, Laurence, Bradley and Gerald, 2004). If the prior information of active constraints were known at the solution to the optimization problem, the solution would be a local maximum point of the problem defined by ignoring the inactive constraint and treating all active constraint as equality constraints (Luenberger, 2003; Silvapulle, 2006) which reflects that changing the right-hand side by a small amount will not affect the optimal solution. To maximize the objective function, firstly need to differentiate (3.4) with respect to $\boldsymbol{\beta}$ and λ and setting $\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}}$ and $\frac{\partial \mathcal{L}}{\partial \lambda}$ equals zero, we derive the modified maximum likelihood estimator of $\boldsymbol{\beta}$ as

$$\widehat{\boldsymbol{\beta}}_{\text{mmle}} = \boldsymbol{\beta}_{\text{mle}} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta}_{\text{mle}} - \boldsymbol{\xi}). \tag{3.5}$$

for $\hat{\lambda} = -2[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{\xi} - \boldsymbol{R}\boldsymbol{\beta}_{\text{mle}})$

3.3.1 Properties of Modified Maximum Likelihood Estimator (MMLE)

The expected value of modified maximum likelihood estimator (MMLE) is given below

$$\begin{aligned} E(\widehat{\boldsymbol{\beta}}_{\boldsymbol{mmle}}) &= E[\boldsymbol{\beta}_{\mathbf{mle}} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta}_{\mathbf{mle}} - \boldsymbol{\xi})] \\ &= E[\boldsymbol{\beta}_{\mathbf{mle}}] - E[(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta}_{\mathbf{mle}} - \boldsymbol{\xi})] \\ &= \boldsymbol{\beta} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}E[\boldsymbol{\beta}_{\mathbf{mle}}] - \boldsymbol{\xi}) \end{aligned}$$

Since maximum likelihood is unbiased (Johnson and Wichern, 2013)and$\boldsymbol{R}\boldsymbol{\beta} = \boldsymbol{\xi}$,

$$E(\widehat{\boldsymbol{\beta}}_{\boldsymbol{mmle}}) = \boldsymbol{\beta} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta} - \boldsymbol{\xi}) = \boldsymbol{\beta}$$

Corollary 3.3.1(a): *MMLE is an unbiased estimator i.e.* $E(\widehat{\boldsymbol{\beta}}_{mmle}) = \boldsymbol{\beta}$.

The covariance matrix of modified maximum likelihood estimator (MMLE) is given as

$$\begin{aligned} E\left[(\widehat{\boldsymbol{\beta}}_{\text{mmle}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}}_{\text{mmle}} - \boldsymbol{\beta})^t\right] &= E[(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{\varepsilon})(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{\varepsilon})^t] \\ &= \boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t)^t \otimes E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\boldsymbol{t}}] \\ &= \boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t)^t \otimes \boldsymbol{\Sigma} \\ &= \boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{X}(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1})^t \otimes \boldsymbol{\Sigma} \end{aligned}$$

$$= \boldsymbol{M}(\boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1})^t \otimes \boldsymbol{\Sigma}$$

$$Cov(\widehat{\boldsymbol{\beta}}_{\boldsymbol{mmle}}) = (\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma}$$

Where $\boldsymbol{\beta}_{\boldsymbol{mmle}} - \boldsymbol{\beta} = \boldsymbol{M}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{\varepsilon}$

$$\boldsymbol{M} = \boldsymbol{I} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}\boldsymbol{R}$$

Corollary 3.3.1(b): The covariance matrix of MMLE is smaller than MLE because of $Cov(\widehat{\boldsymbol{\beta}}_{\boldsymbol{mle}}) = (\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma}$.

3.4 Likelihood Ratio Test (LRT) for Overall Regression Parameters with Exact Linear Restriction

Let $\boldsymbol{Y}$ be a matrix of p continuous multivariate response variables from $MND(\boldsymbol{X\beta}_{n\times p}, \boldsymbol{\Sigma}_{p\times p})$ where $\boldsymbol{\Sigma}$is unknown. $\boldsymbol{X}$ be a design matrix of $n \times (k+1)$ nonstochastic predictor variables with rank $k \leq n$ where n is the sample size.

i) $\boldsymbol{\mathcal{H}_0}: \boldsymbol{R\beta} = \boldsymbol{0}$ against $\boldsymbol{\mathcal{H}_1}: \boldsymbol{R\beta} \geq \boldsymbol{0}$

Since Σ is unknown, the log likelihood discarding the constant is given by

$$\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma}) = \sum_{i=1}^{l} \left(-\frac{1}{2}\right)\{\ln|\boldsymbol{\Sigma}| + (\boldsymbol{Y_i} - \boldsymbol{X\beta})^t\boldsymbol{\Sigma}^{-1}(Y_i - X\boldsymbol{\beta})\}.$$

Hence the likelihood ratio can be obtained as

$LRT = 2\left[\max_{\boldsymbol{R\beta}\geq\boldsymbol{0}}\{\ell(\boldsymbol{\beta},\boldsymbol{\Sigma})\} - \max_{\boldsymbol{R\beta}=\boldsymbol{0}}\{\ell(\boldsymbol{\beta},\boldsymbol{\Sigma})\}\right]$where$\boldsymbol{\Sigma} > \boldsymbol{0}$ and$\boldsymbol{\Sigma}$ is positive definite.The estimator of $\boldsymbol{\Sigma}$ will be $\widehat{\boldsymbol{\Sigma}} = n^{-1}\left(\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}_r\right)^t\left(\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}_r\right)$.

ii) $\boldsymbol{\mathcal{H}_0}: \boldsymbol{R\beta} = \boldsymbol{0}$ against $\boldsymbol{\mathcal{H}_1}: \boldsymbol{R_1\beta} \geq \boldsymbol{0}$

If $\boldsymbol{Y} \sim MND(\boldsymbol{X\beta}, \boldsymbol{\Sigma})$where $\boldsymbol{\Sigma}$ is positive definite matrix but unknown, R be matrix of order $q \times p, \text{rank}(R) = q < p$, and let R_1 be a sub matrix of R of order $r < p$. Then, the LRT for testing the hypothesis is given below

$$LRT = 2\left[\max_{R_1\beta \geq 0}\{\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma})\} - \max_{R\beta=0}\{\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma})\}\right].$$

To find the sampling distribution of LRT under the null hypothesis is a challenging issue. The common approach of identifying null distribution is to

$\Pr(LRT \leq c|\mathcal{H}_0) = \sum_{i=0}^{r} w_i\,(r, R_i \Sigma R_i^t)\Pr\,(\chi^2_{q-r+i} \leq c)$ where w_i is the weight (Silvapulle, 2006), suggested an estimation method for these weights but that still have room to improvements.

3.5 Modified t test (t_{mod}) for Individual Regression Parameters with Exact Linear Restriction

With underlying assumption of normality, the joint distribution of $\hat{\beta}$and $\hat{\Sigma}$ in classical multivariate regression are $\widehat{\boldsymbol{\beta}} \sim \text{MND}(\boldsymbol{\beta}, (\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma})$and $(n-k)\widehat{\boldsymbol{\Sigma}} \sim W_p(n-k, \boldsymbol{\Sigma})$ where $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\Sigma}}$ are independent.

Hence, the test statistic for testing the hypothesis $\mathcal{H}_0: \beta = 0$ against $\mathcal{H}_1: \beta \neq 0$ is $t = SE(\hat{\beta})^{-1} \times (\hat{\beta} - \beta)$ which follows the multivariate t distribution (Kotz and Nadarajah, 2004).

But, the difficulty is higher if the regression parameters are restricted and the hypotheses related to the parameters are one sided. The hypothesis under consideration is

$$\mathcal{H}_0: \beta_i = 0 \ \text{ against } \mathcal{H}_1: \beta_i > 0 \ or\ \mathcal{H}_1: \beta_i < 0.$$

At statistic modified for testing the above hypothesis can be given as

$$\boldsymbol{t}_{\mathbf{mod}} = \frac{(\widehat{\boldsymbol{\beta}}_{\text{mmle}} - \boldsymbol{\beta})}{\boldsymbol{SE}(\widehat{\boldsymbol{\beta}}_{\text{mmle}})} \tag{3.6}$$

where $\widehat{\boldsymbol{\beta}}_{\text{mmle}} = \boldsymbol{\beta}_{\text{mle}} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}R^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta}_{\text{mle}} - \boldsymbol{\xi})$ and the standard error of each $\beta_{\text{mmle}(i)}$ is the square root of the diagonal element of the variance covariance matrix-

$$\text{Cov}\left(\hat{\beta}_{\text{mmle}(i)}, \hat{\beta}_{\text{mmle}(j)}\right) = (\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma}.$$

While the distribution of t_{mod} in equation 3.6 does not match with any conventional distribution, a Monte Carlo simulation is used to generate the simulated critical value of $t_{\text{mod}}$and to compare the power of t_{mod} and multivariate t test.

3.6 Joint Confidence Region for Regression Parameters with Exact Linear Restriction

Joint confidence regions constitute confidence intervals for a vector of parameters, comprising individual intervals for the separate components, with a coverage probability of the simultaneous correctness of all the statements involved. The exact distribution function of the restricted statistic is not always attainable, and then quantile of the statistic can be calculated by using Monte Carlo Simulation. The length of modified joint confidence interval using quantile points will be shorter than conventional methods and maintain highest coverage probability.

The modified joint confidence regions with $(1-\alpha)\%$ level of confidence for multiple comparisons of restricted parameters in multivariate regression with continuous responses are derived as

$$Pr\left[\{\hat{\beta}_{\text{mmle}(i)} - q_{(1-\alpha)}(t_{\text{mod}}) \times \text{SE}(\hat{\beta}_{\text{mmle}(i)})\} \leq \beta \leq \{\hat{\beta}_{\text{mmle}(i)} + q_{(1-\alpha)}(t_{\text{mod}}) \times \text{SE}(\hat{\beta}_{\text{mmle}(i)})\}\right] = 1 - \alpha$$

where $\widehat{\boldsymbol{\beta}}_{\text{mmle}} = \boldsymbol{\beta}_{\text{mle}} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}R^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}(\boldsymbol{R}\boldsymbol{\beta}_{\text{mle}} - \boldsymbol{\xi})$,

$q_{(1-\alpha)}(\text{mod } t) = (1-\alpha)\%$ quantile of modified multivariate t statistic, and

$\mathrm{Cov}\left(\hat{\beta}_{\mathrm{mmle}(i)}, \hat{\beta}_{\mathrm{mmle}(j)}\right) = (\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma} - (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t[\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t]^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1} \otimes \boldsymbol{\Sigma}$, and

$\mathrm{SE}(\hat{\beta}_{\mathrm{mmle}(i)})$ refers to the square root of the i^{th} diagonal element of $\mathrm{Cov}\left(\hat{\beta}_{\mathrm{mmle}(i)}, \hat{\beta}_{\mathrm{mmle}(j)}\right)$.

3.7 Monte Carlo Experiments

Monte Carlo experiments have been conducted to evaluate the performance of the modified statistic for estimating and testing the restricted parameters of the multivariate regression model with continuous responses.

The study considered a multivariate regression model with bivariate responses ($p = 2$). The multivariate linear model was-

$$\left[\boldsymbol{Y}_{(1)} \middle| \boldsymbol{Y}_{(2)}\right] = \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \text{where, } \boldsymbol{\beta} = \left[\boldsymbol{\beta}_{(1)} \middle| \boldsymbol{\beta}_{(2)}\right], \qquad \boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_{(1)} | \boldsymbol{\varepsilon}_{(2)}] \sim MND(\boldsymbol{0}, \boldsymbol{\Sigma})$$

Assumptions:

a) The predictor variable's values of the design matrix are fixed.
b) The exact linear restriction exists among the subset of the parameters which is believed to be true for each response.
c) The distribution of Y is multivariate normal (MND) with mean $X\beta$ and the covariance matrix $\boldsymbol{\Sigma} = \{\sigma_{ij}\}$ for all i, j.

Since the application of multivariate regression analysis depends on the correlation among the response variables, the different trials have been conducted for different arbitrary value of correlation coefficient namely, $\rho = 0.00, 0.25, 0.75, 0.80, 0.90$ where $\sigma_{11} = 100$ *and* $\sigma_{22} = 81$ including sample sizes of different order, $n = 25, 50, 100, 200, 400$ and 1000.

The generation of multivariate response variables also depends on the parameter values of the regression coefficients taken to be

$$[\boldsymbol{\beta}_{(1)}|\boldsymbol{\beta}_{(2)}] = \begin{bmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} = \begin{bmatrix} 25 & 175 \\ 3.5 & 2.5 \\ -1.75 & -1.25 \\ -1.5 & -1.25 \end{bmatrix} \text{where restriction matrix will be}$$

$$i)\ \boldsymbol{R} = [0 \quad 1 \quad 2 \quad 0] \quad 2)\ \boldsymbol{R} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ for } \boldsymbol{\xi} = [0 \quad 0]\ and\ \boldsymbol{\xi} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{respectively.}$$

The simulated performance of modified maximum likelihood (MMLE) estimate and maximum likelihood estimator (MLE) depends on the number of iteration of the trials. Sample size and information of the test increase for increasing number of iteration, and then the estimated error will be reduced (Koçak, 2020). So, it is important to estimate the required number of iteration. Hence, we determine the minimum number of iteration required using the following formula given by Banks *et.al.* (2001) for vector valued parameter considering minimum loss of information

$$\text{Number of iteration} \geq \left(\frac{Z\alpha_{/2} \times \text{Sample Variance}}{\text{Margin of error}}\right)^2.$$

Theoretically, it is proved in section 3 that modified maximum likelihood estimate is unbiased and the variance of MMLE is lower than general maximum likelihood estimate. But the experiment has been revealed that simulated expected value of the modified maximum likelihood estimate is not exactly equal to their parameter value. Table 3.1 reveals that the amount of bias will be reduced if correlation among the response variables is high which may also true for increased sample size.

Table 3.1: Measuring Bias of Modified Maximum Likelihood Estimate for Different ρ and n

Sample Size, n	ρ_{12} *	Relative Bias (RB**) of Modified Maximum Likelihood Estimate							
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	**0.00**	6.6404	0.1473	0.1473	0.5377	-5.2375	-1.203	-0.1803	-4.6609
	0.25	24.2352	0.5605	0.5605	1.8764	-3.3801	-0.7833	-0.3889	-2.9772
	0.75	18.942	0.436	0.436	1.4745	-1.292	-0.3064	-0.4245	-1.1068
	0.80	17.7314	0.4075	0.4075	1.3824	-1.0514	-0.2513	-0.4235	-0.8919
	0.90	14.6055	0.3341	0.3341	1.1446	-0.4734	-0.1189	-0.4173	-0.376
50	**0.00**	17.6768	0.4008	0.4008	1.5293	0.2276	0.0957	4.5037	-0.8778
	0.25	14.9969	0.324	0.324	1.6292	1.6676	0.4046	2.7583	0.8496
	0.75	16.2572	0.3575	0.3575	1.6354	2.1768	0.5016	0.9044	1.7582
	0.80	16.483	0.3638	0.3638	1.6318	2.2066	0.5063	0.6935	1.836
	0.90	16.9722	0.3777	0.3777	1.6151	2.2581	0.513	0.1891	2.0042
100	**0.00**	15.99	0.3236	0.3236	1.6247	-1.7463	-0.4463	-2.5659	-0.9345
	0.25	20.3877	0.4482	0.4482	1.7313	0.0391	-0.0674	-3.4087	0.9011
	0.75	19.4129	0.4175	0.4175	1.7377	1.1954	0.1975	-3.0772	1.867
	0.80	19.142	0.4095	0.4095	1.7339	1.3065	0.2238	-3.0057	1.9498
	0.90	18.3708	0.3875	0.3875	1.7161	1.5579	0.2842	-2.8108	2.1287
200	**0.00**	4.1745	0.1015	0.1015	0.5722	-0.7765	-0.1531	-2.5992	0.2477
	0.25	6.443	0.1453	0.1453	0.437	-0.2311	-0.031	-2.6131	0.7515
	0.75	5.8412	0.1342	0.1342	0.4928	0.185	0.0567	-1.946	0.8862
	0.80	5.6924	0.1313	0.1313	0.5035	0.2278	0.0656	-1.8514	0.8904
	0.90	5.2918	0.1236	0.1236	0.5279	0.3271	0.0859	-1.6116	0.8929
	0.00	-4.5287	-0.0874	-0.0874	-0.144	1.32	0.3049	0.9239	0.9663
	0.25	-8.6586	-0.1868	-0.1868	-0.4181	0.6102	0.157	0.0942	0.5851
400	**0.75**	-7.4883	-0.1581	-0.1581	-0.3366	-0.0114	0.0198	-0.4721	0.1847
	0.80	-7.2108	-0.1514	-0.1514	-0.3178	-0.0784	0.0047	-0.5279	0.1393
	0.90	-6.4791	-0.1337	-0.1337	-0.2692	-0.2363	-0.0309	-0.6551	0.0307

*ρ_{12} *refers the correlation among responses* ** $RB = \frac{E(\hat{\beta})-\beta}{\beta} \times 100$

The relative efficiency Table 3.2 sorts out that variance of modified maximum likelihood estimate (MMLE) are smaller than that of maximum likelihood estimate (MLE) for each ρwhich fulfill the property of minimum variance unbiased estimate (MVUE).

Table 3.2: Relative Efficiency of MMLE on MLE

Sample Size, n	$\rho_{12}{}^{*}$	Relative Efficiency (RE**) of Modified Maximum Likelihood Estimate							
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	**0.00**	1.679558	1.447613	11.088879	1.026724	1.613201	1.49928	11.281796	1.014476
	0.25	1.657591	1.463613	11.143086	1.022805	1.652568	1.47003	11.17942	1.021568
	0.75	1.666887	1.456668	11.11863	1.024485	1.676913	1.450689	11.10538	1.02611
	0.80	1.668688	1.455341	11.11404	1.024808	1.678341	1.449482	11.10042	1.026386
	0.90	1.672776	1.452357	11.10391	1.025538	1.680607	1.447465	11.09164	1.026835
50	**0.00**	2.136489	1.529128	14.739342	1.071524	2.077175	1.540895	14.564368	1.06151
	0.25	2.12429	1.52299	14.58073	1.070763	2.103163	1.546549	14.795936	1.064258
	0.75	2.130568	1.52401	14.631702	1.071479	2.12754	1.538789	14.825957	1.068817
	0.80	2.131671	1.524366	14.6432	1.071578	2.12943	1.537682	14.821051	1.069247
	0.90	2.133976	1.52546	14.672225	1.071731	2.13309	1.534996	14.803747	1.070162
100	**0.00**	2.225337	1.455111	14.14655	1.096848	2.215036	1.450167	13.996744	1.09577
	0.25	2.228641	1.456161	14.174738	1.097077	2.21285	1.449773	13.989252	1.095685
	0.75	2.228311	1.456145	14.175223	1.097074	2.218804	1.452457	14.069438	1.09627
	0.80	2.228139	1.456099	14.174059	1.097063	2.219587	1.452789	14.079184	1.096342
	0.90	2.227555	1.455916	14.169174	1.097024	2.221451	1.453565	14.101883	1.096511
200	**0.00**	2.202896	1.394682	13.25396	1.108771	2.209436	1.365277	12.8409	1.116077
	0.25	2.230978	1.401563	13.49708	1.11162	2.174876	1.362831	12.63286	1.111277
	0.75	2.223106	1.401294	13.45411	1.110467	2.180418	1.378635	12.89957	1.108743
	0.80	2.22117	1.400971	13.43966	1.110238	2.182329	1.380608	12.93893	1.108615
	0.90	2.216049	1.399761	13.39599	1.109709	2.18767	1.385256	13.03577	1.10844
400	**0.00**	1.969029	1.448207	12.7128	1.062011	1.992177	1.444276	12.7798	1.065618
	0.25	1.945278	1.445951	12.53647	1.058923	2.017806	1.447845	12.98951	1.068841
	0.75	1.950422	1.446752	12.57995	1.05956	1.998552	1.448951	12.89688	1.066053
	0.80	1.951913	1.446922	12.59151	1.059751	1.995402	1.448952	12.87861	1.065614
	0.90	1.956184	1.447337	12.62338	1.060306	1.987509	1.448861	12.83125	1.064523

* ρ_{12} *refers the correlation among responses* ** $RE = \frac{Var(\hat{\beta}_{MLE})}{Var(\hat{\beta}_{MMLE})}$

After estimating the parameters, testing the significance of individual regression coefficient are essential. Multivariate t statistic can be used to test the regression coefficient. But, the Table 3.3 and figure 3.1 demonstrate that t_{mod} is more powerful than multivariate t statistic for each ρ and sample size.

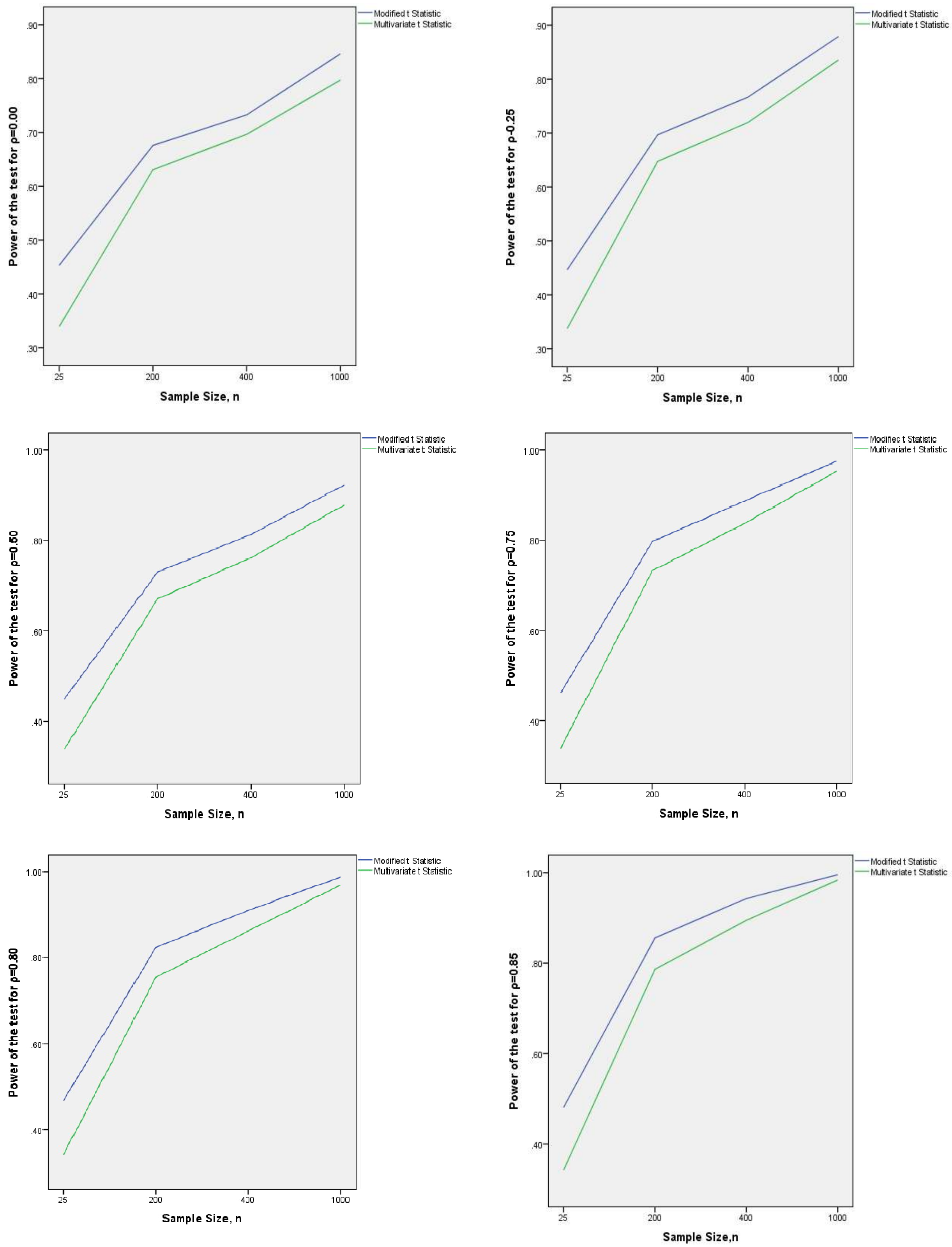
Table 3.3: Power of the test statistics in different sample size and ρ

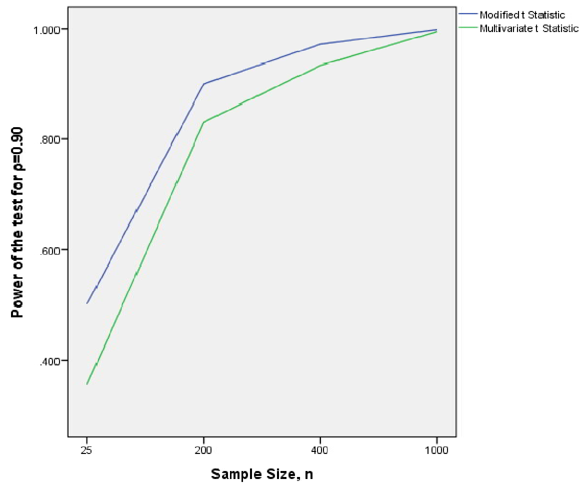
n^*	Test Statistic**	$\rho = 0.00$	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$	$\rho = 0.80$	$\rho = 0.85$	$\rho = 0.90$	$\rho = 0.95$
25	t_{mod}	0.4531	0.4465	0.4486	0.4617	0.4683	0.481	0.501	0.5465
	Multivariate t	0.3393	0.3374	0.3371	0.3387	0.3426	0.3425	0.3551	0.380
200	t_{mod}	0.6759	0.6967	0.7302	0.7978	0.8241	0.8558	0.900	0.9532
	Multivariate t	0.6308	0.6474	0.6709	0.7331	0.7551	0.7863	0.831	0.8939
400	t_{mod}	0.7326	0.7667	0.8133	0.8875	0.9102	0.9429	0.972	0.9946
	Multivariate t	0.6965	0.7197	0.7618	0.8378	0.8621	0.8948	0.9335	0.9751
1000	t_{mod}	0.8459	0.879	0.9231	0.9757	0.9874	0.9955	0.9987	0.9999
	Multivariate t	0.7972	0.8354	0.8788	0.9533	0.9688	0.9837	0.9951	0.9998

*n, **t_{mod} *and* ρ *refer sample size, modified* t *statistic and population correlation coefficient among responses, respectively*

It is need to address that the power of the modified test will be increased for increasing sample size and also increasing correlation among response variables. The simulated critical values for t_{mod} at different levels are given in appendix 1.

Figure 3.1: Power Comparison between$t_{\mathrm{mod}}$and multivariate t statistic





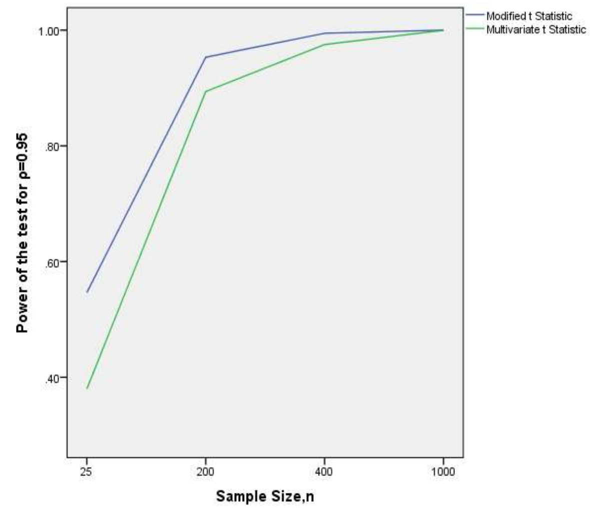


Table 3.4 reveals that the modified joint confidence regions provide shorter intervals (modified joint confidence interval/length of multivariate t interval) for correlated response variables from small to large sample where multivariate t intervals are slightly better for uncorrelated responses with small sample.

Table 3.4: Evaluation of modified joint confidence interval with multivariate t interval

Sample Size	ρ_{12}	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	0	1.693398	2.869468	0.453826	1.505744	2.551487	0.403535
	0.75	0.771601	1.30748	0.206787	0.765985	1.297964	0.205282
	0.9	0.551472	0.934471	0.147793	0.550375	0.932612	0.147499
200	0	0.009438	0.018844	0.005731	0.009523	0.019014	0.005783
	0.75	0.010051	0.02007	0.006103	0.010632	0.021229	0.006456
	0.9	0.010317	0.020599	0.006264	0.010715	0.021394	0.006506
400	0	0.129823	0.239941	0.05763	0.128323	0.237168	0.056964
	0.75	0.142323	0.263045	0.063179	0.14708	0.271835	0.06529
	0.9	0.146152	0.27012	0.064878	0.149512	0.27633	0.06637
1000	0	0.068795	0.133994	0.038015	0.068979	0.134353	0.038116
	0.75	0.000244	0.219251	0.061231	40.53882	0.05751	0.11379
	0.9	0.146152	0.27012	0.064878	0.149512	0.27633	0.06637

3.8 Conclusion

Parameter estimation and significant variable selection are two important goals in multivariate analysis. This chapter reviewed systematically the previous research and proposed-i) a minimum variance unbiased estimator namely modified MLE, ii) modified multivariate t test statistic whose power is comparatively better than traditional multivariate t test statistic and iii) modified joint confidence region considering exact linear restriction of multivariate regression parameters for small to big data. Monte Carlo simulation has been used to evaluate the performance of proposed modified methods and construct the quantile values of t_{mod} test statistic.

Chapter 4

Modified Inferential Approach for Multivariate Continuous Responses with Stochastic Linear Restrictions

4.1 Multivariate Continuous Responses with Stochastic Linear Restrictions

The prior information's about the population parameter in linear regression analysis is well known to provide more efficient estimators of regression coefficients. Such prior information can be obtainable in different forms from different sources especially past experience of the researcher or similar kind of researches conducted in the past. When the prior information is available in the form of stochastic restrictions, then in many practical situations a systematic bias can arise, due to various reasons like personal judgments of the person involved in the experiment, in the testing of general linear hypothesis in linear models when the null hypothesis is rejected or in missing values imputation through regression approach. Another problem in multiple linear regression models, close linear dependency among the predictors causes the problem of multicollinearity, which reduces the efficiency of the ordinary least squares (OLS) estimator. Total inferential procedure both estimation technique and testing procedure need to be addressed these shorts of phenomena in multivariate regression model to give best policy options. Whatever, addressing stochastic constraint of unknown parameters and multicollinearity of the predictors, we consider a modified approach for MRA with continuous responses where can be considered both sample and non-sample information in this chapter.

4.2 Model Specification

The exact linear restrictions assume that there is no randomness involved in the prior information. Sometimes in real life, the truthfulness of this assumption can be suspected and accordingly an element of uncertainty can be introduced within the parameters. Stochastic restrictions on unknown parameters are one of the alternative techniques in the linear regression model to tackle the multicollinearity among the predictors.

A multivariate multiple linear regression model is considered in the form

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \tag{4.1}$$

where $\boldsymbol{Y} \sim MND(\boldsymbol{X\beta}, \boldsymbol{\Sigma} \otimes \boldsymbol{I})$ is $n \times p$ random matrix (observation matrix) of p continuous multivariate responses, $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots, \boldsymbol{Y}_p), \boldsymbol{Y}_i \sim \boldsymbol{ND}(\boldsymbol{X\beta}_i, \sigma_{ii}\boldsymbol{I}_{\boldsymbol{nn}})$, $i = 1,2,\dots,p, \boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_p)$, $\boldsymbol{Cov}(\boldsymbol{Y}_i, \boldsymbol{Y}_j) = \sigma_{ij}\boldsymbol{I}_{nn}$,$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}\sigma_{12}\cdots\sigma_{1p} \\ \sigma_{21}\sigma_{22}\cdots\sigma_{2p} \\ \cdots\ \cdots\ \cdots\ \cdots \\ \sigma_{p1}\sigma_{p2}\cdots\sigma_{pp} \end{bmatrix}$and $\boldsymbol{X\beta}$is the mean of the observation matrix $E(\boldsymbol{Y}) = \boldsymbol{X\beta}$**,** $\boldsymbol{X}$is an $n \times (k+1)$ design matrix of nonstochastic predictor variables with $k \leq n$ where n is the sample size. $\boldsymbol{\beta}$is a $(k+1) \times p$ matrix of unknown parameters. $\boldsymbol{\Sigma} \otimes \boldsymbol{I}$is the covariance matrix of the observation vector $vec(\boldsymbol{Y}) = (\boldsymbol{Y}_1^t, \boldsymbol{Y}_2^t, \cdots, \boldsymbol{Y}_p^t)^t$.

In addition to model (4.1), it is supposed that there some prior information about $\boldsymbol{\beta}$in the form of a set of independent stochastic linear restrictions

$$\boldsymbol{\xi} = \boldsymbol{R\beta} + \boldsymbol{V} \tag{4.2}$$

where $\boldsymbol{\xi}$ is a vector of known elements, $\boldsymbol{R}$ is a full rank matrix with known elements and $\boldsymbol{V}$ is a vector of stochastic elements assumed to be distributed with zero mean and variance-covariance matrix Ψ with known elements. It is also assumed that the element of $\boldsymbol{V}$ are stochastically independent of the elements of $\boldsymbol{\varepsilon}$**.**

4.3 Parameter Estimation of Multivariate Regression with Stochastic Linear Restrictions

The least squares estimator of multivariate regression (MLSE) of the parameter, $\boldsymbol{\beta}$ in model (4.1) is given by

$$\widehat{\boldsymbol{\beta}}_{MLSE} = (\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{Y} \text{ where } t \text{ is the symbol of transpose} \tag{4.3}$$

by minimizing the objective function (Johnson and Wichern, 2013)

$$\Phi_1 = \mathrm{tr}[(\boldsymbol{Y}-\boldsymbol{X\beta})^t(\boldsymbol{Y}-\boldsymbol{X\beta})] \tag{4.4}$$

The MLSE estimator is the widely known estimator of the coefficients in a linear regression model since it is unbiased and has the minimum variance among all linear unbiased estimator (Wu, 2014). When the stochastic linear restriction as prior information on the unknown parameters assumed to be held, Schafrin and Toutenberg (1990) introduced the method of weighted mixed regression and developed the weighted mixed estimator (WME) where sample information and the prior information are not equally likely based on some extraneous consideration in the estimation of regression parameters. Considering sample information in equation (4.1) and the prior information in equation (4.2), the form of objective function for minimization is given as

$$\Phi_2 = \mathrm{tr}[(\boldsymbol{Y}-\boldsymbol{X\beta})^t(\boldsymbol{Y}-\boldsymbol{X\beta})] + w(\boldsymbol{\xi}-\boldsymbol{R\beta})^t\Psi^{-1}(\boldsymbol{\xi}-\boldsymbol{R\beta}) \tag{4.5}$$

where$w$ is a nonstochastic and nonnegative scalar weight. Since $w$is $0 \le w \le 1$, the value of w specifies an estimator in which the prior information receives less weight in comparison to the sample information (Liu, Yang and Wu, 2013).

Differentiating of equation (4.5) with respect to $\boldsymbol{\beta}$lead to the normal equations

$$\boldsymbol{X}^t\boldsymbol{X\beta} - \boldsymbol{X}^t\boldsymbol{Y} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R\beta} - w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi} = \boldsymbol{0}, \tag{4.6}$$

From equation (5.6), the estimator of $\boldsymbol{\beta}$is as follows-

$$\widehat{\boldsymbol{\beta}}_{WME} = (\boldsymbol{X}^t\boldsymbol{X} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R})^{-1}(\boldsymbol{X}^t\boldsymbol{Y} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi}), \tag{4.7}$$

And observing

$$(\boldsymbol{X}^t\boldsymbol{X} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R})^{-1} = (\boldsymbol{X}^t\boldsymbol{X})^{-1} - w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}$$

and $((\boldsymbol{X}^t\boldsymbol{X})^{-1} - w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1})w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi} = w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{\xi}$,

After simplification of equation (4.7), the modified multivariate extension of the weighted mixed estimator is

$$\widehat{\boldsymbol{\beta}}_{MMWME} = \widehat{\boldsymbol{\beta}}_{MLSE} + w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t)^{-1}(\boldsymbol{\xi} - \boldsymbol{R}\widehat{\boldsymbol{\beta}}_{MLSE}) \tag{4.8}$$

The regression model faces the challenge to handle multicollinearity problems in real life experiment. When the problem of multicollinearity is present, the MLSE estimator may be statistically insignificant with wrong sign and large variances; hence the biased estimation as an alternative to the MLSE estimator is recommended in order to obtain some reduction in variance (Özkale, 2014; Özbay and Kaçiranlar, 2017). Özkale and Kaçiranlar (2007) introduced two parameter estimators to overcome the problem of multicollinearity for univariate regression. The objective function of multivariate multiple regression

$$\Phi_3 = \mathrm{tr}[(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})^t(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})] + \mathbf{K}\left[\left(\boldsymbol{\beta} - \mathbf{d}\widehat{\boldsymbol{\beta}}\right)^{\boldsymbol{t}}\left(\boldsymbol{\beta} - \mathbf{d}\widehat{\boldsymbol{\beta}}\right) - \boldsymbol{C}\right] \tag{4.9}$$

Where $\mathbf{K}$ is the Lagrangian multiplier, $\boldsymbol{C}$ is a constant and $\mathbf{d}$ is a Liu (Li and Yang, 2010) biasing parameter lies between $0 < \mathbf{d} < 1$. Differentiating both sides with respect to $\boldsymbol{\beta}$and$\mathbf{K}$ for minimizing the objective function (4.9)

$$\boldsymbol{X}^t\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}^t\boldsymbol{Y} + \mathrm{K}\left(\boldsymbol{\beta} - \mathbf{d}\widehat{\boldsymbol{\beta}}\right) = \mathbf{0} \tag{4.10}$$

By solving the equation (4.10), the multivariate approach of the two-parameter estimator is given as

$$\widehat{\boldsymbol{\beta}}_{MTPE} = (\boldsymbol{X}^t\boldsymbol{X} + \mathbf{K}\otimes\mathbf{I})^{-1}(\boldsymbol{X}^t\boldsymbol{Y} + \mathbf{K}\mathbf{d}\widehat{\boldsymbol{\beta}}) \tag{4.11}$$

Combining the objective function Φ_2 in equation (4.5) and Φ_3 in equation (4.9), the modified objective function

$$\Phi_4 = \mathrm{tr}[(\boldsymbol{Y}-\boldsymbol{X\beta})^t(\boldsymbol{Y}-\boldsymbol{X\beta})] + \mathbf{K}\left[\left(\boldsymbol{\beta}-\mathbf{d}\widehat{\boldsymbol{\beta}}\right)^{\boldsymbol{t}}\left(\boldsymbol{\beta}-\mathbf{d}\widehat{\boldsymbol{\beta}}\right)-\boldsymbol{C}\right] + w(\boldsymbol{\xi}-\boldsymbol{R\beta})^t\Psi^{-1}(\boldsymbol{\xi}-\boldsymbol{R\beta}), \tag{4.12}$$

where $\mathbf{K}, \mathbf{d}, w$ are Lagrangian multiplier, biasing parameter and nonstochastic scalar respectively. Differentiating both sides with respect to $\boldsymbol{\beta}$and$\mathbf{K}$ for minimizing the objective function (4.12) and putting $\frac{\delta\Phi_3}{\delta\boldsymbol{\beta}} = 0$ and $\frac{\delta\Phi_3}{\delta\boldsymbol{K}} = 0$

$$\boldsymbol{X}^t\boldsymbol{X\beta} - \boldsymbol{X}^t\boldsymbol{Y} + \mathrm{K}\left(\boldsymbol{\beta}-\mathbf{d}\widehat{\boldsymbol{\beta}}\right) + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R\beta} - w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi} = \mathbf{0} \tag{4.13}$$

$$\left(\boldsymbol{\beta}-\mathbf{d}\widehat{\boldsymbol{\beta}}\right)^{\boldsymbol{t}}\left(\boldsymbol{\beta}-\mathbf{d}\widehat{\boldsymbol{\beta}}\right) - \boldsymbol{C} = \mathbf{0} \tag{4.14}$$

The simplified form of equation (4.13) is

$$\boldsymbol{X}^t\boldsymbol{X}\widetilde{\boldsymbol{\beta}}_{MTPWE} + \mathbf{K}\widetilde{\boldsymbol{\beta}}_{MTPWE} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R}\widetilde{\boldsymbol{\beta}}_{MTPWE} = \boldsymbol{X}^t\boldsymbol{Y} + \mathbf{Kd}\widehat{\boldsymbol{\beta}} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi}$$

$$\widetilde{\boldsymbol{\beta}}_{MTPWE} = (\boldsymbol{X}^t\boldsymbol{X} + \mathbf{K}\otimes\mathbf{I} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}(\boldsymbol{X}^t\boldsymbol{Y} + \mathbf{Kd}\widehat{\boldsymbol{\beta}} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi}) \tag{4.15}$$

And observing

$$(\boldsymbol{X}^t\boldsymbol{X} + w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{R})^{-1} = (\boldsymbol{X}^t\boldsymbol{X})^{-1} - w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1} \tag{4.16}$$

$$((\boldsymbol{X}^t\boldsymbol{X})^{-1} - w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X})^{-1})w\boldsymbol{R}^t\boldsymbol{\Psi}^{-1}\boldsymbol{\xi} = w(\boldsymbol{X}^t\boldsymbol{X})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}\boldsymbol{\Psi}^{-1}\boldsymbol{R}^t)^{-1}\boldsymbol{\xi} \tag{4.17}$$

Using (4.16) and (4.17) in (4.15);

$$\widetilde{\boldsymbol{\beta}}_{MTPWE} = \widehat{\boldsymbol{\beta}}(\mathbf{K},\mathbf{d}) + w(\boldsymbol{X}^t\boldsymbol{X} + \mathbf{K}\otimes\mathbf{I})^{-1}\boldsymbol{R}^t(\boldsymbol{\Psi} + w\boldsymbol{R}(\boldsymbol{X}^t\boldsymbol{X} + \mathbf{K}\otimes\mathbf{I})^{-1}\boldsymbol{R}^{\boldsymbol{t}})^{-1}(\boldsymbol{\xi} - \boldsymbol{R}\widehat{\boldsymbol{\beta}}(\mathbf{K},\mathbf{d})) \tag{4.18}$$

Assumptions:

e) The predictor variables' values of the design matrix are fixed.

f) The distribution of random disturbances is multivariate normal i.e. $\boldsymbol{\varepsilon} \sim MND(\mathbf{O}_{n\times p}, \boldsymbol{\Sigma}_{p\times p})$.

g) $\boldsymbol{\beta}$, $\boldsymbol{\Sigma}$ are the unknown parameters where $\boldsymbol{\beta}$ is a $(k+1)\times p$ matrix of regression coefficient.

h) The exact linear restriction exists among the subset of the parameters which is believed to be true for each response.

4.3.1 Optimum Value of $\boldsymbol{k}$ and $\boldsymbol{d}$

In multivariate regression, orthogonal transformation is used to convert classical regression model to canonical form.

$$y = ZQ + \in$$

Where Z=XA, Q= $A^tB$and A is a orthogonal matrix such that $Z^tZ = A^tX^tXA = \Lambda =$ dies $(\lambda_1, \lambda_2, \cdots, \lambda_p)$ $= \Lambda$ dies (λ_1,λ_2,$\cdots$,λ_p) where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ where λ_i are the ordered eigen value of X^tX.The selection of the estimators of the parameters d and k in $\hat{\beta}(k,d)$ can be obtained by applying the following literature procedure-

Step 1: Calculate $\hat{d}$ where $\hat{d} < min\left\{\frac{\hat{\theta}_\iota^{\,2}}{\frac{\hat{\sigma}^2}{\lambda_1}+\hat{\theta}_\iota^{\,2}}\right\}$

$\hat{\theta}_\iota^{\,2}$, $\hat{\sigma}^2$ are the unbiased estimate of $\hat{\theta}_\iota^{\,2}$and σ^2

Step2: Estimate $\widehat{K}_{op}$ by using $\hat{d}$ in step 1 then

$$\widehat{K}_{op} = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{\hat{\sigma}^2}{\hat{\theta}_i^{\,2} - d(\frac{\hat{\sigma}^2}{\lambda_i} + \hat{\theta}_i^{\,2}}\right\}$$

It is supposed that X^tX and $R^tW^{-1}R$ are commutative (Liu, Yang and Wu, 2013), then $Q^tR^tW^{-1}RQ = \Psi = diag(\xi_1, \dots, \xi_p)$ for $k > 0$. Since the value of w is weight level to the sample information and the prior information, the formula to choose optimum w for the fixed value of $\widehat{K}_{op}$ is

$$\widehat{w} = \frac{\sqrt{\sum_{i=1}^{p} \widehat{h}_{2i}^2 + 8(\sum_{i=1}^{p} \widehat{h}_{1i})^2} - \sum_{i=1}^{p} \widehat{h}_{2i}}{\sum_{i=1}^{p} 4\widehat{h}_{1i}}$$

Where $\widehat{h}_{1i} = \hat{\sigma}^2 \xi_i (\lambda_i + \widehat{K}_{op})^2$ and $\widehat{h}_{2i} = 2\hat{\sigma}^2 \lambda_i^3 (\lambda_i + 2\widehat{K}_{op})^2 + 2\widehat{K}_{op}^2 \lambda_i \hat{\hat{\beta}}_i^2 - \hat{\sigma}^2 \xi_i^3 (\lambda_i + \widehat{K}_{op})^2$

Step 3: Obtain $\hat{d}_{op}$ by using the optimum value of k

$$\hat{d}_{opt} = \sum_{i=1}^{p} \frac{(K\widehat{\theta}_i^{\,2} - \hat{\sigma}^2) \big/ (\lambda_i + k)^2}{\frac{K(\hat{\sigma}^2 + \widehat{\theta}_i^{\,2} \widehat{\lambda}_i^{\,2})}{\lambda_i (\lambda_i + K)}}$$

Step 4: If $\hat{d}_{opt}$ is negative used $\hat{d}_{opt} = \hat{d}.\widehat{d}_{opt}$ is always less than one, but is bigger than zero.

4.4 Monte Carlo Experiment

In this section, the Monte Carlo simulation study has been conducted to examine the performance of three estimation methods described in the previous section. First, the study has been compared to relative bias by looking at the average parameter estimates over the replications. Second, the study also has been examined the relative efficiency of three modified statistics with a number of restricted parameters used in the simulation estimates.

The study planned a multivariate regression model with bivariate continuous responses ($p = 2$). Four different sets of correlation coefficients$\rho = 0.25, 0.75, 0.80, 0.9$ have been considered to examine the consistency and efficiency of the estimators where $\sigma_{11} = 100$ and $\sigma_{22} = 81$ including sample sizes of different order, $n = 25, 50, 100, 200, 400$. The response matrix were generated by multivariate regression model with continuous responses which is given below

$$[\boldsymbol{Y}_{(1)}|\boldsymbol{Y}_{(2)}] = \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \text{ where, } \boldsymbol{\beta} = [\boldsymbol{\beta}_{(1)}|\boldsymbol{\beta}_{(2)}], \qquad \boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_{(1)}|\boldsymbol{\varepsilon}_{(2)}] \sim MND(\boldsymbol{0}, \boldsymbol{\Sigma})$$

Where $\boldsymbol{\varepsilon}$ are independent normal pseudo-random numbers with mean $\boldsymbol{0}$ and constant variance$\boldsymbol{\Sigma}$.

The simulation study has been continued considering the following three assumptions.

a) The predictor variable's values of the design matrix are fixed.

b) The stochastic linear restriction exists among the subset of the parameters which is believed to be true for each response.

c) The distribution of Y is multivariate normal distribution (MND) with mean $\boldsymbol{X\beta}$and the covariance matrix$\boldsymbol{\Sigma} = \{\sigma_{ij}\}$ for all i, j.

For each choice of ρ and n, the multivariate continuous responses have been generated which also depends on the parameter values of the regression coefficients taken to be

$$[\boldsymbol{\beta}_{(1)}|\boldsymbol{\beta}_{(2)}] = \begin{bmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} = \begin{bmatrix} 25 & 175 \\ 3.5 & 2.5 \\ -1.75 & -1.25 \\ -1.5 & -1.25 \end{bmatrix}$$

The study has been also considered the arbitrary value of restriction matrix, $R = [0 \quad 1 \quad 2 \quad 0]$for stochastic restriction, $\boldsymbol{\xi} = \boldsymbol{R\beta} + \boldsymbol{V}$ in the multivariate regression where$\boldsymbol{V}$is a vector of stochastic elements assumed to be distributed with zero mean and variance-covariance matrix $\hat{\Sigma}$ (Yang, Chang and Liu, 2009; Li and Yang, 2010). The weight of the prior information w is chosen as 0.20, 0.50 and 0.50. Further for the Lagrangian multiplier,K and the biasing parameter, d some selected values are chosen that $0 \le k \le 1$and $0 \le d \le 1$. The simulated performance of Multivariate Least Squares Estimate (MLSE) estimate, Modified Restricted Least Squares Estimate (MRLE), Multivariate Weighted Mixed estimator and Multivariate Two Parameter Weighted Estimator depends on the number of iteration of the trials. The experiment is repeated 10000 times by generating new pseudo-random numbers before determining the minimum number of iteration required using the following formula given by Banks *et.al.* (2001)

$$\text{Number of iteration} \geq \left(\frac{Z\alpha/_{2} \times \text{Sample Variance}}{\text{Margin of error}}\right)^2.$$

Theoretically, the ordinary least square (OLS) estimator is the best linear unbiased estimator (Montgomery and Peck, 1992).Sometimes stochastic linear restrictions exhibit the instability of the estimate. The Monte Carlo experiment has been revealed that simulated expected value of the MLSE is not exactly equal to their parameter value. Table 4.1 and 4.2 reveals that the amount of bias for both multivariate least squares estimate and modified restricted least squares estimate will be reduced if correlation among the response variables is high.

Table 4.1: Measuring Relative Bias (RB) of Multivariate Least Squares Estimate (MLSE) for Different ρ and n

Sample Size, n	ρ_{12} *	RB** of MLSE							
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	0.25	32.8218	0.3964	-0.2084	-1.8764	3.8295	0.6991	-0.3889	-2.9772
	0.75	26.556	0.2904	-0.2458	-1.4745	1.1574	0.3316	-0.4245	-1.1068
	0.80	25.1104	0.2665	-0.2533	-1.3824	0.8552	0.2881	-0.4235	-0.8919
	0.90	21.3593	0.205	-0.2707	-1.1446	0.1334	0.1826	-0.4173	-0.376
50	0.25	47.7633	0.189	-2.286	-1.6292	1.3476	0.8672	-2.7583	-0.8496
	0.75	43.743	0.0728	-1.8318	-1.6354	1.6609	0.5808	-0.9044	-1.7582
	0.80	42.7352	0.0472	-1.7273	-1.6318	1.9668	0.5431	-0.6935	-1.836
	0.90	40.0004	0.0172	-1.4566	-1.6151	2.6731	0.4493	-0.1891	-2.0042
100	0.25	35.343	0.2392	-0.6761	-1.7313	4.5744	0.6886	-3.4087	-0.9011
	0.75	39.0907	0.1425	-1.0618	-1.7377	5.6402	0.4113	-3.0772	-1.867
	0.8	39.7899	0.1209	-1.1427	-1.7339	5.69	0.3765	-3.0057	-1.9498
	0.9	41.3584	0.0663	-1.3406	-1.7161	5.7588	0.2912	-2.8108	-2.1287
200	0.25	5.4331	0.1587	-0.2196	-0.437	3.3508	0.4989	-2.6131	-0.7515
	0.75	9.2037	0.0893	-0.1132	-0.4928	2.9632	0.3061	-1.946	-0.8862
	0.8	9.9979	0.074	-0.1854	-0.5035	2.887	0.2818	-1.8514	-0.8904
	0.9	11.9459	0.0349	-0.3658	-0.5279	2.682	0.2217	-1.6116	-0.8929
500	0.25	11.4336	0.1409	-0.8269	-0.2682	0.0497	0.2768	-0.7302	-0.4779
	0.75	10.8005	0.1679	-0.8753	-0.2053	0.6137	0.2980	-1.0410	-0.1997
	0.80	10.6300	0.1733	-0.8833	-0.1910	0.6782	0.2968	-1.0625	-0.1673
	0.90	10.1513	0.1860	-0.899	-0.1541	0.8251	0.2915	-1.1042	-0.0893

* ρ_{12} *refers the correlation among responses and* $**RB = \frac{E(\hat{\beta})-\beta}{\beta} \times 100$

Table 4.2: Measuring Relative Bias of Modified Restricted Least Squares Estimator(MRLE) for Different ρ and n

Sample Size, n	ρ_{12} *	RB** of MRLE							
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	0.25	32.8676	0.3955	-0.2125	1.8771	-3.7991	-0.7048	-0.4155	-2.9724
	0.75	26.4615	0.2922	-0.2374	1.4732	-1.1245	-0.3378	-0.4534	-1.1017
	0.80	24.931	0.2699	-0.2372	1.3800	-0.8107	-0.2964	-0.4626	-0.8849
	0.90	21.1273	0.2094	-0.2499	1.1415	-0.0928	-0.1902	-0.4529	-0.3696
50	0.25	47.5621	-0.1859	-2.2700	1.6256	-1.3192	0.8629	2.7362	0.8556
	0.75	43.3734	-0.067	-1.8024	1.6287	1.7071	0.5737	0.8683	1.7679
	0.80	42.2954	-0.0404	-1.6923	1.6239	2.0235	0.5344	0.6493	1.8479
	0.90	39.3567	0.0273	-1.4053	1.6035	2.7606	0.4359	0.1207	2.0226
100	0.25	35.3263	0.2395	-0.6748	1.7310	4.5649	-0.6873	-3.4017	0.8988
	0.75	39.0945	0.1424	-1.0621	1.7378	5.6294	-0.4098	-3.0692	1.8645
	0.80	39.7982	0.1208	-1.1434	1.7341	5.6788	-0.3750	-2.9974	1.9471
	0.90	41.3897	0.0659	-1.3429	1.7167	5.7440	-0.2891	-2.7998	2.1251
200	0.25	5.4405	0.1586	0.2190	0.4371	3.3465	-0.4983	-2.6100	0.7504
	0.75	9.2301	0.0890	-0.1151	0.4933	2.9570	-0.3053	-1.9415	0.8846
	0.80	10.0303	0.0735	-0.1877	0.5042	2.8800	-0.2808	-1.8463	0.8887
	0.90	11.9952	0.0343	-0.3695	0.5290	2.6723	-0.2204	-1.6047	0.8905
500	0.25	11.4300	-0.1408	-0.8266	0.2681	-0.0499	-0.2768	-0.7301	-0.4780
	0.75	10.7977	-0.1678	-0.8751	0.2053	0.6134	-0.2980	-1.0408	-0.1997
	0.80	10.6272	-0.1732	-0.8830	0.1909	0.6780	-0.2968	-1.0623	-0.1673
	0.90	10.1484	-0.1859	-0.8988	0.1541	0.8249	-0.2915	-1.1041	-0.0893

* ρ_{12} r*efers the correlation among responses and* $**RB = \frac{E(\hat{\beta})-\beta}{\beta} \times 100$

Table 4.3: Relative Efficiency (RE) of Modified Restricted Least Squares Estimator (MRLE) with respect to MLSE for Different ρ and n

Sample Size, n	ρ_{12}*	RE** of MRLE							
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	0.25	1.007884	1.006286	1.018122	1.000488	1.009729	1.007614	1.022311	1.000552
	0.75	1.007895	1.00648	1.018142	1.000542	1.009815	1.007727	1.022314	1.000628
	0.80	1.0078951	1.006512	1.01843	1.000527	1.009843	1.007757	1.022334	1.000645
	0.90	1.00796	1.00655	1.018433	1.000578	1.009848	1.007758	1.022638	1.000691
50	0.25	1.005096	1.003302	1.008901	1.000634	1.006222	1.004111	1.011007	1.000729
	0.75	1.005127	1.003381	1.00891	1.00065	1.006256	1.004103	1.010866	1.000737
	0.80	1.00514	1.003381	1.008989	1.000654	1.006280	1.004096	1.01082	1.000798
	0.90	1.005148	1.00339	1.008991	1.000666	1.006874	1.004081	1.0107	1.000798
100	0.25	1.002603	1.001461	1.004403	1.000416	1.003206	1.001818	1.005452	1.000507
	0.75	1.002609	1.001467	1.004407	1.000422	1.003212	1.001839	1.005473	1.000508
	0.80	1.002616	1.001469	1.004481	1.000426	1.00322	1.001848	1.00548	1.0005081
	0.90	1.002625	1.001469	1.00482	1.000432	1.003223	1.001852	1.00548	1.000509
200	0.25	1.001131	1.000595	1.001912	1.000203	1.00135	1.000697	1.002356	1.000241
	0.75	1.001137	1.000596	1.001915	1.000205	1.001351	1.0007	1.002357	1.000241
	0.80	1.001138	1.000598	1.001916	1.000206	1.001351	1.0007	1.002358	1.000241
	0.90	1.001147	1.000599	1.001927	1.000208	1.001352	1.0007	1.0023581	1.000242
500	0.25	1.000379	1.000244	1.000722	1.000042	1.00048	1.000305	1.000894	1.000054
	0.75	1.000374	1.000246	1.000723	1.000044	1.000489	1.000307	1.000902	1.000056
	0.80	1.000372	1.000247	1.00073	1.000045	1.000491	1.000307	1.000905	1.000056
	0.90	1.000366	1.000249	1.000731	1.000049	1.000499	1.000308	1.000914	1.000058

*ρ_{12} refers the correlation among responses and $**RE = \frac{Var(\hat{\beta}_{MLSE})}{Var(\hat{\beta}_{MRLE})}$

In Table 4.3, the performance of modified restricted least squares estimator (MRLE) is relatively more efficient than MLSE for different sample size where the stochastic restriction presents in regression parameters of the multivariate regression model.

Table 4.4: Relative Efficiency (RE) of Modified Multivariate Weighted Least Square (MMWLS) Estimate with respect to MLSE for Different ρ, W and n

Sample Size, n	ρ_{12} *	W **	**RE** ** **of MMWLS**							
			β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	**0.25**	0.2	1.001557	1.001266	1.003617	1.000084	1.001962	1.001537	1.004453	1.000112
		0.5	1.003883	1.003156	1.009049	1.000209	1.004889	1.003829	1.011141	1.000279
		0.8	1.006197	1.005037	1.01449	1.000333	1.007799	1.006105	1.017839	1.000443
	0.75	0.2	1.001494	1.001308	1.00363	1.00007	1.00199	1.00151	1.004463	1.000128
		0.5	1.003721	1.003258	1.009074	1.000173	1.00495	1.003752	1.011139	1.000318
		0.8	1.005931	1.005196	1.014516	1.000275	1.007879	1.005966	1.017792	1.000505
	0.80	0.2	1.001483	1.001316	1.003635	1.000067	1.002002	1.001502	1.004463	1.000132
		0.5	1.00369	1.003278	1.009079	1.000166	1.004974	1.003726	1.011124	1.000327
		0.8	1.005877	1.005224	1.014513	1.000263	1.007908	1.005916	1.017744	1.000519
	0.90	0.2	1.001439	1.001336	1.003627	1.000058	1.002036	1.001568	1.004444	1.000143
		0.5	1.003567	1.003316	1.009026	1.000143	1.005028	1.003731	1.011002	1.000353
		0.8	1.005658	1.005266	1.014376	1.000225	1.007945	1.00594	1.017429	1.000557
50	**0.25**	0.2	1.001021	1.000662	1.001798	1.000127	1.001247	1.000825	1.002198	1.000147
		0.5	1.00255	1.001654	1.004499	1.000318	1.003115	1.00206	1.005497	1.000366
		0.8	1.004079	1.002643	1.007202	1.000508	1.00498	1.003291	1.008802	1.000584
	0.75	0.2	1.001028	1.000659	1.001796	1.000131	1.001217	1.000826	1.002176	1.000143
		0.5	1.002567	1.001644	1.00449	1.000326	1.003038	1.00206	1.005437	1.000356
		0.8	1.004104	1.002627	1.007186	1.000521	1.004851	1.003287	1.008696	1.000567
	0.80	0.2	1.001031	1.000659	1.001798	1.000132	1.00121	1.000826	1.00217	1.000141

		0.5	1.002575	1.001645	1.004495	1.000328	1.003017	1.002058	1.005419	1.000351
		0.8	1.004115	1.002627	1.007191	1.000524	1.004814	1.003283	1.008662	1.00056
	0.90	0.2	1.001037	1.000654	1.001793	1.000134	1.001193	1.000829	1.002163	1.000137
		0.5	1.002585	1.00163	1.004477	1.000335	1.002965	1.00206	1.005386	1.000339
		0.8	1.004125	1.002599	1.007152	1.000534	1.004717	1.003277	1.008583	1.000538
100	**0.25**	0.2	1.000521	1.000297	1.00088	1.000083	1.000642	1.000364	1.001089	1.000102
		0.5	1.001302	1.000741	1.0022	1.000208	1.001604	1.00091	1.002724	1.000254
		0.8	1.002083	1.001185	1.003521	1.000333	1.002565	1.001455	1.00436	1.000407
	0.75	0.2	1.000522	1.000294	1.000877	1.000085	1.000644	1.000369	1.001095	1.000102
		0.5	1.001305	1.000734	1.002193	1.000211	1.001608	1.000921	1.002737	1.000254
		0.8	1.002088	1.001174	1.003509	1.000338	1.002571	1.001472	1.004378	1.000406
	0.80	0.2	1.000524	1.000293	1.000876	1.000085	1.000643	1.000371	1.001097	1.000101
		0.5	1.001309	1.000731	1.002191	1.000213	1.001606	1.000926	1.002742	1.000252
		0.8	1.002093	1.001168	1.003505	1.000341	1.002567	1.00148	1.004385	1.000403
	0.90	0.2	1.000527	1.000291	1.000876	1.000087	1.00064	1.000373	1.001096	1.0001
		0.5	1.001315	1.000725	1.002188	1.000217	1.001595	1.00093	1.002736	1.000248
		0.8	1.002102	1.001158	1.003498	1.000346	1.002546	1.001484	1.00437	1.000395
200	**0.25**	0.2	1.000226	1.000119	1.000382	1.000041	1.00027	1.00014	1.000471	1.000048
		0.5	1.000565	1.000298	1.000956	1.000102	1.000675	1.000349	1.001177	1.000121
		0.8	1.000905	1.000476	1.00153	1.000163	1.00108	1.000558	1.001884	1.000193
	0.75	0.2	1.000227	1.000118	1.000383	1.000041	1.00027	1.00014	1.000466	1.000048
		0.5	1.000569	1.000296	1.000957	1.000103	1.000676	1.00035	1.001164	1.00012
		0.8	1.00091	1.000474	1.001532	1.000164	1.001081	1.00056	1.001863	1.000192
	0.80	0.2	1.000228	1.000118	1.000383	1.000041	1.00027	1.00014	1.000465	1.000048
		0.5	1.000569	1.000296	1.000958	1.000103	1.000676	1.00035	1.001162	1.00012
		0.8	1.000911	1.000473	1.001533	1.000165	1.00108	1.00056	1.001858	1.000193
	0.90	0.2	1.00023	1.000118	1.000386	1.000042	1.000269	1.00014	1.000461	1.000048
		0.5	1.000574	1.000296	1.000964	1.000104	1.000671	1.000349	1.001151	1.000119
		0.8	1.000918	1.000473	1.001542	1.000167	1.001073	1.000558	1.001841	1.000191

500	**0.25**	0.2	1.000076	1.000049	1.000144	1.000008	1.000096	1.000061	1.000179	1.000011
		0.5	1.00019	1.000122	1.000361	1.000021	1.00024	1.000153	1.000447	1.000027
		0.8	1.000303	1.000195	1.000577	1.000033	1.000384	1.000244	1.000716	1.000043
	0.75	0.2	1.000075	1.000049	1.000142	1.000008	1.000098	1.000061	1.00018	1.000011
		0.5	1.000187	1.000121	1.000356	1.00002	1.000245	1.000153	1.000451	1.000028
		0.8	1.000299	1.000194	1.00057	1.000033	1.000391	1.000245	1.000722	1.000045
	0.80	0.2	1.000074	1.000048	1.000142	1.000008	1.000098	1.000061	1.000181	1.000011
		0.5	1.000186	1.000121	1.000355	1.00002	1.000246	1.000153	1.000452	1.000028
		0.8	1.000298	1.000194	1.000568	1.000032	1.000393	1.000245	1.000724	1.000045
	0.90	0.2	1.000073	1.000048	1.00014	1.000008	1.0001	1.000062	1.000183	1.000012
		0.5	1.000183	1.000121	1.000351	1.00002	1.000249	1.000154	1.000457	1.000029
		0.8	1.000293	1.000193	1.000561	1.000031	1.000399	1.000246	1.000731	1.000046

* ρ_{12} *refers the correlation among responses,* ** *W refers nonnegative scalar weight and***$RE = \frac{Var(\hat{\beta}_{MLSE})}{Var(\hat{\beta}_{MNWLS})}$*

Table 4.4 has been represented that the modified multivariate weighted mixed estimator (MMWME) is shows better performance rather than MLSE. It is also stated in the simulation experiment that if the correlation between predictors inflates, then the estimated relative efficiency values of the modified multivariate weighted mixed estimator (MMWME) also increase.

Table 4.5: Relative Efficiency (RE) of modified two parameters weighted mixed estimator (MTPWME) with respect to MLSE for Different ρ, W and n

Sample Size, n	**ρ_{12}** *	W**	**RE*** of Modified Multivariate Two Parameter Weighted Estimator**					
			β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)
25	0.25	0.2	2.487068	2.469872	1.910375	2.554078	2.560669	1.912374
		0.5	2.487264	2.470203	1.91036	2.554253	2.561108	1.912318
		0.8	2.48746	2.470534	1.910345	2.554429	2.561547	1.912262
	0.75	0.2	2.692266	2.68273	1.920508	2.769392	2.749747	1.918376
		0.5	2.692476	2.683073	1.920498	2.769546	2.750175	1.918321
		0.8	2.692686	2.683417	1.920488	2.769701	2.750604	1.918266
	0.80	0.2	2.739507	2.730736	1.926606	2.820264	2.797978	1.924713
		0.5	2.739724	2.731085	1.926598	2.820411	2.798404	1.924657
		0.8	2.73994	2.731435	1.926589	2.820558	2.79883	1.924602
	0.90	0.2	2.877911	2.870047	1.95087	2.969743	2.94189	1.949232
		0.5	2.878164	2.870423	1.950869	2.969849	2.942297	1.94917
		0.8	2.878416	2.870799	1.950869	2.969954	2.942705	1.949108
50	0.25	0.2	2.601416	2.407231	2.027656	2.703683	2.532185	1.98785
		0.5	2.601578	2.407415	2.027644	2.703838	2.532463	1.987793
		0.8	2.60174	2.407599	2.027631	2.703993	2.532741	1.987737
	0.75	0.2	2.828575	2.572447	2.035532	2.957099	2.679187	2.003298
		0.5	2.828755	2.572634	2.035524	2.957232	2.679439	2.003245
		0.8	2.828935	2.572822	2.035517	2.957365	2.679691	2.003192
	0.8	0.2	2.881635	2.610218	2.041818	3.017018	2.716301	2.010998
		0.5	2.881824	2.610409	2.041813	3.017144	2.716548	2.010945
		0.8	2.882013	2.610599	2.041808	3.017269	2.716794	2.010892
	0.9	0.2	3.034665	2.72106	2.065738	3.192632	2.82745	2.039173
		0.5	3.0349	2.721268	2.065742	3.192718	2.827681	2.039113
		0.8	3.035135	2.721476	2.065746	3.192803	2.827912	2.039053
100	0.25	0.2	2.326808	2.262891	1.980358	2.50151	2.501585	1.901793
		0.5	2.326914	2.263014	1.980352	2.501609	2.501805	1.901749
		0.8	2.32702	2.263136	1.980346	2.501708	2.502025	1.901704
	0.75	0.2	2.479874	2.428602	1.963829	2.683715	2.629919	1.918062
		0.5	2.479999	2.428733	1.963831	2.683787	2.630118	1.918016
		0.8	2.480125	2.428863	1.963833	2.683858	2.630317	1.917971
	0.8	0.2	2.516164	2.46414	1.965517	2.727515	2.660732	1.923332

		0.5	2.516297	2.464274	1.965522	2.727579	2.660927	1.923286
		0.8	2.51643	2.464407	1.965526	2.727643	2.661122	1.923239
	0.9	0.2	2.619273	2.564953	1.977005	2.852575	2.750268	1.942597
	0.9	0.5	2.619448	2.565104	1.97702	2.852599	2.750452	1.942541
	0.9	0.8	2.619623	2.565255	1.977036	2.852623	2.750636	1.942486
200	0.25	0.2	1.931704	1.998775	1.869967	2.048222	2.329264	1.721134
		0.5	1.931773	1.99886	1.869965	2.048285	2.329439	1.721106
		0.8	1.931842	1.998946	1.869963	2.048348	2.329615	1.721079
	0.75	0.2	2.011653	2.159665	1.826335	2.179246	2.434512	1.730851
		0.5	1.000296	1.000957	1.000103	1.00035	1.001164	1.00012
		0.8	2.011827	2.15986	1.82635	2.179322	2.434828	1.730783
	0.8	0.2	2.030232	2.190971	1.822922	2.209592	2.457739	1.734727
		0.5	2.030326	2.191072	1.822932	2.209623	2.457895	1.734691
		0.8	2.03042	2.191173	1.822942	2.209654	2.458051	1.734655
	0.9	0.2	2.081067	2.274436	1.819145	2.292133	2.52014	1.747612
		0.5	2.081193	2.274554	1.819167	2.292132	2.520295	1.747562
		0.8	2.08132	2.274671	1.819189	2.29213	2.52045	1.747513
500	0.25	0.2	1.520573	1.660035	1.711523	1.613199	1.988925	1.48605
		0.5	1.520622	1.660098	1.711529	1.613244	1.989041	1.486038
		0.8	1.52067	1.660161	1.711534	1.61329	1.989156	1.486026
	0.75	0.2	1.543241	1.788547	1.641626	1.663116	2.028195	1.483066
		0.5	1.543302	1.788622	1.64164	1.663145	2.028305	1.483045
		0.8	1.543363	1.788698	1.641655	1.663174	2.028415	1.483024
	0.8	0.2	1.548852	1.80998	1.632819	1.675161	2.037024	1.483189
		0.5	1.548917	1.810059	1.632836	1.675185	2.037134	1.483165
		0.8	1.548983	1.810137	1.632854	1.675209	2.037244	1.483141
	0.9	0.2	1.562962	1.862002	1.612019	1.704516	2.053631	1.483965
		0.5	1.563049	1.862093	1.61205	1.704516	2.053745	1.483924
		0.8	1.563136	1.862184	1.61208	1.704516	2.05386	1.483884

*ρ_{12} *refers the correlation among response,* **W *refers nonnegative scalar weight and****$RE = \frac{Var(\hat{\beta}_{MLSE})}{Var(\hat{\beta}_{MTPWME})}$

According to table 4.5, it is observed that the modified two parameters weighted mixed estimator (MTPWME) is superior to the MLSE. It is also seen that an increase in W generally increase the relative efficiency of MTPWME, which leads to the results that an increment to the weight of the prior information increases the dominance of the modified two parameters weighted mixed estimator (MTPWME) over the MLSE which is true for small to large sample size.

4.5 Conclusion

In this chapter, multivariate two-parameter weighted (MTPWE) estimator has been proposed for the estimation of multivariate regression models with stochastic linear regression. Moreover, a Monte Carlo simulation is done to ensure a comparison of the proposed multivariate two-parameter weighted (MTPWE) estimator to the other modified methods for different sample size and various levels of different parameters. Based on the Monte Carlo simulation, the study reveals that the MTPWE always outperforms for multicollinearity aspects.

Chapter 5

Modified Inferential Approach for Multivariate Mixed Responses

5.1 Introduction

In applied research especially socio-demographic, epidemiological and agricultural study, the researcher needs to find out the influential factors for the target multivariate responses where the response variables may follow the mixed distribution and the parameters of the model may be restricted. However, the concept of multivariate regression faces hurdles when the response matrix is a combination of both categorical and numerical variables which is a common scenario in socio-economic and demographic analysis.

Hence, multivariate regression analysis faces a problem to estimate regression parameters or testing it because of the error distribution may not be multivariate normal. One example of the response vectors may be that it consists of desired family size, mothers weight and contraceptive use whereas, covariate matrix consists of age, income, education, sex, number of children, employment status, migration status etc. Hence, the challenge for the social statistician is to develop the appropriate model by addressing challenging issues and seek out powerful estimation and hypothesis testing procedure to fit model correctly and predicting or forecasting about the future phenomenon. In this chapter, the study has been tried to propose an estimation technique for the restricted parameters of multivariate regression with mixed responses. Here, the multivariate multiple regression models is as follows-

$$Y_{n\times p} = Z_{n\times(r+1)}\beta_{(r+1)\times p} + \epsilon_{n\times p}$$

$$E\left(\in_{(i)}\right) = 0, \quad Cov\left(\in_{(i)}, \in_{(j)}\right) = \sigma_{ik} I \quad ; \quad i.k = 1,2,\dots,p$$

where, $Y_{n\times p}$ is the vector of the n measurements of the mixed variables; $\beta_{(r+1)\times p}$refers the matrix of regression coefficients with restriction; $\epsilon_{n\times p}$is the vector of error which follows multivariate mixed distribution; and $Z_{n\times(r+1)}$is the design matrix.

5.2 Review of Literature

When categorical and continuous responses occur simultaneously then influential factors on responses can't be assessed jointly through separate analysis of those responses (Cox and Wermuth, 1992). Furthermore, the separate analysis gives biased estimates for the parameters and misleading inference. So, for multivariate mixed categorical and continuous responses, a joint model with appropriate distribution pattern is necessary for precise analysis (Samani and Ganjali, 2008). Heckman (1978) was initiated to develop a general model for simultaneously analyzing two mixed correlated responses. The joint model considers that categorical responses are inter-correlated and also are dependent on continuous responses. Simultaneous modeling of categorical and continuous variables can be described in terms of a correlated multivariate normal distribution for the underlying latent variables of ordinal responses and continuous responses (Samani and Ganjali, 2008).

Though very little research has been addressed the problem of estimating joint density function with mixed categorical and continuous responses; but still established parametric or Semiparametric procedure are not available to estimate joint density functions for mixed responses, calculating the multivariate regression coefficients for mixed responses, model specification test, and the test of the individual parameters. In this study, a modified maximum likelihood estimator has been proposed to estimate the restricted parameters of multivariate regression with continuous responses.

5.3 Proposed Approach of Parameter Estimation in Multivariate Regression with Mixed Responses

5.3.1 Model Specification

Let $\boldsymbol{Y}$ be an $n \times 2$ observation matrix of mixed response variables and $\boldsymbol{X}$ be a design matrix of $n \times (k+1)$ nonstochastic predicted variables with rank $k \leq n$ where n is the sample size. A multivariate regression model with linear restriction is given as

$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ with prior restriction $\boldsymbol{R\beta}_1 = \boldsymbol{\xi}_1$ and $\boldsymbol{R\beta}_2 = \boldsymbol{\xi}_2$

where $\boldsymbol{Y} = \begin{pmatrix} Y_{11} & Y_{21} \\ Y_{12} & Y_{22} \\ \cdots & \cdots \\ Y_{1n} & Y_{2n} \end{pmatrix}$ is the mixed response with categorical response $\boldsymbol{Y}_1 = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$ and continuous response $\boldsymbol{Y}_2$ which lies between $-\infty$ to $+\infty$. Here, the design matrix, disturbance term and matrix of regression coefficient are $\boldsymbol{X} = \begin{pmatrix} X_{01} & X_{11} & \cdots & X_{k1} \\ X_{02} & X_{12} & \cdots & X_{k2} \\ \cdots & \cdots & \cdots & \cdots \\ X_{0n} & X_{1n} & \cdots & X_{kn} \end{pmatrix}$, $\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{21} \\ \varepsilon_{12} & \varepsilon_{22} \\ \cdots & \cdots \\ \varepsilon_{1n} & \varepsilon_{2n} \end{pmatrix}$,

$\boldsymbol{\beta} = \begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \\ \cdots & \cdots \\ \beta_{1n} & \beta_{2n} \end{pmatrix}$ respectively. The restricted parameters are $\boldsymbol{R} = \begin{pmatrix} R_{01} & R_{11} & \cdots & R_{k1} \\ \cdots & \cdots & \cdots & \cdots \\ R_{0r} & R_{1r} & \cdots & R_{rr} \end{pmatrix}_{r<k}$ and $\boldsymbol{\xi_i} = (\xi_{i1} \quad \xi_{i2})$.

The model considered the following assumption to estimate the parameters.

i) The sample observation for each unit is independent.

ii) Each restriction is same for both dependent variables.

5.3.2 Parameter Estimation

The random component identifies the probability distribution function of response variable. The joint distribution of the response variables $\boldsymbol{Y_1}$and $\boldsymbol{Y_2}$ given the design matrix $\boldsymbol{X}$ without restriction is given below

$$f(Y_1, Y_2 | X_1, \dots, X_k) = \frac{1}{\sigma\sqrt{2\pi}} \frac{\left[e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]^{Y_{1i}}\left[e^{-\frac{1}{2\sigma^2}(Y_{2i}-\beta_{02}-\beta_{12}X_{1i}-\cdots-\beta_{k2}X_{ki})^2}\right]}{\left[1+e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]} \quad (5.1)$$

So, the likelihood function of the parameters for $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ given $\boldsymbol{X}$ is

$$L(\beta_{j1}, \beta_{j2}) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \frac{\left[e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]^{Y_{1i}}\left[e^{-\frac{1}{2\sigma^2}(Y_{2i}-\beta_{02}-\beta_{12}X_{1i}-\cdots-\beta_{k2}X_{ki})^2}\right]}{\left[1+e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]} \quad (5.2)$$

for $j = 1,2, \dots k$

Taking logarithm both sides in equation (5.2), the log likelihood function will be

$$Ln\ L = ln\left[\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \frac{\left[e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]^{Y_{1i}}\left[e^{-\frac{1}{2\sigma^2}(Y_{2i}-\beta_{02}-\beta_{12}X_{1i}-\cdots-\beta_{k2}X_{ki})^2}\right]}{\left[1+e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right]}\right]$$

In order to estimate the regression coefficient, the following log-likelihood function will be

$$Ln\ L = \sum_{i=1}^{n} Y_{1i}(\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}) - \sum_{i=1}^{n} ln\left[1+e^{\beta_{01}+\beta_{11}X_{1i}+\cdots+\beta_{k1}X_{ki}+\gamma_{12}Y_{2i}}\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_{2i}-\beta_{02}-\beta_{12}X_{1i}-\cdots-\beta_{k2}X_{ki})^2 + n ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) \quad (5.3)$$

Though the link functions play a vital role of linking the random component with systemic component for exponential family but the link function of the joint distribution containing binary and continuous responses will be mixed, namely logit link and identity link bridging by an association parameters (Islam and Chowdhury, 2017). Here, γ_{12} is the association parameter.

Consider the following restrictions on the parameter space of the coefficient vector β,$R\beta_{j1} = \xi_1$ and $R\beta_{j1} = \xi_2$, the modified maximum likelihood estimate is proposed by imposing restriction on the log likelihood function (5.3). Therefore, the following objective function should be maximized.

$$\tilde{\ell}(\beta_{j1}, \beta_{j2}, \lambda_1, \lambda_2) = Ln\, L + \lambda_1(R\beta_{j1} - \xi_1) + \lambda_2(R\beta_{j2} - \xi_2) \quad (5.4)$$

where λ_1 and λ_2 are two Lagrangian multipliers for different restrictions, respectively.

Using matrix approach in equation (5.4) and taking differentiation with respect to β_{j1} to find the first normal equation (5.5)

$$\frac{\delta\tilde{\ell}}{\delta\beta_{j1}} = X'\left[Y_1 - \frac{exp(X'\beta_{j1}+Y_2\gamma_{12})}{1+\exp(X'\beta_{j1}+Y_2\gamma_{12})}\right] + R'\lambda_1 = 0 \quad (5.5)$$

Hence, Newton-Raphson method has been used to estimate the parameters for categorical response given design matrix $\boldsymbol{X}$and continuous response$\boldsymbol{Y}_2$.

$$\hat{\beta}_{j1(mmle)}^{(t+1)} = \hat{\beta}_{j1(mle)}^{t+1} + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}[\xi_1 - R\hat{\beta}_{j1(mle)}^{t+1}]$$

$$\therefore \hat{\beta}_{j1(mmle)} = \hat{\beta}_{j1(mle)} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}[R\hat{\beta}_{j1(mle)} - \xi_1] \quad (5.6)$$

Using matrix approach in equation (5.4) and taking differentiation with respect toβ_{j2} to find the second normal equation (5.7).

$$\frac{\delta\tilde{\ell}}{\delta\beta_{j2}} = -2\, X'Y_2 + 2X'X\, \beta_{j2(mmle)} + R'\lambda_2 = 0 \quad (5.7)$$

Now, restriction $R\beta_{j2} - \xi_2 = 0$ has been used to estimate the parameters of β_{j2}, Hence, minimization of the objective function with respect to β_{j2} and λ_2, we derived the modified likelihood estimate for β_{j2} .

$$\hat{\beta}_{j2(mmle)} = \hat{\beta}_{j2(mle)} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}\left[R\hat{\beta}_{j2(mle)} - \xi_2\right] \quad (5.8)$$

Now, differentiating both sides with respect toγ_{12} for obtain (Islam and Chowdhury, 2017) the normalized equation (5.9)

$$\frac{\delta\tilde{\ell}}{\delta\gamma_{j2}} = \sum_{i=1}^{n}\left[Y_{1i}Y_{2i} - \frac{Y_{2i}\,exp\left(X\beta_{j1} + Y_{2i}\gamma_{12}\right)}{1 + exp\left(X\beta_{j1} + Y_{2i}\gamma_{12}\right)}\right] = 0 \quad (5.9)$$

Since$p_i = \pi = \frac{exp\left(X\beta_{j1} + Y_2\gamma_{12}\right)}{1 + exp\left(X\beta_{j1} + Y_2\gamma_{12}\right)}$, the equation (4.9) has been reduced as

$\sum_{i=1}^{n} Y_{2i}[Y_{1i} - P_i] = 0$, hence iterative weighted least square has been used to find out the modified maximum likelihood estimate for γ_{12}.

$$\hat{\gamma}_{12}^{(t+1)} = \hat{\gamma}_{12}^{t} + \left(Y_2'\widehat{W}^tY_2\right)^{-1}Y_2\left(Y_1 - \hat{P}_i'\right) \quad (5.10)$$

where $\hat{P}_i'$ is the estimated value of P_i using $\hat{\beta}_{j1}^{t}$ and$\widehat{W}'$ = diag $\hat{P}_j\left(1 - \hat{P}_i\right)$ such that $\hat{P}_j$ in the j^{th}element of the $\hat{\pi}'$.Again, differentiating both sides with respect to σ^2 and putting the function equal zero for measuring σ^2 .

$$\frac{\delta\tilde{\ell}}{\delta\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\left(Y_{2i} - X\beta_{j2}\right)'\left(Y_{2i} - X\hat{\beta}_{j2}\right) \quad (5.11)$$

$$\hat{\sigma}^2 = \frac{1}{n}\left(Y_2 - X\beta_{2j}\right)'\left(Y_2 - X\beta_{2j}'\right)$$

5.4 Monte Carlo Experiment

Monte Carlo experiments have been conducted to examine the performance of the newly proposed modified maximum likelihood estimator for estimating and testing the restricted parameters of the multivariate regression model with mixed responses. The study considered a multivariate regression model with mixed responses ($p = 2$) where categorical response $\boldsymbol{Y}_1 = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$ and continuous response $\boldsymbol{Y}_2$ lies between $-\infty$ to $+\infty$.

Since the application of multivariate regression analysis depends on the correlation among the response variables, the different trials have been conducted for different arbitrary value of correlation coefficient namely, $\rho = 0.00, 0.25, 0.75, 0.80, 0.90$ where $\sigma_{11} = 100\ and\ \sigma_{22} = 81$ including sample sizes of different order, $n = 25,50,100, 200, 400$. The generation of multivariate responses also depends on the parameter values of the regression coefficients taken to be

$\beta_{j2} = [25 \quad 3.5 \quad -1.75 \quad -1.5]$ and $\beta_{j2} = [175 \quad 2.5 \quad -1.25 \quad -1.25 \quad 5.5]$

$i)\boldsymbol{R_2} = [0 \ \ 1 \ \ 2 \ \ 0]$ 2) $\boldsymbol{R_1} = [0 \ \ 1 \ \ 2 \ \ 0 \quad 0]$ for $\boldsymbol{\xi} = [0 \ \ 0]$ respectively.

The relative efficiency (RE) has been used to diagnosis the simulated performance of modified maximum likelihood estimate and maximum likelihood estimator for restricted coefficients of multivariate regression with continuous responses. Hence, the study used 10,000 times iteration to find the results.

Table 5.1: Relative Efficiency (RE) of MMLE on MLE for Multivariate regression with mixed responses

Sample Size, n	ρ_{12} *	Relative Efficiency (RE**) of Modified Maximum Likelihood Estimate								
		β_{01} (25)	β_{11} (3.5)	β_{21} (-1.75)	β_{31} (-1.5)	β_{02} (175)	β_{12} (2.5)	β_{22} (-1.25)	β_{32} (-1.25)	γ (5.5)
25	**0.00**	1.6132	1.4993	11.2818	1.0145	3.1512	0.4455	19.9002	1.2766	0.8789
	0.25	1.6526	1.47	11.1794	1.0216	1.7119	1.449	11.5114	1.0388	1.0753
	0.75	1.6769	1.4507	11.1054	1.0261	1.722	1.4028	10.8732	1.0445	1.0893
	0.80	1.6783	1.4495	11.1004	1.0264	1.1873	2.3213	12.0493	1.7593	1.9428
	0.90	1.6806	1.4475	11.0916	1.0268	2.25	12.2113	16.6991	1.848	1.958
50	**0.00**	2.0772	1.5409	14.5644	1.0615	61.4582	0.0007	17.8633	2.1534	1.9384
	0.25	2.1032	1.5465	14.7959	1.0643	2.0753	2.9331	18.6844	2.7262	1.9616
	0.75	2.1275	1.5388	14.826	1.0688	2.2655	3.0195	18.7296	2.7251	1.9896
	0.80	2.1294	1.5377	14.8211	1.0692	9.3336	6.1609	18.8757	2.762	1.9964
	0.90	2.1331	1.535	14.8038	1.0702	3.5262	4.0497	18.8803	2.3288	1.0543
100	**0.00**	2.215	1.4502	13.9967	1.0958	1.2048	1.168	4.2097	1.1962	1.9932
	0.25	2.2129	1.4498	13.9893	1.0957	1.3984	1.3585	1.6842	1.4101	1.9797
	0.75	2.2188	1.4525	14.0694	1.0963	1.8569	1.3765	3.7339	1.8469	1.9953
	0.80	2.2196	1.4528	14.0792	1.0963	1.8784	1.9254	4.9135	1.85321	1.9755
	0.90	2.2215	1.4536	14.1019	1.0965	1.9008	1.6353	7.3219	1.2576	1.9109
200	**0.00**	2.2094	1.3653	12.8409	1.1161	1.6232	1.1627	2.884	1.2841	1.0535
	0.25	2.1749	1.3628	12.6329	1.1113	1.4371	1.2263	3.8123	1.294	1.0927
	0.75	2.1804	1.3786	12.8996	1.1087	1.596	2.7186	6.2365	1.7699	1.098
	0.80	2.1823	1.3806	12.9389	1.1086	2.6294	2.7918	6.6784	2.1096	1.0937
	0.90	2.1877	1.3853	13.0358	1.1084	1.6509	2.6491	7.4321	2.7842	1.0979
400	**0.00**	1.9922	1.4443	12.7798	1.0656	1.6509	2.6491	1.4321	2.7842	1.0179
	0.25	2.0178	1.4478	12.9895	1.0688	1.4389	2.7429	2.891	1.1362	1.0019
	0.75	1.9986	1.449	12.8969	1.0691	1.6897	2.7864	43.4435	1.5493	1.0098
	0.80	1.9954	1.449	12.8786	1.0696	1.7795	2.7957	6.4671	1.3749	1.0068
	0.90	1.9875	1.4489	12.8313	1.0699	1.8718	2.4598	41.9685	1.321	1.0054

*ρ_{12} *refers the correlation among response, and* $^{**}RE = \frac{Var(\hat{\beta}_{MLE})}{Var(\hat{\beta}_{MMLE})}$

According to Table 5.1, it is observed that the modified maximum likelihood estimate is superior to the MLE for the restricted parameters of multivariate regression with mixed responses.

Theorem 5.1: Let $Y_{1i} \sim Bernouli(P_i)$and $Y_{2i} \sim Normal\ (\mu, \sigma^2)$ are to be random variables which are interrelated. Y_1 and Y_2 are the categorical and continuous variables respectively and $X_{1i}, X_{2i}, \dots, X_{ki}$ are the explanatory variables. Hence the joint density function of $Y_1$and Y_2 given the explanatory variables $X_{1i}, X_{2i}, \dots, X_{ki}$ is

$P[Y_1, Y_2 | X_1, X_2, X_3, \dots X_k]$

$$= \frac{1}{\sigma\sqrt{2\pi}} \frac{[\exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]]^{Y_{1i}} \times \exp[-\frac{1}{2\sigma^2}(y_{2i} - \beta_{02} - \beta_{12}x_{1i} - \cdots - \beta_{k2}x_{ki})^2]}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}$$

Let Y_2 is a continuous random variable with density function

$P[Y_2 = y_2 | X_1 = x_1, X_2 = x_2, \dots X_k = x_k] = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(y_{2i} - \beta_{02} - \beta_{12}x_{1i} - \cdots - \beta_{k2}x_{ki})^2$ whose distribution is normal with expectation $\beta_{02} + \beta_{12}x_{1i} + \beta_{22}x_{2i} + \cdots + \beta_{k2}x_{ki}$ and variance σ^2.

$$P[Y_1 | Y_2, X_1, \dots, X_k] = P[Y_1 = y_1 | Y_2 = y_2, X_1 = x_1, \dots X_k = x_k] = P_i^{Y_{1i}}(1 - P_i)^{1 - Y_{1i}}$$

where$P_i = \frac{\exp[\beta_{01} + \beta_{11}x_{1i} + \beta_{21}x_{2i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \beta_{21}x_{2i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}$ (Bel *et. al.*, 2018).

$P[Y_1 | Y_2, X_1, X_2, X_3, \dots X_k]$

$$= \left[\frac{\exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}\right]^{Y_{1i}} \left[\frac{1}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}\right]^{1 - Y_{1i}}$$

$$= \frac{[\exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]]^{y_{1i}}}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}}$$

The mathematical approach to formulate joint distribution function of $Y_1$and Y_2 given a set of explanatory variables (Islam and Chowdhury, 2017) is

$P(Y_1, Y_2 | X_1, X_2, \dots, X_k) = P(Y_1 | Y_2, X_1, \dots, X_k)\ P(Y_2 | X_1, \dots, X_k)$. Hence, the joint distribution of two correlated variables $Y_1$and Y_2 given a set of explanatory variables is formulated as

$P[Y_1, Y_2|X_1, X_2, X_3, \dots X_k]$

$$= \frac{1}{\sigma\sqrt{2\pi}} \frac{[\exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]]^{Y_{1i}} \times \exp[-\frac{1}{2\sigma^2}(y_{2i} - \beta_{02} - \beta_{12}x_{1i} - \cdots - \beta_{k2}x_{ki})^2]}{1 + \exp[\beta_{01} + \beta_{11}x_{1i} + \cdots + \beta_{k1}x_{ki} + \gamma_{12}y_{2i}]}$$

5.5 Conclusion

This study is the initial initiative to find out a uniform approach for estimating the restricted parameter of multivariate regression with mixed responses especially the mixed of binary and continuous responses. Based on the Monte Carlo simulation, the study reveals that the variance of modified maximum likelihood estimator is lower and relative efficiency is higher than current methods.

Chapter 6

Socio-economic Determinants of Households Food Expenditure in Haor Areas of Bangladesh: A Restricted Multivariate Regression Approach

6.1 Background

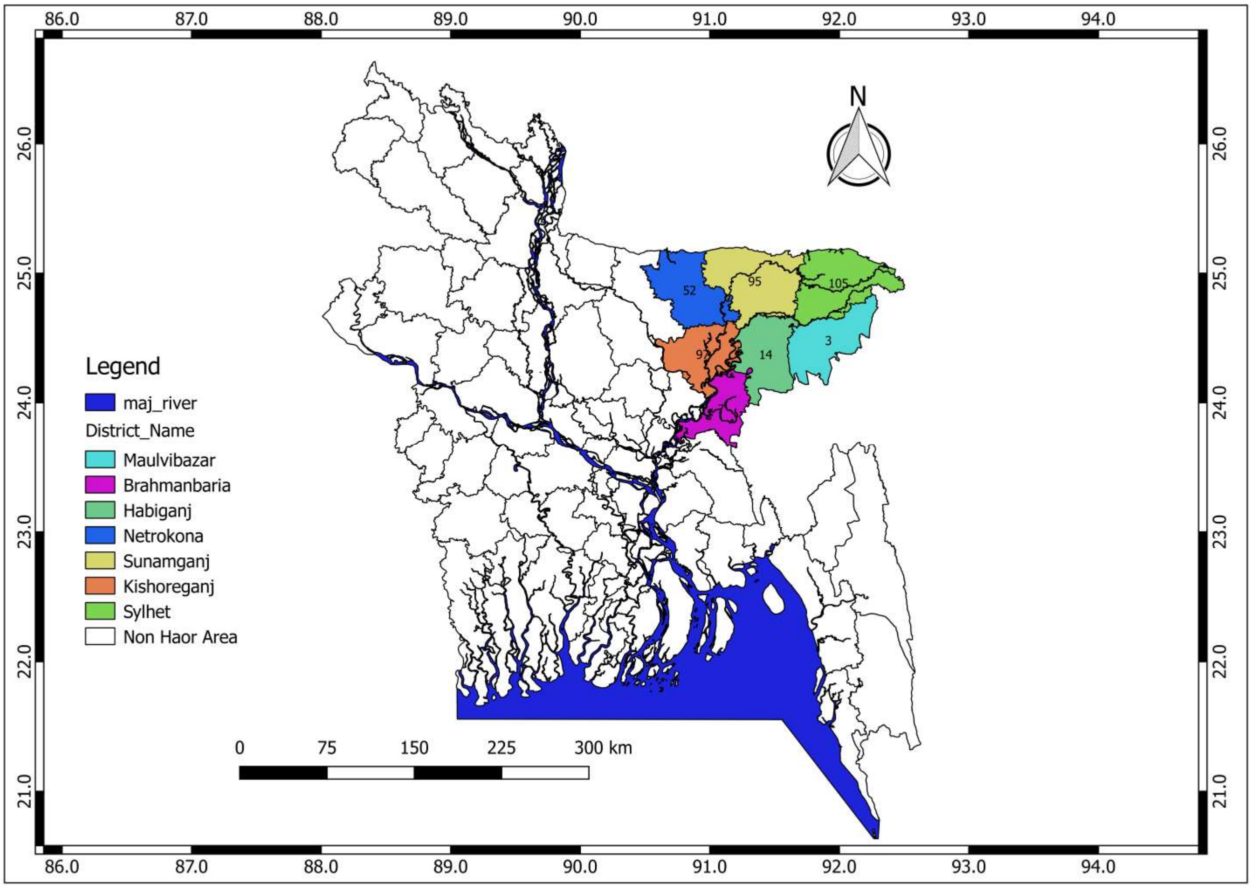
Bangladesh has experienced positive improvements in social, economic, and health sectors. Still, the progress is not up to the mark in Haor areas. Haors are wetland ecosystem located in the northeastern region of Bangladesh which is physically a bowl or saucer-shaped shallow depressions. Haor is particularly low lying basin area below the level of the flood plain. These areas are also similar to swampland covered by water almost six months of a year starting from the monsoon. The total number of 373 Haors situated in the districts of Sunamganj, Habiganj, Maulabibazar, Sylhet, Mymensingh, Bramanbaria and kishorganj which covered 1.99 million ha. of areas.

Table 6.1: Descriptions of the Haor Areas in Bangladesh

District	**Total Area of the District** (in ha.)	**Haor Area** (in ha.)	**No. of Haor**
Sunamganj	367,000	268,531	95
Sylhet	349,000	189,909	105
Habiganj	263,700	109,514	14
Maulvibazar	279,900	47,602	3
Netrokona	274,400	79,345	52
Kishoreganj	273,100	133947	97
Brahmanbaria	192700	29616	7
Total	1999800	858,460	373

Source: Report on Classification of Wetlands of Bangladesh, Department of Bangladesh Haorand Wetlands Development, Ministry of Water Resources, Bangladesh, 2016

Figure 6.1: Study Areas (Haor Areas) in Bangladesh



The economic progress of Bangladesh is moving steadily. Yet, the Haor regions have long been lagging behind mainstream national development. It is difficult to anticipate the country's overall progress without the development of the Haor region due to it covers the major part of the country and the population which deserves special development initiatives.

The Haor people's livelihood strategy is still neither viable nor sustainable (Gardener and Ahmed, 2006). Seasonality is closely correlated with uncertain fluctuation of food security and situational poverty in low-income peoples which impacts on four major broad indicators of food security: food access, availability, distribution and utilization. Food consumption and food security became a top priority concern for the governments and socio-demographic research because of increasing input cost in the cereal production and dramatic rise in the prices of food throughout the world. This chapter has been tried to find out the implication of applying a restricted multivariate regression approach for finding socio-economic determinants of household's food consumption in Haor Areas of Bangladesh.

6.2 Sources of Data

Bangladesh Household Income Expenditure Survey (HIES) 2016 data set has been used to detect the consequence of proposed inferential approach for multivariate regression analysis. The study has been selected socio-economic data namely social safety net programme, wage employment, food and non-food expenditure data of 2280 agriculture and nonagricultural workers in Haor regions (Sunamganj, Habiganj, Maulabibazar, Sylhet, Mymensingh, Bramanbaria and kishorganj districts) as a target sample.

6.3 Variable Selection

The variables have been selected based on the review of the literature and field experience. Monthly food consumption, total monthly expenditure, family size, age, employment status, marital status and educational attainment of household head are logically interrelated variables (Shekhampu, 2012). The study has been assumed that household total expenditure and food consumption to be function of total monthly income, family size, total operating land and other predictors. The description of the explanatory variables is given below.

Total monthly income: Total monthly income of a respondent was measured by summing of all income earned by a respondent and other member of the family in a month from agriculture sector (crop farming, livestock rearing, fisheries, farm labour, homestead forestry sector) and nonagricultural sector (service, business, social benefits scheme, relief and driving boat) which expressed in taka. The study assumes that food expenditure is positively and significantly influenced by household income (Babalola and Isitor, 2014).

Family size: Family size of the respondent is the total members of the family including the respondent himself, spouse, children and other dependants who use to live, eat and act together in a family. The expected sign of the relationship between family size and food consumption is positive (Alam, Alam and Mustaq, 2018).

Total operating land: Total operating land is the most important factor in agricultural sector. Access of total operating land is considered the key determinant of the livelihood strategy of rural low income people by influencing household's crop production capacity leading to increase food availability and also enhance extra income from marketing the surplus production. Again, the expected sign of the relationship between family size and food consumption is positive (Alam, Alam and Mustaq, 2018).

The Logarithm functional form is used to explain responses in household total and food expenditure socio-economic predictors. The logarithmic transformed variables are given below.

TME=Logarithm of Total Monthly Expenditure

MFC= Logarithm of Monthly Food Consumption

FS= Logarithm of Family Size

TMI= Logarithm of Total Monthly Income as a Worker

TOL=Total Operating Land

Table 6.2: Association among logarithm function of monthly total expenditure, food expenditure and socio-economic predictors

Pearson Correlation Coefficient (r)	TME	MFC	FS	TMI	TOL
TME		.960** (.000)	.487** (.000)	.346** (.000)	.056** (.000)
MFC	.960** (.000)		.503** (.000)	.302** (.000)	.039 (.060)
FS	.487** (.000)	.503** (.000)		.090** (.000)	.035 (.091)
TMI	.346** (.000)	.302** (.000)	.090** (.000)		-.004 (.840)
TOL	.056** (.007)	.039 (.060)	.035 (.091)	-.004 (.840)	

Food consumption and household income are positively correlated (Talukder and Chile, 2013). Though logarithm functional form of total monthly expenditure and monthly food consumption are highly($r = 0.96$) and significantly ($p < 0.00$) correlated, "TME" and "MFC" can be used as a multivariate response variable.

Sensitivity analysis is important to quantify how the uncertainty in the output of a model is related to the uncertainty in its inputs (Salciccioli, Crutain, Komorowski and Marshall, 2016). The previous literature reveals that household total and food expenditure is positively influenced by family size, total household income and total operating land. It also found out that the influence of family size is higher than household income on food expenditure as well as family income. Scatter matrix (figure A1 in appendix A) as a method of sensitivity analysis (Bells, Alary, Laguerre, Fanke, 2018) and correlation matrix is also support this conditions. FS, TMI and TOL are significantly related to both "TME" and "MFC". This study has considered FS, TMI and TOL as the predictors.

6.4 Multivariate Analysis

Multivariate data analysis is the statistical methodologies that allow simultaneous investigation of more than two interrelated variable. It attempts to explain or predict the multiple response variables on the basis of fixed predictors. The challenge in this regard is to develop a logical frame of the multivariate regression model.

6.4.1 Model Selection

Multivariate regression model has been used to find the degree of dependency among multivariate response and predictors. The two different models are assumed of these issues.

Model 1: $[\mathbf{TME} \vdots \mathbf{MFC}] = [\mathbf{1FS} \quad \mathbf{TMI} \quad \mathbf{TOL}] \times \begin{bmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} + [\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}]$

Assumptions:

1. The random disturbance $[\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}]$ follows multivariate normal distribution with $E[\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}] = [\mathbf{0} \vdots \mathbf{0}]$ and covariance matrix, $\Sigma = \begin{bmatrix} \sigma_{\mathbf{TME}} & \sigma_{\mathbf{TME,MFC}} \\ \sigma_{\mathbf{TME,MFC}} & \sigma_{\mathbf{MFC}} \end{bmatrix}$.
2. The observations in different trials are independent.
3. $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ are unknown parameters of the design matrix $[\mathbf{FS} \quad \mathbf{TMI} \quad \mathbf{TOL}]$.

Model 2: $[\mathbf{TME} \vdots \mathbf{MFC}] = [\mathbf{1FS} \quad \mathbf{TMI} \quad \mathbf{TOL}] \times \begin{bmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} + [\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}]$

with respect to $[R_1 \quad R_2 \quad R_3 \quad R_4] \times \begin{bmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} = [r_1 \quad r_2]$

The restriction $[R_1 \quad R_2 \quad R_3 \quad R_4] = [0 \quad 1 \quad -2 \quad 0]$ and $[r_1 \quad r_2] = [0 \quad 0]$ considered subjectively on the basis of prior information.

Assumptions:

1. The random disturbance $[\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}]$ follows multivariate normal distribution with $E[\boldsymbol{\varepsilon}_{\mathbf{TME}} \vdots \boldsymbol{\varepsilon}_{\mathbf{MFC}}] = [\mathbf{0} \vdots \mathbf{0}]$ and covariance matrix, $\Sigma = \begin{bmatrix} \sigma_{\mathbf{TME}} & \sigma_{\mathbf{TME,MFC}} \\ \sigma_{\mathbf{TME,MFC}} & \sigma_{\mathbf{MFC}} \end{bmatrix}$.
2. The observations in different trials are independent.
3. $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ are unknown parameters of the design matrix $[\mathbf{FS} \quad \mathbf{TMI} \quad \mathbf{TOL}]$.
4. The restriction of the model is exact linear.

6.4.2 Parameter Estimation and Testing Methods

Both maximum likelihood and modified maximum likelihood method are used to estimate the parameters of the model 1 and model 2. Model 2 is the modification of the model 1 on the basis of prior information.

Modified restricted likelihood ratio test has been used to evaluate the relative performance of the two models. Proposed modified multivariate t test has been applied to test the individual parameter of the required model. Modified confidence intervals are used to obtain interval estimation of the parameters. Stata 14 and R program are used for computing the datasets.

6.5 Results and Discussion

The value of restricted likelihood ratio test for comparing model 1 to model 2 is 1.030216 with $\mathrm{p-value(Lr\ statistic)} = 0.09481$ which indicate that model 2 is better at 10% level of significance.

Table 6.3: Parameter estimation and testing of model 1

					95% Conf. Interval	
Model 1	**Coefficient**	**Std. Err.**	**t**	**P>t**	**LCL**	**UCL**
TME						
FS	0.588836	0.022123	26.62	0.000	0.5455	0.6322
TMI	0.270384	0.015293	17.68	0.000	0.2404	0.3004
TOL	0.002279	0.000949	2.4	0.016	0.00042	0.00413
_cons	6.949503	0.136465	50.93	0.000	6.6819	7.2171
MFC						
FS	0.602938	0.021871	27.57	0.000	0.56005	0.6458
TMI	0.22565	0.015119	14.93	0.000	0.1960	0.2554
TOL	0.001279	0.000938	1.36	0.173	-0.00056	0.0031
_cons	7.131675	0.134908	52.86	0.000	6.86712	7.3962

The results of the multivariate regression model on the factors that affect household food consumption are shown in the Table 6.3. The results of the study shows that total monthly income, family size, total operating land have a significant influence on monthly food expenditure.

These predictors were found to exert a positive impact on both food consumption and total monthly expenditure. The study reveals that logarithm form of total monthly expenditure and food consumption as multivariate continuous responses are significantly related to total operating land, logarithm form of family size and total monthly income $(p < 0.01)$ considering a restriction on the parameters at 5% level of significance.

Table 6.4: Parameter Estimation and Testing of Model 2

Model 2	Coefficient	Std. Err.	t_{mod}	p>t_{mod}	95% Modified Confidence Interval	
TME					LCL	UCL
Constant	6.7813693	0.090	75.30593	0	6.6077	6.95499
FS	0.5489785	0.014	39.57764	0.00	0.5198	0.57819
TMI	0.2744892	0.013	21.86325	0.00	0.2599	0.2891
TOL	0.0013737	0.000	56.14568	0.00	-0.00024	0.00299
MFC						
Constant	6.8384578	0.091	75.29916	0	6.6634	7.0136
FS	0.5717308	0.014	40.87016	0.00	0.5423	0.6012
TMI	0.2858654	0.013	22.57726	0.00	0.2711	0.3006
TOL	0.0023087	0.000	93.56499	0.00	0.00068	0.0039

6.6 Conclusion

The proposed inferential approach has been also used to detect the numerical nexus among socio-demographic predictors, food expenditure and total monthly expenditure in Haor regions of Bangladesh. An increase in monthly household income, household size and total operating land of the household is associated with a positive increase in household food expenditure and total expenditure.

Chapter 7

Conclusions and Further Research

The aim of this dissertation was to search appropriate inferential approach for analyzing multivariate regression considering prior information about the parameters. The appropriate inferential approach was proposed after checking the utility of different conventional parameter estimation and testing methods, and modifying and developing new techniques or methods if necessary. This chapter summarizes the overall work in this thesis, highlighting the key findings, and outlining possible areas of interest for future research.

7.1 Multivariate Regression with Exact Linear Restriction

Parameter estimation and significant variable estimation are two important goals in regression modeling. In chapter 2, related published scientific papers have been reviewed for finding the research gap, justification of the study, developing the objectives and the framework to fulfill the objectives. In chapter 3, Modified maximum likelihood estimator has been proposed to estimate the exact restricted parameters of multivariate regression with continuous responses. The proposed estimator is unbiased, consistent and relatively efficient than the classical maximum likelihood estimator. In chapter 5, Modified maximum likelihood estimator has been proposed to estimate the exact restricted parameters of multivariate regression with mixed responses. The performance of modified estimator is relatively efficient than the maximum likelihood estimator. In chapter 3, modified likelihood ratio test, modified Akaike information criterion has been applied to select the related variables of multivariate responses. Modified multivariate $'t'$ test has been proposed to check the significance of individual restricted parameters. Modified joint confidence region has been developed to obtain joint confidence interval for restricted parameters. Based on the simulation study, this research concludes that the proposed estimation technique and hypothesis testing methods are more appropriate for multivariate regression with exact linear restriction.

The proposed modified inferential approach has been also applied to detect the numerical nexus among socio-demographic determinants, food expenditure and total monthly expenditure in "Haor" areas of Bangladesh. The study has been revealed that total monthly expenditure and food expenditure are significantly related to total operating land, family size and total monthly income ($p < 0.01$) considering restricted parameters. Based on the simulation study and empirical application, the performance of the modified inferential approach is better than the existing methodology.

7.2 Multivariate Regression with Stochastic Restriction

Most of the situations in real life, the prior information of the parameter of multivariate regression model are not exact. In chapter 4, modified restricted least squares estimator (MRLSE), modified multivariate weighted least squares (MMWLS), modified two parameters weighted mixed estimator (MTPWME) have been proposed to estimate the stochastic linear restricted parameters. The study has revealed that the proposed estimator MRLSE is relatively unbiased, consistent and relatively efficient than multivariate ordinary least square. However, to overcome the multicollinearity problems arise in the classical ordinary least squares estimation procedure, the MTPWME has been proposed. Moreover, A Monte Carlo simulation experiment has been done to create confirms comparison of the MTPWME to the MMWLS, MRLSE and MLSE for the various levels of different parameters. The simulation study has recommended that the MTPWME always shows better performance towards the MMWLS and MRLSE with the given stochastic restrictions and multicollinearity for the multivariate regression model.

7.3 Limitations and Directions for Further Research

There are numerous ways in which the results developed in this thesis can be extended.

i) In socio-demographic research, many of the times response matrices are categorical. Multivariate logit models are widely used to describe the correlated binary decision data. If there have been any prior information regarding the degree of dependency among response matrix and predictors or multicollinearity has been attained in the model, the proposed modified statistic can be extended to find the relative efficient estimators. So, the proposed modified statistics can be extended for multivariate regression with categorical (either ordinal or nominal) responses.

ii) Sometimes, statistically, the problem introduced by the presence of multicollinearity in the data matrix and also the existence of the stochastic restrictions in the parameters, modified multivariate two-parameter weighted estimate can be used to estimate the parameter. However, the opportunity is still there to research further for finding out the individual and overall test of the parameters.

iii) Nowadays, big data is another issue for the statisticians. The modified inferential approach can be used or extended to estimate the restricted parameters in big data and also for missing data.

iv) The proposed modified approach can also be used or extended for restricted parameters of multivariate regression for mixed responses. The powerful testing procedure for the restricted parameters is also needed to be developed.

7.4 Concluding Remarks

Complex natures of real life data have widened the scope of devising estimators with restrictions. This comes with the complex nature and structure of data and mixed type of the response vectors in regression analysis. Furthermore, these pose serious challenge to the existing hypothesis testing methodology. This thesis has tried to shed light on such aspects and suggested a way forward to solve complex estimation and testing problems. More should be done in this area, and inferential statistics should be ready to deal with emerging complex problems, especially with the emergence of big data sets on various fields.

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Appendix A

Appendix A.1: Simulated Quantile (Critical Value) point for t_{mod} at different level

Sample Size, n	ρ_{12}	**0.9**	**0.95**	**0.975**	**0.985**	**0.99**	**0.9975**
25	0	4.822417	7.155769	8.322446	8.905784	9.022451	9.372454
	0.25	2.953842	3.165833	3.271828	3.324825	3.335425	3.367224
	0.75	2.860572	3.067135	3.170417	3.222058	3.232386	3.26337
	0.8	2.832061	3.036853	3.139249	3.190448	3.200687	3.231406
	0.9	2.747137	2.946526	3.046221	3.096068	3.106037	3.135946
200	0	2.145359	2.552059	2.755409	2.857085	2.87742	2.93842
	0.25	2.315094	2.380149	2.412676	2.428939	2.432192	2.44195
	0.75	2.686368	2.846584	2.926693	2.966747	2.974757	2.99879
	0.8	2.732407	2.881341	2.955808	2.993042	3.000489	3.0228
	0.9	2.829456	2.942884	2.999598	3.027956	3.033627	3.050641
400	0	2.455515	2.771747	2.929863	3.008921	3.024732	3.072167
	0.25	2.275267	2.469454	2.566547	2.615094	2.624803	2.653931
	0.75	2.871769	2.875078	2.876732	2.877559	2.877725	2.878221
	0.8	2.916904	2.918396	2.919143	2.919516	2.91959	2.919814
	0.9	3.000161	3.007663	3.011415	3.013291	3.013666	3.014791
1000	0	0.9548702	1.009798	1.037262	1.050994	1.053741	1.06198
	0.25	0.9665599	1.177758	1.283358	1.336157	1.346717	1.378397
	0.75	0.9566973	1.148151	1.243878	1.291741	1.301314	1.330032
	0.8	0.9444962	1.131594	1.225143	1.271917	1.281272	1.309337
	0.9	0.981476	1.119708	1.188824	1.223382	1.230294	1.251029

Appendix A.2: Measuring Relative Bias of Multivariate Weighted Mixed estimator Estimate for Different ρ and n

Sample Size, n	ρ_{12}	W	Relative Bias of Multivariate Weighted Mixed estimator Estimate							
			β_{01}	β_{11}	β_{21}	β_{31}	β_{02}	β_{12}	β_{22}	β_{32}
25	0	0.2	11.7059	0.0504	-0.3064	0.5378	-6.3991	-0.9853	-0.1835	-4.6603
25	0	0.5	11.7237	0.0501	-0.3079	0.5381	-6.3937	-0.9863	-0.1883	-4.6594
25	0	0.8	11.7402	0.0498	-0.3094	0.5383	-6.3883	-0.9873	-0.193	-4.6586
25	0.25	0.2	32.8316	0.3962	-0.2093	1.8766	-3.8233	-0.7003	-0.3943	-2.9762
25	0.25	0.5	32.8457	0.3959	-0.2105	1.8768	-3.8141	-0.702	-0.4024	-2.9748
25	0.25	0.8	32.8591	0.3957	-0.2117	1.8769	-3.805	-0.7037	-0.4103	-2.9734
25	0.75	0.2	26.5371	0.2908	-0.2441	1.4742	-1.1507	-0.3329	-0.4304	-1.1058
25	0.75	0.5	26.5088	0.2913	-0.2416	1.4738	-1.1408	-0.3347	-0.4391	-1.1042
25	0.75	0.8	26.4804	0.2919	-0.2391	1.4735	-1.131	-0.3366	-0.4477	-1.1027
25	0.8	0.2	25.0745	0.2672	-0.25	1.3819	-0.8462	-0.2898	-0.4314	-0.8905
25	0.8	0.5	25.0206	0.2682	-0.2452	1.3812	-0.8328	-0.2923	-0.4432	-0.8884
25	0.8	0.8	24.9668	0.2692	-0.2404	1.3805	-0.8195	-0.2948	-0.4549	-0.8863
25	0.9	0.2	21.3125	0.2059	-0.2665	1.144	-0.1251	-0.1842	-0.4245	-0.3747
25	0.9	0.5	21.2427	0.2072	-0.2603	1.143	-0.1129	-0.1865	-0.4353	-0.3728
25	0.9	0.8	21.1733	0.2085	-0.254	1.1421	-0.1008	-0.1887	-0.4459	-0.3709
50	0	0.2	32.2452	0.1728	-0.7596	1.5289	-5.4107	0.9608	4.497	-0.876
50	0	0.5	32.214	0.1732	-0.7571	1.5284	-5.3979	0.9589	4.4871	-0.8734
50	0	0.8	32.1827	0.1737	-0.7546	1.5278	-5.3852	0.9569	4.4772	-0.8707
50	0.25	0.2	47.7231	-0.1884	-2.2828	1.6285	-1.342	0.8664	2.7539	0.8508
50	0.25	0.5	47.6628	-0.1875	-2.278	1.6274	-1.3334	0.8651	2.7473	0.8526
50	0.25	0.8	47.6024	-0.1865	-2.2732	1.6263	-1.3249	0.8637	2.7406	0.8544
50	0.75	0.2	43.6691	-0.0716	-1.826	1.6341	1.6701	0.5794	0.8972	1.7601
50	0.75	0.5	43.5583	-0.0699	-1.8171	1.6321	1.684	0.5773	0.8864	1.763
50	0.75	0.8	43.4474	-0.0682	-1.8083	1.6301	1.6978	0.5751	0.8755	1.7659
50	0.8	0.2	42.6473	-0.0459	-1.7203	1.6302	1.9781	0.5414	0.6847	1.8384
50	0.8	0.5	42.5154	-0.0438	-1.7098	1.6279	1.9951	0.5388	0.6714	1.8419
50	0.8	0.8	42.3834	-0.0417	-1.6993	1.6255	2.0121	0.5362	0.6581	1.8455
50	0.9	0.2	39.8718	0.0192	-1.4463	1.6128	2.6906	0.4467	0.1754	2.0079

50	0.9	0.5	39.6787	0.0222	-1.4309	1.6093	2.7168	0.4426	0.1549	2.0134
50	0.9	0.8	39.4855	0.0253	-1.4156	1.6059	2.7431	0.4386	0.1344	2.0189
100	0	0.2	43.9731	-0.0675	-1.7801	1.6244	1.1296	-0.8401	-2.565	-0.9348
100	0	0.5	43.9559	-0.0672	-1.7788	1.6241	1.1276	-0.8399	-2.5635	-0.9352
100	0	0.8	43.9388	-0.067	-1.7775	1.6237	1.1257	-0.8396	-2.5621	-0.9357
100	0.25	0.2	35.3397	0.2393	-0.6758	1.7313	4.5725	-0.6883	-3.4073	0.9007
100	0.25	0.5	35.3346	0.2393	-0.6754	1.7312	4.5696	-0.6879	-3.4052	0.9
100	0.25	0.8	35.3296	0.2394	-0.675	1.7311	4.5668	-0.6875	-3.4031	0.8993
100	0.75	0.2	39.0915	0.1424	-1.0619	1.7378	5.6381	-0.411	-3.0756	1.8665
100	0.75	0.5	39.0926	0.1424	-1.062	1.7378	5.6348	-0.4105	-3.0732	1.8658
100	0.75	0.8	39.0937	0.1424	-1.0621	1.7378	5.6315	-0.4101	-3.0708	1.865
100	0.8	0.2	39.7916	0.1209	-1.1429	1.7339	5.6878	-0.3762	-3.004	1.9492
100	0.8	0.5	39.7941	0.1209	-1.143	1.734	5.6844	-0.3758	-3.0015	1.9484
100	0.8	0.8	39.7966	0.1208	-1.1432	1.734	5.681	-0.3753	-2.999	1.9476
100	0.9	0.2	41.3646	0.0662	-1.341	1.7162	5.7559	-0.2908	-2.8086	2.128
100	0.9	0.5	41.374	0.0661	-1.3417	1.7164	5.7514	-0.2902	-2.8053	2.1269
100	0.9	0.8	41.3834	0.0659	-1.3425	1.7166	5.747	-0.2895	-2.802	2.1258
200	0	0.2	16.3149	-0.0603	-0.7915	0.5721	2.616	-0.5962	-2.5986	0.2475
200	0	0.5	16.3114	-0.0603	-0.7913	0.572	2.6148	-0.596	-2.5977	0.2472
200	0	0.8	16.3078	-0.0602	-0.791	0.572	2.6136	-0.5959	-2.5969	0.2469
200	0.25	0.2	5.4345	0.1587	0.2194	0.437	3.35	-0.4988	-2.6125	0.7513
200	0.25	0.5	5.4368	0.1587	0.2193	0.437	3.3486	-0.4986	-2.6116	0.751
200	0.25	0.8	5.439	0.1586	0.2191	0.4371	3.3473	-0.4984	-2.6106	0.7506
200	0.75	0.2	9.209	0.0893	-0.1136	0.4929	2.962	-0.306	-1.9451	0.8858
200	0.75	0.5	9.2169	0.0892	-0.1141	0.4931	2.9601	-0.3057	-1.9437	0.8854
200	0.75	0.8	9.2248	0.0891	-0.1147	0.4932	2.9582	-0.3055	-1.9424	0.8849
200	0.8	0.2	10.0043	0.0739	-0.1858	0.5036	2.8856	-0.2816	-1.8504	0.8901
200	0.8	0.5	10.0141	0.0737	-0.1866	0.5038	2.8835	-0.2813	-1.8488	0.8896
200	0.8	0.8	10.0238	0.0736	-0.1873	0.504	2.8814	-0.281	-1.8473	0.889
200	0.9	0.2	11.9558	0.0348	-0.3666	0.5281	2.68	-0.2214	-1.6103	0.8924
200	0.9	0.5	11.9706	0.0346	-0.3677	0.5285	2.6771	-0.2211	-1.6082	0.8917
200	0.9	0.8	11.9853	0.0344	-0.3687	0.5288	2.6743	-0.2207	-1.6061	0.891

500	0	0.2	8.708	-0.2114	-0.9117	0.0606	-1.0454	-0.1348	0.0346	-0.7145
500	0	0.5	8.7069	-0.2114	-0.9116	0.0606	-1.0454	-0.1348	0.0346	-0.7145
500	0	0.8	8.7057	-0.2114	-0.9115	0.0606	-1.0454	-0.1348	0.0346	-0.7145
500	0.25	0.2	11.4329	-0.1408	-0.8268	0.2682	-0.0497	-0.2768	-0.7302	-0.4779
500	0.25	0.5	11.4318	-0.1408	-0.8267	0.2682	-0.0498	-0.2768	-0.7302	-0.4779
500	0.25	0.8	11.4307	-0.1408	-0.8266	0.2681	-0.0499	-0.2768	-0.7301	-0.4779
500	0.75	0.2	10.7999	-0.1679	-0.8753	0.2053	0.6136	-0.298	-1.0409	-0.1997
500	0.75	0.5	10.7991	-0.1679	-0.8752	0.2053	0.6136	-0.298	-1.0409	-0.1997
500	0.75	0.8	10.7983	-0.1678	-0.8752	0.2053	0.6135	-0.298	-1.0408	-0.1997
500	0.8	0.2	10.6295	-0.1733	-0.8832	0.191	0.6782	-0.2968	-1.0624	-0.1673
500	0.8	0.5	10.6286	-0.1732	-0.8832	0.191	0.6781	-0.2968	-1.0624	-0.1673
500	0.8	0.8	10.6278	-0.1732	-0.8831	0.191	0.6781	-0.2968	-1.0623	-0.1673
500	0.9	0.2	10.1507	-0.186	-0.899	0.1541	0.825	-0.2915	-1.1042	-0.0893
500	0.9	0.5	10.1498	-0.1859	-0.8989	0.1541	0.825	-0.2915	-1.1041	-0.0893
500	0.9	0.8	10.149	-0.1859	-0.8988	0.1541	0.8249	-0.2915	-1.1041	-0.0893

Appendix A.3: Measuring Relative Bias of Multivariate Two Parameter Weighted Estimator Estimate for Different ρ and n

Sample Size, n	ρ_{12}	W	Relative Bias of Multivariate Two Parameter Weighted Estimator							
			β_{01}	β_{11}	β_{21}	β_{31}	β_{02}	β_{12}	β_{22}	β_{32}
25	0	0.2	-11.37272	-27.06629	-71.57447	-38.42419	-31.77158	-25.65577	-56.80517	-57.4178
25	0	0.5	-11.37273	-27.06455	-71.56967	-38.42184	-31.77158	-25.65439	-56.80134	-57.41558
25	0	0.8	-11.37273	-27.06281	-71.56487	-38.41949	-31.77158	-25.653	-56.79752	-57.41337
25	0.25	0.2	7.977683	-27.229025	-72.76695	-37.867542	-29.69459	-25.68563	-57.67678	-56.48681
25	0.25	0.5	7.97768	-27.22739	-72.76244	-37.86533	-29.69459	-25.68482	-57.67453	-56.4855
25	0.25	0.8	7.977677	-27.225757	-72.75794	-37.863123	-29.69459	-25.684	-57.67228	-56.4842
25	0.75	0.2	41.01671	-29.5614	-80.23684	-39.81449	-31.2368	-27.55408	-62.1332	-60.28908
25	0.75	0.5	41.0167	-29.55862	-80.22918	-39.81073	-31.2368	-27.55566	-62.13753	-60.29161
25	0.75	0.8	41.0167	-29.55585	-80.22152	-39.80697	-31.2368	-27.55723	-62.14186	-60.29413
25	0.8	0.2	45.98043	-30.0742	-81.79102	-40.33483	-31.87679	-28.00758	-63.11206	-61.33818
25	0.8	0.5	45.98042	-30.07087	-81.78185	-40.33033	-31.87679	-28.01001	-63.11874	-61.34208
25	0.8	0.8	45.98042	-30.06754	-81.77268	-40.32583	-31.87679	-28.01243	-63.12542	-61.34598
25	0.9	0.2	59.49417	-31.50447	-86.11373	-41.79917	-33.90442	-29.31022	-65.85101	-64.44126
25	0.9	0.5	59.49416	-31.49835	-86.09686	-41.79089	-33.90442	-29.3169	-65.86944	-64.45202
25	0.9	0.8	59.49414	-31.49223	-86.07999	-41.78262	-33.90441	-29.32359	-65.88788	-64.46278
50	0	0.2	-7.635661	-27.952472	-73.766003	-39.68146	-33.16646	-25.99221	-56.93009	-58.79376
50	0	0.5	-7.635663	-27.95148	-73.763306	-39.6801	-33.16646	-25.99126	-56.92753	-58.79221
50	0	0.8	-7.635664	-27.950489	-73.76061	-39.67874	-33.16646	-25.99031	-56.92496	-58.79066
50	0.25	0.2	15.62719	-28.22495	-75.34713	-39.14749	-30.46805	-26.05206	-58.07295	-57.64649
50	0.25	0.5	15.62718	-28.22403	-75.34464	-39.14624	-30.46805	-26.05152	-58.07149	-57.64561
50	0.25	0.8	15.62718	-28.22311	-75.34214	-39.14498	-30.46805	-26.05099	-58.07004	-57.64474
50	0.75	0.2	46.86404	-30.32095	-82.09825	-40.84675	-31.76755	-28.11262	-63.04917	-61.70493
50	0.75	0.5	46.86403	-30.31927	-82.09371	-40.84445	-31.76755	-28.11368	-63.05203	-61.70666
50	0.75	0.8	46.86403	-30.31759	-82.08916	-40.84214	-31.76755	-28.11473	-63.05489	-61.70839
50	0.8	0.2	51.50635	-30.78653	-83.51099	-41.31576	-32.45009	-28.56571	-63.99821	-62.78091
50	0.8	0.5	51.50634	-30.78448	-83.50544	-41.31295	-32.45009	-28.56734	-64.00263	-62.78358
50	0.8	0.8	51.50634	-30.78244	-83.4999	-41.31014	-32.45009	-28.56897	-64.00704	-62.78626
50	0.9	0.2	63.8518	-32.07753	-87.40668	-42.63816	-34.60208	-29.81943	-66.52759	-65.8802
50	0.9	0.5	63.85179	-32.07361	-87.39605	-42.63276	-34.60208	-29.82396	-66.53987	-65.88764
50	0.9	0.8	63.85179	-32.06969	-87.38542	-42.62736	-34.60208	-29.82849	-66.55215	-65.89507
100	0	0.2	1.672889	-27.008333	-71.489211	-38.054342	-27.89175	-25.95942	-58.47291	-56.49156
100	0	0.5	1.672888	-27.007623	-71.487297	-38.053358	-27.89175	-25.95862	-58.47078	-56.49025
100	0	0.8	1.672887	-27.006913	-71.485383	-38.052375	-27.89175	-25.95783	-58.46866	-56.48894
100	0.25	0.2	3.178136	-26.929186	-71.373138	-37.863089	-26.16888	-25.89567	-58.90841	-55.60537
100	0.25	0.5	3.178135	-26.928552	-71.371427	-37.862211	-26.16888	-25.89518	-58.90711	-55.60457

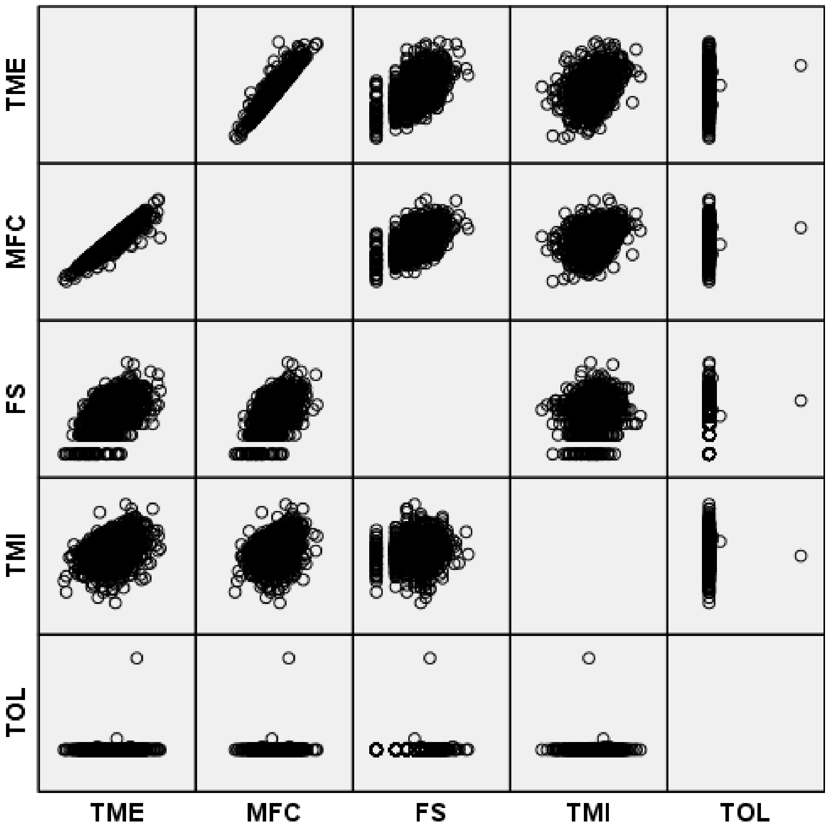
100	0.25	0.8	3.178134	-26.927918	-71.369717	-37.861332	-26.16888	-25.8947	-58.90581	-55.60377
100	0.75	0.2	42.51916	-28.97808	-78.28393	-39.18461	-29.54313	-27.32587	-61.45764	-59.5666
100	0.75	0.5	42.51916	-28.97687	-78.28066	-39.18292	-29.54313	-27.32664	-61.4597	-59.56787
100	0.75	0.8	42.51916	-28.97565	-78.2774	-39.18124	-29.54313	-27.32741	-61.46176	-59.56914
100	0.8	0.2	48.83766	-29.41007	-79.66599	-39.54216	-30.45841	-27.67537	-62.04692	-60.5768
100	0.8	0.5	48.83766	-29.40858	-79.66197	-39.54008	-30.45841	-27.67658	-62.05017	-60.5788
100	0.8	0.8	48.83766	-29.40709	-79.65796	-39.53801	-30.45841	-27.67779	-62.05342	-60.58081
100	0.9	0.2	65.05775	-30.57638	-83.36543	-40.541	-33.13212	-28.6569	-63.66369	-63.46185
100	0.9	0.5	65.05774	-30.57343	-83.35752	-40.53691	-33.13212	-28.66037	-63.673	-63.4676
100	0.9	0.8	65.05774	-30.57049	-83.34961	-40.53282	-33.13212	-28.66384	-63.68232	-63.47334
200	0	0.2	-13.85445	-23.24759	-61.3082	-33.1827	-27.49176	-22.46651	-49.64615	-50.27287
200	0	0.5	-13.85445	-23.24707	-61.3068	-33.18198	-27.49176	-22.46594	-49.64462	-50.27192
200	0	0.8	-13.85445	-23.24655	-61.30541	-33.18125	-27.49176	-22.46536	-49.64309	-50.27097
200	0.25	0.2	-11.3304	-23.35796	-61.68373	-33.24403	-26.77865	-22.3795	-49.68011	-49.79613
200	0.25	0.5	-11.3304	-23.35749	-61.68247	-33.24338	-26.77865	-22.37916	-49.6792	-49.79557
200	0.25	0.8	-11.3304	-23.35701	-61.6812	-33.24272	-26.77865	-22.37881	-49.67829	-49.795
200	0.75	0.2	26.57132	-24.86055	-67.12676	-33.83547	-30.99382	-23.39827	-50.8146	-53.46699
200	0.75	0.5	26.57131	-24.85963	-67.12431	-33.8342	-30.99382	-23.39891	-50.8163	-53.46804
200	0.75	0.8	26.57131	-24.85872	-67.12187	-33.83293	-30.99382	-23.39954	-50.818	-53.4691
200	0.8	0.2	32.51279	-25.17157	-68.18105	-34.03397	-31.9384	-23.65976	-51.15797	-54.34445
200	0.8	0.5	32.51279	-25.17045	-68.17804	-34.0324	-31.9384	-23.66073	-51.16058	-54.34606
200	0.8	0.8	32.51279	-25.16932	-68.17502	-34.03084	-31.9384	-23.66171	-51.16319	-54.34768
200	0.9	0.2	47.63424	-25.97664	-70.89957	-34.55841	-34.58161	-24.37416	-52.07312	-56.77064
200	0.9	0.5	47.63423	-25.9744	-70.89357	-34.55529	-34.58161	-24.37688	-52.08041	-56.77516
200	0.9	0.8	47.63423	-25.97216	-70.88758	-34.55217	-34.58161	-24.3796	-52.0877	-56.77968
500	0	0.2	-22.0819	-17.72896	-46.34055	-25.75307	-30.28171	-17.06643	-34.15259	-42.60725
500	0	0.5	-22.0819	-17.72862	-46.33966	-25.75261	-30.28171	-17.06615	-34.15184	-42.60679
500	0	0.8	-22.0819	-17.72829	-46.33877	-25.75214	-30.28171	-17.06587	-34.15109	-42.60632
500	0.25	0.2	-5.819548	-17.688937	-46.851264	-25.047696	-29.15708	-17.08936	-34.64839	-42.11162
500	0.25	0.5	-5.819548	-17.688619	-46.850414	-25.047251	-29.15708	-17.08921	-34.648	-42.11138
500	0.25	0.8	-5.819549	-17.6883	-46.849564	-25.046806	-29.15708	-17.08907	-34.64761	-42.11114
500	0.75	0.2	25.93832	-18.46847	-50.13209	-24.87271	-31.83975	-17.73834	-35.3846	-44.44087
500	0.75	0.5	25.93831	-18.46786	-50.13044	-24.87185	-31.83975	-17.73883	-35.3859	-44.44168
500	0.75	0.8	25.93831	-18.46724	-50.1288	-24.87099	-31.83975	-17.73931	-35.3872	-44.44249
500	0.8	0.2	30.71173	-18.60849	-50.68602	-24.87847	-32.51126	-17.8745	-35.4976	-44.9808
500	0.8	0.5	30.71173	-18.60774	-50.68401	-24.87742	-32.51126	-17.87521	-35.49949	-44.98198
500	0.8	0.8	30.71173	-18.60699	-50.682	-24.87637	-32.51126	-17.87592	-35.50138	-44.98316
500	0.9	0.2	43.02567	-18.89748	-51.9228	-24.79201	-34.31055	-18.1798	-35.64152	-46.32838
500	0.9	0.5	43.02567	-18.89601	-51.91889	-24.78996	-34.31055	-18.18164	-35.64643	-46.33144
500	0.9	0.8	43.02566	-18.89455	-51.91497	-24.7879	-34.31055	-18.18347	-35.65133	-46.3345

Table A.4: Measuring Relative Efficiency of Multivariate Two Parameter Weighted Estimator Estimate for Different ρ and n

Sample Size, n	ρ_{12}	W	Relative Efficiency of Multivariate Two Parameter Weighted Estimator							
			β_{01}	β_{11}	β_{21}	β_{31}	β_{02}	β_{12}	β_{22}	β_{32}
25	0	0.2	1.757071	2.458332	2.420542	1.898276	1.783417	2.525884	2.537792	1.926081
25	0	0.5	1.757071	2.458517	2.420863	1.898257	1.783417	2.526071	2.538231	1.926029
25	0	0.8	1.757071	2.458702	2.421183	1.898238	1.783417	2.526259	2.538671	1.925977
25	0.25	0.2	1.770389	2.487068	2.469872	1.910375	1.769599	2.554078	2.560669	1.912374
25	0.25	0.5	1.770389	2.487264	2.470203	1.91036	1.769599	2.554253	2.561108	1.912318
25	0.25	0.8	1.770389	2.48746	2.470534	1.910345	1.769599	2.554429	2.561547	1.912262
25	0.75	0.2	1.756908	2.692266	2.68273	1.920508	1.743657	2.769392	2.749747	1.918376
25	0.75	0.5	1.756908	2.692476	2.683073	1.920498	1.743657	2.769546	2.750175	1.918321
25	0.75	0.8	1.756908	2.692686	2.683417	1.920488	1.743657	2.769701	2.750604	1.918266
25	0.8	0.2	1.758128	2.739507	2.730736	1.926606	1.744138	2.820264	2.797978	1.924713
25	0.8	0.5	1.758128	2.739724	2.731085	1.926598	1.744138	2.820411	2.798404	1.924657
25	0.8	0.8	1.758128	2.73994	2.731435	1.926589	1.744138	2.820558	2.79883	1.924602
25	0.9	0.2	1.768865	2.877911	2.870047	1.95087	1.75316	2.969743	2.94189	1.949232
25	0.9	0.5	1.768865	2.878164	2.870423	1.950869	1.75316	2.969849	2.942297	1.94917
25	0.9	0.8	1.768865	2.878416	2.870799	1.950869	1.75316	2.969954	2.942705	1.949108
50	0	0.2	1.892117	2.580916	2.36801	2.037584	1.878182	2.701369	2.549141	1.999127
50	0	0.5	1.892117	2.581082	2.368193	2.037571	1.878182	2.701534	2.549433	1.999072
50	0	0.8	1.892117	2.581248	2.368376	2.037558	1.878182	2.701699	2.549724	1.999017
50	0.25	0.2	1.891122	2.601416	2.407231	2.027656	1.864124	2.703683	2.532185	1.98785
50	0.25	0.5	1.891122	2.601578	2.407415	2.027644	1.864124	2.703838	2.532463	1.987793
50	0.25	0.8	1.891122	2.60174	2.407599	2.027631	1.864124	2.703993	2.532741	1.987737
50	0.75	0.2	1.877665	2.828575	2.572447	2.035532	1.843793	2.957099	2.679187	2.003298
50	0.75	0.5	1.877665	2.828755	2.572634	2.035524	1.843793	2.957232	2.679439	2.003245
50	0.75	0.8	1.877665	2.828935	2.572822	2.035517	1.843793	2.957365	2.679691	2.003192
50	0.8	0.2	1.879115	2.881635	2.610218	2.041818	1.844573	3.017018	2.716301	2.010998
50	0.8	0.5	1.879115	2.881824	2.610409	2.041813	1.844573	3.017144	2.716548	2.010945
50	0.8	0.8	1.879115	2.882013	2.610599	2.041808	1.844573	3.017269	2.716794	2.010892
50	0.9	0.2	1.890589	3.034665	2.72106	2.065738	1.854574	3.192632	2.82745	2.039173
50	0.9	0.5	1.890589	3.0349	2.721268	2.065742	1.854574	3.192718	2.827681	2.039113
50	0.9	0.8	1.890589	3.035135	2.721476	2.065746	1.854575	3.192803	2.827912	2.039053
100	0	0.2	1.923627	2.312587	2.242411	1.982694	1.86866	2.52942	2.511451	1.910002
100	0	0.5	1.923627	2.312691	2.242533	1.982687	1.86866	2.52953	2.511674	1.90996
100	0	0.8	1.923627	2.312795	2.242655	1.98268	1.86866	2.529639	2.511897	1.909919
100	0.25	0.2	1.925052	2.326808	2.262891	1.980358	1.868779	2.50151	2.501585	1.901793
100	0.25	0.5	1.925052	2.326914	2.263014	1.980352	1.868779	2.501609	2.501805	1.901749

100	0.25	0.8	1.925052	2.32702	2.263136	1.980346	1.868779	2.501708	2.502025	1.901704
100	0.75	0.2	1.912153	2.479874	2.428602	1.963829	1.858907	2.683715	2.629919	1.918062
100	0.75	0.5	1.912153	2.479999	2.428733	1.963831	1.858907	2.683787	2.630118	1.918016
100	0.75	0.8	1.912153	2.480125	2.428863	1.963833	1.858907	2.683858	2.630317	1.917971
100	0.8	0.2	1.91274	2.516164	2.46414	1.965517	1.858608	2.727515	2.660732	1.923332
100	0.8	0.5	1.91274	2.516297	2.464274	1.965522	1.858608	2.727579	2.660927	1.923286
100	0.8	0.8	1.91274	2.51643	2.464407	1.965526	1.858608	2.727643	2.661122	1.923239
100	0.9	0.2	1.921193	2.619273	2.564953	1.977005	1.863804	2.852575	2.750268	1.942597
100	0.9	0.5	1.921193	2.619448	2.565104	1.97702	1.863804	2.852599	2.750452	1.942541
100	0.9	0.8	1.921193	2.619623	2.565255	1.977036	1.863804	2.852623	2.750636	1.942486
200	0	0.2	1.916736	1.910708	1.978683	1.873374	1.816132	2.048215	2.281368	1.731134
200	0	0.5	1.916736	1.910774	1.978767	1.873372	1.816132	2.04829	2.281544	1.731109
200	0	0.8	1.916736	1.910839	1.978851	1.87337	1.816132	2.048365	2.28172	1.731083
200	0.25	0.2	1.910528	1.931704	1.998775	1.869967	1.82054	2.048222	2.329264	1.721134
200	0.25	0.5	1.910528	1.931773	1.99886	1.869965	1.82054	2.048285	2.329439	1.721106
200	0.25	0.8	1.910528	1.931842	1.998946	1.869963	1.82054	2.048348	2.329615	1.721079
200	0.75	0.2	1.898524	2.011653	2.159665	1.826335	1.801331	2.179246	2.434512	1.730851
200	0.75	0.5	1.000569	1.000296	1.000957	1.000103	1.000676	1.00035	1.001164	1.00012
200	0.75	0.8	1.898524	2.011827	2.15986	1.82635	1.801331	2.179322	2.434828	1.730783
200	0.8	0.2	1.89848	2.030232	2.190971	1.822922	1.798975	2.209592	2.457739	1.734727
200	0.8	0.5	1.89848	2.030326	2.191072	1.822932	1.798975	2.209623	2.457895	1.734691
200	0.8	0.8	1.89848	2.03042	2.191173	1.822942	1.798975	2.209654	2.458051	1.734655
200	0.9	0.2	1.90232	2.081067	2.274436	1.819145	1.796264	2.292133	2.52014	1.747612
200	0.9	0.5	1.90232	2.081193	2.274554	1.819167	1.796264	2.292132	2.520295	1.747562
200	0.9	0.8	1.90232	2.08132	2.274671	1.819189	1.796264	2.29213	2.52045	1.747513
500	0	0.2	1.914487	1.547322	1.648532	1.761425	1.692166	1.627719	1.991869	1.494697
500	0	0.5	1.914487	1.54737	1.648592	1.76143	1.692166	1.62777	1.991988	1.494687
500	0	0.8	1.914487	1.547417	1.648652	1.761435	1.692166	1.62782	1.992107	1.494677
500	0.25	0.2	1.895905	1.520573	1.660035	1.711523	1.699139	1.613199	1.988925	1.48605
500	0.25	0.5	1.895905	1.520622	1.660098	1.711529	1.699139	1.613244	1.989041	1.486038
500	0.25	0.8	1.895905	1.52067	1.660161	1.711534	1.699139	1.61329	1.989156	1.486026
500	0.75	0.2	1.879919	1.543241	1.788547	1.641626	1.666859	1.663116	2.028195	1.483066
500	0.75	0.5	1.879919	1.543302	1.788622	1.64164	1.666859	1.663145	2.028305	1.483045
500	0.75	0.8	1.879919	1.543363	1.788698	1.641655	1.666859	1.663174	2.028415	1.483024
500	0.8	0.2	1.876881	1.548852	1.80998	1.632819	1.660052	1.675161	2.037024	1.483189
500	0.8	0.5	1.876881	1.548917	1.810059	1.632836	1.660052	1.675185	2.037134	1.483165
500	0.8	0.8	1.876881	1.548983	1.810137	1.632854	1.660052	1.675209	2.037244	1.483141
500	0.9	0.2	1.867854	1.562962	1.862002	1.612019	1.642194	1.704516	2.053631	1.483965
500	0.9	0.5	1.867854	1.563049	1.862093	1.61205	1.642194	1.704516	2.053745	1.483924
500	0.9	0.8	1.867854	1.563136	1.862184	1.61208	1.642194	1.704516	2.05386	1.483884

Figure A.1: Scatter Matrix among TME, MFC, FS, TMI and TOL



Appendix B

Appendix B.1: # Monte Carlo Simulation of Multivariate Regression with Continuous Responses for Stochastic Restriction#

```
it<-10000                                  # Number of iterations
p<-2                                       # Number of responses
n<-500                                     #Sample size
rho<-0.9                                   # Value of correlation coefficient among responses
sigma11<-100                               # Variance of First response variable
sigma22<-81                                # Variance of Second response variable
sigma12<-rho*sqrt(sigma11)*sqrt(sigma22) # Covariance between First and Second responses variables
sigma21<-sigma12                           # Covariance between Second and First responses variables

b01<-matrix(data=NA, nrow = it, ncol = 1)
b11<-matrix(data=NA, nrow = it, ncol = 1)
b21<-matrix(data=NA, nrow = it, ncol = 1)
b31<-matrix(data=NA, nrow = it, ncol = 1)
b02<-matrix(data=NA, nrow = it, ncol = 1)
b12<-matrix(data=NA, nrow = it, ncol = 1)
b22<-matrix(data=NA, nrow = it, ncol = 1)
b32<-matrix(data=NA, nrow = it, ncol = 1)

rb01<-matrix(data=NA, nrow = it, ncol = 1)
rb11<-matrix(data=NA, nrow = it, ncol = 1)
rb21<-matrix(data=NA, nrow = it, ncol = 1)
rb31<-matrix(data=NA, nrow = it, ncol = 1)
rb02<-matrix(data=NA, nrow = it, ncol = 1)
rb12<-matrix(data=NA, nrow = it, ncol = 1)
rb22<-matrix(data=NA, nrow = it, ncol = 1)
rb32<-matrix(data=NA, nrow = it, ncol = 1)

library (MASS)
for (i in 1:it) {
mu<- matrix(c(0,0),nrow=2,ncol=1)
sigma<- matrix(c(sigma11,sigma12,sigma21,sigma22),nrow=p,ncol=p)
set.seed(950+i)
error<- mvrnorm(n,mu,sigma)
set.seed(950)
  X1<-round(abs(rnorm(n,mean=1000,sd=60)),0)
set.seed(950)
  X2<-round(abs(rnorm(n,mean=375,sd=30)),0)
set.seed(950)
  X3<-round(abs(rnorm(n,mean=500,sd=20)),0)
xbeta<-matrix(c((25+3.5*(X1)-1.75*(X2)-2.5*(X3)),(175+2.5*(X1)-1.25*(X2)-1.5*(X3))),nrow=n,ncol=p)

  X0<-matrix(c(1), nrow = n, ncol = 1)
  X<-cbind(X0,X1,X2,X3)              # Design matrix
  Y<-xbeta+error                     # Response Matrix

estbeta<-(solve(t(X)%*%X))%*%(t(X)%*%Y)
b01[i]<-estbeta[1,1]
b11[i]<-estbeta[2,1]
b21[i]<-estbeta[3,1]
```

```
b31[i]<-estbeta[4,1]
b02[i]<-estbeta[1,2]
b12[i]<-estbeta[2,2]
b22[i]<-estbeta[3,2]
b32[i]<-estbeta[4,2]
  xbeta1<-X%*%estbeta
est_error<-(Y-xbeta1)
est_sigma<-(1/n)*(t(est_error)%*%est_error)
  R<- matrix(c(0,1,2,0),nrow=1,ncol=4,byrow=TRUE)
set.seed(950+i)
  V <- mvrnorm(n,mu,est_sigma)
beta<-(matrix(c(25,3.5,-1.75,-2.5,175,2.5,-1.25,-1.5),nrow=4,ncol=2))
 r<-R%*%beta+colMeans(V)
  A<-((R%*%estbeta)-r)
  A2<-(solve(t(X)%*%X))
  B1<-(A2%*%t(R))
  B2<-(R%*%B1)
  B21<-matrix(c(B2),nrow=2,ncol=2,byrow=TRUE)+est_sigma
  B3<-solve(B21)
  C<-((solve(t(X)%*%X))%*%t(R))
restbeta<-estbeta-(C%*%(A%*%B3))
rb01[i]<-restbeta[1,1]
rb11[i]<-restbeta[2,1]
rb21[i]<-restbeta[3,1]
rb31[i]<-restbeta[4,1]
rb02[i]<-restbeta[1,2]
rb12[i]<-restbeta[2,2]
rb22[i]<-restbeta[3,2]
rb32[i]<-restbeta[4,2]
}
rbols<-matrix(c(abs(mean(b01)-25)/25*100,abs(mean(b11)-3.5)/3.5*100,abs(mean(b21)-(-1.75))/(-
1.75)*100,abs(mean(b31)-(-2.5))/(-2.5)*100,abs(mean(b02)-175)/175*100,abs(mean(b12)-
2.5)/2.5*100,abs(mean(b22)-(-1.25))/(-1.25)*100,abs(mean(b32)-(-1.5))/(-1.5)*100),nrow=4,ncol=2)

rbrols<-matrix(c((mean(rb01)-25)/25*100,(mean(rb11)-3.5)/3.5*100,(mean(rb21)-(-1.75))/(-
1.75)*100,(mean(rb31)-(-2.5))/(-2.5)*100,(mean(rb02)-175)/175*100,(mean(rb12)-2.5)/2.5*100,(mean(rb22)-(-
1.25))/(-1.25)*100,(mean(rb32)-(-1.5))/(-1.5)*100),nrow=4,ncol=2)

varols<-matrix(c(var(b01),var(b11),var(b21),var(b31),var(b02),var(b12),var(b22),var(b32)),nrow=4,ncol=2)
varrols<-matrix(c(var(rb01),var(rb11),var(rb21),var(rb31),var(rb02),var(rb12),var(rb22),var(rb32)),nrow=4,ncol=2)
re<-varols/varrols
```

Appendix B.2:# Simulation of Multivariate Regression with Continuous Responses with exact restriction #

```
library (MASS)
p<-2
n<-400
it<-10000
rho<-0.9
sigma11<-100
sigma22<-81
sigma12<-rho*sqrt(sigma11)*sqrt(sigma22)
sigma21<-sigma12
b01<-matrix(data=NA, nrow = it, ncol = 1)
b11<-matrix(data=NA, nrow = it, ncol = 1)
b21<-matrix(data=NA, nrow = it, ncol = 1)
b31<-matrix(data=NA, nrow = it, ncol = 1)
b02<-matrix(data=NA, nrow = it, ncol = 1)
b12<-matrix(data=NA, nrow = it, ncol = 1)
b22<-matrix(data=NA, nrow = it, ncol = 1)
b32<-matrix(data=NA, nrow = it, ncol = 1)
rb01<-matrix(data=NA, nrow = it, ncol = 1)
rb11<-matrix(data=NA, nrow = it, ncol = 1)
rb21<-matrix(data=NA, nrow = it, ncol = 1)
rb31<-matrix(data=NA, nrow = it, ncol = 1)
rb02<-matrix(data=NA, nrow = it, ncol = 1)
rb12<-matrix(data=NA, nrow = it, ncol = 1)
rb22<-matrix(data=NA, nrow = it, ncol = 1)
rb32<-matrix(data=NA, nrow = it, ncol = 1)
for (i in 1:it) {
mu<- matrix(c(0,0),nrow=2,ncol=1)
sigma<- matrix(c(sigma11,sigma12,sigma21,sigma22),nrow=p,ncol=p)
set.seed(950+i)
error<- mvrnorm(n,mu,sigma)
set.seed(950)
  X1<-round(abs(rnorm(n,mean=1000,sd=60)),0)
set.seed(950)
  X2<-round(abs(rnorm(n,mean=375,sd=30)),0)
set.seed(950)
  X3<-round(abs(rnorm(n,mean=500,sd=20)),0)
xbeta<-matrix(c((25+3.5*(X1)-1.75*(X2)-2.5*(X3)),(175+2.5*(X1)-1.25*(X2)-1.5*(X3))),nrow=n,ncol=p)
  Y<-xbeta+error
  X0<-matrix(c(1),nrow = n,ncol = 1)
  X<-cbind(X0,X1,X2,X3)
estbeta<-(solve(t(X)%*%X))%*%(t(X)%*%Y)
b01[i]<-estbeta[1,1]
b11[i]<-estbeta[2,1]
b21[i]<-estbeta[3,1]
b31[i]<-estbeta[4,1]
b02[i]<-estbeta[1,2]
b12[i]<-estbeta[2,2]
b22[i]<-estbeta[3,2]
```

```
b32[i]<-estbeta[4,2]
  R<- matrix(c(0,1,2,0),nrow=1,ncol=4,byrow=TRUE)
  r<- matrix(c(0,0),nrow=1,ncol=2,byrow=TRUE)
  A<-((R%*%estbeta)-r)
  A2<-(solve(t(X)%*%X))
  B1<-(A2%*%t(R))
  B2<-(R%*%B1)
  B3<-solve(B2)
  C<-((solve(t(X)%*%X))%*%t(R))
restbeta<-estbeta-(C%*%(B3%*%A))
rb01[i]<-restbeta[1,1]
rb11[i]<-restbeta[2,1]
rb21[i]<-restbeta[3,1]
rb31[i]<-restbeta[4,1]
rb02[i]<-restbeta[1,2]
rb12[i]<-restbeta[2,2]
rb22[i]<-restbeta[3,2]
rb32[i]<-restbeta[4,2]
  }
####### Properties of the Modified Maximum Likelihood Estimator   #######
bols<-matrix(c((mean(b01)-25)/25*100,(mean(b11)-3.5)/3.5*100,(mean(b21)-(-1.75))/(-1.75)*100,(mean(b31)-(-
2.5))/(-2.5)*100,(mean(b02)-175)/175*100,(mean(b12)-2.5)/2.5*100,(mean(b22)-(-1.25))/(-1.25)*100,(mean(b32)-
(-1.5))/(-1.5)*100),nrow=4,ncol=2)
rbrls<-(matrix(c((mean(rb01)-25)/25*100,(mean(rb11)-3.5)/3.5*100,(mean(rb21)-(-1.75))/(-1.75)*100,(mean(rb31)-
(1.5))/(1.5)*100,(mean(rb02)-175)/175*100,(mean(rb12)-2.5)/2.5*100,(mean(rb22)-(-1.25)/(-
1.25)*100),(mean(rb32)-(1.75)/(1.75)*100)),nrow=4,ncol=2))
varianceoLs<-matrix(c(var(b01),var(b11),var(b21),var(b31),var(b02),var(b12),var(b22),var(b32)),nrow=4,ncol=2)
varianceMLE<-
matrix(c(var(rb01),var(rb11),var(rb21),var(rb31),var(rb02),var(rb12),var(rb22),var(rb32)),nrow=4,ncol=2)
re<-varianceoLs/varianceMLE
```

Appendix B.3: # Monte Carlo Simulation of testing the power of the modified test

```
library (MASS)
p<-2
n<-25
it<-10000
rho<-0.9
alpha<-.99
sigma11<-100
sigma22<-81
iv<-3
sigma12<-rho*sqrt(sigma11)*sqrt(sigma22)
sigma21<-sigma12
beta<-matrix(c(25,3.5,-1.75,-2.5,175,2.5,-1.25,-1.5), nrow = 4, ncol = 2)
beta1<-matrix(c(0),nrow=4,ncol=2,byrow=TRUE)
b01<-matrix(data=NA, nrow = it, ncol = 1)
b11<-matrix(data=NA, nrow = it, ncol = 1)
b21<-matrix(data=NA, nrow = it, ncol = 1)
b31<-matrix(data=NA, nrow = it, ncol = 1)
b02<-matrix(data=NA, nrow = it, ncol = 1)
b12<-matrix(data=NA, nrow = it, ncol = 1)
b22<-matrix(data=NA, nrow = it, ncol = 1)
b32<-matrix(data=NA, nrow = it, ncol = 1)
rb01<-matrix(data=NA, nrow = it, ncol = 1)
rb11<-matrix(data=NA, nrow = it, ncol = 1)
rb21<-matrix(data=NA, nrow = it, ncol = 1)
rb31<-matrix(data=NA, nrow = it, ncol = 1)
rb02<-matrix(data=NA, nrow = it, ncol = 1)
rb12<-matrix(data=NA, nrow = it, ncol = 1)
rb22<-matrix(data=NA, nrow = it, ncol = 1)
rb32<-matrix(data=NA, nrow = it, ncol = 1)
t01<-matrix(data=NA, nrow = it, ncol = 1)
t11<-matrix(data=NA, nrow = it, ncol = 1)
t21<-matrix(data=NA, nrow = it, ncol = 1)
t31<-matrix(data=NA, nrow = it, ncol = 1)
t02<-matrix(data=NA, nrow = it, ncol = 1)
t12<-matrix(data=NA, nrow = it, ncol = 1)
t22<-matrix(data=NA, nrow = it, ncol = 1)
t32<-matrix(data=NA, nrow = it, ncol = 1)
o01<-matrix(data=NA, nrow = it, ncol = 1)
o11<-matrix(data=NA, nrow = it, ncol = 1)
o21<-matrix(data=NA, nrow = it, ncol = 1)
o31<-matrix(data=NA, nrow = it, ncol = 1)
o02<-matrix(data=NA, nrow = it, ncol = 1)
o12<-matrix(data=NA, nrow = it, ncol = 1)
o22<-matrix(data=NA, nrow = it, ncol = 1)
o32<-matrix(data=NA, nrow = it, ncol = 1)
oo01<-matrix(data=NA, nrow = it, ncol = 1)
oo11<-matrix(data=NA, nrow = it, ncol = 1)
```

```
oo21<-matrix(data=NA, nrow = it, ncol = 1)
oo31<-matrix(data=NA, nrow = it, ncol = 1)
oo02<-matrix(data=NA, nrow = it, ncol = 1)
oo12<-matrix(data=NA, nrow = it, ncol = 1)
oo22<-matrix(data=NA, nrow = it, ncol = 1)
oo32<-matrix(data=NA, nrow = it, ncol = 1)
beta<-matrix(c(0,0),nrow=4,ncol=2)
ttt01<-matrix(data=NA, nrow = it, ncol = 1)
ttt11<-matrix(data=NA, nrow = it, ncol = 1)
ttt21<-matrix(data=NA, nrow = it, ncol = 1)
ttt31<-matrix(data=NA, nrow = it, ncol = 1)
ttt02<-matrix(data=NA, nrow = it, ncol = 1)
ttt12<-matrix(data=NA, nrow = it, ncol = 1)
ttt22<-matrix(data=NA, nrow = it, ncol = 1)
ttt32<-matrix(data=NA, nrow = it, ncol = 1)

for (i in 1:it) {
mu<- matrix(c(0,0),nrow=2,ncol=1)
sigma<- matrix(c(sigma11,sigma12,sigma21,sigma22),nrow=p,ncol=p)
set.seed(950+i)
error<- mvrnorm(n,mu,sigma)
set.seed(950)
  X1<-round(abs(rnorm(n,mean=1000,sd=60)),0)
set.seed(950)
  X2<-round(abs(rnorm(n,mean=375,sd=30)),0)
set.seed(950)
  X3<-round(abs(rnorm(n,mean=500,sd=20)),0)
xbeta<-matrix(c((25+3.5*(X1)-1.75*(X2)-2.5*(X3)),(175+2.5*(X1)-1.25*(X2)-1.5*(X3))),nrow=n,ncol=p)
  Y<-xbeta+error
  X0<-matrix(c(1),nrow = n,ncol = 1)
  X<-cbind(X0,X1,X2,X3)
estbeta<-(solve(t(X)%*%X))%*%(t(X)%*%Y)
b01[i]<-estbeta[1,1]
b11[i]<-estbeta[2,1]
b21[i]<-estbeta[3,1]
b31[i]<-estbeta[4,1]
b02[i]<-estbeta[1,2]
b12[i]<-estbeta[2,2]
b22[i]<-estbeta[3,2]
b32[i]<-estbeta[4,2]
  R<- matrix(c(0,1,2,0),nrow=1,ncol=4,byrow=TRUE)
  r<- matrix(c(0,0),nrow=1,ncol=2,byrow=TRUE)
  A<-(R%*%estbeta)-r
  A2<-(solve(t(X)%*%X))
  B1<-(A2%*%t(R))
  B2<-(R%*%B1)
  B3<-solve(B2)
  C<-((solve(t(X)%*%X))%*%t(R))
restbeta<-estbeta-(C%*%(B3%*%A))
```

```
rb01[i]<-restbeta[1,1]
rb11[i]<-restbeta[2,1]
rb21[i]<-restbeta[3,1]
rb31[i]<-restbeta[4,1]
rb02[i]<-restbeta[1,2]
rb12[i]<-restbeta[2,2]
rb22[i]<-restbeta[3,2]
rb32[i]<-restbeta[4,2]

  xrbeta<-
matrix(c((restbeta[1,1]+restbeta[2,1]*(X1)+restbeta[3,1]*(X2)+restbeta[4,1]*(X3)),(restbeta[1,2]+restbeta[2,2]*(X1
)+restbeta[3,2]*(X2)+restbeta[4,2]*(X3))),nrow=n,ncol=2)
rerror<-(Y-xrbeta)
rsigma<-(1/(n-iv-1))*(t(rerror)%*%rerror)
  cr1<-A2%*%t(R)%*%B3%*%R%*%A2
  sig1<-kronecker(rsigma,A2)
  sig2<-kronecker(rsigma,cr1)
varrbeta<-sig1-sig2
ttt<-matrix(c((diag(varrbeta))),nrow=4,ncol=2)
varsqt<-sqrt(ttt)
  s<-(restbeta-beta1)/varsqt
t01[i]<-s[1,1]
t11[i]<-s[2,1]
t21[i]<-s[3,1]
t31[i]<-s[4,1]
t02[i]<-s[1,2]
t12[i]<-s[2,2]
t22[i]<-s[3,2]
t32[i]<-s[4,2]

cl<-(restbeta)-quantile(s, probs = c(alpha))*varsqt
cu<-(restbeta)+quantile(s, probs = c(alpha))*varsqt
  #bb<-(qmvt(0.90, df = n,corr=rho, tail = "both")$quantile)
  #cl<-restbeta-bb*varsqt
  #cu<-restbeta+bb*varsqt
out<-ifelse(beta1<cl|beta1>cu,1,0)
o01[i]<-out[1,1]
o11[i]<-out[2,1]
o21[i]<-out[3,1]
o31[i]<-out[4,1]
o02[i]<-out[1,2]
o12[i]<-out[2,2]
o22[i]<-out[3,2]
o32[i]<-out[4,2]

  xebeta<-
matrix(c((estbeta[1,1]+estbeta[2,1]*(X1)+estbeta[3,1]*(X2)+estbeta[4,1]*(X3)),(estbeta[1,2]+estbeta[2,2]*(X1)+est
beta[3,2]*(X2)+estbeta[4,2]*(X3))),nrow=n,ncol=2)
oerror<-(Y-xebeta)
```

```
osigma<-(1/(n-iv-1))*(t(oerror)%*%oerror)
  sig01<-kronecker(osigma,A2)
varols<-sqrt(matrix(c((diag(sig01))),nrow=4,ncol=2))
os<-(estbeta-beta1)/varols
ttt01[i]<-os[1,1]
ttt11[i]<-os[2,1]
ttt21[i]<-os[3,1]
ttt31[i]<-os[4,1]
ttt02[i]<-os[1,2]
ttt12[i]<-os[2,2]
ttt22[i]<-os[3,2]
ttt32[i]<-os[4,2]
cll<-estbeta-quantile(os, probs = c(alpha))*varols
cuu<-estbeta+quantile(os, probs = c(alpha))*varols
outt<-ifelse(beta1<cll|beta1>cuu,1,0)
oo01[i]<-outt[1,1]
oo11[i]<-outt[2,1]
oo21[i]<-outt[3,1]
oo31[i]<-outt[4,1]
oo02[i]<-outt[1,2]
oo12[i]<-outt[2,2]
oo22[i]<-outt[3,2]
oo32[i]<-outt[4,2]
   }
power1<-max(mean(o01),mean(o11),mean(o21),mean(o31),mean(o02),mean(o12),mean(o22),mean(o32))
power2<-max(mean(oo01),mean(oo11),mean(oo21),mean(oo31),mean(oo02),mean(oo12),mean(oo22),mean(oo32))
power<-c(power1,power2)
View(t(power))
```

Appendix B.4: # Monte Carlo Simulation for comparing modified joint confidence region

```
library (MASS)
library(expm)
p<-2
n<-25
it<-10000
rho<-.90
sigma11<-100
sigma22<-81
sigma12<-rho*sqrt(sigma11)*sqrt(sigma22)
sigma21<-sigma12
lr<-matrix(data=NA, nrow = it, ncol = 1)
pvalue<-matrix(data=NA, nrow = it, ncol = 1)
chi<-matrix(data=NA, nrow = it, ncol = 1)
b01<-matrix(data=NA, nrow = it, ncol = 1)
b11<-matrix(data=NA, nrow = it, ncol = 1)
b21<-matrix(data=NA, nrow = it, ncol = 1)
b31<-matrix(data=NA, nrow = it, ncol = 1)
b02<-matrix(data=NA, nrow = it, ncol = 1)
b12<-matrix(data=NA, nrow = it, ncol = 1)
b22<-matrix(data=NA, nrow = it, ncol = 1)
b32<-matrix(data=NA, nrow = it, ncol = 1)
rb01<-matrix(data=NA, nrow = it, ncol = 1)
rb11<-matrix(data=NA, nrow = it, ncol = 1)
rb21<-matrix(data=NA, nrow = it, ncol = 1)
rb31<-matrix(data=NA, nrow = it, ncol = 1)
rb02<-matrix(data=NA, nrow = it, ncol = 1)
rb12<-matrix(data=NA, nrow = it, ncol = 1)
rb22<-matrix(data=NA, nrow = it, ncol = 1)
rb32<-matrix(data=NA, nrow = it, ncol = 1)
for (i in 1:it) {
mu<- matrix(c(0,0),nrow=2,ncol=1)
sigma<- matrix(c(sigma11,sigma12,sigma21,sigma22),nrow=p,ncol=p)
set.seed(950+i)
error<- mvrnorm(n,mu,sigma)
set.seed(950)
  X1<-round(abs(rnorm(n,mean=1000,sd=60)),0)
set.seed(950)
  X2<-round(abs(rnorm(n,mean=375,sd=30)),0)
set.seed(950)
  X3<-round(abs(rnorm(n,mean=500,sd=20)),0)
xbeta<-matrix(c((25+3.5*(X1)-1.75*(X2)-2.5*(X3)),(175+2.5*(X1)-1.25*(X2)-1.5*(X3))),nrow=n,ncol=p)
  Y<-xbeta+error
  X0<-matrix(c(1),nrow = n,ncol = 1)
  X<-cbind(X0,X1,X2,X3)
estbeta<-(solve(t(X)%*%X))%*%(t(X)%*%Y)
b01[i]<-estbeta[1,1]
```

```
b11[i]<-estbeta[2,1]
b21[i]<-estbeta[3,1]
b31[i]<-estbeta[4,1]
b02[i]<-estbeta[1,2]
b12[i]<-estbeta[2,2]
b22[i]<-estbeta[3,2]
b32[i]<-estbeta[4,2]
 R<- matrix(c(0,1,2,0),nrow=1,ncol=4,byrow=TRUE)
 r<- matrix(c(0,0),nrow=1,ncol=2,byrow=TRUE)
 A<-((R%*%estbeta)-r)
 A2<-(solve(t(X)%*%X))
 B1<-(A2%*%t(R))
 B2<-(R%*%B1)
 B3<-solve(B2)
 C<-((solve(t(X)%*%X))%*%t(R))
restbeta<-estbeta-(C%*%(B3%*%A))
rb01[i]<-restbeta[1,1]
rb11[i]<-restbeta[2,1]
rb21[i]<-restbeta[3,1]
rb31[i]<-restbeta[4,1]
rb02[i]<-restbeta[1,2]
rb12[i]<-restbeta[2,2]
rb22[i]<-restbeta[3,2]
rb32[i]<-restbeta[4,2]
 xrbeta<-
matrix(c((restbeta[1,1]+restbeta[2,1]*(X1)+restbeta[3,1]*(X2)+restbeta[4,1]*(X3)),(restbeta[1,2]+restbeta[2,2]*(X1
)+restbeta[3,2]*(X2)+restbeta[4,2]*(X3))),nrow=n,ncol=2)
rerror<-(Y-xrbeta)
rsigma<-(1/n)*(t(rerror)%*%rerror)
 xbeta<-
matrix(c((estbeta[1,1]+estbeta[2,1]*(X1)+estbeta[3,1]*(X2)+estbeta[4,1]*(X3)),(estbeta[1,2]+estbeta[2,2]*(X1)+est
beta[3,2]*(X2)+estbeta[4,2]*(X3))),nrow=n,ncol=2)
olserror<-(Y-xbeta)
olssigma<-(1/n)*(t(olserror)%*%olserror)
lr[i]<-det(rsigma)/det(olssigma)
chi[i]<-dchisq(lr[i], 2)
pvalue[i]<-pchisq(lr[i], 2, lower.tail=TRUE)
}
xrbeta<-
matrix(c((restbeta[1,1]+restbeta[2,1]*(X1)+restbeta[3,1]*(X2)+restbeta[4,1]*(X3)),(restbeta[1,2]+restbeta[2,2]*(X1
)+restbeta[3,2]*(X2)+restbeta[4,2]*(X3))),nrow=n,ncol=2)
rerror<-(Y-xrbeta)
rsigma<-(1/n)*(t(rerror)%*%rerror)
cr1<-A2%*%t(R)%*%B3%*%R%*%A2
cr11<-A2-cr1
varrbeta<-kronecker(rsigma,cr11)
varrbetasqr<-matrix(c(sqrt(diag(varrbeta))),nrow=4,ncol=2)
t<-restbeta/varrbetasqr
rucl<-(restbeta)+(qt(0.95, 21))*varrbetasqr
```

```
rull<-(restbeta)-(qt(0.95, 21))*varrbetasqr
xbeta<-
matrix(c((estbeta[1,1]+estbeta[2,1]*(X1)+estbeta[3,1]*(X2)+estbeta[4,1]*(X3)),(estbeta[1,2]+estbeta[2,2]*(X1)+est
beta[3,2]*(X2)+estbeta[4,2]*(X3))),nrow=n,ncol=2)
olserror<-(Y-xbeta)
olssigma<-(1/n)*(t(olserror)%*%olserror)
olsvarrbeta<-kronecker(olssigma,A2)
lr<-det(rsigma)/det(olssigma)
```