Chaotic features of the generalized shift map and the complemented shift map

Ph.D. THESIS

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Chaotic features of the generalized shift map and the complemented shift map

BY

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A DISSERTATION

submitted in partial fulfillment of the requirements for the degree of

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Declaration

I do hereby declare that the submitted thesis entitled "Chaotic features of the generalized shift map and the complemented shift map" has been composed by me and all the works presented herein are of my own findings. I further declare that this work has not been submitted anywhere for any academic degree, prize or scholarship.

(Hena Rani Biswas)

Dedicated to my

Beloved Parents and Husband

Certificate

I have much pleasure to certify that the research work presented in this dissertation entitled "**Chaotic features of the generalized shift map and the complemented shift map"** has been performed by **Hena Rani Biswas.** She accomplished all sorts of research activities under my supervision and guidance. The part of this dissertation has not been submitted elsewhere for any degree or diploma. It is further certified that the work presented herewith is original and very suitable for submission for the award of the degree of Ph.D.

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The Author

ABSTRACT

Now a days, the discussion of chaotic dynamics has become increasingly popular. A particular class of dynamical systems is defined as chaotic dynamical systems. The idea of dynamical systems has gone through phases and has even been given different names. Chaos is gradually becoming a part of our daily life. Devaney's definition of chaos is considered a general and strong definition of chaos. It is based on the strength of topological transitivity in the discovery of chaos.

In this research, we reviewed the Proximity theorem 3.3.1 and its proof and using this theorem we have found strong chaotic features of the shift map σ, on $\sum_m$. We have also proved the chaotic features of the generalized shift map. We have found the essential chaotic properties of the complemented shift map and confirmed that σ^c is chaotic on $\sum_m$.

Presently, we are using the shift map as a chaotic model of a dynamical system. This research has established that the generalized shift map and the complemented shift map are chaotic. So, we can use the above two maps as constituting the new model for chaotic dynamical systems.

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INTRODUCTION

Dynamical systems are part of life. Quite often, it has been studied as an abstract concept in mathematics. For every dynamical system, a state-space represents the set of values for which iterations of the system are generated. The state space is given by a set of real numbers or possibly a vector at any given time. Every point or vector used can be represented by a point in the appropriate state space. The path of a dynamical system [39] (trajectory or orbit) is relevant if only individual systems could be obtained and comprehended. It is often difficult due to the complexity of many dynamical systems.

Chaos theory was discovered in 1963 though Lorenz had observed the phenomenon two years earlier. Quite a number of things could contribute to this irregularity which would later become a giant concept of study years later on. Lorenz is believed to propound the idea, but he is certainly not the first to associate chaos with the phenomenon under study. Alexander Lyapunov also made some contributions in the early stages. He was in the study of the instability of fluids and turbulence in fluids or gases. He tried to measure the transition from order to chaos. Chaos is one of the few concepts in mathematics which cannot usually be defined in a word or statement. Chaos theory is a branch of mathematics focusing on dynamical systems that are highly sensitive to initial conditions. Chaos theory is the study of how systems that follow simple, straightforward, deterministic laws can exhibit very complicated and seemingly random long-term behavior. In mathematics, chaos cannot usually be defined in a word or statement. Chaotic features depend on either the topological or metric properties of the system. In mathematics, Chaos theory is presented through a brief analysis of some interesting dynamical systems that exist one-, two-, and three-dimensional maps like logistic, tent, doubling, Smale's horseshoe map, Hénon map and Lorenz model which exhibit chaotic behavior. This behavior can be studied through the analysis of a chaotic mathematical model.

Symbolic dynamics gives a method for converting natural system trajectories into symbol sequences and answering how much the underlying system can be deduced from these sequences. Symbolic dynamics shows how one may convert a dynamical system into an extended technique, studies the simplified dynamics in sequences space, and then take the results back to the state-space of the original system. The symbolic dynamical systems form a significant category of dynamical systems. We may consider a shift operator that shifts a sequence one symbol left in such a place.

As first mathematicians, Li and Yorke [15] assembled the term chaos with a map. According to the definition of Devaney's chaos [1], topological transitive map is chaotic. The concept of chaos accompanied with the concept of transitivity and sensitivity by J. Auslander and J.A. Yorke [20] in 1980. There are various ways of assessing the complicated or chaotic nature of the dynamics. Akin [57] proposed connection between sensitivity and Li-Yorke version of chaos. L. Snoha [24] induced the conception of dense chaos and dense δ- chaos.

Thompson [78] describes how Dynamical systems can be studied from a distinct point of view of which one dominant area is topological dynamics. Topological dynamics deals with a space, a topology, and a function that acts on it. One of the two components of Devaney chaos is topologically transitive . It is well known that having the property of transitivity is sufficient enough for a system on the interval [52] and, on the infinite shift space; it is obvious that transitivity implies dense periodic points and therefore implies SDIC [22]. This is not true if we replace the property of transitivity with the other ingredient of Devaney chaos, dense periodicity property yields different results.

Topological transitivity for transitive maps is quite similar to topological mixing. In the case of interval maps [55], weakly mixing, transitivity, and topological mixing are equivalent. Transitivity guarantees sensitivity dependence for interval maps; the converse is true. Horseshoe maps have positive topological entropy. Positive entropy and homoclinic points are equivalent properties.

The symbol space we conduct here has a metric that is defined naturally. Hence the study can be related to the metric space using the standard concepts of dynamical systems. This discussion aims to show the chaoticity and related properties of σ. There are many concepts of chaoticity defined in metric spaces. Biswas H. R. [16] expanded the concept of the shift map to the generalized shift map σ_n in the symbol space Σ_2 and showed that the generalized shift map is chaotic on Σ_2. Ju H., Shao H., Choe Y., and Shi Y. [17] gave an idea of the conditions for any maps to be conjugate or semi-conjugate (topologically) to subshifts of finite type. In recent times [14, 18, 19], some attractive research works have been done on the particular property sensitive dependence on initial conditions. Ruette [55] showed that shift maps are topologically mixing, have sensitive dependence on initial conditions, and dense periodic points on an invariant subset supporting topological semi-conjugacy. Dutta T. K. and Burhagohain A. in [33] used topological conjugacy to prove chaoticity. The basic ingredients of Li-Yorke chaos are uncountable scrambled sets called Li-York pairs. Construction of Li-Yorke pairs has been done [21, 54] in a clear cut-way. In 1993, Li Si. [81] introduced the notion of ω-chaos through the introduction of ω-scrambled set.

Mathematica and Matlab software are useful for programming, and so to analyze the chaotic behaviors of different maps, we have used these software's to perform our research.

Organisation of the thesis

The chapter of the thesis is organized as follows:

Chapter-1: Some basic and general concepts of dynamical systems, types of dynamical systems, different types of orbits, several definitions of chaos and related theorem, chaotic maps in the interval which are required in the subsequent chapters of this thesis are presented in this chapter.

Chapter-2: How symbolic dynamics work for one-dimensional, and two-dimensional maps such as quadratic map, logistic map, tent map, Smale's horseshoe map, and a specific horseshoe map is discussed here. This chapter shows the chaotic significance by comparing one map with another map with the help of topological conjugacy.

Chapter-3: Important strong chaotic properties of the shift map σ on the generalized one-sided symbol space $\sum_m,\ m\ (\geq 2) \in N$ is explained in chapter 3.

Chapter-4: Strong chaotic features of the generalized shift map σ_n is discussed in this chapter.

Chapter-5: Complemented shift map and its essential chaotic features are established here. It is proved that σ^c on $\sum_m$ is Devaney as well as Auslander-York Chaotic.

CHAPTER -1

DYNAMICAL SYSTEMS AND DIFFERENT TYPES OF CHAOS

1.1 Introduction

A dynamical system is a mathematical attempt to describe the phenomenon of change. In terms of modern dynamical systems and their ideas, a relatively brief historical past exists. Considered the prominent founding father of dynamical systems, the French mathematician Henri Poincaré (1854-1912) developed the study of non-linear differential equations by presenting the qualitative approaches of geometry and topology instead of analytic methods to discuss the general properties of solutions of these systems. As a mathematician, a world and international appreciation and acceptance of the behavior of all system solutions were equally more important than just the solved analytically-precise solutions. Birkhoff appreciated the significance of the study of mappings and accentuated discrete dynamics as a method of figuring out the continuous dynamics arising from differential equations. As times progressed, the subject of dynamical systems has benefited from a combination of interest and techniques and methods and applications from all sorts of fields [12].

Science and Mathematics have always been interrelated. Perhaps this could be due to the fact that mathematical expressions can present almost every idea in science. Science helps interpret nature, whereas mathematics enables us to solve real-life problems that are usually difficult to solve or deal with directly. These expressions are typically in the form of equations and, more often, differential equations. It is generally done using the concept of modeling. In models, real-life science is described with purely mathematical language. Most often, these are considered to be adequate and accurately such that solutions to the mathematics model imply the problem in science is solved. It was born out of a real-life challenge some years ago, somewhere in the eighteenth century. Once in Russia, in Koenigsberg, the river Pregel had over flown its banks and run through the city. There existed some seven bridges that connected the regions in this city. People wanted to find out the possibility of going through the city but crossing each bridge only once. Given their position, even Euler believed it was impossible to walk across the bridges.

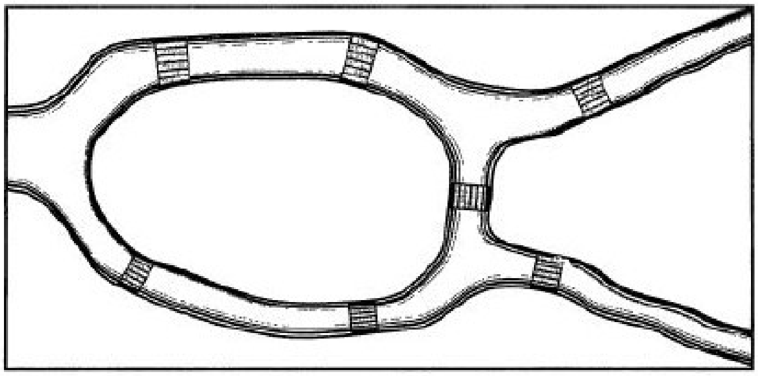


Figure 1.1: The Koenigsberg bridges (Adapted from an illustration in Newman,1983)

Mathematicians believed that what is now being defined as chaos is not really anything new. It has been with us all these years, and we are only renaming it to make it look more mathematical. Perhaps they would agree with Henry Poncaire's quote that "Mathematics is the art of giving new names to old things". The first experience with what is now called chaos was with Henry Poincaré, the famous French mathematician in the early 1900's. Poincaré is considered the last of the universalist (people who made major contributions to all major and known areas) in mathematics. He studied what was called the three-body problem (motion of the solar system) by Newton. In Poncaire's view, there is always a small cause (infinitely small) that we usually are not aware of, and irrespective of the fact that we overlook it, it has a big and noticeable effect that we cannot afford to overlook. Often, we attribute this cause to chance.

1.2 General Overview

A dynamical system can be thought of as repeating events once and again. We consider anything that evolves (changes over time) as a dynamical system. Since life is full of changing (non-constant events), so life could be considered a dynamical system. In dynamical systems, the starting point, the journey along the line as well as the finishing points are all relevant, and hence we pay attention to each of them as such. One of the ways of describing the passage in time of points in a given space S is a dynamical system. The space S varies depending on the area of dynamics [2]. Dynamical systems help appreciate the relationships between mathematics and various aspect of science.

1.3 Dynamical System

A dynamical system is best to describe in terms of these three words

(i) Phase space
(ii) Time
(iii) Law of evolution

(i) Phase space: The phase space captures the various structures of a dynamical system. The different aspects of dynamical systems are obtained based on these structures. They could either be differentiable, topological or considered preserving (ergodic).

(ii) Time: Time is expressed either as discrete when the values are integers, whereas it is considered continuous when the set of values are real numbers. Time considered here is either reversible or irreversible depending on its domain.

(iii) Law of Evolution: This is the rule that allows us to determine the state of a system at any moment, given its current state.

Mathematically, (X, G, Ψ) is a dynamical system if

(i) X is a non-empty set.
(ii) G is a group or semi-group.
(iii) Ψ is a map where $\Psi: X \times G \to X$ satisfies $\Psi(\Psi(x, g_1), g_2) = \Psi(x, g_1 \times g_2)$.

Here X is called state-space or phase space. We give some examples of dynamical systems which shown in our real-life [4].

Example:

(i) Population growth,
(ii) A swinging pendulum,
(iii) The motions of celestial bodies,
(iv) Atmosphere (weather),
(v) Economy (stock market).

1.3.1 Types of Dynamical Systems

Here, we will discuss different dynamical systems, such as discrete and continuous dynamical systems.

1.3.1.1 Discrete Dynamical System

Given the current or present state of a system, we expect to know the state of the system given a change in time. A discrete dynamical system [1] is defined as a sequence X_n with $X_{n+1} = f(X_n)$ where $f: \mathbb{R} \to \mathbb{R}$.

Actually, we have a discrete dynamical system when time is a sequence of separate chucks, each of the next like beads on a string. In such cases, one can really distinguish between the position of the bead in front and the bead behind without confusion or ambiguity. In discrete dynamical systems, usually preceding states can be obtained depending on computations of the current state. It is always important to know where a system will be in the next instant. Also, there are intervals (big) between two distinct time intervals; hence, we say discrete dynamical systems change in cycles after the expected time periods.

1.3.1.2 Continuous Dynamical System

A continuous dynamical system [2] is mostly represented by differential equations. It is usually expressed as $X^{'} = f(x)$ which describes the rate at which the system changes with time. Here our interest is how quickly the system changes with time.

Example 1.3.1.1 An orange is thrown up in the air. It will be unfortunate to ask where the mango will be at the next instant, though we have every reason to know how the height and velocity of the mango changes with time.

We can describe such a system by a vector representation of its height or position and velocity or speed. Velocity here is simply the rate of change of position relative to time. As the mango falls back from up there (return to its starting point), it obtains a velocity against gravity.

Mathematically,

$$X = [h, v]\ \frac{dh}{dv} = v \text{ and } \frac{dv}{dt} = -g$$

The solution of this system indicates the height and velocity of the mango at any time (t). One area or type of system where continuous systems really appear most is in chemical reactions. It deals with the response of several components and can usually be modeled as a differential equation. We note that both discrete and continuous dynamical systems can appear beyond the one-dimensional form. As stated earlier, there are various aspects of the dynamical system due to the behavior of their state space.

1.4 Other Concepts in Dynamical Systems

The nature of its orbit [1] is always important and worth noting in a dynamical system. There is always the tendency to have repetitions in orbit.

Definition 1.4.1 (Orbits)

Given $x_o \in \mathbb{R}$, we define the orbit of x_o under f to be the sequence of points.

$$x_0, x_1 = f(x_0), x_2 = f^2(x_0), \dots\dots\dots, x_n = f^n(x_0), \dots\dots$$

That is, $x_{n+1} = f(x_n) = f^{n+1}(x_0)$. The point x_0 is the seed of the orbit [1].

Definition 1.4.2 (Fixed Point): If $f(x_0) = x_0$ then x_0 is a fixed point [1], where $f: X \to X$ is a continuous map.

Definition 1.4.3 (Periodic Point):

A point $x_0 \in X$ is periodic point [1] of a function f of period n if $f^n(x_0) = x_0$, $n > 1$ is the order of f.

Definition 1.4.4 (Forward Asymptotic point):

If $\lim_{i\to\infty} f^{in}(x) = p$ then x is an asymptotic forward point [1] where p is the periodic point of period n. $W^s(p)$ is a stable set of consisting of all points forward asymptotic to p.

Definition 1.4.5 (Backward Asymptotic point):

We can define forward asymptotic points by $\left|f^i(x) - f^i(p)\right| \to 0$ as $i \to \infty$ where p is a non-periodic point. If f is invertible, then we have backward asymptotic points [1] by putting $i \to -\infty$. $W^u(p)$ is an unstable set of p consisting of all points that are backward asymptotic to p.

1.5 Chaos and Chaotic dynamics

In this section we will discuss about chaos and chaotic dynamics. Chaos represents one of the interesting behaviors of dynamical system and it shows movement of sets from their existing position or location. Chaos theory as an idea in non-linear mathematics is applicable in both social sciences and natural science.

1.5.1 General Discussion

Most Mathematicians and Physicist like Newton and Laplace made indirect contributions to chaos theory. The pair believed in the same cause being equal to the same effect. They pointed out that there are always clear rules of life (cause and effect), which brought about predictability and could always be controlled. They believed systems behaved nicely once we repeated doing the same thing, expecting the same results. Though Newton and his colleague believed in predictability, it had challenges in predicting systems like the weather. The orbits of the weather or solar system created a gap in what they believed. Basically, they meant that, given two bodies in motion from similar points, we should be able to trace one orbit using the other. In this height, he desired to see the three-body problem solved. He discovered in his study that there are orbits of systems that are not periodic and yet never move closer or converge to any fixed point. Though Poncaire never solved Newton's three-body problem, he made significant remarks in that direction. His solution was considered as a partial solution to the problem. It was still awarded for it, perhaps because other legendary Mathematicians like Euler, Laplace, Lagrange, and others could not help out. In Poincare's solution, he did approximate orbits in the form of series. He later realized he had made a mistake, and it was the genuineness of mind in admitting gave rise to what will now be termed as chaos. He realized that little changes had more than just a little effect over time. His idea on chaos almost never fell through because people had lived with Newtonian science for long and maybe because other mathematics like Laplace, Leibniz, and others still believed in Newtonian science. Almost every system is linear and could be predicted to a point unknown and perhaps unseen.

Edward Lorenz from MIT is known and acknowledged as the father (modern) of chaos theory. He was a meteorologist and had so much interest in long-term predictions of the weather. This happened during one of his routine computations trying to predict the weather. He did the same computations but with different input values. This was because he continued one of his after-break sessions using input values from his computer. The difference in the values was small such that he thought was negligible and insignificant. Edward Lorenz describes chaos in these words "Chaos is when the present determines the future, but the approximate present does not approximately determine the future". He further explained that the unpredictability nature of the weather is because we can only measure the weather approximately. He only realized from his work graphically that though they have almost identical starting points, the difference in their final points given the same number of iterations was wide and unimaginable [2]. After careful consideration and scrutiny, Edward Lorenz realized that the two initial input values differ by decimal points. The output from his machine had three decimal places compared to the six decimals of his original inputs. This small numerical difference has contributed to the great difference in his computations. If his computer is not faulty, then mathematicians are failing to acknowledge; a small change in input produces a significant difference in the end.

1.5.2 Three main characteristics of chaos

Three main characteristic behaviors are associated with a chaotic system. They are (i) sensitivity to initial condition, (ii) density of unstable periodic orbits in a chaotic attractor, (iii) Topological transitivity.

(i) Sensitive Dependence to Initial Conditions:

The characteristic of sensitivity is usually observed during the iterations. The idea of sensitivity dependence allows the orbits to be far apart as the number of iterations increases. Usually, sensitivity dependence holds for large time values. Suppose a system is sensitive depending on initial conditions [19]. In that case, our observation does not necessarily start from the first iterate, or better still, we are not only interested and expecting a change in trajectory due to the initial condition but that the slightest perturbation causes preceding values to differ from the expected. The idea of sensitivity dependence is otherwise called the butterfly effect. Generally, this is experienced in non-linear science. The butterfly effect is one of the few ideas in mathematics that directly refer to the non-scientific world.

Mathematically, A dynamical system (X, T) has sensitive dependence on initial conditions if $\exists$ some $\delta > 0$ such that for $x \in X$ and $\varepsilon > 0$, $\exists\, y \in X$ with $d(x, y) < \varepsilon$ and $\exists\, n \in N$ such that $d\big(T^n(x), T^n(y)\big) > \delta$, where X is a compact metric space and T is a continuous map.

(ii) Density of Periodic Orbits:

A dynamical system (X, f) has a dense orbit [1] if and only if $\exists\, x \in X{:}\, \forall\, y \in X$ $\forall \varepsilon > 0\ \exists\, n \in N{:}\, d(f^n(x), y) < \varepsilon$ where x and y represent the distinct initial points for the iteration, $f^n(x)$ represents a specific iteration. The orbit of x moves arbitrarily close to another orbit at a given in time such that the metric between them is significantly small. As the iteration continues $(n \to \infty)$, the possibility of every other point experiencing this is high. It implies that $d(f^n(x), y) < \varepsilon$.

(iii) Topologically Transitivity:

A dynamical system (X, f) is topologically transitive [1] if and only if all open subsets (non-empty) $U, V \subset X, \exists\, k > 0$ such that $f^k(U) \cap V \neq \emptyset$. Generally, transitivity implies the existence of dense orbit.

1.5.3 Some Useful Definitions

We present some definitions and results in this section which are essentials for establishing the theorem in the next chapters.

Definition 1.5.3.1 (Invariant)

A set $\wedge$ is invariant [1] for a function f on $\wedge$ if we have $f \wedge = \wedge$.

Note that $f{:}\ \wedge \to \wedge$ follows if $\wedge$ is an invariant. Also $\wedge$ is negative invariant if $\wedge \in f^{-1} \wedge$ and $\wedge$ is a positive invariant if $f \wedge \in \wedge$.

Definition 1.5.3.2 (ω-limit set)

Let $f: X \rightarrow X$ be a continuous map, where (X, d) is a compact metric space. Then the ω-limit set [27] of points of $x \in X$ is the set of all limit points of the orbit of x. We denote the ω-limit set of points of x by $\omega_f(x)$.

Hence $\omega_f(x) = \bigcap_{n \in N} \{\overline{f^k(x)} : k \geq n\}$.

Definition 1.5.3.3 (Recurrent point)

A point $x \in X$ is called recurrent [81] if $x \in \omega_f(x)$. Hence we can say that orbit of x returns to an arbitrarily small neighborhood of x infinitely often.

Definition 1.5.3.4 (Non-wandering point)

A point $x \in X$ is called non-wandering point [81] if, for every neighborhood $N(x)$ of x, ∃ an integer $k \geq 1$ such that, $N(x) \cap f^k(N(x)) \neq \emptyset$.

Definition 1.5.3.5 (Wandering point)

A point $x \in X$ is called a wandering point [81] if there exists a neighborhood $N(x)$ of $x$is disjoint from all $f^k(N(x))$, $k \geq 1$.

Definition 1.5.3.6

For a given positive number $\delta > 0$, a pair $(x, y) \in (X, X)$ is said to be δ-scrambled if

(i) $\lim_{n \to \infty} sup\, d(f^n(x), f^n(y)) > \delta$

(ii) $\lim_{n \to \infty} inf d(f^n(x), f^n(y)) = 0$

1.5.4 Different Types of Chaos and related theorems

In this section, we now discuss various types of chaos as defined by mathematicians of the world.

Definition 1.5.4.1 (Devaney's Chaos)

A continuous map $f: X \rightarrow X$ is said to chaotic [1] if

(i) f has sensitive dependence on initial conditions.

(ii) f are transitive in X.

(iii) the periodic Points of f are dense in X.

where X is a metric space.

This definition of chaos is one of the widely known and accepted definitions in chaos theory. This makes the foundation of the definition redundant. The property of transitivity and periodic orbits being dense in the set X are invariant under topological conjugation. It points out the fact that the two are topological properties. A property is considered topological and preserved under topological conjugation only if the space X is defined in topology and as a compact space. Sensitivity to initial conditions is usually expressed in metric spaces and is therefore non-invariant under topological conjugation.

Definition 1.5.4.2 (Li-Yorke chaos)

Let $x, y \in X$. The pair $(x, y) \in (X, X)$ is a Li-Yorke scrambled pair [15] if

(i) $\lim_{n\to\infty} supd\big(f^n(x), f^n(y)\big) > 0$

(ii) $\lim_{n\to\infty} inf d\big(f^n(x), f^n(y)\big) = 0$

Since d is a metric imposed on the iterates as n varies through to infinity, we could simply consider the distance between the iterate at some n.
That is $d\big(f^n(x), f^n(y)\big) = |f^n(x) - f^n(y)|$.

Generally, x and y are different points. The trajectory path for the two points, irrespective of how close they may apply to each other at the beginning, grows to a positive non-zero value. The closest distance at any point in the iteration is very small and equivalent to zero. A map with points of discontinuity is usually not Li-Yorke chaotic.

Definition 1.5.4.3 (Wiggins Chaos)

In the sense of Wiggins [10] chaos, any map $f: X \to X$ is chaotic if it is

(i) topologically transitive

(ii) has sensitivity dependence on initial conditions.

Definition 1.5.4.4 (Lyapunov Chaos)

If $f: \mathbb{R} \to \mathbb{R}$ is the differentiable and continuous map, then this map f is said to be Lyapunov chaos [30] if it is

(i) transitive (topologically)

(ii) has positive Lyapunov exponent.

Definition 1.5.4.5 (Knudsen Chaos)

According to Knudsen [84], the dynamical system (X, f) is chaotic if it

(i) has dense orbits

(ii) has the sensitivity to the initial condition

where $f: X \to X$ is a continuous map on a metric space (X, d).

Definition 1.5.4.6 (Kato's Chao)

A map $f: X \to X$ is said to be chaotic in the sense of Kato [58] if f is both sensitive and accessible. Kato introduced it in 1966.

Definition 1.5.4.7 (Martelli's Chaos)

In 1999 Martelli introduced another type of chaos is called Martelli's [59] chaos. A map $f: X \to X$ is said to be chaotic in the sense of Martelli if $\exists\, x_0 \in X$ such that the orbit of x_0 is dense in X and unstable.

Definition 1.5.4.8 (Auslander-Yorke Chaos)

A continuous map $f: X \to X$ is said to be chaotic in the sense of Auslander-Yorke [20] if it

(i) has a point x whose orbit is dense,

(ii) has sensitive dependence on initial conditions.

Definition 1.5.4.9 (Bau-Sen Du)

According to the definition of Bau-sen Du [19], f is chaotic if there exists a positive number δ such that for any point x and any non-empty open set V in X there is a point y in V such that

$\lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) \geq \delta$ and $\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$

Theorem 1.5.4.1

Let $f: I \to I$ be a continuous map on a compact interval I, then Martelli's and Devaney's chaos are equivalent. We know that Martelli's chaos implies topological transitivity; hence the result follows from [41].

Recently, Y. Shi and P. Yu [40] discussed some features of turbulent maps in non-compact sets of a complete metric space. In [42], the authors discussed Li-York chaotic sets of continuous and discontinuous maps. In [50], J. Canovas and R. Haric proved that any continuous map $f: T \to T$ is distributionally chaotic if and only if its topological entropy is positive, where T is a finite tree, and any vertex of T is a fixed of T is a fixed point of f. W. Huang and X. Ye [7] proved that Devaney's chaos is stronger than Li-Yorke's chaos. In [5], Aulbach and Kieninger discussed three types of chaos: Devaney, Li-Yorke, and Block-Coppel. Also, various types of chaos have been discussed in [56, 62] and [37]. Y. Shi and G. Chen [43] examined the chaos of discrete dynamical systems in complete metric spaces and introduced several new concepts. In [40], the same authors discussed chaotification of discrete dynamical systems.

From the following example, we observe that if any function has periodic points which are dense, but it has no sensitive dependence on initial conditions and no transitive points.

Example 1.5.4.1

Consider $g_1(x) = x$ be a map on I. Here x is a fixed point for g_1. So, g_1 has dense periodic points. since every interval is invariant under g_1, so g_1 is not transitive. Also, for every $\delta > 0$, there is a nbd of x, for any $x \in I$, such as an open ball of x with radius $\delta/3$, $B\left(x, \frac{\delta}{3}\right)$ such that for every y in the ball,

$$|g_1^n(x) - g_1^n(y)| = |x - y| < \delta/3 \text{ for all } n.$$

So, g_1 is not transitive, and it does not have sensitive dependence on initial conditions.

1.5.5 Characterization of topological transitivity and mixing

In a dynamical system (X, f), where X is a compact metric space and f is onto, topological transitive is equivalent to the existence of a dense orbit in X. i.e., there is a point $x \in X$ such that orbit of x is dense in X.

Example 1.5.5.1

Take $X = [0, 2]$ and $f: X \to X$ defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2x + 3 & \text{if } \frac{1}{2} \leq x \leq 1 \\ -x + 2 & \text{if } 1 \leq x \leq 2 \end{cases}$$

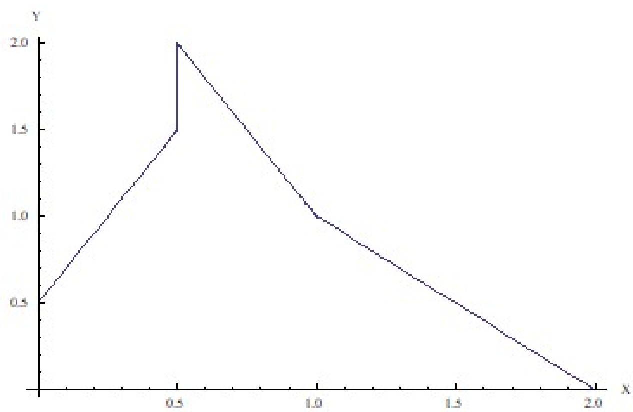


Figure 1.5.1: Graph of $f(x)$

Then f is topologically transitive but f^2 is not. Also, $f \times f$ is not transitive.

The next result gives a sufficient condition under which totally transitivity implies weakly mixing.

A totally transitive dynamical system (X, f) with dense set of periodic points is weakly mixing.

The following result gives a sufficient condition under which weakly mixing implies strongly mixing.

A totally transitive dynamical system (X, f) where X is compact with an open interval J having a dense set of periodic points is strongly mixing.

So, in general, we have the following implications

Strongly mixing $\Rightarrow$Weakly mixing $\Rightarrow$ totally transitive $\Rightarrow$ transitive.

Example 1.5.5.2

Let,

$$H(x) = \begin{cases} \frac{1}{2} + 2x & \text{if } 0 \leq x \leq \frac{1}{4} \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ 1 - x & \text{if} \frac{1}{2} \leq x \leq 1 \end{cases}$$

The above function $H: [0,1] \to [0,1]$ is transitive but not mixing. Since H is transitive, then the set of periodic points of H is dense in X. It implies that $([0,1], H)$ is Devaney chaotic. Because H has positive topological entropy than is Li-Yorke ε-chaotic for some positive ε but not chaotic.

1.5.6 Chaotic maps in the interval

Here, we will discuss different chaotic maps such as a logistic, tent, and expanding maps.

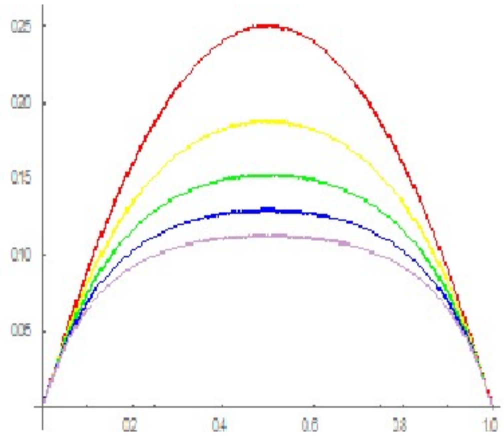
1.5.6.1 Logistic Map

Define the function

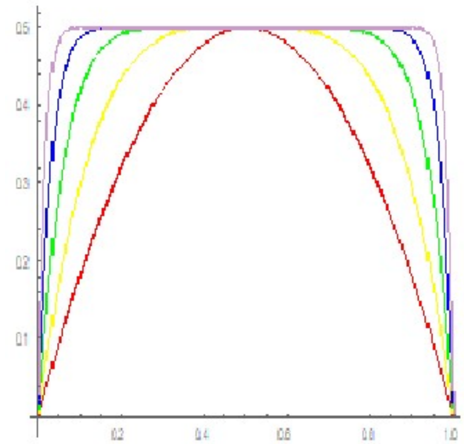
$f_a(x) = ax(1-x), x \in [0,1]$, $a > 0$ for given $x \in [0,1]$ is called logistic map.

B. Aulbach and B. Kieninger have given a simple proof for hyperbolicity and chaos of the logistic map in [5]. Also, in [69], P. Glendinning has investigated hyperbolicity of the invariant set for the logistic map when $a > 4$.

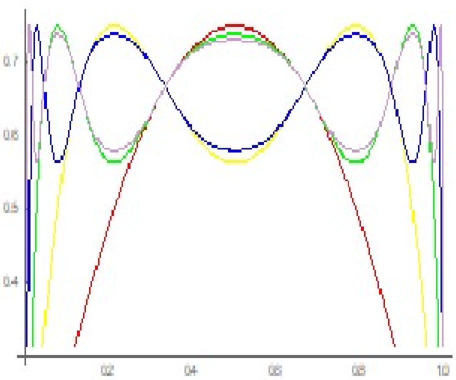
Now, the graphical representation of the dynamical behavior is as follows:



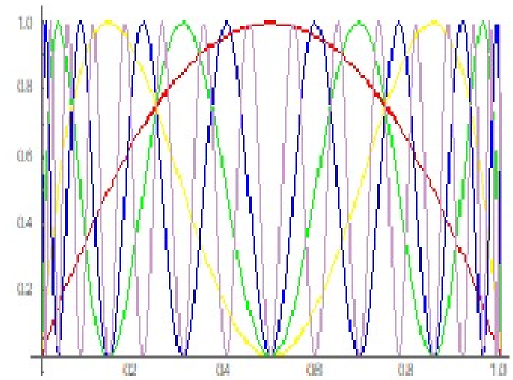
(i) $a = 1$



(ii) $a = 2$



(iii) $a = 3$



(iv) $a = 4$

Figure 1.5.2: Behavior of the logistic map for (i) $a = 1$, (ii) $a = 2$ (iii) $a = 3$, (iv) $a = 4$

1.5.6.2 Bifurcation Diagram of Logistic Map:

From the Figure 1.5.3 (a), 1.5.3 (b), 1.5.3 (c), we see that for $a < 1$, all the points are plotted at 0, for $1 < a < 3$, we still have one-point attractors, but the attracted value of x increases as a increases, at least to $a = 3$. Bifurcation occurs at $a = 3$, 3.449, 3.54409, 3.5644, 3.56875 (approximately) until just beyond 3.57, where the system is chaotic. But for $a > 3.57$, the system is not chaotic.

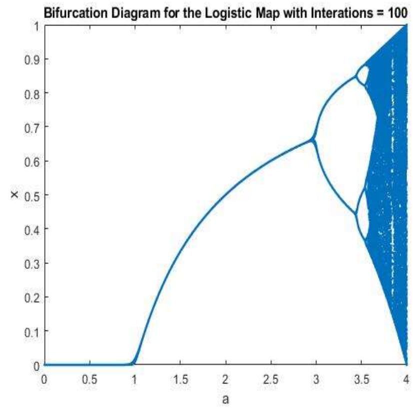


Figure 1.5.3 (a)

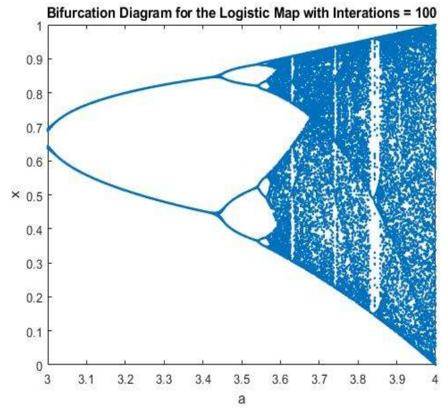


Figure 1.5.3 (b)

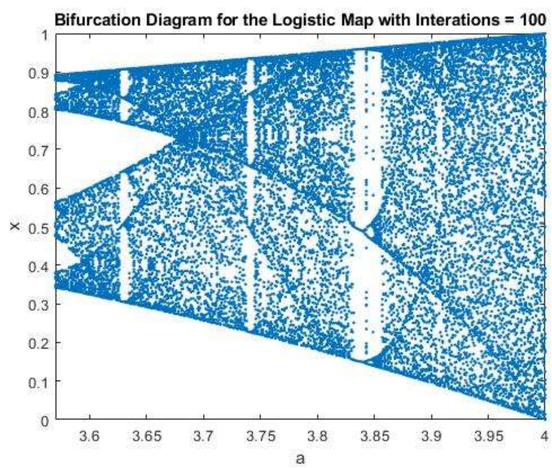


Figure 1.5.3 (c)

1.5.6.3 Tent map

Consider the function

$$T(x) = \begin{cases} 2x, & if\ 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & if\ \frac{1}{2} < x \leq 1 \end{cases}$$

The given tent map is also chaotic in the closed interval [0, 1].

Consider the iterations of the tent function as n gets larger (approaches infinity), the pattern in the diagram is lost gradually, and it gets worse and worse. This is observed very clearly up to the seventh iteration. We notice the interval in the diagram reducing as the iteration furthered on. In iterations eight and nine, the behavior of the orbits seems to have changed entirely from the one we could predict as half of the previous iteration. At this point, the regular periodicity is getting lost, and chaos is imminent. Chaos, just like its routes, is experienced over time.

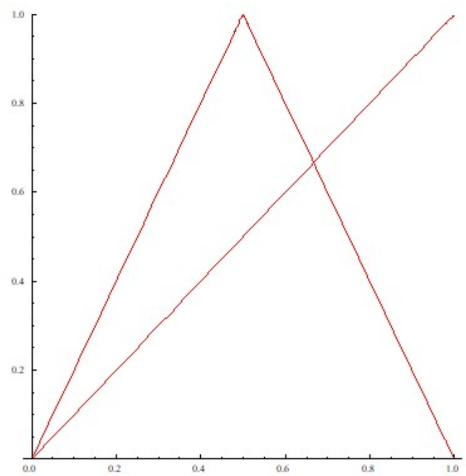


Figure 1.5.4: The Tent function

From the iterations below, there is an observed pattern. The preceding figure for the iteration is obtained by dividing the existing figure into two. It implies the number of fixed points increases as well. The fixed point is the point of intersection of the line $y = x$ with the orbits of the diagram.

This behavior continues clearly through from iteration one to tenth.

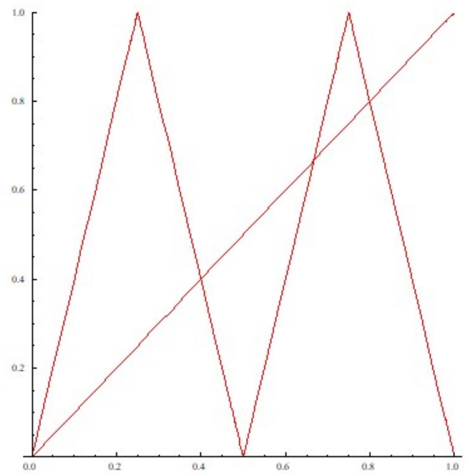


Figure 1.5.5: Second iteration of the tent function

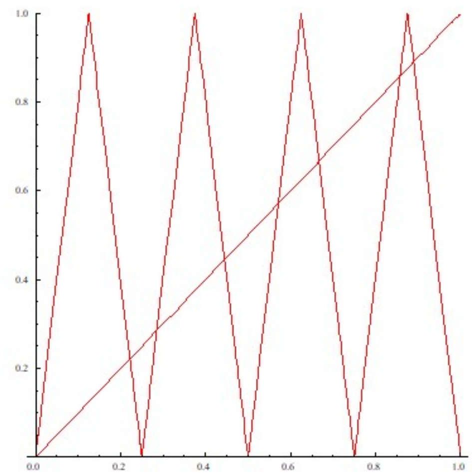


Figure 1.5.6: Third iteration of the tent function

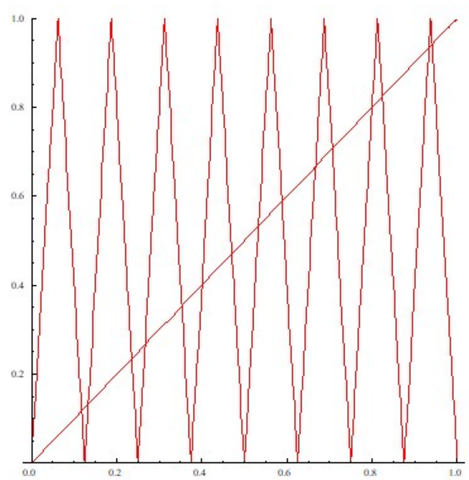


Figure 1.5.7: Fourth iteration of the tent function

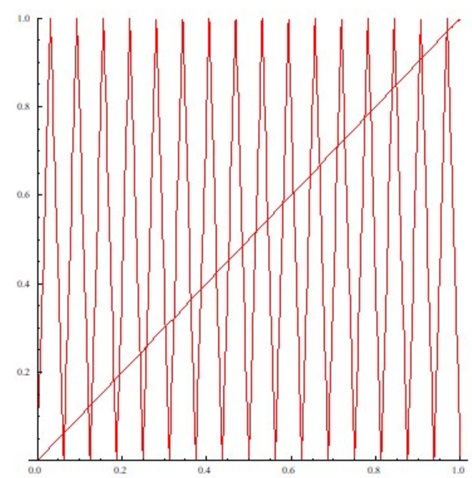


Figure 1.5.8: Fifth iteration of the tent function

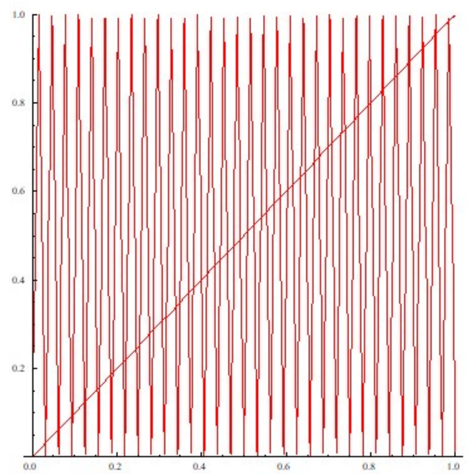


Figure 1.5.9: Sixth iteration of the tent function

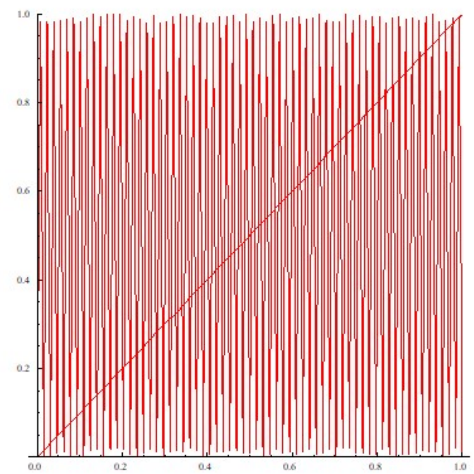


Figure 1.5.10: Seventh iteration of the tent function

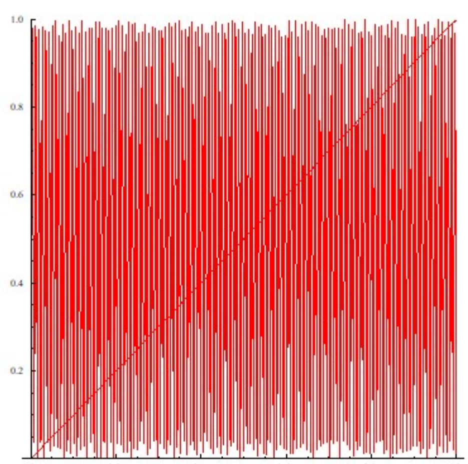


Figure 1.5.11: Eighth iteration of the tent function

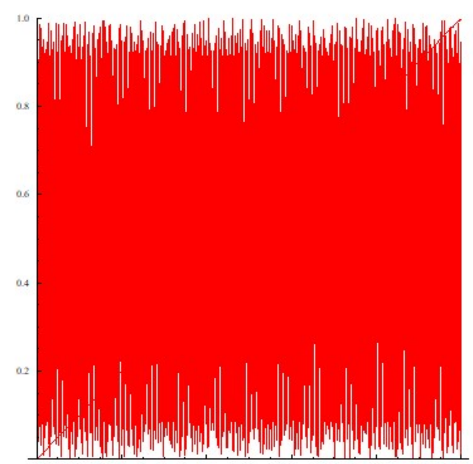


Figure 1.5.12: Nineth iteration of the tent function

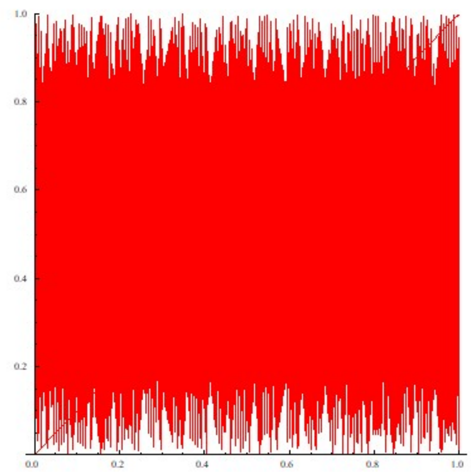


Figure 1.5.13: Tenth iteration of the tent function

Both iteration nine and ten behave in the same pattern. The movement of their orbits has changed. The regular periodicity is lost. The routes to chaos at this point are becoming obvious and visible.

Graphically, the tent map is chaotic as the iterations increase. This is typical of almost every chaotic map such that the phenomenon is observed for considerably long iterations depending on the nature of the map. Here we observed lost patterns. The various trajectories or orbits cannot be distinguished. Periodic orbits have become so dense. Hence chaos is in motion, and the tent map is considered chaotic.

1.5.6.4 Expanding map

If $f: I \rightarrow I$ is piecewise monotone mapping on [0,1], then the mapping is said to be expanding [49] if $\exists$ a constant $\lambda > 1$ such that $|f(x) - f(y)| \geq \lambda |x - y|$. Here the constant λ is said to be expanding constant for f.

Expanding maps play a significant role in the context of chaotic interval maps.

1.6 Summary and Conclusion

Dynamical systems are about the result trends and changes observed over time concerning a particular real-life scenario and practice. Time is a significant factor in the study of dynamical systems. The different behavior of various systems has become relevant to study and understand. The idea of dynamical systems has gone through phases and has even been accorded different names until now. We still don't have one principled definition for chaos, but it is defined based on the set of conditions it satisfies in terms of topology or metric. Different routes lead to the conclusion of systems being chaotic. The routes in terms of metrics are usually measurable, whiles the topological routes are usually analytical. The topological routes of a known map can be interrelated to the properties of an unknown map such that the conclusion for the two maps is the same. Topological conjugation preserves topological properties, but the same can not be said for all metric properties of chaos. It was observed that the tent map exhibits either non-periodicity at higher iterations or a different kind of periodicity. The tent has shown some properties of chaoticity.

The mathematical language expressed by chaotic systems (especially sensitive dependence, transitivity, and dense orbits) guarantees that a chaotic system passes the element of regularity, unpredictability, and indecomposability. The chaos of a map cannot be soley based on sensitive dependence or its equivalent relation of positive Lyapunov exponent. Transitivity is about the strongest property among all the conditions. Perhaps it is the reason most definition has an aspect of transitivity. For most maps, when transitivity fails, it is likely if not evident, that the condition of dense orbits might fail. Of course, positive Lyapunov does not depend on transitivity.

CHAPTER -2
SYMBOLIC DYNAMICS AND TOPOLOGICAL CONJUGACY FOR CHAOTIC MAPS

2.1 Introduction

The key idea of this chapter is to show the usefulness of symbolic dynamics in proving some simple mappings are chaotic. The methodology of symbolic dynamics was used as far back as Hadamard in 1898. Symbolic dynamics grew into its discipline in the 1930s with the works of Birkhoff, Morse, and Hedlund, which incidentally inspired Smale to construct his horseshoe map. J. Hadamar [74] illustrated the first application of symbolic dynamics. He applied trajectory coding to represent the universal behavior of geodesics on surfaces with negative curvature. C. Hsu enlarged the cell-to-cell mapping method. Brin [50] discussed cells of a given partition and their images under the action of a system. G. Osipenko [67] introduced the method of a symbolic image in which an oriented graph represents the transitions of a trajectory of partition elements. This method is effectively useful to the cost of production of invariant sets and Morse spectrum. Symbolic dynamics attempts to answer how much actuality about a dynamical system can be drawn from a data sequence produced by system measurements.

Topological conjugacy feature has an essential role in studying the chaotic behavior of a map. With the help of this feature, we can explore the chaotic significance by comparing one map with another map. Topological conjugacy [64] has such importance as it can protect many topological dynamical properties. In discussing any chaotic dynamical system, topological conjugacy between maps is a potential tool. Topological conjugacy is a significant notion about knowledge of dynamical systems. This essential tool makes predictions about a dynamical system's behavior by comparing this with another dynamical system whose specific properties are recognized. We get the basic idea about symbolic dynamics from many papers and books. There are some interesting applications of a symbolic dynamical system. Such as the symbolic representation of the Cantor set. From the article of H.R. Biswas and Monirul Islam [18], we get the basic idea of shift maps and their chaotic properties.

This chapter will discuss the symbolic dynamics of different types of maps. First, we will discuss the Quadratic family of functions $f_c(x) = x^2 + c$, where c is a constant. While this function looks simple enough, we will see their dynamics are amazingly complicated. Indeed, their behavior is not entirely understood for certain c-values. In section 2.4 and section 2.5, we will show in what way symbolic dynamics work on logistic function and Smale's Horseshoe map. In theorem 2.5.3.3, we will establish that Cantor map or big tent map is chaotic. We know that conjugacy (topological) and symbolic dynamics are an exclusive incorporation of tools of dynamical systems. This chapter gives some important examples of conjugacy between different maps.

2.2 Basic Concepts

Symbolic dynamical systems are suitable for exalted generalization and abstraction of the original dynamical systems based on the topological conjugacy between the continuous manifestations of the dynamical systems. When the original dynamical systems are hard to be resolved, symbolic dynamics can provide a hopeful direction. Symbolic dynamical systems are space of sequences $\sum_2$={s: s =($s_0s_1s_2$......), $s_i =$ 0 or $s_i = 1$ for all i} together with the shift map defined on it. Symbolic dynamics are trembling with maps on sets.

German mathematician Georg Cantor in 1883 was made famous by introducing the Cantor set in his works of mathematics. The ternary Cantor set [86] is the most well-known of the Cantor sets and can be best described by its construction. This set starts with the closed interval zero to one and is constructed in iterations. The first iteration requires deleting the middle third of this interval, and the second iteration will delete the middle third of each of these two remaining intervals. These iterations continue in this fashion infinitely. Finally, the ternary Cantor set is described as the intersection of these intervals. This set is particularly interesting because its unique properties are uncountable, closed, length of zero, and more. A more general Cantor set is created by taking the intersection of iterations that remove any middle portion during each iteration.

We are giving below some important definitions which are essential for this chapter.

Definition 2.2.1 (Shift map):

The map $\sigma: \sum_2 \to \sum_2$ is defined by

$$\sigma(s_i) = s_{i+1}$$

that is $\sigma(s_0 s_1 s_2 \ldots\ldots) = (s_1 s_2 s_3 \ldots \ldots)$ is called the shift map.

Definition 2.2.2 (Cantor set):

A non-empty set C is said to be a Cantor set provided it is (i) perfect, (ii) totally disconnected, and (iii) compact.

Definition 2.2.3 (Cantor's Middle-Thirds Set):

From $I = [0, 1]$, delete the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Again delete the middle thirds, i.e., the pair of open intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from the remaining part, which is the closed interval $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. We continue removing middle thirds in this fashion infinitely many times. The remaining set is called Cantor's middle thirds set [88]. At the kth step, the total length of the 2^k closed intervals is $\left(\frac{2}{3}\right)^k$ which tends to zero as $k \to \infty$. If we go up to n times iterations, we can write

$$C_n = \left[0, \frac{1}{3^n}\right] \cup \left[\frac{2}{3^n}, \frac{3}{3^n}\right] \cup \ldots . \cup \left[\frac{3^n-3}{3^n}, \frac{3^n-2}{3^n}\right] \cup \left[\frac{3^n-1}{3^n}, 1\right], \text{where } n \geq 0.$$

Hence, we get, the Cantor middle $\frac{1}{3}$ set is $C = \cap_{n=0}^{\infty} C_n$.

2.3 Symbolic Dynamics for Quadratic family

Symbolic Dynamics is the most potential tool for understanding the chaotic behavior of the dynamical system. In this section, we want to show that quadratic map $f_c(x) = x^2 + c$ is chaotic on its invariant set.

We get the following two roots by solving the equation $x^2 + c - x = 0$:

$$r_1 = \frac{1}{2}(1 - \sqrt{1 - 4c})$$

$$r_2 = \frac{1}{2}(1 + \sqrt{1 - 4c})$$

Let $I = [-r_2, r_2]$ and $x_0 \notin I \Rightarrow x_n \to \infty$. Now let the invariant set of the quadratic map, $\Lambda = \{x \in I |\, f_c^n(x) \in I\ \forall\, n\}$. Consider $A_1 = (-\sqrt{-c - r_2}, \sqrt{-c - r_2})$ is the open interval of I that contains all the points x_0 such that $x_1 < -r_2$; that is the orbit of x_0 under $f_c(x)$ escapes to infinity after one iteration. A_1 divides I into two disjoint closed subintervals I_0 and I_1.

Definition 2.3.1 (Itineraries)

Let $x_0 \in \ \Lambda \subset I_0 \cup I_1$. Then$\forall\, n$, we can write $x_n = f_c^n(x_0) \in I_0 \cup I_1$.

Definition 2.3.2

The itinerary of x_0 is the sequence $S(x_0)$ of 0's and 1's given by

$$S(x_o) = (s_0 s_1 s_2 \ldots s_n \ldots) \text{ such that } s_n = \begin{cases} 0 & \text{if } x_n \in I_0 \\ 1 & \text{if } x_n \in I_1 \end{cases}.$$

The itinerary of x_0 is a simplified, symbolic representation of the orbit of x_0 under f_c.

2.3.1 Proposition (Density)

Suppose a continuous map $F: X \to Y$ is onto. If D is a dense subset of X, We can say $F(D)$ is a dense subset of Y.

The following theorem shows that the quadratic map $f_c(x)$ is chaotic on its invariant set.

Theorem 2.3.1: The quadratic map $f_c(x)$ is chaotic on its invariant set Λ for $c < -\frac{5+2\sqrt{5}}{4}$.

Proof: Using the given condition c the itinerary map $H^{-1}: \sum_2 \to \Lambda$ is a conjugacy, then $H^{-1}: \sum_2 \to \Lambda$ is a homeomorphism.

Now $H^{-1}(h)$ is a periodic point in Λ if and only if h is a periodic point in $\sum_2$. Now using the Density Proposition 2.3.1, H^{-1} maps the dense set of periodic points for σ in $\sum_2$ to a dense set of periodic points for f_c in Λ.

Secondly, since orbit of $\hat{h}$ under σ is dense in $\sum_2$, the Density Proposition 2.3.1 ensures that the orbit of $H^{-1}(\hat{h})$ under f_c is also dense in Λ. It means that f_c is also transitive.

Now we need to show that f_c is sensitive to initial conditions. I_0 and I_1 are the closed disjoint subintervals of $I = [-r_2, r_2]$, produced by discarding all the points in the open interval A_1. Let β be the length of the interval A_1. Now let $x, y \in \Lambda, x \neq y$.

Since H is bijective $H(x) \neq H(y)$, and there is a k such that the k-th entries of $H(x)$ and $H(y)$ differ. It means that both $f_c^k(x)$ and $f_c^k(y)$ are not in the same interval, I_0 or I_1. Consequently, $|f_c^k(x) - f_c^k(y)| \geq \beta$. Therefore, the orbit of y under f_c for any $y \neq x$ eventually separates from the orbit of x under f_c by at least β.

Hence $f_c(x)$ is chaotic on the set Λ.

2.4 Symbolic Dynamics for the Logistic Function:

Let $F_\mu(x) = \mu x(1-x)$ and $\Lambda = \{ x | F^n(x) \text{ is in} [0,1] \ \forall\, n \}$. In this section, we will see that for $\mu > 4$, the non-wandering set of the logistic function is a Cantor set.

Here we solve a problem about the Cantor set of Logistic Functions.

Problem 2.4.1 (Cantor set of the logistic map):

Consider the logistic map $F_\mu = \mu x(1-x)$ on $I = [0,1]$, where $\mu > 4$.

Note that in this case $F_\mu\left(\frac{1}{2}\right) > 1$. Since $F_\mu(0) = 0$, we can write using intermediate value theorem, $\exists \alpha_0 \in \left(0, \frac{1}{2}\right)$ such that $F_\mu(\alpha_0) = 1$. Since F_μ is monotone on $\left[0, \frac{1}{2}\right]$, the interval $I_0 = [0, \alpha_0]$ of all points the left of $\frac{1}{2}$ where $F_\mu(x) \in I$. Similarly, there exists $\alpha_1 > \frac{1}{2}$ with $F_\mu(x)$ such that $F_\mu(x) \in I, \forall x \in I_1 = [\alpha_1, 1]$ [See Figure 2.4.1 (a) and 2.4.1 (b)].

Consider $A_1 = I_0 \cup I_1$.Then $A_1 = \{x \in I: F_\mu(x) \in I\}$.

In the same way, we can write that A_2 consists of the four closed intervals $A_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$ (See Figure 2.4.2) where

$I_{00} = \{x: x \in I_0 \text{ and } F_\mu(x) \in I_0 \}$,

$I_{01} = \{x: x \in I_0 \text{ and } F_\mu(x) \in I_1\}$,

$I_{11} = \{x: x \in I_1 \text{ and } F_\mu(x) \in I_1\}$,

$I_{10} = \{x: x \in I_1 \text{ and } F_\mu(x) \in I_0\}$.

Continuing this process, we construct $A_n = \cup I_{s_0 s_1 \dots s_{n-1}}$, where s_i is either 0 or 1, and

$$I_{s_0 s_1 \dots s_{n-1}} = \left\{ x \in I : x \in I_{s_0}, F_\mu(x) \in I_{s_{1,\dots,}} F_\mu^j(x) \in I_{s_j} \right\}$$
$$= \cap_{k=0}^{j} F_\mu^{-k}(I_{s_k})$$
$$= I_{s_0} \cap F_\mu^{-1}(I_{s_0 s_1 \dots s_j})$$

We first note that $A_n = \{x \in I : F_\mu^n(x) \in I\}$. Furthermore,

$$I_{s_0 s_1 \dots s_{n-1}} = I_{s_0 s_1 \dots s_{n-1}} \cap F_\mu^{-n}(I_{s_n}) \subset I_{s_0 s_1 \dots s_{n-1}}$$

Hence $A_{n+1} \subset A_n$. Define the set

$$\Lambda = \bigcap_{n=1}^{\infty} A_n$$

Now, we have to show that Λ is a Cantor set.

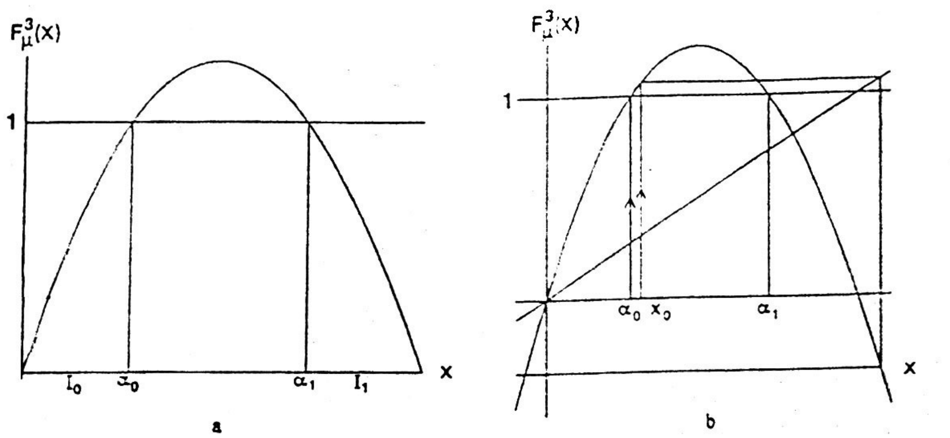


Figure 2.4.1: $A_1 = I_0 \cup I_1$. If $x_0 \notin I$.

We begin this task by computing points $\alpha_0 \in I_0$ and $\alpha_1 \in I_1$. After solving $\mu x(1-x) = 1$, we get,

$$\alpha_0 = \frac{1}{2} - \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}$$

and $\alpha_1 = \frac{1}{2} + \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}$.

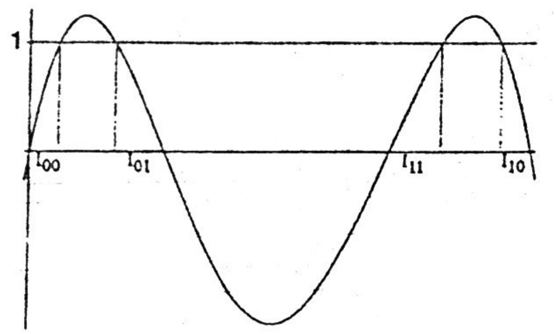


Figure 2.4.2: $A_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$

To establish that Λ is a Cantor set, we now assume that $\mu > 2 + \sqrt{5}$. We need to show that Λ is closed, totally disconnected, and perfect set:

(i) Λ is closed. We observed that A_0 is open. Furthermore, we know $A_{n+1} = F_\mu^{-1}(A_n)$. F_μ is a continuous function, so the pre-image of an open set is open.
Hence, $A_1 = F_\mu^{-1}(0)$ is open, and by induction, every A_n is open. So, $\cup_{n=1}^{\infty} A_n$ is open. Its complement, $I - \cup_{n=1}^{\infty} A_n = \Lambda$ is thus closed.

(ii) Λ is totally disconnected. We know that $|F_\mu'(x)| > 1$ for $\mu > 2 + \sqrt{5}$ and $x \in I - A_0$. There exists $\lambda > 1$ such that $|F_\mu'(x)| > \lambda > 1 \forall x \in \Lambda$. Then $|F_\mu^n(x)| > \lambda^n$ by the chain rule. Assume $\exists\, x, y \in \Lambda$ such that $x \neq y$ that form a closed interval $[x, y] \subset \Lambda$. Then $|F_\mu^n(\alpha)| > \lambda^n \;\forall\, \alpha \in [x, y]$. We can choose n in such a way that $\lambda^n |x - y| > 1$. Then we can apply the Mean value theorem $|F_\mu^n(x) - F_\mu^n(y)| > \lambda |x - y| > 1$. This implies that the distance between $F_\mu^n(x)$ and $F_\mu^n(y)$ is larger than 1, and thus at least one of them must be outside of I. This is contradicting with $x, y \in \Lambda$. This implied that x and y can never I. Therefore, there are no intervals in Λ, and it must be totally disconnected.

(iii) Λ is perfect. Λ set is perfect if all its point is limit points. So, we have to prove that $\forall x_0 \in \Lambda$ and all $\varepsilon > 0$, there is a $y \in \Lambda$ such that $x_0 \neq y$ and $|x_0 - y| < \varepsilon$. Now using I_0 and I_1 where $I_0 = [0,\ \frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}]$ and $I_1 = \left[\frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, 1\right]$, the restrictions $F_{\mu/I_0}: I_0 \to [0, 1]$ and $F_{\mu/I_1}: I_1 \to [0, 1]$ are homeomorphisms, and thus, there exist inverse maps $h_0: I_0 \to [0, 1]$ and $h_1: I_1 \to [0, 1]$ such that $x = F_\mu(h_1(x))$. The orbits of $x \in \Lambda$ never leave [0,1], so if $x \in \Lambda$, we know that $h_0(x), h_1(x) \in \Lambda$.

We know that $x_{n+1} = F_\mu(x_n)$ so $x_n = h_0(x_{n+1})$ or $x_n = h_1(x_{n+1})$ depending on whether x_n is in I_0 or I_1. There exists $\lambda > 1$ such that, for $a, b \in \Lambda$:

$$|h_0(a) - h_0(b)| \leq \frac{1}{\lambda}|a - b| \text{ and } |h_1(a) - h_1(b)| \leq \frac{1}{\lambda}|a - b|.$$

Now we can write $x_0 = h \circ h \circ \dots \dots \circ h$. Now we choose $y' = 0$, which is a fixed point in Λ. For any $\varepsilon > 0 \exists$ as n such that $\frac{1}{\lambda^n} < \varepsilon$. Then for some $y \neq x_0$ converging to y' we find that $|x_0 - y| \leq \frac{1}{\lambda^n}|x_n - y| \leq \frac{1}{\lambda^n} < \varepsilon$, which completes the proof.

Although this is true for $\mu > 4$, the proof becomes very much involved for $4 < \mu \leq 2 + \sqrt{5}$.

2.5 Symbolic Dynamics and the Smale Horseshoe

The methodology of symbolic dynamics was used as far back as Hadamard in 1898. Symbolic dynamics grew into its own discipline in the 1930's with the works of Birkhoff, Morse, and Hedlund which incidentally inspired Smale to construct his horseshoe map. In this section we establish that the horseshoe map f, is chaotic on its invariant set.

2.5.1 The Bernoulli shift map

The phase space is given by

$\sum_2$ = {bi-infinite sequences of 0's and 1's}

Thus, any element of $\sum_2$ is of the form $s = (\dots s_{-n} \dots s_{-1}.s_0 s_1 \dots s_n \dots)$, such that $s_1 = 0$ or $s_i = 1$, for all $i \in \mathbb{N}$. In this notation, the "." is used to denote the central element in the sequence. The phase space $\sum_2$ becomes a normed space with the norm

$$d(s, \bar{s}) = \sum_{i=-\infty}^{\infty} \frac{|s_i - \bar{s}_i|}{2^{|i|}}$$

Now, we define the Bernoulli shift map:

$$\alpha(s) = (\dots s_{-n} \dots s_{-1}.s_0 s_1 \dots s_n \dots)$$

Thus, the shift map moves all elements to the left, one position:

$(\alpha(s))_i = s_{i+1}$.

Clearly, the shift map is invertible.

Theorem 2.5.1.1: The Bernoulli shift map α on $\sum_2$ has

(i) countably infinitely many periodic orbits,

(ii) uncountably many non-periodic orbits, and

(iii) s dense orbit.

Proof: (i) Consider any $s \in \sum_2$ which consists of a repeating sequence of digits. Suppose that

$s_{i+k} = s_i$, $i \in \mathbb{Z}$, for some positive integer k.

Then $\alpha^k s = s$ which means that s is a k-periodic point of α. The collection of all such points is countable. Indeed, they may be enumerated as

$(.\overline{0}), (.\overline{1})$,

$(.\overline{01})$,

$(.\overline{001}), (.\overline{011})$,

$(.\overline{0001}), (.\overline{0011}), (.\overline{0111})$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

(ii) Any sequence s that does not repeat gives rise to a non-periodic orbit. Such sequences are uncountable; consider all real numbers between 0 and 1. These numbers may be represented in binary form. Those numbers whose binary from repeats are rational numbers. The others are irrational. We know that irrational numbers are a full measure uncountable set. It follows that αhas an uncountable number of non-periodic orbits.

(iii) Consider $s =$ concatenation of the root part of all periodic orbits.

Then s is dense in s, by construction: it will approach any sequence arbitrarily close, as all central parts occur somewhere down the line in s.

The following theorem shows that the Bernoulli shift map α is sensitive dependence on initial conditions, in $\sum_2$.

Theorem 2.5.1.2 The Bernoulli shift map α is sensitively dependent on initial conditions, in $\sum_2$.

Proof: Consider $s, \overline{s}$ be two points such that

$$d(s, \bar{s}) < \varepsilon$$

for $\varepsilon > 0$. Otherwise, s and $\bar{s}$ are arbitrary. It implies that the central digits of s and $\bar{s}$ agree, but it puts no constraint on their digits sufficiently far from the center. It follows that a large enough number of shifts will make this difference arbitrarily large. This concludes the proof.

2.5.2 Smale's Horseshoe:

Stephen Smale [35] introduced the horseshoe map at the time of discussion of the behavior of the orbits of the van der Pol oscillator. The action of map is defined geometrically. Firstly it squeezing to square, then extending into a long strip, and lastly folding the strip into the shape of a horseshoe. Squeezing and stretching are uniforms in case of horseshoe map.

Smale's horseshoe is a naturalistic, higher dimensional version of the open binary shifts. Let us consider a map of rectangular set into a "Horseshoe" shaped set covering entire breadth of the original in two places, then the set surviving for infinite time is a Cantor set in the unstable direction and smooth in the stable direction. It will be also a Cantor set and labeled by the shift on the full space, if the set staying infinite times in the positive and negative direction, which will be intersection of two Cantor sets [86]. Also, this is structurally stable.

Before we get started, it should be pointed out that it is possible to give an analytical description of the Smale's Horseshoe, just like we'll provide a geometrical one. We will stick to the geometrical picture.

Let us define a map f by,

$$f : D \rightarrow D : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] .$$

It is possible to give an analytical description of this map, but the illustrations Figure 2.5.1 may be more useful for our purposes. The map f is referred to as Baker's map. It is a composition of three elementary steps: (a) a stretching step, where the unit square is stretched to a long vertical strip; (b) a folding step, where this long vertical strip is bent in the shape of a horseshoe; and (c) an overlay step, where the map is restricted to its original domain.

The inverse map is remarkably similar: it also consists of a composition of three elementary steps, given by (a) a stretch, where the unit square is horizontally stretched; (b) a folding step, where a horseshoe is created, once again; and (c) an overlay step where the result is restricted to the unit square. All of this is illustrated in Figure 2.5.2.

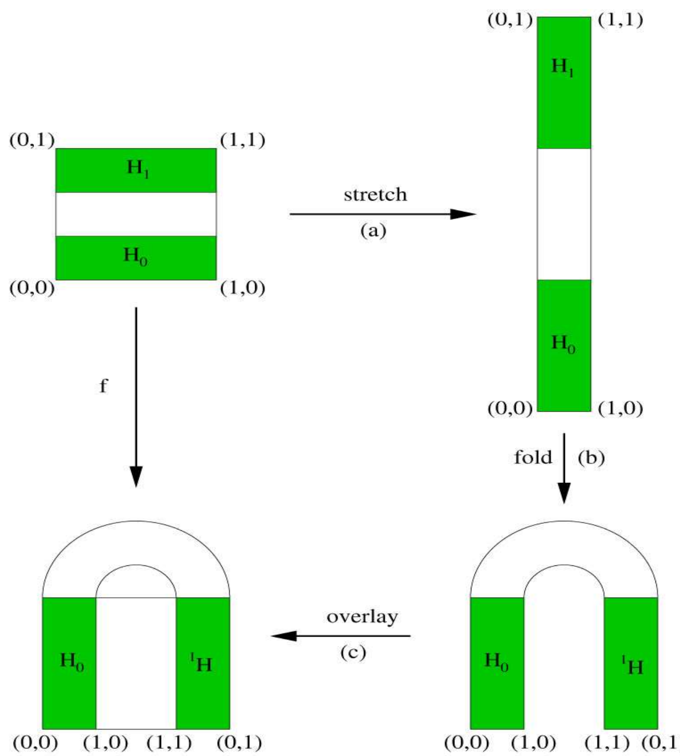


Figure 2.5.1: The action of Baker's map

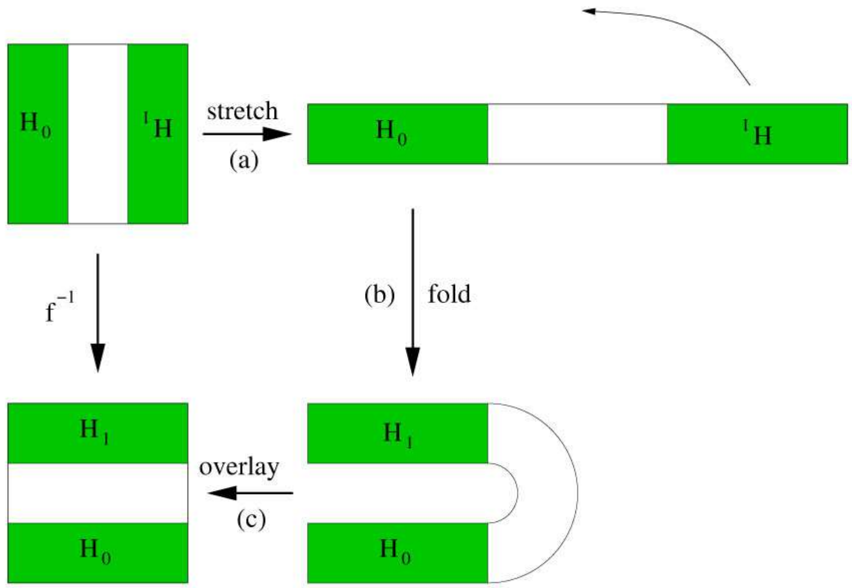


Figure 2.5.2: The action of the Baker's inverse map

2.5.2.1 The Forward iterates:

The first few forward iterates of the map are shown in Figure 2.5.3. It follows from this figure that all points in the invariant set are confined to a middle-third Cantor set collection of vertical line segments. We associate an address with each of the iterates, as shown in Figure 2.5.3.

Here's how the assigning of the address is done:

After one iteration, the left strip is assigned a zero, the right strip is a one.

After two iterations, we transport addresses assigned at the previous iteration to their new location. In other words, the left-most strip inherits a zero, as does the right-most strip, since they both originate from the previous left strip. The other two strips inherit a one, as they originate from the last right strip. We modify these inherited addresses: the two on the left get an extra zero (since they are at the left; the zero is added to the front). The two on the right get an extra one (since they are at the right; the one is added to the front).

After three iterations, we repeat this: after using the map again, all new strips inherit their old address, taking into account where they originate from. The strips left of the center get an extra zero tagged to their front; the strips right of the center get an extra up front.

The number of digits in the address equals the number of iterates of f. It should also be clear that an address uniquely determines which vertical line we mean.

If we consider an infinite number of forwarding iterates, then in the limit, every strip has associated with it an infinite sequence of zero's and one's.

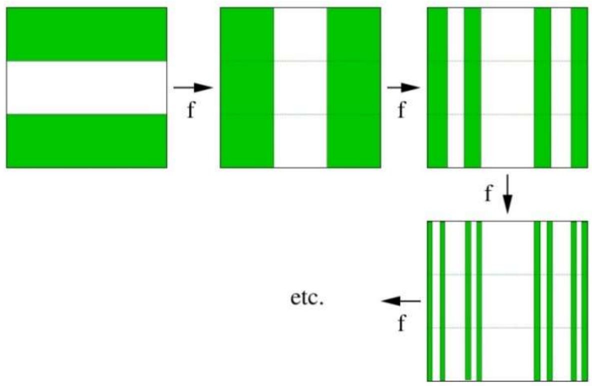


Figure 2.5.3: Several forward iterates of the Baker's map

2.5.2.2 The backward iterates:

This part is very similar to the previous part. The first few backward iterates of the map are shown in Figure 2.5.4. It follows from this figure that all points in the invariant set are confined to a middle-third Cantor set collection of horizontal line segments. With each of the iterates, we associate an address, as shown in Figure 2.5.4

Here's how the assigning of the address is done:

After one iteration, the bottom strip is assigned a zero, the top strip is a one.

After two iterations, we transport the address assigned at the previous iteration to their new location. In other words, the bottom-most strip. The other two strips inherit a one, as they originate from the previous top strip. We modify these inherited addresses: the two on the bottom get an extra zero (since they are at the bottom; the zero is added to the front). The two on top get an extra one (since they are at the top; the one is added to the front).

After three iterations, we repeat this: after using the map again, all new strips inherit their old address, taking into account where they originate from. The strips below center get an extra zero tagged to their front; the strips above center get an extra one up front.

The number of digits in the address is equal to the number of iterates of f^{-1}. It should also be clear that an address uniquely determines which horizontal line we mean. If we consider an infinite number of backward iterates, then in the limit, every strip has associated with it an infinite sequence of zero's and one's.

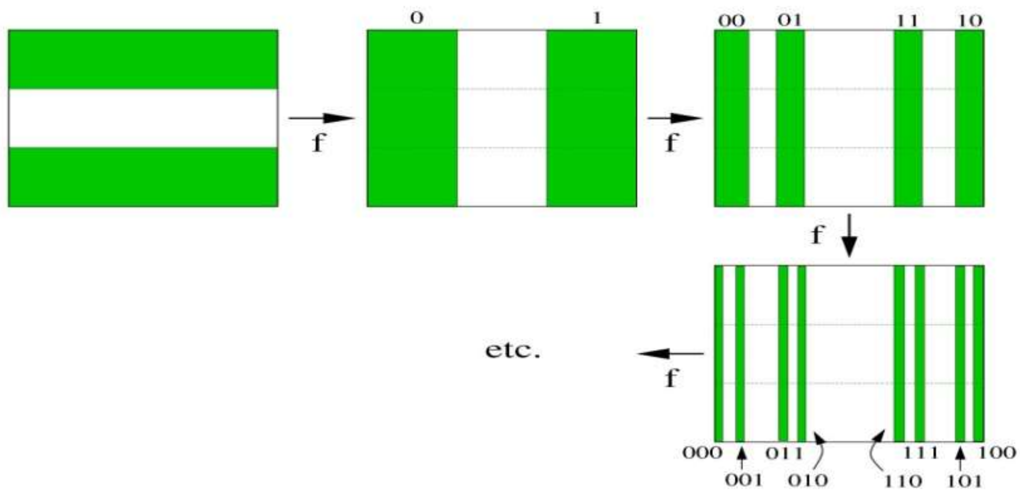


Figure 2.5.4: Associate an address with different forward iterates of the Baker's map

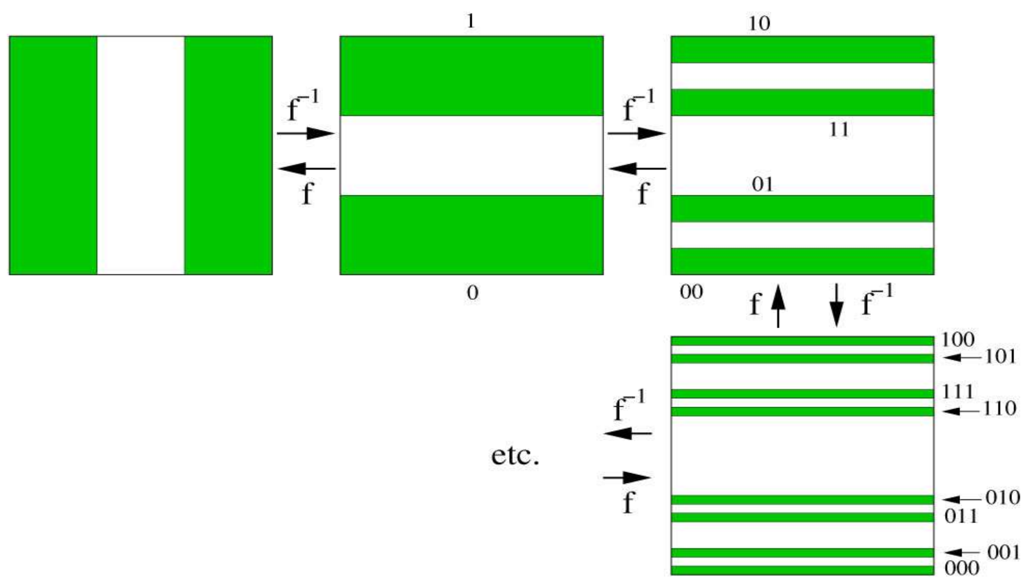


Figure 2.5.5: Associate an address with different, forward iterates of the Baker's map

2.5.2.3 The invariant set

The invariant set is the intersection of the forward and backward invariant sets. It consists of the intersection of two Cantor line sets, one horizontal and one vertical. Thus, the invariant set is a point set with an uncountable number of entries. We should remember that the points of the invariant set are not fixed points of the map f. Rather, they belong to a set of points which is invariant under the action of f, that is, one point of the invariant set is mapped to another, under either a forward or backward iterate of f.

Taking the addresses, we have constructed for the forward and backward iterates, we may assign a unique bi-infinite address with every point P in the invariant set. We do this as follows: let

$$V_{s_{-1}s_{-2}\ldots\ldots}$$

denotes a unique vertical line. Similarly, let

$$H_{s_0 s_1 s_2\ldots\ldots}$$

denotes a unique horizontal line. Their intersection point P is assigned the address

$$\sigma(P) = (\ldots\ s_{-2}s_{-1}.s_0 s_1 s_2 \ldots).$$

As before, we use the '.' to denote the central element. Thus, with every address σ there corresponds a unique point P in the invariant set, and for every point in the invariant set, we have a unique address. In other words, through the assigning of addresses, we may identify the invariant set of the Baker's map with $\sum_2$, the phase space for the Bernoulli shift map.

Theorem 2.5.2.1: The restriction of the Baker's map to its invariant set satisfies

$$\alpha(\sigma(P)) = \sigma(f(P)),$$

where α is the Bernoulli shift map, P is a point of the invariant set, and σ is an address assigned as described above.

Proof: Since the Bernoulli shift map only shifts one digit, it suffices to consider the digits of $\sigma(P)$ within 2 positions of the central marker. This means that we have to consider a finite number of possibilities in effect. Again, we proceed by doing this graphically. Since we only worry about the central part of the addresses, we can consider the first two layers of the invariant set, as shown in Figure 2.5.6

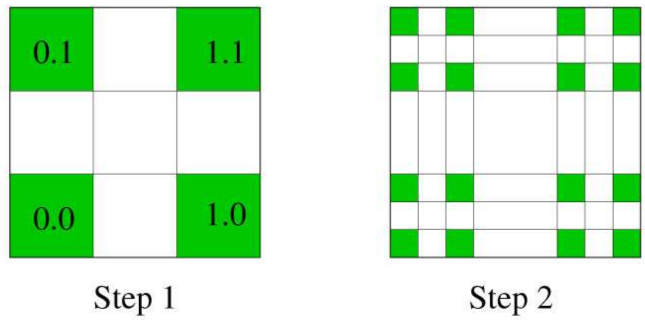


Figure 2.5.6: The first two layers, with some addressing, of the invariant set of f.

Let us check this for the red small square indicated in Figure 2.5.7. The central part of the address of any point in this square is $\sigma\ (P) = (\dots 00\ .01) \dots)$. Thus

$$\alpha(\sigma(P)) = (\dots 000\ .1 \dots).$$

On the other hand, Baker's map f maps the small red square to the stretched blue rectangle. The addresses of all points on the invariant set in the blue rectangle are

$$\sigma(f(P)) = (\dots 0\ .1 \dots),$$

which agrees with the action of the shift map.

The reasoning is very similar to that for the first case. Consider the small red square. The central part of the address of any point in this square is $\sigma(P) = (\dots 00\ .10 \dots)$.

Thus, $\alpha(\sigma(P)) = (\dots 001\ .0 \dots).$,

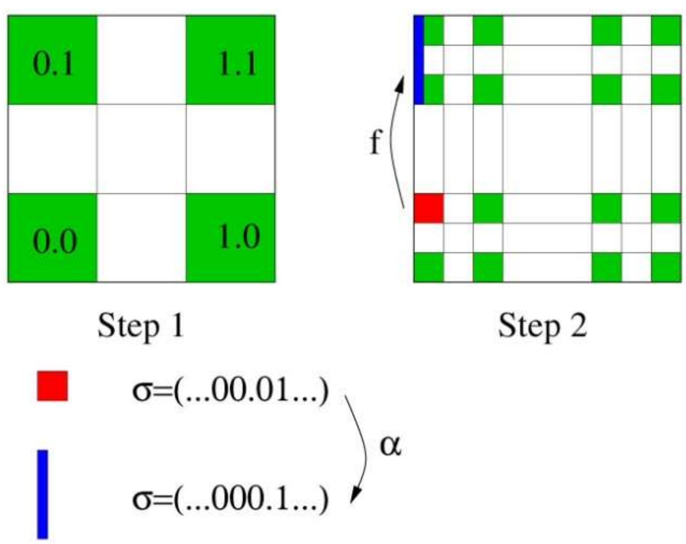


Figure 2.5.7: Checking the first of 16 cases

On the other hand, Baker's map f maps the small red square to the stretched blue rectangle. The addresses of all points on the invariant set in the blue rectangle are

$$\sigma(f(P)) = (\dots 1\,.0 \dots),$$

which agrees with the action of the shift map.

As we said, there is 14 more cases to check.

We immediately conclude that the Baker's map is chaotic, restricted to its invariant set. This is also a justification for why we investigated the Bernoulli shift map in the first place. It was pretty easy to prove that it was chaotic. Now we have shown that Baker's map restricted to its invariant set is chaotic. This has huge consequences for what follows: in the sense of this construction, the presence of a horseshoe in a dynamical system is enough to consider the map to be chaotic, as least on some part of its phase space.

We show the action of the Smale horseshoe [36], which is based on the Baker's map, the Bernoulli shift map has countably infinitely many periodic orbits, uncountably many non-periodic orbits, and σ dense orbit on $\sum_2$. We also proved that the Bernoulli shift map is sensitively dependence on initial conditions in $\sum_2$.

Theorem 2.5.2.2: The Horseshoe map f, is chaotic on its invariant set.

Proof: Let $P \in \wedge$. The itinerary of P is the bi-infinite sequence of 0's and 2's given by

$$\varphi(P) = (\dots\, s_{-2}s_{-1}.\, s_0 s_1 s_2 \dots)$$

where $s_i = k$ if $f^i(P) \in V_k$, $i \in Z$, $k = [0,1]$.

Since each horizontal line intersects with a specific vertical line at a unique point, P then there is a well-defined map from points in the invariant set to bi-infinite sequences of 0's and 1's called φ. The decimal point in the bi-infinite sequence separates the past iterates from the future iterates. Thus, we can find the sequence for an associated $f^k(P)$ by shifting the decimal point until it is immediately to the left of S_k.

Now, we need to prove that the map $\varphi{:}\wedge\rightarrow \sum_2$ is a homeomorphism.

(i) φ is one-to-one: Suppose $P, P' \in \wedge$, where $P \neq P'$, and $\varphi(P) = \varphi(P') = \{\dots\, s_{-n} \dots s_{-1}.\, s_0 \dots s_n \dots\}$. But we defined $P \in \wedge$ as being the unique point resulting from the intersection of a vertical line and a horizontal line. Since there can only be one point at an intersection with a uniquely associated bi-infinite sequence then $P = P'$, which is a contradiction.

(ii) φ is onto: It was expressly set up so that each vertical line is associated with a specific sequence of 0's and 1's $\{.s_0 s_1 s_2 \dots s_n\}$. Each horizontal line is also associated with a specific sequence of 0's and 1's $\{\dots\ s_{-n} \dots s_{-1}.\}$. Concatenating the sequences together gives us a unique bi-infinite sequence corresponding to a unique point in the invariant set proving onto.

(iii) φ is continuous: Let $P \in \wedge$ and $\varepsilon > 0$ be given. There is a $\delta > 0$,
$|P - P'| < \delta \Rightarrow d(\varphi(P), \varphi(P')) < \varepsilon$. In order to have $d(\varphi(P), \varphi(P')) < \varepsilon$, there must be an $n \in N$ such that if $\varphi(P) = \{\dots\ s_{-n} \dots s_{-1}.s_0 \dots s_n \dots\}$ and
$\varphi(P') = \{\dots s'_{-n} \dots s'_{-1}.s'_{-0} \dots s'_{-n} \dots\}$ for $s_i = s'_i, i = 0, \pm 1, \dots, \pm N, N \in Z$. Based on the construction of $\wedge$, this means that P and P' lie in the same rectangular region created by the intersection of a horizontal and vertical rectangle after some iteration of the horseshoe map. The width and height of such a rectangular region would be λ^{N+1} and $\frac{1}{\mu^N}$ to ensure continuity.

Hence φ:$\wedge\rightarrow$ $\sum_2$ is a homeomorphism. So φ is a conjugacy.

Suppose a continuous map $F: X \rightarrow Y$ is onto. If D is a dense subset of X, We can write $F(D)$ is a dense subset of Y. So, the periodic points of the horseshoe map are dense and that the mapping also has a dense orbit, which is equivalent to transitivity, satisfying the first two criteria of Devaney's definition. The only thing remaining to prove is sensitive dependence.

Choose $\beta > 0$ that is smaller than the minimum distance between the closed, disjoint intervals, V_0 and V_1. Since all orbits of our horseshoe mapping, regardless of how close the seeds may be, are unique, they will eventually separate by at least β proving sensitive dependence.

Hence horseshoe map f, is chaotic on its invariant set.

2.5.3 A specific Horseshoe Map

In the general horseshoe case, points in a two-dimensional plane were represented as bi-infinite sequences in sequence space. In this specific case, the height will be contracted by one-third its original height, and the width will be expanded by three times the actual width before wrapping it back around into the unit square D.

Definition 2.5.3.1:

Setting the origin as the bottom left corner of a unit square then the horseshoe map, H described above is,

$$H(x,y) = \begin{cases} \left(3x, \frac{1}{3}y\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \left(3-3x, 1-\frac{1}{3}y\right) & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.5.3.1)$$

The inverse is horizontally shrinking the unit square to one-third of its original width. It is stretched to three its actual height followed by bending itself back into the unit square.

$$\text{Thus, } H^{-1}(x,y) = \begin{cases} \left(\frac{1}{3}x, 3y\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \left(1-\frac{1}{3}x, 3-3y\right) & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.3.2)$$

We see that the contracting / expanding of the Smale horseshoe map in the specific case can be modeled by the Cantor middle-thirds set via ternary numbers. Since the invariant set is equivalent to the Cantor set and future iterates remove all one's from their ternary expansions, one would still end up with a bi-infinite sequence of 0's and 2's.

Definition 2.5.3.2:

Let $y = 0.y_1y_2y_3\ldots$ in ternary [36] as defined for x and denote $\bar{y}$ by $\bar{y} = 2 - y_i$. That is, multiplying by 3 shifts the decimal one place to the right and inversely multiplying by $\frac{1}{3}$ shifts the decimal one place to the left.

$$H(0.x_1x_2x_3\ldots, 0.y_1y_2y_3\ldots) = \begin{cases} (0.x_2x_3x_4\ldots, 0.0y_1y_2\ldots) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ (0.\bar{x}_2\bar{x}_3\bar{x}_4\ldots, 0.2\,\bar{y}_1\bar{y}_2\ldots) & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

$$= \begin{cases} 0.x_2x_3x_4\ldots, 0.0y_1y_2\ldots & \text{if } x_1 = 0 \\ 0.\bar{x}_2\bar{x}_3\bar{x}_4\ldots, 0.x_1\bar{y}_1\bar{y}_2\ldots & \text{if } x_1 = 2 \end{cases} \quad \ldots\ldots\ldots (2.5.3.3)$$

It becomes necessary that the second component of the specific Horseshoe map be defined in terms of the first component. If this can be accomplished, then $H(x,y)$ can be proven to be chaotic directly on the space defined by the Cantor set.

The first three iterates of $H(x, y)$ indicate a pattern that accomplishes this goal.

First iterate: $H(0.x_1x_2x_3 \ldots, 0.y_1y_2y_3 \ldots \ldots) = \begin{cases} (0.x_2 \ldots, 0.x_1y_1 \ldots) & \text{if} \quad x_1 = 0 \\ (0.\bar{x}_2 \quad \ldots, 0.x_1\bar{y}_1 \quad \ldots) & \text{if} \quad x_1 = 2 \end{cases}$

Second iterate: $H^2(0.x_1x_2x_3 \ldots, 0.y_1y_2 \ldots) = \begin{cases} (0.x_3 \ldots, 0.x_1x_2y_1y_2 \ldots) & \text{if } x_2 = 0 \\ (0.\bar{x}_3 \quad \ldots, 0.\bar{x}_1\ x_2\bar{y}_1\bar{y}_2 \ldots) & \text{if } x_2 = 2 \end{cases}$

Third iterate: $H^3(0.x_1x_2x_3x_4 \ldots, 0.y_1y_2y_3 \ldots) =$

$$\begin{cases} (0.x_4 \ldots, 0.x_1x_2x_3y_1y_2y_3 \ldots) & \text{if } x_3 = 0 \\ (0.\bar{x}_4 \quad \ldots, 0.x_2\bar{x}_1\ x_3\bar{y}_1\bar{y}_2\bar{y}_3 \ldots) & \text{if } x_3 = 2 \end{cases}$$

We observe that there are 2^n horizontal bands following n iterations that are now $\frac{1}{3^n}$ high. We can see the following the first iteration that both components of $H(x, y)$ are the same or, specifically, x_1 as seen in equation (2.5.3.3). Upon further iterations, we observe that there are 2^n vertical bands with 2^n-1 inner thirds removed. Now the horseshoe map is more of a snake map as it bends over itself, essentially 2^n-1 times. There are 2^n binary combinations of 0's and 2'sover n places past the decimal point, which can be assigned simply by ordering the bands by location.

Proposition 2.5.3.1:

$$H^n(x) = \begin{cases} 0.x_{n+1}x_{n+2} \ldots, 0.x_{n-1}x_{n-2} \ldots x_1x_ny_1y_2 \ldots & \text{if} \quad x_n = 0 \\ 0.\bar{x}_{n+1}\bar{x}_{n+2} \quad \ldots, 0.\bar{x}_{n-1}\bar{x}_{n-2} \ldots \bar{x}_1x_n\bar{y}_1\bar{y}_2 \ldots & \text{if} \quad x_n = 2 \end{cases} \quad \ldots\ldots(2.5.3.4)$$

Proof: The first component of the horseshoe map is just the iterative ternary map. However, remember the second component is dependent upon the first component so both components are needed in this proof. Let the second component be

$$W_n(x, y) = (0.x_1x_2x_3x_4 \ldots, 0.y_1y_2y_3 \ldots)$$
$$= \begin{cases} 0.x_{n-1}x_{n-2} \ldots x_1x_ny_1y_2 \ldots & \text{if } x_n = 0 \\ 0.\bar{x}_{n-1}\bar{x}_{n-2} \ldots \bar{x}_1x_n\bar{y}_1\bar{y}_2 \ldots & \text{if } x_n = 2 \end{cases}$$

Looking at the iterative horseshoe map, when $n = 1$, the original definition from equation (2.5.3.3) must be true. Assume that $W_k(x, y)$ is true for some $k > 1 \in N$ and shows that it is valid for $k + 1$. Concentrating on the second component of the horseshoe map,

$$W_{k+1}(x, y) = \begin{cases} W(0.x_{k+1}x_{k+2} \ldots, 0.x_{k-1}x_{k-2} \ldots x_1x_ky_1y_2 \ldots) \text{if } x_k = 0 \\ W(0.\bar{x}_{k+1}\bar{x}_{k+2} \ldots, 0.\bar{x}_{k-1}\bar{x}_{k-2} \ldots \bar{x}_1x_k\bar{y}_1\bar{y}_2 \ldots) \text{ if } x_k = 2 \end{cases} \ldots(2.5.3.5)$$

Solving Equation (2.5.3.5) for x_k and x_{k+1}, we have

(i) $x_k = 0$ and $x_{k+1} = 0$

(ii) $x_k = 0$ and $x_{k+1} = 2$

(iii) $x_k = 2$ and $x_{k+1} = 0$

(iv) $x_k = 2$ and $x_{k+1} = 2$

Applying all of this to equation (2.5.3.5) yields,

$$W_{k+1}(x,y) = \begin{cases} 0.x_{k+1}x_{k-1} \dots x_1 x_k y_1 \dots & \text{if } x_k = 0 \text{ and } x_{k+1} = 0 \\ 0.x_{k+1}\bar{x}_{k-1} \dots \bar{x}_1 \bar{x}_k \bar{y}_1 \dots & \text{if } x_k = 0 \text{ and } x_{k+1} = 2 \\ 0.\bar{x}_{k+1}x_{k-1} \dots x_1 \bar{x}_k y_1 \dots & \text{If } x_k = 2 \text{ and } x_{k+1} = 0 \\ 0.\bar{x}_{k+1}\bar{x}_{k+2} \dots \bar{x}_1 x_k \bar{y}_1 \dots & \text{if } x_k = 2 \text{ and } x_{k+1} = 2 \end{cases}$$

which can be simplified further to

$$W_{k+1}(x,y) = \begin{cases} 0.x_k x_{k-1} \dots x_1 x_{k+1} y_1 \dots & \text{if } x_{k+1} = 0 \\ 0.\bar{x}_k \bar{x}_{k-1} \dots \bar{x}_1 x_{k+1} \bar{y}_1 \dots & \text{if } x_{k+1} = 2 \end{cases}$$

Since this inductively proves that the second component of the Horseshoe map is correct, it is asserted with confidence that equation (2.5.3.4) is correct. Smale's Horseshoe map's iterated is now all expressed as ternary expansion with only 0's and 2's.

Proposition 2.5.3.2:

Let $p = (p_0 \dots p_n \dots) \in \sum$, $q = (q_0 \dots q_n \dots) \in \sum$ then $d(p,q) = \sum_{i=0}^{\infty} \frac{|p_i - q_i|}{3^{|i|}}$ is a metric on $\sum$.

Theorem 2.5.3.1: (Proximity theorem):

Let $p, q \in \Gamma$ and suppose $p_i = q_i$ for $i = 0,1,\dots,k$, then $d(p,q) \le \frac{2}{3^k}$. Conversely, if $d(p,q) < \frac{2}{3^k}$, then $p_i = q_i$ for $i \le n$.

Theorem: 2.5.3.2: The horseshoe map, H is chaotic.

Proof: (i) The horseshoe map, H has a dense set of periodic points.

Here the periodic points of the form

$x = 0.\overline{x_1 x_2 \dots x_{n-1} 0}$ and $y = 0.\overline{x_{n-1} x_{n-2} \dots x_1 0}$ or

$x = \overline{x_1 x_2 \dots x_{n-1} 2 \bar{x}_1 \bar{x}_2 \dots \dots \bar{x}_{n-1}\, 0}$ or $y = \overline{x_{n-1} x_{n-2} \dots x_1 2\, \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_1 0}$

Let $(p, q) \in \Gamma$, let $\varepsilon > 0$, and choose $r \in N$ such that $\frac{2}{3^{r+1}} < \varepsilon$. Choose a periodic point $x = 0.\overline{p_1 p_2 \dots p_r q_r q_{r-1} \dots q_1 0}$ and $y = 0.\overline{q_1 q_2 \dots q_r p_r p_{r-1} \dots p_1 0}$ with period $n = 2r + 1$. Since (x, y) agrees with (p, q) to r ternary places $\|(x, y) - (p, q)\| \leq \frac{2\sqrt{2}}{3^r} < \varepsilon$ proving that the horseshoe map has a dense set of periodic points.

(ii) The horseshoe map, H has a dense orbit.

Let $x = B_2 B_4 B_6 \dots$ contains all blocks of the form $p_1 p_2 \dots p_n 0 q_1 q_2 \dots q_n$ where the p's and q's are 0's and 2's. Let $p = 0.p_1 p_2 \dots, q = 0.q_1 q_2 \dots \in \Gamma$, let $\varepsilon > 0$, choose $y \in \Sigma$, choose $k \in N$ such that $\frac{2}{3^k} < \varepsilon$. There exists a block $q_k q_{k-1} \dots q_1 0\ p_1 p_2 \dots p_k$ in x where the middle zero is the n^{th} digit of x.

Then $H^n(x) = (0.p_1 p_2 \dots p_k, 0.q_1 q_2 \dots q_k \dots)$ and so $\|H^n(x, y) - (p, q)\| \leq \frac{2\sqrt{2}}{3^r}$. Thus, the orbit of (x, y) is dense.

(iii) Let $\beta > 0$ which is smaller than the minimum distance between the closed, disjoint intervals remaining after the middle third has been removed. Since all orbits of the tent map, regardless of how close the seeds may be, are unique, they will eventually separate by at least β proving sensitive dependence.

Hence the horseshoe map is chaotic.

Theorem 2.5.3.3: Cantor map or Big Tent map T, is chaotic.

Proof: The big tent map, given below, will be used to connect the Cantor set and the x dimensional of Smale's horseshoe map. In fact, it will be proven that the sequence space representing $x \in \Gamma$ actually is the Cantor set. They are utilizing the fact that the tent map is the Cantor set.

$$\text{Let } T: \mathbb{R} \to \mathbb{R} \text{ be } T(x) = \begin{cases} 3x & \text{if } x \leq \frac{1}{2} \\ 3 - 3x & \text{if } x > \frac{1}{2} \end{cases}$$

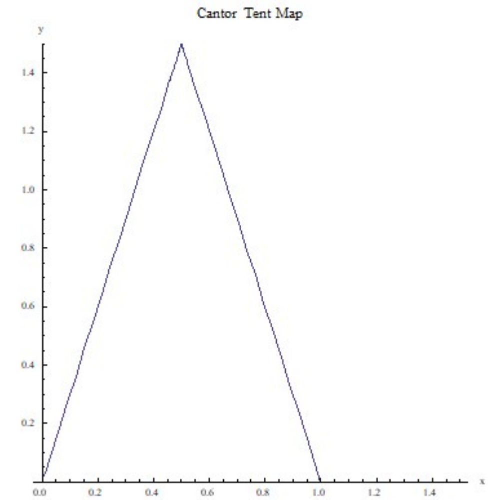


Figure: 2.5.8 Cantor Tent map acting on unit interval

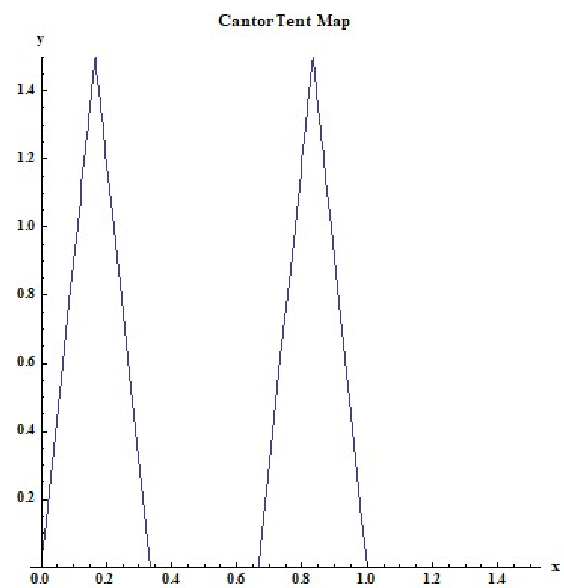


Figure: 2.5.9 Cantor Tent map for T^2 showing the middle third has been removed

Let all the points that aren't removed with future iterations of $T^i(x) \in [0,1]$ for $i \in N$ be Ω and let $T(x): \Omega \to \Omega$.

It is evident that $T(x)$ removes the inner third of the interval, following each iteration, provided $x \in [0,1]$thus yielding the Cantor set. Let $x = 0.x_1x_2x_3 \ldots \ldots$ and denote $\bar{x}$ by $\bar{x} = 2 - x$, then

$$T(x) = \begin{cases} x_1.x_2x_3 \ldots \text{if} \ \ 0 \leq x \leq \frac{1}{2} \\ \bar{x}_1.\ \bar{x}_2\,\bar{x}_3 \ldots.\ \text{if} \ 1 \geq x > \frac{1}{2} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.3.6)$$

Due to our mapping restrictions, $T(x)$ must be in [0,1], but clearly, this is not the case if $x_1 = 1$ or $\bar{x}_1 = 1$ thus,

$$T(x) = \begin{cases} 0.x_2x_3 \ldots \text{if} x_1 = 0 \\ 0.\bar{x}_2\bar{x}_3 \ldots \text{if} \ x_1 = 2 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.3.7)$$

Since all ternary expansions of the Cantor set have only '0' or '2' in it as all the ternary expansions with a 1 escape after some iterate, then the ternary expansion of the Cantor set is only comprised of '0's and '2' s, proves that all ternary expansions of the tent map also only contain '0's and '2'. In other words, $T^i(x) \in [0,1]$ if and only $x_i = 0$ or 2. This proves that $x \in \Omega$ and that Ω is the Cantor set showing that T maps Ω into itself.

Visually, a picture of Ω is that of a disconnected Cantor set (see Figure 2.5.10)

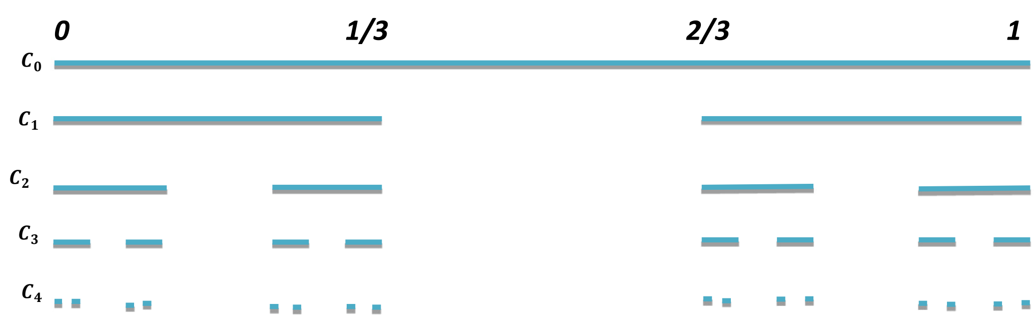


Figure 2.5.10: Cantor set

Now for $n \geq 1$, we assume that $T^n(0.x_1x_2 \ldots) = \begin{cases} 0.x_{n+1}x_{n+2} \ldots \text{if} \ \ x_n = 0 \\ 0.\bar{x}_{n+1}\bar{x}_{n+2} \ldots \ \text{if} \ \ x_n = 2 \end{cases}$

For proving tent map is chaotic, we need to verify that the following three properties:

(i) Tent map, T has a dense set of periodic points.

Following one iteration of $T^n(x)$ a point $x \in \Omega$ of period n can be defined as

$$0.x_1x_2 \ldots = \begin{cases} 0.x_{n+1}x_{n+2} \ldots \text{if} \ \ x_n = 0 \\ 0.\bar{x}_{n+1}\bar{x}_{n+2} \ldots \text{if} \ \ x_n = 2 \end{cases}$$

Hence, there are periodic points of the form

$$x = 0..\overline{x_1x_2 \ldots x_{n-1}0} \ \text{ or } x = .\overline{x_1x_2 \ldots x_{n-1}2\,\bar{x}_1\bar{x}_2 \ \ldots \bar{x}_{n-1}0}$$

Let $P = 0.p_1p_2 \dots \in \Omega$, $\varepsilon > 0$, and choose $k \in N$ such that $\frac{2}{3^k} < \varepsilon$. Choose a periodic point $x = 0.p_1p_2 \dots p_{k-1}0$ which yields $|x - P| \leq \frac{2}{3^k} < \varepsilon$ prove that the tent map has a dense set of periodic points.

(ii) Tent map has a dense orbit.

Let $x = A_1A_2 \dots$, where each A_n is a block of length n that contains every combination of 0's and 2's preceded by a 0. Let $P = 0.p_1p_2 \dots \in \Omega$, $\varepsilon > 0$, and choose $k \in N$ such that $\frac{2}{3^k} < \varepsilon$. A finite sequence of digits $x = 0.p_1p_2 \dots p_k$ thus occurs in block A_k. If the sequence starts at the m^{th} digit of x then $T^m(x) = 0.p_1p_2 \dots p_k$ and so in the interval $T^m(x) - P$ the first k digits are zero, yielding $|T^m(x) - P| \leq \frac{2}{3^k}$. Thus, the orbit of x is dense.

(iii) Tent map is sensitive to initial conditions.

Let $\beta > 0$ which is smaller than the minimum distance between the closed, disjoint intervals remaining after the middle third has been removed. Since all orbits of the tent map, regardless of how close the seeds may be, are unique, then eventually they will separate by at least β proving sensitive dependence.

So the tent map is chaotic.

2.5.4 Topological Conjugacy for one-dimensional map

This section aims to study the chaotic significance by comparing one map with another map with the help of topological conjugacy. Topological conjugacy [44] is a proper equivalence relation. Number and type of periodic points will be same in case of two conjugate (topologically) maps. This section shows that the tent map is conjugate to logistic map, doubling map is conjugate to logistic map and doubling map is also conjugate to shift map. Using this essential tool, we can predict behavior of a dynamical system comparing it with another dynamical system whose specific properties are known.

Definition 2.5.4.1 A function $k: X \to Y$ is a homeomorphism if

(i) k is injective

(ii) k is surjective

(iii) k is continuous

(iv) k^{-1} is also continuous.

Difinition 2.5.4.2

A map $k: X \to Y$ is a conjugacy (topological) from $f_1: X \to X$ to $f_2: Y \to Y$ if

(i) k is a homomorphism from X to Y and

(ii) $kf_1 = f_2k$ or equivalently $f_2 = kf_1k^{-1}$ or $f_1 = k^{-1}f_2k$.

Commuted diagram relates more commonly with Semi-conjugacy and conjugacy which is given to the Figure 2.5.11. The Diagram A commutes in case of k is a semi-conjugacy from f_1 to f_2. However, k and k^{-1} is a semi-conjugacy from f_1 to f_2 and f_2 to $f_1$respectively, if k is a conjugacy and the Diagram A and Diagram B commute representing semi-conjugation for k and k^{-1}.

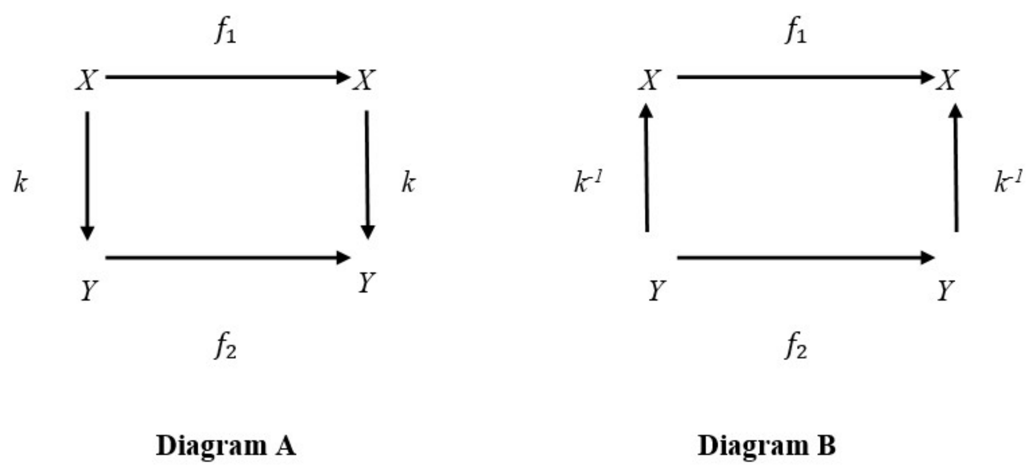


Figure 2.5.11 Semi-conjugacy and conjugacy between $f_1$and f_2

Some useful lemmas and theorems are presented in the next context.

Lemma 2.5.4.1

(i) $f_1: X \to X$ is conjugate to itself.

(ii) If $k: X \to Y$ is a conjugacy (topological) from $f_1: X \to X$ to $f_2: Y \to Y$, then k^{-1} is a conjugacy from $f_2: Y \to Y$ to $f_1: X \to X$.

(iii) If $k$is a conjugacy (topological) from $f_1: X \to X$ to $f_2: Y \to Y$ and k_1 is a conjugacy (topological) from $f_2: Y \to Y$ to $f_3: Z \to Z$, then kk_1 is a conjugacy from $f_1: X \to X$ to $f_3: Z \to Z$.

Lemma 2.5.4.2: If h is a conjugacy from$f_1$to f_2, then k is also a conjugacy from ${f_1}^n$ to ${f_2}^n$ for every positive integer.

Theorem 2.5.4.1

If $f: A \to A$ is topologically conjugate to $g: B \to B$ via topological conjugacy $h: A \to B$, then f is transitive if and only if g is transitive. That is, topological conjugacy [44] preserves transitivity.

Theorem 2.5.4.2

If $h: A \to B$ is an onto continuous mapping [2], then the image under h of a set dense in A is a set dense in B.

Theorem 2.5.4.3

Let us consider that f is a map of the set of periodic points of period n and it is denoted by $\text{Per}_n(f)$. If f and $g$are topologically conjugate, then $\text{Per}_n(f)$ is dense [28] if and only if $\text{Per}_n(g)$ is dense.

The following example shows that the logistic map is topologically conjugate to the tent map.

Example 2.5.4.1 The tent map T, defined by $T(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x < 1 \end{cases}$ is conjugate to the logistic map, $F_4 = 4x(1-x)$, via the conjugacy $k(x) = \frac{2}{\pi}\sin^{-1}\sqrt{x}$.

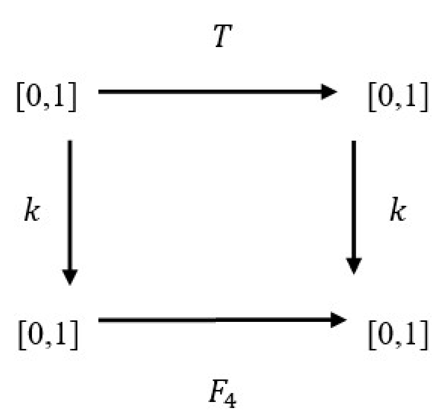


Figure 2.5.12 Conjugacy between tent map and logistic map

Solution: The function k is a homeomorphism on $I = [0,1]$. Then it follows that

$k^{-1}(x) = \left(\sin\frac{\pi}{2}x\right)^2$. We have to show that $F_4(x) = k^{-1}\text{o}\, T\text{o}\, k(x)$.

The first case, $0 \leq x \leq \frac{1}{2}$;

$$F_4(x) = k^{-1}\text{o}T\text{o}\, k(x)$$

$$= \left(\sin\left(\frac{\pi}{2}\left(\frac{4}{\pi}\sin^{-1}\sqrt{x}\right)\right)\right)^2$$

$$= \left(\sin\left(2\sin^{-1}\sqrt{x}\right)\right)^2$$

$$= \left(2\sin\left(\sin^{-1}\sqrt{x}\right)\cos\left(\sin^{-1}\sqrt{x}\right)\right)^2$$

$$= \left(2\sqrt{x}\cos\left(\sin^{-1}\sqrt{x}\right)\right)^2$$

$$= 4x\cos^2\left(\sin^{-1}\sqrt{x}\right)$$

$$= 4x\left(1 - \sin^2\left(\sin^{-1}\sqrt{x}\right)\right)$$

$$= 4x(1 - x)$$

The second case, $\frac{1}{2} \leq x \leq 1$;

$$F_4(x) = k^{-1}\text{o}T\text{o}\, k(x)$$

$$= \left(\sin\left(\frac{\pi}{2}\left(-\frac{4}{\pi}\sin^{-1}\sqrt{x} + 2\right)\right)\right)^2$$

$$= \left(\sin\left(-2\sin^{-1}\sqrt{x} + \pi\right)\right)^2$$

$$= \left(2\sin\left(-\sin^{-1}\sqrt{x} + \frac{\pi}{2}\right)\cos\left(-\sin^{-1}\sqrt{x} + \frac{\pi}{2}\right)\right)^2$$

$$= \left(2\cos\left(\sin^{-1}\sqrt{x}\right)\sin\left(\sin^{-1}\sqrt{x}\right)\right)^2$$

$$= 4x\left(\cos\left(\sin^{-1}\sqrt{x}\right)\right)^2$$

$$= 4x(1 - x)$$

Hence, we get, $F_4(x) = k^{-1} \text{o}\, T \text{o}\, k(x)$ for all $x \in [0,1]$.

Therefore, all topological properties of F_4 and T are the same, and we can concentrate on the piecewise linear tent map T.

Here we give another example that proves that the Doubling map is conjugate to logistic Map.

Example 2.5.4.2 The Doubling map $\mathrm{D}(x) = \begin{cases} 2x & \text{if } 0 < x < \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} < x < 1 \end{cases}$ is conjugate to logistic Map.

Poof: Let us consider $J(x) = \sin^2 2\pi x$ be a homeomorphism and putting

$$\varphi(x) = (J \circ D \circ J^{-1})(x)$$

Hence,

$$\varphi(x) = J\left(D\left(J^{-1}(x)\right)\right) = J\left(D\left(\frac{1}{2\pi}\sin^{-1}\sqrt{x}\right)\right), 0 \leq \frac{\sin^{-1}\sqrt{x}}{2\pi} < 1 \text{ (i.e. } 0 \leq x < 1)$$

$$= J\left(2 \times \frac{1}{2\pi}\sin^{-1}\sqrt{x}\right), 0 \leq \frac{\sin^{-1}\sqrt{x}}{2\pi} < \frac{1}{2} \text{ (i.e. } 0 \leq x \leq \frac{1}{2})$$

$$= J\left(\frac{1}{\pi}\sin^{-1}\sqrt{x}\right), 0 \leq \frac{\sin^{-1}\sqrt{x}}{2\pi} < \frac{1}{2} \left(\text{i.e. } 0 \leq x < \frac{1}{2}\right)$$

$$= \left\{\sin\left(2\pi \times \frac{1}{\pi}\sin^{-1}\sqrt{x}\right)\right\}^2$$

$$= \{\sin(2\sin^{-1}\sqrt{x})\}^2$$

$$= \left\{2\sin\left(\sin^{-1}\sqrt{x}\right).\cos\left(\sin^{-1}\sqrt{x}\right)\right\}^2$$

$$= \left(2\sqrt{x}\right)^2\{1 - \sin^2\left(\sin^{-1}\sqrt{x}\right)\}$$

$$= 4x(1 - x)$$

Similarly,

$$\varphi(x) = J\left(D\left(J^{-1}(x)\right)\right)$$

$$= J\left(D\left(\frac{1}{2\pi}\sin^{-1}\sqrt{x}\right)\right), 0 \le \frac{\sin^{-1}\sqrt{x}}{2\pi} < 1 \text{ (i.e. } 0 \le x < 1)$$

$$= J\left(2 \times \frac{1}{2\pi}\sin^{-1}\sqrt{x} - 1\right), 0.5 \le \frac{\sin^{-1}\sqrt{x}}{2\pi} < 1 \text{ (i.e. } 0 \le x < 1)$$

$$= J\left(\frac{1}{\pi}\sin^{-1}\sqrt{x} - 1\right), 0.5 \le \frac{\sin^{-1}\sqrt{x}}{2\pi} < 1 \left(\text{i.e. } 0 \le x < \frac{1}{2}\right)$$

$$= \left\{\sin 2\pi\left(\frac{1}{\pi}\sin^{-1}\sqrt{x} - 1\right)\right\}^2$$

$$= \left\{\sin\left(2\sin^{-1}\sqrt{x} - 2\pi\right)\right\}^2$$

$$= \left\{\sin\left(2\sin^{-1}\sqrt{x}\right)\right\}^2$$

$$= 2\sin\left(\sin^{-1}\sqrt{x}\right).\cos(\sin^{-1}\sqrt{x})\}\text{^}2$$

$$= \left(2\sqrt{x}\right)^2\{1 - \sin^2\left(\sin^{-1}\sqrt{x}\right)\}$$

$$= 4x(1 - x)$$

So, D is conjugate to logistic map.

Example 2.5.4.3: Doubling map $D(x) = \begin{cases} 2x & if\ 0 \le x < \frac{1}{2} \\ 2 - 2x, & if\ \frac{1}{2} \le x < 1 \end{cases}$ is conjugate to shift map.

Proof: Doubling map is defined by $D(x) = \begin{cases} 2x & if\ 0 \le x < \frac{1}{2} \\ 2 - 2x, & if\ \frac{1}{2} \le x < 1 \end{cases}$

Let's represent each $x \in [0,1]$ by its binary expansion:
$x = 0_{.2}b_1b_2b_3 \ldots = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$ where each $b_i \in \{0,1\}$. For $x = \frac{1}{2^n}$, represent x with a binary expansion ending in 0's rather than 1's. Then if $b_1 = 0$, we

know $x \in [0, \frac{1}{2})$. Similarly, $b_1 = 1$, we know $x \in \left[\frac{1}{2}, 1\right)$. Suppose $x \in [0, \frac{1}{2})$ is given $x = 0_{.2}0b_2b_3b_4 \ldots = \frac{0}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$.

Then $D(x) = 2x = 0_{.2}0b_2b_3b_4 \ldots = \frac{0}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$.

Consider $x = 0_{.2}01010110 \ldots$. Then $x = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{128} + \cdots$.

$$D(x) = 2x = \frac{2}{4} + \frac{2}{16} + \frac{2}{64} + \frac{2}{128} = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{64} + \cdots \frac{1}{64} = 0_{.2}01010110 \ldots$$

Now suppose $x \in \left[\frac{1}{2}, 1\right)$ is given. Then $x = 0_{.2}1b_2b_3b_4 \ldots = \frac{1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$.

Thus $D(x) = 2x - 1 = 1_{.2}b_2b_3b_4 \ldots - 1 = 0_{.2}b_2b_3b_4 \ldots$. Hence on [0,1), $D(x)$ is equivalent to $\sigma(x)$, the shift map on two symbols. Since shift map is chaotic on two symbol space, so Doubling map is also chaotic on the entire space.

From the following theorem, we prove that the itinerary map, π is continuous and surjective.

Theorem 2.5.4.4 The itinerary map π is continuous, surjective but not a homeomorphism.

Proof: We need to show that the itinerary map π is continuous,
so we let $s = s_0 s_1 \ldots . \in \sum_2$ and $\varepsilon > 0$. Pick n such that $\frac{1}{2^{n+1}} < \varepsilon$. Let $t \in \sum_2$ satisfy $d(s,t) < \frac{1}{2^n}$. Then $s_i = t_i$ for every $i \leq n$. Therefore

$$\begin{aligned}
|\pi(s) - \pi(t)| &= \left|\sum_{k=0}^{\infty} \frac{s_i - t_i}{2^{k+1}}\right| \\
&\leq \sum_{k=0}^{\infty} \left|\frac{s_i - t_i}{2^{k+1}}\right| \\
&\leq \sum_{k=n+1}^{\infty} \left|\frac{s_i - t_i}{2^{k+1}}\right| \\
&= \frac{1}{2^n} \leq \varepsilon
\end{aligned}$$

Here we see that π is surjective, so π is not a homeomorphism since it is not injective. Hence π is continuous surjective but not a homeomorphism.

Theorem 2.5.4.5 π is a topologically conjugate for the doubling map, *D*, and the shift map, σ.

From this section, we observe that topologically conjugacy and symbolic dynamics are an illustrious join of tools to discussing dynamical systems.

Example 2.5.4.4 Let $I = [0,1]$ and define $f: I \to I$ by

$$T(x) = \begin{cases} 3x, & x \in \left[0, \frac{1}{3}\right] \\ 3x - 1, & x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ 3x - 2, & x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

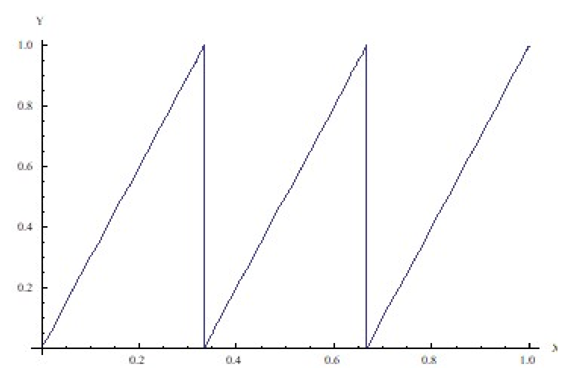


Figure: 2.5.13 Function of $T(x)$ acting on [0, 1]

Then let $X = \left\{x \in I: T^n(x) \notin \left(\frac{1}{3}, \frac{2}{3}\right), n \geq 0\right\}$, X is a standard Cantor set that is invariant under T.

Here it is easy to see that (X, T) is topologically conjugate to a symbolic system (Σ, σ) of two symbols.

The functions given below are not conjugate topologically:

Example 2.5.4.5 The maps $f, g: [0,1] \to [0,1]$ given by $f(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ -2x + 2 & \text{if } x \geq 0.5 \end{cases}$ and $g(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ 2x - 1 & \text{if } x \geq 0.5 \end{cases}$ are not topologically conjugate.

Solution: A topological conjugacy consists of a homeomorphism $h: [0,1] \rightarrow [0,1]$ which in particular must send endpoints. Moreover, it must send orbits to orbits and thus, in particular fixed points to fixed points. However, the two fixed points of g are the two endpoints, whereas the two fixed points of f are one of the endpoints and the other is in the interior; thus, h cannot be a conjugacy.

2.6 Summary and Conclusion

This chapter has discussed the symbolic dynamics of one and two-dimensional maps. Symbolic dynamics evolved into discussing standard dynamical systems; the skills and concepts have found significant applications in data storage and transmission and linear algebra. For $\mu > 4$, we observed that specific points of I leave the interval after the first iteration. The points that never leave are called the non-wandering set. This non-wandering set is a Cantor set. Furthermore, we have looked at symbolic dynamics. We used symbolic dynamics to see that the logistic function is chaotic for $\mu > 4$ on its Cantor sets. There is some application of symbolic dynamical systems such as the Cantor set and its ternary representation with symbolic behavior and Cantor set of the logistic map. We believe that several critical applications of the symbolic dynamics for a two-dimensional chaotic map can be explored. In this chapter we have proved that horseshoe map is chaotic on its invariant set. It is also established that Big Tent map is chaotic. We have shown that the action of the Smale's horseshoe, which is based on the Baker's map, the Bernoulli shift map has countably infinitely many periodic orbits, uncountably many non-periodic orbits, and σ dense orbit on $\sum_2$.

In this chapter, we see that the tent map, T_b and the logistic map, F_μ are conjugate. From another example, we find that the doubling map is conjugate to logistic map and the doubling map is also conjugate to shift map. But in Example 2.5.4.5, we see that the given two maps are not conjugate.

CHAPTER -3
CHAOTIC CHARACTERISTICS OF THE SHIFT MAP ON THE GENERALIZED m-SYMBOL SPACE

3.1 Introduction

Chaotic dynamical systems have been extensively discussed and investigated its characteristic in a distinguished extent of studies. In modern times, the chaoticity of a dynamical system is a more needful and requiring area for both mathematicians and physicists.

Symbolic dynamical systems are beautiful examples of topological dynamical systems. Robert L. Devaney [1] has given an unforgettable representation of space Σ_2 . By symbolic dynamical systems, we know that the space of sequences $\Sigma_2 = \{\alpha : \alpha = (\alpha_0 \alpha_1 \dots \dots \dots), \alpha_i = 0$ or $1\}$ along with the shift map defined on it. From [1], we know that Σ_2 is a compact metric space where $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^{i+1}}$, $s = (s_0 s_1 \dots \dots \dots)$ and $t = (t_0 t_1 \dots \dots \dots)$. At the beginning of the sequence, we can think of s and t as close if their sequences are similar.

This chapter aims to investigate some chaos-related characteristics of shift map on $\sum_m$, $m\ (\geq 2) \in N$. We assume several concepts of chaos available in modern literature. In section 3.2, we have proved that shift map is topological dynamical systems (TDS). The Proximity Theorem is established in section 3.3, and using this theorem, we have shown that shift map is transitive (topologically), topologically mixing. It is seen that σ is generically δ-chaotic. We have also shown that $(\sum_m, \sigma)$ has modify weakly chaotic dependence on initial conditions. In section 3.4, we have shown that symbol space $\sum_m$ is Cantor set. We have established that $\sigma: \sum_m \to \sum_m$ is topologically conjugate to map $f_m(x) = mx(mod 1)$on the space R/Z. In theorem 3.5. the shift map is exact Devaney chaotic.

3.2 Useful Definitions and Theorems

In this section we have discussed some important definitions which are useful to prove chaotic properties.

Definition 3.2.1: (Full Shift space)

If $\sum$ is a finite alphabet, then the full $\sum$- shift is the collection of all bi-infinite sequences of symbols from $\sum$. The full $\sum$- shift is denoted by $\sum^Z = \{x = (x_i)_{i \in Z}: x_i \in \sum \forall\, i \in Z\}$.

Here each sequence $x \in \sum^Z$ is called a point of the full shift. A block or word over $\sum$ is a finite sequences of symbols from $\sum$.

Definition 3.2.2: (Language)

Consider X be a subset of a full shift, and let $B_n(x)$ denote the set of all n- block that occur in points in X. The language [68] of X is the collection $B(X) = \bigcup_{n=0}^{\infty} B_n(X)$.

Definition 3.2.3

For each n we can construct the following function $\sigma: \sum_m \to \sum_m$ by

$$\sigma\big((a_i)\big) = (a_{i+1}).$$

In other words, if $(a_i) = (a_0 a_1 a_2 \dots \dots)$ then $\sigma\big((a_i)\big) = \big((a_1 a_2 a_3 \dots \dots)\big)$.

To define following definitions we consider (X, d) is a compact metric space, $g: X \to X$ is a continuous map and two non-empty open sets U, V.

Definition 3.2.4 (Transitive (Topologically)):

For $U, V \subset X\ \exists\ k \geq 0$ such that $g^k(U) \cap V \neq \phi$ then $g: X \to X$ is called transitive (topologically) [1].

Definition 3.2.5 (Strictly topologically transitive): Let $g^n: X \to X, n \geq 1$ be a sequence of continuous maps. If $U, V \subset X\ \exists\ k \geq 0$ such that $g^k(U) \cap V \neq \emptyset$, the sequence $\{g^n\}_{n=1}^{\infty}$ is said to be strictly topologically transitive [1].

Definition 3.2.6 (Totally Transitive):

If g^n is totally transitive $\forall\ n \geq 1$, then $g: X \to X$ is called totally transitive [2].

Definition 3.2.7 (Transitive Point):

Any point on a compact metric space (X, d) is said to be a transitive point [1] if it has a dense orbit.

Definition 3.2.8 (Topologically Mixing):

The map g is said to be topologically mixing [27] if $U, V \subset X \; \exists \; m \geq 0$ such that $\forall \; n \geq m, g^n(U) \cap V \neq \phi$.

Definition 3.2.9 (Strong Sensitive Dependence on Initial Conditions):

A continuous map $g: X \to X$ has strong sensitive dependence on initial conditions [64] if for any $x \in X$ and non-empty open set $U$of X, $\exists \; y \in U$ and $n \geq 0$ such that $d\big(g^n(x), g^n(y)\big)$ is maximum in X. It is evident that all strong, sensitive dependence maps have sensitive dependence.

Definition 3.2.10 (Generically δ-chaotic):

If $LY(g, \delta)$ is residual in X^2 then $g: X \to X$ on a compact metric space X is called generically δ-chaotic [13].

We also need the following Proposition, Lemma and Theorem.

Problem 3.2.1 Prove that d_m is a metric on $\sum_m$.

Proof: Here d_m is non-negative since $|s_i - t_i| \geq 0 \; \forall \; i$, and it vanishes if and only if $s_i = t_i$. So we need to show it is symmetry and triangle inequality. Symmetry follows since $|s_i - t_i| = |t_i - s_i|$ for s_i and t_i. Also triangle inequality follows since

$$d_m(s,t) + d_m(t,u) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{m^i} + \sum_{i=0}^{\infty} \frac{|t_i - u_i|}{m^i}$$

$$= \sum_{i=0}^{\infty} \frac{|s_i - t_i| + |t_i - u_i|}{m^i}$$

$$\geq \sum_{i=0}^{\infty} \frac{|s_i - u|}{m^i}$$

$$= d_m(s,u)$$

Then d_m is a metric and $(d_m, \sum_m)$ is a metric space.

Now we find the maximum distance between a pair of sequences in $\sum_m$:

The maximum value of $|s_i - t_i|$ is $m-1$, and therefore, the maximum distance between two sequences in $\sum_m$ is $d_m(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{m^i}$

$$\leq \sum_{i=0}^{\infty} \frac{m-1}{m^i}$$

$$= (m-1)\frac{1}{1-{}^{1}/_{m}} = m.$$

So, in general, the maximum distance can be no more than the size of the alphabet.

Problem 3.2.2 How many fixed points does the shift map have? How many 2-cycles? How many cycles of prime period 2?

Solution: Let $\sigma: \sum_m \to \sum_m$ with $\sigma(s_0 s_1 s_2 \ldots\ldots) = (s_1 s_2 s_3 \ldots\ldots)$.

Now, σ has m fixed points, indeed,

fix $\sigma = \{(000 \ldots.), (111 \ldots), \ldots\ldots, (kkk \ldots)\}$ where $k = m-1$. Recall that the shift map on $\sum_2$ has two points of prime period 2. It turns out that this is not the case since any sequence of the form $(\overline{s_0 s_1})$ is of period 2, and there are m^2 such points. But m of these are fixed, and so there are $m^2 - m = m(m-1)$ points of prime period 2.

Problem 3.2.3 How many periodic points in $\sum_m$?

Solution: We have that $per_3\sigma = \{(\overline{s_0 s_1 s_2}) | \; s_0, s_1, s_2 \in \sum_m\}$ and so $|per_3\sigma| = m^3$. Of these, $m^3 - m = m(m^2-1)$ are of prime period three since m of them are fixed. Similarly, we can write in general $|per_n\sigma| = m^n$

Remark 3.2.1 It is evident that σ has periodic orbits of every length. There are exactly m^k periodic orbits of length k since for each $k \in Z^+$ there are m^k different blocks of length k consisting of integers from $\{0,1,2, \ldots\ldots, m-1\}$. Repetitions of such a block define a point in $\sum_m$ which is on a periodic orbit of length k and any sequence (a_i) with $\sigma^k\big((a_i)\big) = (a_i)$ is a sequence consisting of repeated blocks of length k.

Proposition 3.2.1 If g is topologically weak mixing [41] then it is generically δ-chaotic on X with $\delta = diam(X)$.

Theorem 3.2.1 If g is transitive (topologically) [1] on X and the periodic points of g are dense in X, then f is chaotic on X, where X is an infinite subset of metric space.

Theorem 3.2.2 If the periodic points of g are dense in X and there is a point whose orbit under iteration of g is dense in the set X, then $g$is topologically transitive [2] on X.

Lemma 3.2.1: [14] Let $s, t \in \Sigma_2$and $s_i = t_i$, $i = 0,1, \dots \dots \dots, m$. Then $d(s,t) < \frac{1}{2^m}$ and conversely if $d(s,t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0,1, \dots \dots \dots, m$.

Theorem 3.2.3: $(\sum_m, \sigma)$ is topological dynamical systems (TDS).

Proof: Consider the space

$$\sum_m = \{0,1,2, \dots \dots, m-1\}^N = \{u = (u_i)_{i=1}^{\infty} : u_i \in \{0,1,2, \dots \dots, m-1\}\}$$

where $m(\geq 2) \in N$, is a compact metric space under $d: \sum_m \times \sum_m \to R$ defined by $d(u,v) = \sum_{k\geq 1} \frac{|u_k - v_k|}{m^k}$ for $u = (u_1 u_2 u_3 \dots \dots .)$, $v = (v_1 v_2 v_3 \dots \dots) \in \sum_m$. Again, this is explicit that $\sigma: \sum_m \to \sum_m$well-defined by $\sigma(u_1 u_2 u_3 \dots \dots) = (u_2 u_3 u_4 \dots \dots)$ is continuous.

Hence, $(\sum_m, \sigma)$ is topological dynamical system.

Note: For any $m, n (n < m) \in N$, we have that $\sum_n \subseteq \sum_m$. So, if it is true for $\sum_m$ is also true for $\sum_n$.

3.3 Chaotic features of the shift map

Important chaotic features of σ on $\sum_m$ is discussed in this section.

Theorem 3.3.1 [The Proximity Theorem]

Let $u, v \in \sum_m$. Then $d(u,v) \leq \frac{1}{m^n}$ if and only if *u* and *v* agree up to *n*-digits.

Proof: If $u = (u_1u_2u_3 \dots \dots.)$, $v = (v_1v_2v_3 \dots \dots) \in \sum_m$ and *u* and *v* agree up to the *n*-digits then $u_i - v_i = 0$ for $i = 1,2,3, \dots \dots, n$ and so, we get $\sum_{i=1}^{n} \frac{|u_i - v_i|}{m^i} = 0$.

Hence $d(u,v) = \sum_{k\geq 1} \frac{|u_k - v_k|}{m^k} = \sum_{i=1}^{n} \frac{|u_i - v_i|}{m^i} + \sum_{k>n} \frac{|u_k - v_k|}{m^k} = 0 + \sum_{k>n} \frac{|u_k - v_k|}{m^k}$

$= \sum_{k>n} \frac{|u_k - v_k|}{m^k}$.

Here $u_i, v_i \in \{0,1,2,3, \dots, m-1\}$, $\forall i \in N$

$\Rightarrow |u_i - v_i| \leq m - 1$, $\forall i \in N \Longrightarrow \frac{|u_k - v_k|}{m^k} \leq \frac{m-1}{m^k}$, $\forall k > n$

$$\therefore d(u,v) = \sum_{k>n} \frac{|u_k - v_k|}{m^k} \leq \sum_{k=n+1}^{\infty} \frac{m-1}{m^k} = \frac{m-1}{m^{n+1}} \sum_{r=0}^{\infty} \frac{1}{m^r} = \frac{m-1}{m^{n+1}} \cdot \frac{1}{1-\frac{1}{m}} = \frac{1}{m^n}.$$

Conversely, let $d(u,v) \leq \frac{1}{m^n}$. We have to show that *u* and *v* agree up to *n*-digits. Let *u* disagree to *v* at least at one digit that leads the n^{th} digit, say at i^{th} digit where $1 \leq i \leq n-1$ and agrees at all other digits up to *n*-digits. Then

$$d(u,v) = \sum_{k\geq 1} \frac{|u_k - v_k|}{m^k} = \frac{|u_i - v_i|}{m^i} + \sum_{k=n+1}^{\infty} \frac{|u_k - v_k|}{m^k} \geq \frac{|u_i - v_i|}{m^i} \geq \frac{1}{m^i} > \frac{1}{m^n}$$

[Since $i \leq n - 1 < n$].

This contradicts our assumption that $d(u,v) < \frac{1}{m^n}$. So, it follows that *u* and *v* essential agree up to *n*-digit.

Theorem 3.3.2 $\sigma: \sum_m \to \sum_m$ is continuous.

Proof: Let $u = (u_0u_1u_2 \dots \dots) \in \sum_m$ and $v = (v_0v_1v_2 \dots \dots) \in \sum_m$

Then $\sigma(u) = (u_1u_2u_3 \dots \dots)$ and $\sigma(v) = (v_1v_2v_3 \dots \dots)$

$$d(\sigma(u), \sigma(v)) = \sum_{i=0}^{\infty} \frac{|u_{i+1} - v_{i+1}|}{m^i}$$

$$= m \sum_{i=0}^{\infty} \frac{|u_{i+1} - v_{i+1}|}{m^{i+1}}$$

$$= m \sum_{i=1}^{\infty} \frac{|u_i - v_i|}{m^i}$$

$$\leq m \ \sum_{i=0}^{\infty} \frac{|u_i - v_i|}{m^i}$$

$$= m \ d(u, v)$$

Let $\varepsilon > 0$ be given and choose n so that $\frac{1}{m^n} < \varepsilon$. Now suppose $\delta = \frac{1}{m^{n+1}}$ and

$d(u, v) < \delta$. Then $u_i = v_i$ for $i \in \{0,1, \dots \dots, m+1\}$ and

$$d\big(\sigma(u), \sigma(v)\big) \leq m \ d(u, v) < m \ \delta = m \ \frac{1}{m^{n+1}} = \frac{1}{m^n} < \varepsilon$$

$$\Rightarrow d\big(\sigma(u), \sigma(v)\big) < \epsilon$$

Hence σ is continuous on $\sum_m$.

We can prove the following theorem using Theorem 3.3.1.

Theorem 3.3.3 $\sigma: \sum_m \to \sum_m$ is transitive (topologically).

Proof: To establish this theorem,
Let $u = (u_1 u_2 u_3 \dots \dots .) \in U$ and $v = (v_1 v_2 v_3 \dots \dots) \in V$ be arbitrary.
Now, $u \in U$, $v \in V$, and U, V are open sets. So, $\exists$ open balls $B(u, r_1) \subseteq U$ and $B(v, r_2) \subseteq V$. If $r = \min\{r_1, r_2\}$ then $B(u, r) \subseteq U$ and $B(v, r) \subseteq V$. We choose $n \in N$ such that $\frac{1}{m^n} < r$. Consider the point $w = (u_1 u_2 u_3 \dots \dots u_n v_1 v_2 v_3 \dots \dots) \in \sum_m$ and using Theorem 3.3.1, we find that

$$d(u, w) \leq \frac{1}{m^n} < r$$

$$\Rightarrow w \in B(u, r) \subseteq U$$

and consequently, it follows that $\sigma^n(w) \in \sigma^n(U)$.

Also $\sigma^n(w) = (v_1 v_2 v_3 \dots \dots) = v \in V$,
$v = \sigma^n(w) \in \sigma^n(U) \Rightarrow v = \sigma^n(w) \in \sigma^n(U) \cap V$. So it follows that $\sigma^n(U) \cap V \neq \emptyset$ and then $\sigma: \sum_m \to \sum_m$ is transitive (topologically).

We know that the chaotic maps are Li-Yorke sensitive and topologically transitive [2]. The following illustration shows that a continuous function is Li-Yorke sensitive maps or topologically transitive but not chaotic in the sense of B. S. Du.

Example 3.3.1: Consider a function $J(x): [-1,1] \to [-1,1]$ is defined by

$$J(x) = \begin{cases} \frac{11}{10}(x+1), & -1 \le x \le -\frac{1}{11} \\ -11x, & -\frac{1}{11} \le x \le 0 \\ x\,, & 0 \le x \le 1 \end{cases}$$

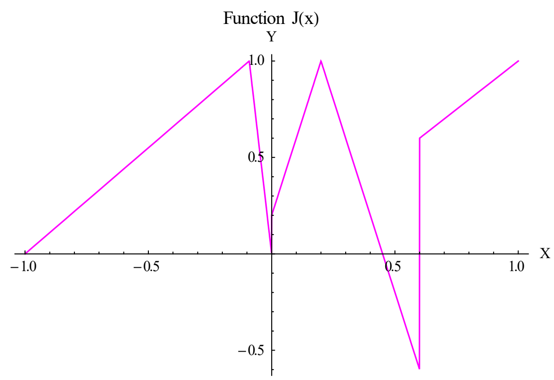


Figure 3.3.1: Function of $J(x)$ on the interval [-1,1]

The above function $J(x)$ is a continuous function. We can simultaneously prove that $J(x)$ is Li-Yorke sensitive and topologically transitive. However, in the sense of B. S. Du [19], it is not chaotic because $\frac{-1}{21}$ (the period two-point) and the interval [0,1] are jumping alternatively [2] and never get close to each other.

Theorem 3.3.4 $\sigma: \sum_m \to \sum_m$ is topologically mixing.

Proof: To prove the above theorem,

consider $u = (u_1 u_2 u_3 \dots \dots.) \in U$ and $v = (v_1 v_2 v_3 \dots \dots) \in V$ be arbitrary. Then since $u \in U, v \in V$and U, V are open sets in $\sum_m$, $\exists$ open balls $B(u, r_{,1}), B(v, r_2)$ such that $B(u, r_1) \subseteq U$ and $B(v, r_2) \subseteq V$.

If $r = \min\{r_1, r_2\}$then $B(u,r) \subseteq U$ and $B(v,r) \subseteq V$ and choose $k \in N$ such that $\frac{1}{m^k} < r$. We then construct a sequence $\{w_n\}$ of points in $\sum_m$ with the help of *k, u,* and v such that

$w_1 = (u_1u_2u_3u_4 \dots u_kv_1v_2v_3v_4 \dots),$

$w_2 = (u_1u_2u_3u_4 \dots u_ka_1v_1v_2v_3v_4 \dots),$

$w_3 = (u_1u_2u_3u_4 \dots u_ka_1a_2v_1v_2v_3v_4 \dots), \dots,$

$z_i = (u_1u_2u_3u_4 \dots u_ka_1a_2 \dots \dots a_{i-1}v_1v_2v_3v_4 \dots), i \geq 2,$

$a_i \in \{0,1,2, \dots \dots, m-1\}.$

Here, every w_i , $i \geq 2$ is created by using the finite word attained by taking first (*i*-1) successive symbols of $a = (a_1, a_2, a_3, \dots \dots a_{i-1}, \dots) \in \sum_m$. More exactly, the first k letters of w_i, for each $i \geq 2$, is the finite word

$u_{[1,k]} = (u_1u_2u_3u_4 \dots u_k)$ taken from $u \in U$ and then

$a_{[1,i-1]} = (a_1, a_2, a_3, \dots \dots, a_{i-1})$ taken from *a* and at last the sequence representing *v*, i.e. $w_i = (u_{[1,k]}, a_{[1,i-1]}, v)$. Now, using $\{0,1,2, \dots, m-1\}$, repeating it for (*i*-1) times rather than using $a_{[1,i-1]}$.

Now, using Theorem 3.3.1, we get, $d(u, w_i) \leq \frac{1}{m^k} < r \; \forall \, i \in N$. So, $w_i \in B(u,r) \subseteq U$ and hence

$\sigma^{k+i-1}(w_i) \in \sigma^{k+i-1}(B(u,r)) \subseteq \sigma^{k+i-1}(U) \forall \, i \in N.$

Also $\sigma^{k+i-1}(w_i) = (v_1v_2v_3 \dots \dots) \in V$, $\sigma^{k+i-1}(w_i) \in \sigma^{k+i-1}(U)$ imply that $\sigma^{k+i-1}(U) \cap V \neq \emptyset$, for all $i \geq 2$.

Therefore, $\sigma^n(U) \cap V \neq \emptyset$, for all $n \geq k$.

So, $\sigma: \sum_m \to \sum_m$ is mixing topologically.

The condition of sensitive dependence on initial conditions (SDIC) is the property that is widely used as one of the important features of chaotic mappings. The following theorem shows that shift map has sensitive dependence on initial conditions (SDIC).

Theorem 3.3.5 $\sigma: \sum_m \to \sum_m$ has sensitive dependence on initial conditions (SDIC).

Proof: Consider $u \in \sum_m$ and $N(u)$ be an arbitrary neighborhood of u. Then $\exists\, U$ (a non-empty open set) such that $u \in U \subseteq N(u)$. Now $u \in U \ \exists\, B(u,r)$ such that $B(u,r) \subseteq U \subseteq N(u)$. Again $v \in B(u,r) \subseteq U \subseteq N(u)$ such that $u \neq v$ and u is very close to v. Now we want to show $\frac{1}{m^k} < r$ for $k \in N$ then the point v to agree with u up to k-digits. So $d(u,v) \leq \frac{1}{m^k} < r$.

Let $d(u,v) = \varepsilon$. Then $\varepsilon > 0$, $\exists$ a significant and unique $n \in N$ such that $\frac{1}{m^{n+1}} < \varepsilon \leq \frac{1}{m^n}$. Consider $d(u,v) = \varepsilon \leq \frac{1}{m^n}$.

Then $d(u,v) \leq \frac{1}{m^n} \Rightarrow u$ and v agree up to the n^{th} digit

$\Rightarrow (n+1)^{th}$ digits of u and v are different

$\Rightarrow$ First digit of $\sigma^n(u)$ and $\sigma^n(v)$ are different

$\Rightarrow d\big(\sigma^n(u), \sigma^n(v)\big) = \sum_{i=1}^{\infty} \frac{|u_{n+i} - v_{n+i}|}{m^i}$

$$= \frac{|u_{n+1} - v_{n+1}|}{m} + \sum_{i=2}^{\infty} \frac{|u_{n+i} - v_{n+i}|}{m^i} \geq \frac{1}{m}$$

From the above relation, it is clear that $\frac{1}{m}$ is equal to the sensitivity constant δ.

Hence for $u \in \sum_m$ and $\exists\, v \in N(u)$ and $n > 0$ satisfying $d\big(\sigma^n(u), \sigma^n(v)\big) \geq \delta$ for $\delta = \frac{1}{m}$.

Therefore, $\sigma: \sum_m \to \sum_m$ has sensitive dependence on initial conditions (SDIC).

Theorem 3.3.6 The set of all the periodic points of σ, $P(\sigma)$ is dense in $\sum_m$.

Proof: At first, we need to show that σ has $m^n - m$ periodic points of period-n in $\sum_m$ for $n \geq 2$. We know that if a definite block of *n*-digits from $\{0, 1, 2, 3, 4, \dots \dots, m-1\}$ repeats indefinitely, then it is a periodic point of σ of period-n in $\sum_m$. A block of *n*-digits can be designed with m distinct digits 0, 1, 2, 3, …., *m*-1 in m^n ways. These blocks cover the m-blocks designed by the same digit, which are not periodic points of period-*n*. If periodic points of period-1, so we take only $(m^n - m)$ numbers of periodic points of period-n in $\sum_m$.

Now, we have to show that for any $\varepsilon > 0$, however small, there is a point $p \in P(\sigma)$ such that $d(u, p) < \varepsilon$. If $u = (u_1 u_2 u_3 \dots \dots .) \in U$, for $\varepsilon > 0$, we get $\frac{1}{m^n} < \varepsilon$ where $n \in N$.

Now, we make a periodic point $p \in P(\sigma)$ of periodic (*n*+1) such that $p = (u_1 u_2 u_3 \dots \dots u_n v\, u_1 u_2 u_3 \dots \dots u_n v\, u_1 u_2 u_3 \dots \dots u_n v \dots \dots)$ i.e. p is made by repeating the word $W = (u_1 u_2 u_3 \dots \dots u_n v)$ infinite number of times so as to it agrees with the digits of u up to *n*-terms and disagrees at (*n*+1)th digit such that $u_{n+1} \neq v$ and $d(u, p) \leq \frac{1}{m^n} < \varepsilon$.

Thus, for every $u \in \sum_m$ and $\varepsilon > 0$, $\exists\, p \in P(\sigma)$ such that $d(u, p) < \varepsilon$. Hence $P(\sigma)$ is dense.

Theorem 3.3.7 σ is Devaney as well as Auslander-Yorke chaotic on $\sum_m$.

Proof: Using Theorem 3.3.3, Theorem 3.3.5, and Theorem 3.3.6, we have established that σ satisfies all the requirements of Devaney's chaotic. Again, according to the definition of Auslander-Yorke chaos [Definition 1.5.4.8], we can say that it is chaotic. Hence it is Devaney as well as Auslander-Yorke chaotic [20].

Theorem 3.3.8 σ is δ-chaotic with $\delta = diam(\sum_m) = 1$.

Proof: In Theorem 3.3.4, it has shown that σ is topologically mixing on $\sum_m$. We know that topologically mixing maps on a compact metric space also weakly topologically mixing, σ is weakly topologically mixing. Using Proposition 3.2.1, we can say the shift map σ on $\sum_m$ is generically δ-chaotic with $\delta = diam(\sum_m) = 1$.

Theorem 3.3.9 $(\sum_m, \sigma)$ has modified weakly chaotic dependence on initial conditions.

Proof: Let $u = (u_1 u_2 u_3 \dots \dots u_n \dots \dots) \in \sum_m$ and $N(u)$ be any neighborhood of u. Then $\exists\, U$ of $\sum_m$ such that $u \in U \subseteq N(u)$.

As $u \in U$ and $\exists$ an open ball $B(u,r)$ with $r > 0$such that $B(u,r) \subseteq U \subseteq N(u)$. Then for $r > 0$, we can choose n such that $\frac{1}{m^n} < r$. We now find two points
$v, w \in B(u,r) \subseteq U \subseteq N(u)$ with $v \neq u, w \neq u$ such that $(v, w) \in \sum_m^2$ is Li-Yorke.

By a word in $\sum_m$ we mean a finite sequence of digits, called letters, from the set $\{0,1,2,3, \dots \dots, m-1\}$. Words are denoted by $A, B, C, \dots, P, Q, R, \dots \dots$ etc. If the words A and B consist of p and n letters respectively such that $A = (a_1 a_2 a_3 \dots \dots a_p)$ and $B = (b_1 b_2 b_3 \dots \dots b_q)$ then by the symbol AB, we mean the composite word $(a_1 a_2 a_3 \dots \dots a_p b_1 b_2 b_3 \dots \dots b_q)$ which consists of $(p+q)$-number of letters. Using the letters in $u = (u_1 u_2 u_3 \dots \dots u_n \dots) \in \sum_m$, we now construct the words $W(u, 3n)$, $W(u, 5n)$, $W(u, 7n)$, … …etc. as follows:

$W(u, 3n) = (u_{3n+1}^* u_{3n+2}^* \dots u_{4n}^* u_{4n+1} u_{4n+2} \dots u_{5n})$,
$W(u, 5n) = (u_{5n+1}^* u_{5n+2}^* \dots u_{6n}^* u_{6n+1} u_{6n+2} \dots u_{7n})$,
$W(u, 7n) = (u_{7n+1}^* u_{7n+2}^* \dots u_{8n}^* u_{8n+1} u_{8n+2} \dots u_{9n})$, …
and so on.

Here it is observed that each of the above words contains $2n$ letters, first n of that are m-nary complements of the letters in the corresponding places of u, and the rest n letters are just the letters in the related areas of u. In all the above words
$u_k^* = (m-1) - u_k, \forall\, k$.

Now take

$$v = (u_1 u_2 u_3 \dots \dots u_n u_{n+1}^* u_{n+2}^* \dots u_{3n}^* u_{3n+1} u_{3n+2} \dots u_{5n} u_{5n+1} u_{5n+2})$$

and

$$w = (u_1 u_2 u_3 \dots \dots u_n (0^*)^n (0)^n W(u, 3n) W(u, 5n) W(u, 7n) W(u, 9n) \dots)$$

where $(0^*)^n = 0^* 0^* 0^* \dots 0^*$, $(0)^n = 000 \dots 0$ and $0^* = (m-1) - 0 = m - 1$.

With these and using Theorem 3.3.1 for $\sum_m$, we now establish the theorem as follows: Since v and w agree with u up to the n^{th} term, we get $d(u,v) \leq \frac{1}{m^n} < r$, $d(u,w) \leq \frac{1}{m^n} < r$ and consequently $v, w \in B(u,r) \subseteq U \subseteq N(u)$.

Here, we look that w holds infinitely many words of $W(u,(2k-1)n)$ containing *2n* letters where $k(\geq 2) \in N$.

Also

$$\sigma^{3n}(v) = (u_{3n+1}u_{3n+2} \dots u_{4n}u_{4n+1}u_{4n+2} \dots u_{5n}u_{5n+1}u_{5n+2} \dots)$$
$$\sigma^{3n}(w) = (u^*_{3n+1}u^*_{3n+2} \dots u^*_{4n}u_{4n+1}u_{4n+2} \dots u_{5n}u^*_{5n+1}u^*_{5n+2} \dots)$$
$$\sigma^{4n}(v) = (u_{4n+1}u_{4n+2} \dots u_{5n}u_{5n+1}u_{5n+2} \dots u_{6n}u_{6n+1}u_{6n+2} \dots)$$
$$\sigma^{4n}(w) = (u_{4n+1}u_{4n+2} \dots u_{5n}u^*_{5n+1}u^*_{5n+2} \dots u^*_{6n}u_{6n+,}u_{6n+2} \dots)$$

Therefore, $sup\, d\big(\sigma^n(v),\sigma^n(w)\big) \geq d(\sigma^{3n}(v),\sigma^{3n}(w))$ and so

$$\lim_{n\to\infty} sup\, d\big(\sigma^n(v),\sigma^n(w)\big) \geq \lim_{n\to\infty} d(\sigma^{3n}(v),\sigma^{3n}(w))$$
$$\geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{|u_{3n+r} - u^*_{3n+r}|}{m^r}$$
$$\geq \lim_{n\to\infty} \left\{\frac{1}{m} + \frac{1}{m^2} + \dots + \frac{1}{m^n}\right\}$$
$$= \frac{1}{m-1}$$

Again, $0 \leq \lim_{n\to\infty} inf d(\sigma^n(v),\sigma^n(w))$

$$\leq \lim_{n\to\infty} d(\sigma^{4n}(v),\sigma^{4n}(w))$$
$$= \lim_{n\to\infty} d((u_{4n+1} \dots u_{5n}u_{5n+1} \dots u_{6n}u_{6n+1} \dots),$$
$$(u_{4n+1} \dots u_{5n}u^*_{5n+1}u^*_{5n+2} \dots \dots u^*_{6n}u_{6n+1} \dots \dots))$$

$$\leq \lim_{n\to\infty} \left\{\left(\frac{m-1}{m^{n+1}} + \frac{m-1}{m^{n+2}} + \dots + \frac{m-1}{m^{2n}}\right) + \left(\frac{m-1}{m^{3n+1}} + \frac{m-1}{m^{3n+2}} + \dots + \frac{m-1}{m^{4n}}\right) + \dots\right\}$$
$$= \lim_{n\to\infty} \left\{\left(\frac{m-1}{m} + \frac{m-1}{m^2} + \dots + \frac{m-1}{m^n}\right).\left(\frac{1}{m^n} + \frac{1}{m^{3n}} + \frac{1}{m^{5n}} + \dots\right)\right\}$$
$$= \lim_{n\to\infty} \left\{\left(1 - \frac{1}{m^n}\right).\frac{1}{m^n}\left(1 + \frac{1}{m^{2n}} + \frac{1}{m^{4n}} + \frac{1}{m^{6n}} \dots \dots\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\left(1-\frac{1}{m^n}\right).\frac{1}{m^n}.\frac{1}{1-\frac{1}{m^{2n}}}\right\}$$

$$= (1-0).0.\left(\frac{1}{1-0}\right)=0$$

Now, $0 \le \lim_{n\to\infty} inf d\big(\sigma^n(v), \sigma^n(w)\big) \le 0 \Rightarrow \lim_{n\to\infty} inf d\big(\sigma^n(v), \sigma^n(w)\big) = 0.$

So, it follows that

$\lim_{n\to\infty} sup\ d(\sigma^n(v), \sigma^n(w)) \ge \frac{1}{m-1}$ and $\lim_{n\to\infty} inf d(\sigma^n(v), \sigma^n(w)) = 0.$

Hence, $(v, w) \in \sum_m^2$ is a Li-Yorke pair with $\delta = \frac{1}{m-1} > 0$. Accordingly, $(\sum_m, \sigma)$ has modified weakly chaotic dependence on initial conditions.

The following theorem proves that shift map has chaotic dependence on initial conditions on $\sum_m$.

Theorem 3.3.10 $(\sum_m, \sigma)$ has chaotic dependence on initial conditions.

Proof: Consider $p = (p_1p_2p_3 \dots \dots) \in \sum_m$ and $p \in U \subseteq N(u)$, where U is an open set. As $p \in U$ so $\exists$ an open ball $B(p, r)$s.t.$B(p, r) \subseteq U \subseteq N(p)$. Then for $r > 0$, consider $\frac{1}{m^n} < r$. Now to find a point $q \in B(p, r) \subseteq U \subseteq N(p)$ such that $(p, q) \in \sum_m^2$ is Li-Yorke.

With letters in $p = (p_1p_2p_3 \dots \dots) \in \sum_m$, we can make the words $W(p, 3n)$, $W(p, 5n)$, $W(p, 7n)$, ...etc. as follows:

$W(p, 3n) = (p^*_{3n+1}p^*_{3n+2} \cdots p^*_{4n}p_{4n+1}p_{4n+2} \cdots p_{5n}),$
$W(p, 5n) = (p^*_{5n+1}p^*_{5n+2} \cdots p^*_{6n}p_{6n+1}p_{6n+2} \cdots p_{7n})$
$W(p, 7n) = (p^*_{7n+1}p^*_{7n+2} \cdots p^*_{8n}p_{8n+1}p_{8n+2} \cdots p_{9n})$, ...
and so on.

We make the point q using the above-defined words as follows:

$$q = (p_1p_2, p_3 \dots p_n(0^*)^n(0)^n W(p, 3n)W(p, 5n)W(p, 7n)W(p, 9n) \dots)$$

where $(0^*)^n = 0^*0^*0^* \dots 0^*, (0)^n = 000 \dots 0$ and $0^* = (m-1) - 0 = m - 1.$

From q, it is clear that $q$agrees with p up to the n[th] term. So using Theorem 3.3.1, we can write $d(p,q) \leq \frac{1}{m^n} < r$ and hence $q \in B(p,r) \subseteq U \subseteq N(p)$.

So, it is observed that q contains infinitely many words of $W(p,(2k-1)n)$ containing *2n* letters where $k \geq 2$.

Also

$$\sigma^{3n}(q) = (p^*_{3n+1}p^*_{3n+2} \cdots p^*_{4n}p_{4n+1}p_{4n+2} \cdots p_{5n}p^*_{5n+1}p^*_{5n+2} \cdots)$$
$$\sigma^{4n}(q) = (p^*_{4n+1}p^*_{4n+2} \cdots p_{5n}p^*_{5n+1}p^*_{5n+2} \cdots p^*_{6n}p_{6n+1}p_{6n+2} \cdots)$$

Therefore, $sup\ d\left(\sigma^n(p),\sigma^n(q)\right) \geq d(\sigma^{3n}(p),\sigma^{3n}(q))$ and so

$$\lim_{n\to\infty} sup\ d\left(\sigma^n(p),\sigma^n(q)\right) \geq \lim_{n\to\infty} d(\sigma^{3n}(p),\sigma^{3n}(q))$$
$$\geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{|p_{3n+r} - q^*_{3n+r}|}{m^r}$$
$$\geq \lim_{n\to\infty} \left\{\frac{1}{m} + \frac{1}{m^2} + \cdots + \frac{1}{m^n}\right\}$$
$$= \frac{1}{m-1}$$

Again, $0 \leq \lim_{n\to\infty} inf d(\sigma^n(p),\sigma^n(q))$

$$\leq \lim_{n\to\infty} d(\sigma^{4n}(p),\sigma^{4n}(q))$$
$$= \lim_{n\to\infty} d((p_{4n+1} \cdots p_{5n}p_{5n+1} \cdots p_{6n}p_{6n+1} \cdots),$$
$$(p_{4n+1} \cdots p_{5n}p^*_{5n+1}p^*_{5n+2} \cdots p^*_{6n}p_{6n+1} \cdots \cdots))$$
$$\leq \lim_{n\to\infty} \left\{\left(\frac{m-1}{m^{n+1}} + \frac{m-1}{m^{n+2}} + \cdots + \frac{m-1}{m^n}\right) + \left(\frac{m-1}{m^{3n+1}} + \frac{m-1}{m^{3n+2}} + \cdots + \frac{m-1}{m^{4n}}\right) + \cdots\right\}$$
$$= \lim_{n\to\infty} \left\{\left(\frac{m-1}{m} + \frac{m-1}{m^2} + \cdots + \frac{m-1}{m^n}\right).\left(\frac{1}{m^n} + \frac{1}{m^{3n}} + \frac{1}{m^{5n}} + \cdots\right)\right\}$$
$$= \lim_{n\to\infty} \left\{\left(1 - \frac{1}{m^n}\right).\frac{1}{m^n}\left(1 + \frac{1}{m^{2n}} + \frac{1}{m^{4n}} + \frac{1}{m^{6n}} \cdots \cdots\right)\right\}$$
$$= \lim_{n\to\infty} \left\{\left(1 - \frac{1}{m^n}\right).\frac{1}{m^n}.\frac{1}{1 - \frac{1}{m^{3n}}}\right\}$$
$$= (1-0).0.\left(\frac{1}{1-0}\right) = 0$$

Now, $0 \leq \lim_{n\to\infty} inf d\left(\sigma^n(p),\sigma^n(q)\right) \leq 0 \Rightarrow \lim_{n\to\infty} inf d\left(\sigma^n(p),\sigma^n(q)\right) = 0.$

So, we get $\lim_{n\to\infty} sup\, d(\sigma^n(a), \sigma^n(b)) \geq \frac{1}{m-1}$ and $\lim_{n\to\infty} inf d(\sigma^n(a), \sigma^n(b)) = 0$.

Hence, $(p, q) \in \sum_m^2$ is a Li-Yorke pair with $\delta = \frac{1}{m-1} > 0$.

So, $(\sum_m, \sigma)$ has chaotic dependence on initial conditions.

Theorem 3.3.11 $\sigma: \sum_m \to \sum_m$ is homeomorphism.

Proof: It is sufficient to show that (i) σ is one-one (ii) σ is onto (iii) σ is continuous (iv) σ^{-1} is continuous.

(i) σ is one-one: Suppose that $\sigma(u) = \sigma(v)$ where $u = (u_n)_{n\epsilon\mathbb{Z}}$, $v = (v_n)_{n\epsilon\mathbb{Z}} \in \sum_m$.

Put $w = \sigma(u) = \sigma(v)$, where $w = (w_n)_{n\epsilon\mathbb{Z}}$; $\mathrm{w_n} = u_{n-1} = v_{n-1}$. $\forall n \in \mathbb{Z}$

Then, $u_n = v_n \forall n \in \mathbb{Z}$. Thus $u = v$ and hence σ is one-one.

(ii) σ is onto: Let $v = (v_n)_{n\epsilon\mathbb{Z}} \in \sum_m$, put $u = (u_n)_{n\epsilon\mathbb{Z}}$ where $u_n = v_{n-1}$

Then $\sigma(u) = v$ and hence σ is onto.

(iii) Continuity of σ: Let, $u' = (u'_n)_{n\in\mathbb{Z}} \in \sum_m$

By the definition of σ, we have, $\sigma(u') = v' = (v'_n)_{n\in\mathbb{Z}} \in \sum_m$,where $v'_n = u'_{n-1}$.

Now $d\left(\sigma(u),\ \sigma(u')\right) = d(v, v')$

$$= \sum_{n=0}^{\infty} \frac{|v_n - v'_n|}{m^n} + \sum_{n=1}^{\infty} \frac{|v_{-n} - v'_{-n}|}{m^n}$$

$$= \sum_{n=0}^{\infty} \frac{|u_{n-1} - u'_{n-1}|}{m^n} + \sum_{n=1}^{\infty} \frac{|u_{-n-1} - u'_{-n-1}|}{m^n}$$

$$= \frac{1}{m}\sum_{n=0}^{\infty} \frac{|u_{n-1} - u'_{n-1}|}{m^{n-1}} + m\sum_{n=1}^{\infty} \frac{|u_{-n-1} - u'_{-n-1}|}{m^{n+1}}$$

$$= \frac{1}{m}\frac{|u_{-1} - u'_{-1}|}{m^{-1}} + \frac{1}{m}\sum_{i=0}^{\infty} \frac{|u_i - u'_i|}{m^i} + m\sum_{l=2}^{\infty} \frac{|u_{-l} - u'_{-l}|}{m^l}$$

$$= \frac{1}{m}\frac{|u_{-1} - u'_{-1}|}{m^{-1}} + m\sum_{l=2}^{\infty} \frac{|u_{-l} - u'_{-l}|}{m^l} + \frac{1}{m}\sum_{i=0}^{\infty} \frac{|u_i - u'_i|}{m^i}$$

$$= m\sum_{l=1}^{\infty}\frac{|u_{-l}-u_{-l}^{'}|}{m^{l}}+\frac{1}{m}\sum_{i=0}^{\infty}\frac{|u_{i}-u_{i}^{'}|}{m^{i}}$$

$$\leq m\sum_{i=0}^{\infty}\frac{|u_{i}-u_{i}^{'}|}{m^{i}}+m\sum_{l=1}^{\infty}\frac{|u_{-l}-u_{-l}^{'}|}{2^{l}}$$

$$= m\, d(u,u^{'})$$

Let $\varepsilon > 0$ be given and choose n so that $\frac{1}{m^n} < \varepsilon$. Now suppose $\delta = \frac{1}{m^{n+1}}$.

If $d(u,u^{'}) < \delta$, then $d\left(\sigma(u),\ \sigma(u^{'})\right) \leq m\, d(u,u^{'}) < m\,\delta = m\ \frac{1}{m^{n+1}} = \frac{1}{m^n} < \varepsilon$

$$\Rightarrow d\left(\sigma(u),\sigma(u^{'})\right) < \epsilon$$

Hence σ is continuous on $\sum_m$.

(iv) Continuity of σ^{-1}: Let, $u^{'} = (u_n^{'})_{n\in\mathbb{Z}} \in \sum_m$ we put $\sigma^{-1}(u) = v$ then $\sigma(v) = u$. Thus, by definition of σ, we have $v_n = u_{n+1}$.

Again, put $\sigma^{-1}(u^{'}) = v^{'} \Rightarrow \sigma(v^{'}) = u^{'}$ and by the definition of σ, we have, $v_n^{'} = u_{n+1}^{'}$.

Now $d\left(\sigma^{-1}(u),\sigma^{-1}(u^{'})\right) = d(v,v^{'})$

$$= \sum_{n=0}^{\infty}\frac{|v_n-v_n^{'}|}{m^n}+\sum_{n=1}^{\infty}\frac{|v_{-n}-v_{-n}^{'}|}{m^n}$$

$$= \sum_{n=0}^{\infty}\frac{|u_{n+1}-u_{n+1}^{'}|}{m^n}+\sum_{n=1}^{\infty}\frac{|u_{-n+1}-u_{-n+1}^{'}|}{m^n}$$

$$= m\sum_{n=0}^{\infty}\frac{|u_{n+1}-u_{n+1}^{'}|}{m^{n+1}}+\frac{1}{m}\sum_{n=1}^{\infty}\frac{|u_{-n+1}-u_{-n+1}^{'}|}{m^{n-1}}$$

$$= m\sum_{i=1}^{\infty}\frac{|u_i-u_i^{'}|}{m^i}+\frac{1}{m}\sum_{l=0}^{\infty}\frac{|u_{-l}-u_{-l}^{'}|}{m^l}$$

$$= m\sum_{i=1}^{\infty}\frac{|u_i-u_i^{'}|}{m^i}+\frac{|u_0-u_0^{'}|}{m}+\frac{1}{m}\sum_{l=1}^{\infty}\frac{|u_{-l}-u_{-l}^{'}|}{m^l}$$

$$\leq \frac{m|u_0-u_0^{'}|}{m}+m\sum_{i=1}^{\infty}\frac{|u_i-u_i^{'}|}{m^i}+\frac{1}{m}\sum_{l=1}^{\infty}\frac{|u_{-l}-u_{-l}^{'}|}{m^l}$$

$$= m\sum_{i=0}^{\infty}\frac{|u_i - u_i^{'}|}{m^i} + \frac{1}{m}\sum_{l=1}^{\infty}\frac{|u_{-l} - u_{-l}^{'}|}{m^l}$$

$$\leq m\sum_{i=0}^{\infty}\frac{|u_i - u_i^{'}|}{m^i} + m\sum_{l=1}^{\infty}\frac{|u_{-l} - u_{-l}^{'}|}{m^l}$$

$$= m\, d(u, u^{'})$$

Let $\varepsilon > 0$ be given and choose n so that $\frac{1}{m^n} < \varepsilon$. We put $\delta = \frac{1}{m^{n+1}}$.

Consider $d(u, u^{'}) < \delta$ then $d\left(\sigma^{-1}(u), \sigma^{-1}(u^{'})\right) < m\delta = m.\frac{1}{m^{n+1}} = \varepsilon$.

Therefore, σ^{-1} is a continuous function.

Hence, we conclude that $\sigma: \sum_m \to \sum_m$is homeomorphism.

3.4 Some features of the symbol space $\sum_m$

In this section, we have discussed some properties of the symbol space $\sum_m$.

Theorem 3.4.1 The sequence space $\sum_m$ is a Cantor set.

Proof: To prove this theorem, we need to show that $\sum_m$ is:

i) sequentially compact

ii) perfect and

iv) totally disconnected.

Proof i) We have to show that $\sum_m$ is sequentially compact. Let $u^{(m)} = (u_n^{(m)})$where $n \in Z, (m = 1,2, \ldots\ldots)$ be a sequence in $\sum_m$, then we need to show that $u \in \sum_m$ and subsequence $u^{m_l} \to u (l = 1,2, \ldots \ldots)$

First, observe that the zeroth term $u_0^m (m = 1,2, \ldots \ldots)$ must take some value in $\{1,2, \ldots . m\}$ infinitely often. Choose such an $u_0 \in \{1,2, \ldots \ldots m\}$ in the $u_0^{(m)} = u_0$for infinitely many. By induction, for $l > 0$ choose $u_l \in \{1,2, \ldots \ldots m\}$ such that $u_0^{(m)} = u_0, \ldots \ldots u_l^{(m)} = u_l$ for finitely many m. Finally, we define $u = (u_l)$ where $l \in \mathbb{Z}$**.**

For such $l \geq 0$, we choose $m_l = m$ such that $u_0^{(m)} = u_0 \ldots \ldots . u_l^{(m)} = u_l$, then

$d\left(u^{(m_l)}, u\right) \leq \frac{1}{m^l}$ and so $d(u^{m_l}, u) \to 0$ as $l \to \infty$

(ii) To see that $\sum_m$ is perfect, fix u and define $u^{(m)}$ s.t. $u_j^{(m)} = u_j$ for $0 \leq j \leq n$ and $u_{n+1}^{(m)} = u_{n+1}$. Hence $u = u^{(m)}$ and $u^{(m)}$ converges to u.

(iii) To see $\sum_m$ is totally disconnected. Let $\delta_{j_0}: \sum_m \to \{0,1,2, \dots \dots m-1\},\ u \to u_{j_0}$ is continuous. Hence the set $U = \{u: u_{j_0} = c\}$ for fix j_0 and c is open and $V = \{u: u_{j_0} \neq c\}$. Now let $u, v \in \sum_m$ if $u \neq v$ then there is j_0 such that $u_{j_0} \neq v_{j_0}$. Now take $c = u_{j_0}$ then from above, we see that U and V are disjoint sets (open) whose union is in $\sum_m$ and which contains u and v.

Theorem 3.4.2 There exists a continuous map on the symbol space $\sum_m$ such that all points of $\sum_m$ are period *m* distinct points by the map.

Proof: We take the map $\varphi: \sum_m \to \sum_m$ as the complement map, that is, $\varphi(s) = (s_0' s_1' \dots \dots)$, where $s = (s_0 s_1 \dots \dots)$ is any point of $\sum_m$.

At first, we need to prove that the complemented map φ is continuous on $\sum_m$.

We pick *n* so large that $\frac{1}{m^n} < \varepsilon$, where $\varepsilon > 0$. Let $u = (u_0 u_1 u_2 \dots \dots \dots)$ and $v = (v_0 v_1 v_2 \dots \dots \dots)$ are any two points of $\sum_m$. Now we choose $\delta = \frac{1}{m^{n+1}}$ and define the complement of α_i by α_i^c. Then

$$d(u,v) < \delta = \frac{1}{m^{n+1}}$$

$$\Rightarrow d\big((u_0 u_1 \dots u_{n+1} \dots), (v_0 v_1 \dots v_{n+1} \dots)\big) < \frac{1}{m^{n+1}}$$

$$\Rightarrow u_i = v_i \text{ for } i = 0,1,2, \dots \dots ., n+1$$

$$\Rightarrow u_i^c = v_i^c \text{ for } i = 1,2, \dots \dots ., n+1$$

$$\Rightarrow d\big((u_0^c u_1^c \dots u_{n+1}^c \dots), (v_0^c v_1^c \dots v_{n+1}^c \dots)\big) < \frac{1}{m^n} < \varepsilon \text{ using Theorem 3.3.1}$$

$$\Rightarrow d\big(\varphi(u), \varphi(v)\big) < \frac{1}{m^n} < \varepsilon$$

which proves that φ is continuous on the symbol space $\sum_m$.

Again by our construction of φ we see that $\varphi^2(s) = \varphi(s_0' s_1' \dots \dots) = (s_0 s_1 \dots \dots) = s$, where $s = (s_0 s_1 \dots \dots)$ is any point of $\sum_2$.

Similarly, we can write $\varphi^m(s) = s, m > 2$, where $s = (s_0 s_1 \dots \dots)$ is any point of $\sum_m$. Hence all points of $\sum_m$ are period *m* distinct points by the continuous map φ.

3.5 Topological Conjugacy and Some Chaotic properties

As referred to dynamical systems, the word conjugacy means the comparison between the dynamical behavior of two mappings. This section proves that σ is exact Devaney chaotic (EDevC), and it is conjugate to the quadratic map.

Theorem 3.5.1 $\sigma\colon \sum_m \to \sum_m$ and $f_m\ : R/Z \to R/Z$ are topologically semi-conjugated where $f_m(u) = mu(mod\ 1)$.

Proof: Let $\Psi\colon \sum_m \to \frac{R}{Z} = I/\sim$ such that $\Psi(u_1u_2u_3 \ldots \ldots) = \sum_{i=1}^{\infty} \frac{u_i}{m^i}$. This map is well defined because the series $\sum_{i=1}^{\infty} \frac{x_i}{m^i} \le \sum_{i=1}^{\infty} \frac{m-1}{m} = 1$ is convergent. We have to show that this mapping is a topological semi-conjugacy between σ and f_m.

To see Ψ is surjective, we consider every real number $x \in I = [0,1]$ has an m-nary expansion, so, we have $u = \sum_{i=1}^{\infty} \frac{u_i}{m^i}$, where $u_i \in \{0,1,2, \ldots \ldots m-1\}$. Then, the digits in the m-nary expansion for u will form the sequences $\bar{u} = (u_1u_2u_3 \ldots \ldots)$.
As $u_i \in \{0,1,2, \ldots \ldots m-1\}$, clearly, $\bar{u} \in \sum_m$. So we can write $\Psi(\bar{u}) = \sum_{i=1}^{\infty} \frac{u_i}{m^i} = u$. Hence, Ψ is surjective.

For any $\bar{u} = (u_1u_2u_3 \ldots \ldots) \in \sum_m$, we have $\sigma(\bar{u}) = (u_2u_3u_4 \ldots \ldots) \in \sum_m$.

Now $(\Psi\ o\ \sigma)(\bar{u}) = \Psi(\sigma(\bar{u})) = \Psi(u_2u_3u_4 \ldots \ldots)$

$$= \sum_{i=1}^{\infty} \frac{u_{i+1}}{m^i}.$$

Also, we get $(f_m\ o\ \Psi)(\bar{u}) = f_m(\Psi(\bar{u}))$

$$= f_m\left(\sum_{i=1}^{\infty} \frac{u_i}{m^i}\right)$$

$$= m\left(\sum_{i=1}^{\infty} \frac{u_i}{m^i}\right)(mod\ 1)$$

$$= \left[u_1 + \sum_{i=1}^{\infty} \frac{u_{i+1}}{m^i}\right](mod\ 1)$$

$$= \sum_{i=1}^{\infty} \frac{u_{i+1}}{m^i}$$

$$= (\Psi \; o \; \sigma)(\bar{u}), \forall \; \bar{u} \in \sum_m$$

Hence, we can conclude that $\Psi \; o \; \sigma = f_m \; o \; \Psi$.

Thus $\Psi: \sum_m \to \frac{R}{Z} = I/\sim$is a semi-conjugacy between σ and f_m.

Theorem 3.5.2 $\sigma: \sum_m \to \sum_m$ is exact Devaney chaotic (EDevC).

Proof: To prove $\sigma: \sum_m \to \sum_m$ is exact [1], consider $U \in \sum_m$,where U is a non-empty open set. For $u \in U, \exists$ an open ball $B(u, r)$ such that $B(u, r) \subseteq U$. Then, we can choose some $k \in N$ such that $\frac{1}{m^k} \leq r$.

Putting $\frac{1}{m^k} = r_1$, we get $r_1 = \frac{1}{m^k} \leq r$ and so $B(u, r_1) \subseteq B(u, r) \subseteq U$.

Then for every $v \in B(u, r_1)$, we can write $d(u, v) < r_1 = \frac{1}{m^k}$.

Hence the k^{th} iterates of all these points in $B(u, r_1)$ constitute the space $\sum_m$, i.e. $\sigma^k(B(u, r_1)) = \sum_m$.

Also, $B(u, r_1) \subseteq U \Rightarrow \sigma^k(B(u, r_1)) \subseteq \sigma^k(U)$
$\Rightarrow \sum_m \subseteq \sigma^k(U)$
$\Rightarrow \sum_m = \sigma^k(U)$, [Since $\sum_m \supseteq \sigma^k(U)$]

So $\sum_m = \sigma^k(U)$ is true for every non-empty open set $U \in \sum_m$. Hence, σ is an exact map on $\sum_m$.

From the above result, we see that σ is exact Devaney chaotic. Sinceσ is exact Devaney chaotic; therefore, it also mixing Devaney chaotic and weak mixing Devaney chaotic.

Note: Since it follows that σ is EDevC. As σ is EDevC, therefore, it also mixing Devaney chaotic and weak mixing Devaney chaotic.

Definition 3.5.1

Let $\Lambda = \{x \in I \; / \; Q_c^n(x) \in I, \forall \; n\}$ be the invariant set of the quadratic map. Here we consider the quadratic family is $Q_c(x) = x^2 + c, c < -\frac{5+2\sqrt{5}}{4}$.

It shows from the following theorem that shift map, σ is conjugate to the quadratic map, $Q_c(x)$ on its invariant set.

Theorem 3.5.3 σ on $\sum_m$ is conjugate to the quadratic map, $Q_c(x)$ on its invariant set, Λ when $c < -2.368$.

Proof: We prove $S: \Lambda \to \sum_m$ homeomorphism. To prove this, we need to show that S is one-to-one and onto and that both S and S^{-1} are continuous.

(i) $S: \Lambda \to \sum_2$ is one-to-one:

Suppose $x, y \in \Lambda$ with $x \neq y$ such that $S(x) = S(y)$; then $Q_c^n(x)$ and $Q_c^n(y)$ are both in the same interval, I_0 or I_1, for each value of n. Now consider the interval $[x, y]$. Then for each n we have the following bijective correspondence

$$[x, y] \to [Q_c^n(x), Q_c^n(y)]$$

Moreover, $|Q_c^{'}(x)| > \mu > 1 \; \forall x \in I_0 \cup I_1$ and some μ. Then by the Mean value theorem, we get $H = [Q_c^n(x), Q_c^n(y)]$ is greater than $\mu^n |x - y|$. But $\mu > 1$.

So as $n \to \infty \Rightarrow$ length $[Q_c^n(x), Q_c^n(y)] \to \infty$, which is a contradiction. Since $x, y \in \Lambda$ with $x \neq y$. Hence $x = y$.

(ii) $S: \Lambda \to \sum_m$ is onto:

Let $s \in \sum_m$, we will construct $x \in \Lambda$ such that $S(x) = s$. Let

$$\begin{aligned} I_{s_0 s_1 s_2 \ldots\ldots s_n} &= \{x \in I \mid x \in I_{s_0}, Q_c(x) \in I_{s_1}, Q_c^2(x) \in I_{s_2}, \ldots \ldots, Q_c^n(x) \in I_{s_n}\} \\ &= I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap \ldots \ldots \cap Q_c^{-n}(I_{s_n}) \\ &= I_{s_0} \cap Q_c^{-1}\left(I_{s_1} \cap \ldots \ldots \cap Q_c^{-(n-1)}(I_{s_n})\right) \\ &= I_{s_0} \cap Q_c^{-1}(I_{s_0 s_1 s_2 \ldots\ldots s_n}) \end{aligned}$$

By induction, on the number of subscripts in $I_{s_0 s_1 s_2 \ldots\ldots s_n}$, we can prove $I_{s_0 s_1 s_2 \ldots\ldots s_n}$ is always a single closed interval. Moreover, these closed subintervals are nested because

$$I_{s_0 s_1 s_2 \ldots\ldots s_n} = I_{s_0 s_1 s_2 \ldots\ldots s_{n-1}} \cap f Q_c^{-n}(I_{s_n}) \subset I_{s_0 s_1 s_2 \ldots\ldots s_{n-1}}$$

Thus $\bigcap_{n \geq 0} I_{s_0 s_1 s_2 \ldots\ldots s_n} \neq \emptyset$; and $x \in \bigcap_{n \geq 0} I_{s_0 s_1 s_2 \ldots\ldots s_n} \Rightarrow x \in \Lambda, S(x) = s$.

So S is onto.

(iii) $S: \Lambda \to \sum_m$ is continuous:

Let $\varepsilon > 0$ be given and pick n such that $\frac{1}{m^n} < \varepsilon$. Let J_n be a closed interval such that $J_n \subset I_{s_n}, Q_c^n(x) \in J_n$. Then $Q_c^{-1}(J_n)$ consists of two closed intervals I_0 and I_1. Let J_{n-1} be the closed interval in $I_{s_{n-1}}$ that contains $Q_c^{n-1}(x)$.

Proceed in this way for $0 \le i \le n$ to obtain closed intervals $J_i \subset I_{s_i}$ such that $Q_c^i(x) \in J_i$.

Then $x, y \in J_0 \Rightarrow Q_c(x), Q_c(y) \in J_1 \Rightarrow \dots \dots \Rightarrow Q_c^n(x), Q_c^n(y) \in J_n$.

Consequently, $x, y \in \Lambda \cap J_0 \Rightarrow d\big(S(x), S(y)\big) \le \frac{1}{m^n} < \varepsilon$.

Therefore, S is continuous at x.

(iv) $S^{-1}: \sum_m \to \Lambda$ exists:

Since $S: \Lambda \to \sum_m$ is one-to -one and onto, $S^{-1}: \sum_m \to \Lambda$ exists.
Hence $Q_c \circ S^{-1} = S^{-1} \circ \sigma$. In general, $Q_c^n \circ S^{-1} = S^{-1} \circ \sigma^n$.

(v) S^{-1} is continuous:

Let $s = S(x)$, $t = S(y)$. As $c < -2.368$, there is $\mu > 1$ such that

$|Q_c^n(x) - Q_c^n(y)| \ge \mu^n |x - y| \Leftrightarrow |x - y| \le \frac{|Q_c^n(x) - Q_c^n(y)|}{\mu^n}$.

But $|Q_c^n(x) - Q_c^n(y)| < L_1$, where L_1 is the length of I_1. Pick n large enough so that $\frac{L_1}{\mu^n} < \varepsilon$.

Then $d\big(S(x), S(y)\big) < \frac{1}{m^n} \Rightarrow |x - y| < \varepsilon \Leftrightarrow \big|S^{-1}\big(S(x)\big) - S^{-1}\big(S(x)\big)\big| < \varepsilon$, which proves the continuity of S^{-1}.

It concludes the proof that S is a homeomorphism.

3.6 Summary and Conclusions

In this chapter, we have corroborated important characteristics of σ on $\sum_m$, we have verified some new results by using the features of conjugacy (topologically). We have established Proximity theorem 3.3.1, and using this theorem proved that chaotic features of σ. We have shown that the shift map is a homeomorphism. It is found that σ and $f_m(u) = mu(mod\ 1)$ are topologically semi-conjugated. We have determined in this chapter that σ is exact Devaney chaotic. In the last part of this section, we have proved that σ is conjugate to $Q_c(x)$ on Λ when $c < -2.368$.

CHAPTER -4
CHAOTICITY OF THE GENERALIZED SHIFT MAP ON $\sum_m$

4.1 Introduction

The aim of this chapter is to introduce a generalization of the shift map. It is well known that the three conditions of Devaney's definition of chaos for continuous mappings are: (a) density of periodic points, (b) topologically transitive, and (c) sensitive dependence on initial conditions. Later it was demonstrated in [1] that the conditions (a) and (b) together imply condition (c). In the previous chapter, we have defined the space $\sum_m$. Properties of the shift map are discussed in Chapter 3. Here we have taken the metric d as defined in Chapter 3.

In section 4.3, we have proved some features of the generalized shift map such as periodic points that are dense in $\sum_m$ and topologically transitive. We have established few stronger characteristics of the generalized shift map related to chaos, such as totally transitive, strong sensitive dependence on initial conditions, topologically mixing, and chaotic dependence on initial conditions in section 4.4. Section 4.4 of this chapter provides an example of a continuous function is chaotic but not topologically transitive.

4.2 Basic Concepts

We start this section by giving some elementary definitions and results, which are the requirements for establishing the main theorem. We first define a generalized shift map and some other definitions and lemma necessary for this chapter.

Definition 4.2.1 Generalized shift map

The generalized shift map $\sigma_n: \sum_m \to \sum_m$ is defined by $\sigma_n(s) = (s_n s_{n+1} \dots \dots \dots)$, where $s = (s_0 s_1 \dots \dots \dots s_n \dots \dots)$ is any point of $\sum_m$ and $n \geq 1$ is any integer.

Definition 4.2.2 (Weakly topologically mixing):

f is said to be weakly topologically mixing if for every non-empty pair $U, V \subset X$, $\exists$ a positive integer k such that $f^k(U) \cap V \neq \emptyset$.

Next, we have to prove that σ_n is continuous on $\sum_m$.

Lemma 4.2.1 $\sigma_n: \sum_m \to \sum_m$ is continuous in the space $\sum_m$.

Proof: Choose p so large that $\frac{1}{m^{np}} < \varepsilon, \varepsilon > 0$. Also, consider $\delta = \frac{1}{m^{n(p+1)}}$ and $s = (s_0 s_1 s_2, \dots \dots), t = (t_0 t_1 t_2, \dots \dots)$ be any two points of $\sum_m$ which satisfies $d(s,t) < \delta$. So, $d(s,t) < \frac{1}{m^{n(p+1)}}$.

Hence by Lemma 3.2.1, we get $s_i = t_i$, for $i = 1,2, \dots \dots \dots, n(p+1)$.

Obviously $d\left(\sigma_n^p(s), \sigma_n^p(t)\right) = d\left((s_{np} s_{np+1} \dots \dots), (t_{np} t_{np+1} \dots \dots)\right)$

$$< \frac{1}{m^{np}} < \varepsilon.$$

So we get our desired result, that is, $d(s,t) < \delta \Rightarrow d\left(\sigma_n^p(s), \sigma_n^p(t)\right) < \varepsilon$.

Hence σ_n is continuous on $\sum_m$.

4.3 Some features of the generalized shift map

Here we discuss some essential characteristics of σ_n. According to the definition of Devaney's chaos, we have established that
$\sigma_n: \sum_m \to \sum_m$ is chaotic. For this, at first, we need to prove that two essential conditions of Devaney's chaos, such as periodic points are dense and topologically transitive.

Theorem 4.3.1 Periodic points are dense in $\sum_m$ for σ_n.

Proof: Consider $s = (s_0 s_1 s_2 \dots \dots)$ be any point of $\sum_m$ and we choose p in a way that $\frac{1}{m^{np-1}} < \varepsilon, \varepsilon > 0$.

Again, choose the point $\alpha_p = (s_0 s_1 s_2 \dots \dots s_{np-1} s_0 s_1 s_2 \dots \dots s_{np-1} s_0 \dots \dots)$.

Then $\sigma_n^p(\alpha_p) = \alpha_p$. Hence α_p is a p-periodic point of σ_n. Now by our construction, we see that s and α_p are similar up to np^{th} entry. Then by Theorem 3.3.1, we can say that $d(\alpha_p, s) < \frac{1}{m^{np-1}} < \varepsilon$. This gives that $\alpha_p \to s$. So, periodic points are dense in $\sum_m$for $\sigma_n: \sum_m \to \sum_m$.

Theorem 4.3.2 $(\sum_m, \sigma_n)$ is topologically transitive on $\sum_m$.

Proof: Let $U, V \in \sum_m$ (U and V are non-empty open sets) and $\varepsilon_1, \varepsilon_2 > 0$. Also, consider $u = (u_0 u_1 \dots\dots\dots) \in U$ such that min $\{d(u, \beta_1)\} \geq \varepsilon_1$, for any β_1 is the boundary of U. Similarly, consider $v = (v_0 v_1 \dots\dots\dots) \in V$ be any point such that min $\{d(u, \beta_2)\} \geq \varepsilon_2$for any β_2 is the boundary of V. Now, choose $k_1$and k_2 so that $\frac{1}{m^{nk_1-1}} < \varepsilon_1$ and $\frac{1}{m^{nk_2}} < \varepsilon_2$, where n is an arbitrary positive integer. Now let $\beta_3 = (u_0 u_1 \dots\dots u_{nk_1-1} v_0 v_1 \dots\dots v_{nk_2} \dots\dots)$. Then by Theorem 3.3.1, we can write, $d(u, \beta_3) < \frac{1}{m^{nk_1-1}} < \varepsilon_1$.

Hence $\beta_3 \in U$, that is $\sigma_n^{k_1}(\beta_3) \in \sigma_n^{k_1}(U)$ ……………………………..………. (4.3.1)

Then again, $\sigma_n^{k_1}(\beta_3) = (v_0 v_1 \dots\dots\dots v_{nk_2} \dots\dots)$.

So$d(\sigma_n^{k_1}(\beta_3), v) < \frac{1}{m^{nk_2}} < \varepsilon_2$, by applying Theorem 3.3.1 again.

Its gives $\sigma_n^{k_1}(\beta_3) \in V$. ……………………………………………………..……(4.3.2)

By (4.3.1) and (4.3.2), we can write $\sigma_n^{k_1}(U) \cap V \neq \emptyset$.

Hence $(\sum_m, \sigma_n)$ is topologically transitive on $\sum_m$.

Theorem 4.3.3 $\sigma_n: \sum_m \to \sum_m$ is exact Devaney chaotic (EDevC).

Proof: To prove σ_n is exact, let $U \in \sum_m$, where U is a non-empty open set. For $u \in U$, $\exists$ an open ball $B(u, r)$ such that $B(u, r) \subseteq U$. Then, we can choose some $k \in N$ such that $\frac{1}{m^{nk}} \leq r$.

Putting $\frac{1}{m^{nk}} \leq r$. we get $r_1 = \frac{1}{m^{nk}} \leq r$ and so $B(u, r_1) \subseteq B(u, r) \subseteq U$.

Then for every $v \in B(u, r_1)$, we can write $d(u, v) < r_1 = \frac{1}{m^{nk}}$.

Hence the k^{th} iterates of all these points in $B(u, r_1)$ constitute the space $\sum_m$.

That is $\sigma_n^k\big(B(u, r_1)\big) = \sum_m$.

Also, $B(u, r_1) \subseteq U \Rightarrow \sigma_n^k(B(u, r_1)) \subseteq \sigma_n^k(U)$
$\Rightarrow \sum_m \subseteq \sigma_n^k(U)$
$\Rightarrow \sum_m = \sigma_n^k(U)$, [Since $\sum_m \supseteq \sigma_n^k(U)$]

So $\sum_m = \sigma_n^k(U)$ is true for every non-empty open set $U \in \sum_m$. Hence, σ_n is an exact map on $\sum_m$.

From the above result, we see that σ_n is exact Devaney chaotic. Since σ_n is exact Devaney chaotic; therefore, it also mixing Devaney chaotic and weak mixing Devaney chaotic.

4.4 Some Stronger chaotic features of σ_n

This section has proved a few more vital chaotic features of σ_n.

Theorem 4.4.1 $\sigma_n: \sum_m \to \sum_m$ is totally transitive on $\sum_m$.

Proof: Consider $U, V \in \sum_m$ (*U* and *V* are two non-empty open sets) and $\varepsilon_1, \varepsilon_2 > 0$. Also, let $u = (u_0 u_1 \dots \dots \dots.) \in U$ such that $\min\{d(u, \beta_1)\} \geq \varepsilon_1$, where $\beta_1$belongs to boundary of *U*. Similarly, let $v = (v_0 v_1 \dots \dots \dots) \in V$ such that $\min\{d(v, \beta_2)\} \geq \varepsilon_2$ where β_2 belongs to boundary of *V*. Next, choose two odd integers $k_1$and k_2 so large that $\frac{1}{m^{nk_1-1}} < \varepsilon_1$ and $\frac{1}{m^{nk_2}} < \varepsilon_2$.

To prove this theorem, consider the following cases:

(i) When *n* is an even integer.

If we take $\alpha = (u_0 u_1 \dots \dots u_{nk_1-1} v_0 v_1 \dots \dots v_{nk_2} \dots \dots)$. Then by Lemma 3.2.1, $d(u, \alpha) < \frac{1}{m^{nk_1-1}} < \varepsilon_1$.

Hence $\alpha \in U$, that is $\sigma_n^{k_1}(\alpha) \in \sigma_n^{k_1}(U)$.
Then again, $\sigma_n^{k_2}(\alpha) = (v_0 v_1 \dots \dots v_{nk_2} \dots \dots)$.

So $d(\sigma_n^{k_2}(\alpha), v) < \frac{1}{m^{nk_2}} < \varepsilon_2$ by applying Lemma 3.2.1 again.

We get $\sigma_n^{k_1}(\alpha) \in V$.

Hence, we can write $\sigma_n^{k_1}(U) \cap V \neq \emptyset$, where n is an even integer.

So, $\sigma_n: \sum_m \to \sum_m$ is totally transitive on $\sum_m$, when n is an even integer.

(ii) Here, we consider n is an odd integer.

Let $\beta = (u_0 u_1 \dots \dots u_{nk_1-1} v_0' v_1' \dots \dots v_{nk_2}' \dots \dots)$.

Then by applying Lemma 3.2.1 again,

we can write $d(u, \beta) < \frac{1}{2^{nk_1-1}} < \varepsilon_1$. Hence $\beta \in U$,

that is $\sigma_n^{k_1}(\beta) \in \sigma_n^{k_1}(U)$. On the other hand, $\sigma_n^{k_1}(\beta) = (v_0 v_1 \dots \dots v_{nk_2} \dots \dots)$.

So, $d(\sigma_n^{k_1}(\beta), v) < \frac{1}{2^{nk_2}} < \varepsilon_2$ by applying Lemma 3.2.1 again.

We have $\sigma_n^{k_1}(\beta) \in V$. Hence we can write $\sigma_n^{k_1}(U) \cap V \neq \emptyset$, where n is an odd integer.

So $\sigma_n: \sum_m \to \sum_m$ is totally transitive on $\sum_m$, when n is an odd integer.

Joining (i) and (ii), we can write σ_n is totally transitive on $\sum_m$.

The following example showing that chaotic functions are not necessarily topologically transitive.

Example 4.4.1 Consider $F_4(x) = 4x(1-x)$ is the logistic map on [0,1] and the map $g(x)$ from [-1,1] to itself defined by $g(x) = \begin{cases} -x \ , -1 \leq x \leq 0 \\ F_4(x) \ , 0 \leq x \leq 1 \end{cases}$.

The map defined above is also continuous. If we choose $U = (0,1)$ and $V = (-1,0)$ then $f^k(U) \cap V = \emptyset \ \forall \ k$ so we see that $g(x)$ is chaotic but not transitive (topologically).

Theorem 4.4.2 $(\sum_m, \sigma_n)$ is topologically mixing.

Proof: To prove the above statement at first take $U, V \in \sum_m$ and $u = (u_0 u_1 u_2 \dots \dots) \in U$ such that min $\{d(u, \beta_1)\} \geq \varepsilon_1$, where $\beta_1$belongs to boundary of U. Again, consider

$v = (v_0 v_1 \dots \dots \dots) \in V$ such that $\min\{d(v, \beta_2)\} \geq \varepsilon_2$ for β_2 belongs to boundary of V. Now, pick two integers $k_1$and k_2 so that $\frac{1}{m^{nk_1-1}} < \varepsilon_1$ and $\frac{1}{m^{nk_2-1}} < \varepsilon_2$, where n is a positive integer.

Now let $\gamma_i = (u_0 u_1 \dots \dots u_{nk_1-1} (0)^{n(i-1)} v_0 v_1 \dots \dots v_{nk_2} \dots \dots)$, for $i = 2,3, \dots ..$

and $\gamma_1 = (u_0 u_1 \dots \dots u_{nk_1-1} v_0 v_1 \dots \dots v_{nk_2} \dots \dots)$.

Then by Theorem 3.3.1,

$d(u, \gamma_i) < \frac{1}{m^{nk_1-1}} < \varepsilon_1$, $i = 1,2, \dots \dots$ ………………………………………..(4.4.1)

Hence $\gamma_i \in U$, $i = 1,2, \cdots \cdots$, that is $\sigma_n^{k_1}(\gamma_i) \in \sigma_n^{k_1}(U)$, for any $k \geq 0$.

On the other hand, $\sigma_n^{k_1}(\gamma_1) = (v_0 v_1 \dots \dots \dots v_{nk_2-1} \dots \dots)$.

Hence by applying theorem 3.3.1, we get

$d(\sigma_n^{k_1}(\gamma_1), v) < \frac{1}{m^{nk_2-1}} < \varepsilon_2$ ……...……………………………………………(4.4.2)

which gives $\sigma_n^{k_1}(\gamma_1) \in V$.

In virtue of (4.4.1) and (4.4.2), we can see that $\sigma_n^{k_1}(U) \cap V \neq \emptyset$. Next, let γ_2 such that $\sigma_n^{k_1+1}(\gamma_2) = (v_0 v_1 \dots \dots \dots v_{nk_2-1} \dots \dots) = V$. Hence $\sigma_n^{k_1+1}(U) \cap V \neq \emptyset$. Finally taking about all γ_i's we get $\sigma_n^{k_1}(U) \cap V \neq \emptyset, \forall\, k \geq k_1$.

So $(\sum_m, \sigma_n)$ is topologically mixing on $\sum_m$.

Theorem 4.4.3 σ_n on $\sum_m$ is generically δ-chaotic with $\delta = diam(\sum_m) = 1$.

Proof: From Theorem 4.4.2, we see that σ_n is topologically mixingon $\sum_m$. As we know that topologically mixing continuous map on a compact metric space is also a topologically weak mixing, so σ_n is also topologically weak mixing. Now, using Proposition 3.2.1, it is established that$\sigma_n$is generically δ-chaotic on $\sum_m$ with
$\delta = diam(\sum_m) = 1$.

Theorem 4.4.4 If $\sigma_n : \sum_m \to \sum_m$ is the generalized shift map. Then $u \in \sum_m$ and any open neighborhood U of u, $\exists$ two non-empty subsets S and T of U, which satisfy the following three conditions:

(i) Both S and T are countable.

(ii) $S \cap T = \emptyset$ and

(iii) $d\left(\sigma_n^{n_j}(s), \sigma_n^{n_j}(u)\right) = 1$, for all $s \in S$ and $d\left(\sigma_n^{m_j}(t), \sigma_n^{m_j}(u)\right) = 0, \forall\, t \in T$, where n_j's and m_j's are different for different points of S and T and depend on the minimum distance of u from the boundary of U.

Proof: Consider $u = (u_0 u_1 \dots \dots) \in \sum_m$ and $\varepsilon > 0$. Choose p such that $\frac{1}{m^{np}} < \varepsilon$, $\forall\, n \geq 1$.

Consider $S = \{s_i : s_i = (u_0 u_1 \dots \dots u_{mni-1} u'_{mni} u'_{mni+1} u'_{mni+2} \dots \dots), i \geq p\}$ and
$T = \{t_j : t_j = (u_0 u_1 \dots \dots u_{mni-1} u'_{mni} u'_{mni+1} u'_{mni+2} \dots \dots), j \geq p\}$.

Now by formation, we follow that all s_i's of S agree with u at least up to u_{np}. So using Lemma 3.2.1, we have that $d(u, s_i) < \frac{1}{m^{np}}$, $\forall\, s_i \in S$. So $s_i \in U$, $\forall\, i \geq p$ and we observe that S is a non-empty subset of U. Similarly, it is proved that T is also a non-empty subset of U. Again by our making structure, it has shown that both S and T are countable. So, (i) is proved.

From two sets S and T we see that after the $(mni + mn)$-th (or $(mnj + mn)$-th) all terms of s_i or t_i are mutually complementary terms for all i (or j). So, $t_j \neq s_i$, $\forall i$ and j, that is, $S \cap T = \emptyset$. It is so proved (ii).

Now
$$d\left(\sigma_n^{mi}(s), \sigma_n^{mi}(u)\right) = d\left((u'_{np} u'_{np+1} \dots \dots), (u_{np} u_{np+1} \dots \dots)\right)$$
$$= \frac{1}{m} + \frac{1}{m^2} + \cdots$$
$$\geq \frac{1}{m-1}, \text{ for all } s_i \in S \text{ and}$$

$$d\left(\sigma_n^{mj+m}(t_j), \sigma_n^{mj+m}(u)\right)$$
$$= d\left((u_{mnj} \quad u_{mnj+mn+1} \dots \dots), (u'_{mnj+mn} u'_{mnj+mn+1} \dots \dots)\right)$$
$$= \frac{0}{m} + \frac{0}{m^2} + \cdots$$
$$= 0, \text{ for all } t_j \in T$$

Also, by the construction of S and T it is observed that $i \geq p$ and $j \geq p$, for S and T respectively, and p are dependent on ε, where ε is the minimum distance of x from the boundary of U. So, we conclude that n_j's and m_j's are depending on the minimum distance of u from boundary of U. So, (iii) is proved.

Hence all the above three conditions of the theorem are proved.

Theorem 4.4.5 $\sigma_n: \sum_m \rightarrow \sum_m$ has strong sensitive dependence on initial conditions.

Proof: Consider $u = (u_0 u_1 \dots \dots) \in \sum_m$ and S be any non-empty open set of $\sum_m$. So we can make an open ball T with radius $\varepsilon > 0$ and center at $\alpha = (\alpha_0 \alpha_1 \dots \dots)$, such that $T \subset S$. If $p > 0$ is an integer such that $\frac{1}{m^{np-1}} < \varepsilon$.

Let $v = \left(\alpha_0 \alpha_1 \dots \dots \alpha_{np-1} u'_{np} u'_{np+1} \dots \dots\right)$. Then v agrees by α up to α_{np-1} and later, all terms of v are the complementary terms of u, starting with u'_{np}.

Using Lemma 3.2.1, we have that $d(\alpha, v) < \frac{1}{m^{np-1}} < \varepsilon$. So $v \in T$, that is $v \in S$ also.

Again, we get $d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = d\left(\left(u_{np} u_{np+1} \dots \dots\right), \left(u'_{np} u'_{np+1} \dots \dots\right)\right)$

$$= \frac{1}{m} + \frac{1}{m^2} + \cdots$$

$$\geq \frac{1}{m-1}$$

From the above relation, it is clear that $\frac{1}{m-1}$ is the role of sensitivity constant δ.

So, for $u \in \sum_m$ and $\exists\, v \in N(u)$ and $n > 0$ satisfying

$d\left(\sigma^n(u), \sigma^n(v)\right) \geq \delta$ for $\delta = \frac{1}{m-1}$.

Hence $\sigma_n: \sum_m \rightarrow \sum_m$ has strong sensitive dependence on initial conditions.

Theorem 4.4.6 Given countable infinite subset $U \in \sum_m$ and $\exists$a dense uncountable

1-scrambled set S of transitive points of $\sum_m$ such that, for $u \in U$ and $v \in P$ we have to prove that

$\lim_{p \to \infty} sup\, d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = \frac{1}{m-1}$ and $\lim_{p \to \infty} inf\, d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = 0.$

Proof: To establish the above theorem 4.4.6, we first require some properties, and we need to prove Lemma 4.4.1.

Properties:

(a) Consider $P = p_0p_1 \dots\dots p_{\mathrm{i}}$ and $Q = q_0q_1 \dots\dots q_m$ are two sequences of 0's and 1's then $PQ = p_0p_1 \dots\dots p_{\mathrm{i}}q_0q_1 \dots\dots q_m$. Again, if $T_1T_2 \dots\dots T_p$ are p finite sequences of 0's and 1's then $T_1T_2 \dots\dots T_p$ can be defined similarly as above.

(b) Let $k \geq 2$ be any integer, $\beta = (\beta_0\beta_1 \dots\dots)$ be any element of $\sum_m$ and $a = (a_0a_1a_2 \dots\dots)$ be a transitive point of $\sum_m$.

We now define,

$$E(\beta, k) = a_0a_1a_2 \dots\dots a_{nk-1}(\beta_0)^{n(k-1)}(\beta_1)^{n(k-1)} \dots\dots$$

$$(\beta_{k-1})^{n(k-1)}(0^n1^n)^{(k-2)!}(0^{mn}1^{mn})^{(k-2)!} \dots\dots \left(0^{n(k-1)}1^{n(k-1)}\right)^{(k-2)!}\beta_0\beta_1$$

$$\dots\dots \beta_{nk-1}.$$

(c) Let $U = \{u_{\mathrm{i}}: u_{\mathrm{i}} = u_{\mathrm{i},0}u_{\mathrm{i},1} \dots\dots, \mathrm{i} \geq 1\} \in \sum_m$.

Now define

$$F(u_{\mathrm{i}}, k) = \left(u_{i,n.t(k,i,j)}\, u_{i,n.t(k,\mathrm{i},\mathrm{j})+1} \dots\dots u_{i,n.t(k,i,j)+nk-1}\right) \text{ and}$$

$$\tilde{F}(u_{\mathrm{i}}, k) = \left(u_{i,n.t(k,i,j)+nk}\, u_{i,n.t(k,i,j)+nk+1} \dots\dots u_{i,n.t(k,i,j)+mnk-1}\right)$$

where $t(k, i, j) = 2k + k^2 + k! + 2k(i - 1) - j$.

Also, we denote $F(u_{\mathrm{i}}, k)\tilde{F}(u_{\mathrm{i}}, k)$ by $\tilde{F}(u_i, k)$.

(d) Define,

$$\tau_\beta = (b_0)^n(b_1)^n \dots\dots (b_{\mathrm{k}-1})^n E(\beta, k)\tilde{F}(x_1, k)\tilde{F}(x_2, k) \dots\dots \tilde{F}(x_k, k)$$

$$E(\beta, k+1)\tilde{F}(x_1, k+1)\tilde{F}(x_2, k+1) \dots\dots \tilde{F}(x_{k+1}, k+1)E(\beta, k+2) \dots\dots,$$

where $b_j (0 \leq j \leq k-1)$ are any fixed k numbers composed of 0's and 1's.

(e) Lastly, consider $P = \{\sigma_n^p(\tau_\beta): p \geq 0, \beta \in \sum_m\}$.

At this time, we also note that,

(i) The length of the finite sequence $E(\beta, k)$ is $nk^2 + nk! + nk$.

(ii) The length of the sequence $(b_0)^n(b_1)^n \dots \dots (b_{k-1})^n$ is nk.

(iii) The length of the sequence

$(b_0)^n(b_1)^n \dots \dots (b_{k-1})^n E(\beta, k)\tilde{F}(x_1, k)\tilde{B}(x_2, k) \dots \dots \tilde{F}(x_k, k)$

is $3nk^2 + nk! + 2nk$.

Lemma 4.4.1 The set $P = \{\sigma_n^p(\tau_\beta): p \geq 0, \beta \in \sum_m\}$ as defined above, is a dense uncountable invariant set of transitive points.

Proof: Now $\sigma_n(P) \subset P$, by our construction. Also, the set is uncountable. We now show that P is a dense subset of $\sum_m$. Consider an arbitrary point $\alpha = (\alpha_0\alpha_1 \dots \dots)$ of $\sum_m$. Then $\sigma_n^{k^2+k!+k}(\tau_\alpha)$ is a point of P. Note that $k \geq 2$ is an integer. Again,

$$d\left(\sigma_n^{k^2+k!+k}(\tau_\alpha), \alpha\right) = d\left((\alpha_0\alpha_1 \dots \dots \alpha_{nk-1} \dots \dots), (\alpha_0\alpha_1 \dots \dots)\right)$$

$$< \frac{1}{m^{nk-1}}, \text{ using Lemma 3.2.1}$$

So, $\sigma_n^{k^2+k!+k}(\tau_\alpha) \to \alpha$, which proves that there is a sequence of points from P, converges to an arbitrary point of $\sum_m$.

Lastly, note that every point of P contains infinitely many finite sequences of fixed transitive points of the type $a_0a_1 \dots \dots a_{nk-1}$ in every $E(\beta, k)$, $k \geq 2$ and for any

$\beta \in \sum_m$. So we can see that orbits of P appear arbitrarily close to any sequence of points from $\sum_m$. Hence S consists of transitive points.

Now we will prove Theorem 4.4.6.

Consider $P = \{\sigma_n^p(\tau_\beta): p \geq 0, \beta \in \sum_m\}$ and $\beta = (\beta_0\beta_1 \dots \dots) \in \sum_m$,

$\gamma = (\gamma_0\gamma_1 \dots \dots) \in \sum_m$ such that $\beta_p \neq \gamma_p$. Then, for two integers i and j such that

$0 \leq i \leq j$ and $j \leq k - 1$.

Using σ_n, $2k + s.(k-1)$ times on the two points τ_β and τ_γ,

We have $\sigma_n^{2k+p.(k-1)}(\tau_\beta) = (\beta_p)^{n(k-1)}(\beta_{p+1})^{n(k-1)} \dots \dots$

and $\sigma_n^{2k+p.(k-1)}(\tau_\gamma) = (\gamma_p)^{n(k-1)}(\gamma_{p+1})^{n(k-1)} \dots \dots$

So,

$$\lim_{p\to\infty} sup\, d\left(\sigma_n^p\left(\sigma_n^i(\tau_\beta)\right), \sigma_n^p\left(\sigma_n^i(\tau_\gamma)\right)\right) \geq \lim_{k\to\infty} d\left(\left(\beta_p\right)^{n(k-1)} \ldots\ldots \left(\gamma_p\right)^{n(k-1)} \ldots\ldots\right)$$

$$\geq \lim_{k\to\infty}\left(\frac{1}{m}+\frac{1}{m^2}+\cdots+\frac{1}{m^{n(k-1)}}\right)$$

$$= \frac{1}{m-1} \qquad \ldots\ldots(4.4.3)$$

Now, let $t_k = k^2 + k + (j-1)\big(j-(i+1)\big).(k-2)!$ then we can write

$$\sigma_n^{t_k}(\tau_\beta) = \left(0^{n(j-i)}1^{n(j-i)}\right)^{(k-2)!} \ldots\ldots$$

$$= 0^{n(j-i)}\left(1^{n(j-i)}0^{n(j-i)}\right)^{(k-2)!-1}1^{n(j-i)} \ldots\ldots$$

and

$$\sigma_n^{t_k}(\tau_\gamma) = \left(0^{n(j-i)}1^{n(j-i)}\right)^{(k-2)!} \ldots\ldots$$

$$= 0^{n(j-i)}\left(1^{n(j-i)}0^{n(j-i)}\right)^{(k-2)!-1}1^{n(j-i)} \ldots\ldots$$

So from the above, we can say that although τ_β and τ_γ are different; after certain number of iterations, they begin with the same sequence. Since $E(\beta,k)$ and $E(\gamma,k)$ both have a finite sequence of the type $\left(0^{n(j-i)}1^{n(j-i)}\right)^{(k-2)!}$ of 0's and 1's.

So,

$$\sigma_n^{t_k}(\tau_\gamma) = \left(1^{n(j-i)}0^{n(j-i)}\right)^{(k-2)!-1}1^{n(j-i)} \ldots\ldots = 1^{n(j-i)}\left(0^{n(j-i)}1^{n(j-i)}\right)^{(k-2)!-1} \ldots\ldots$$

Consequently, $\lim_{p\to\infty} \sup d\left(\sigma_n^p\left(\sigma_n^i(\tau_\beta)\right), \sigma_n^p\left(\sigma_n^i(\tau_\gamma)\right)\right) \geq$

$\lim_{k\to\infty} d\left(\sigma_n^{t_k}(\tau_\beta), \sigma_n^{t_k+(j-i)}(\tau_\gamma)\right)$

$$\geq \lim_{k\to\infty}\left(\frac{1}{m}+\frac{1}{m^2}+\cdots+\frac{1}{m^{n(j-i)+2n(j-i)((k-2)!-1)}}\right)$$

$$= \frac{1}{m-1} \qquad \ldots\ldots(4.4.4)$$

Applying (4.4.3) and (4.4.4), we can write for any $u \neq v$ in S,

$$\lim_{p\to\infty} \sup d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = \frac{1}{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.4.5)$$

Note that by our construction of P, we can always choose two points $\sigma^{n^c}(\tau_\beta)$ and $\sigma^{n^c}(\tau_\gamma)$ for infinitely many integers n^c.

It gives, $\lim_{p\to\infty} inf\ d\left(\sigma_n^p(u), \sigma_n^p(v)\right) \leq \lim_{t\to\infty}\left(\frac{0}{m} + \frac{0}{m^2} + \cdots + \frac{0}{m^t}\right) = 0$, for $u \neq v$ in P and for some positive integer t.

So, $\lim_{p\to\infty} inf\ d\left(\sigma_n^p(x), \sigma_n^p(y)\right) = 0$, for $u \neq v$ in P. ……………………….(4.4.6)

Furthermore, if v is a periodic point of σ_n and for any u in P, choose a positive integer m such that $\sigma_n^m(u), \sigma_n^m(v)$ are different in the first term of the sequence.

So, $\lim_{p\to\infty} \sup d\left(\sigma_n^p(v), \sigma_n^p(u)\right) \geq \lim_{m\to\infty} d\left(\sigma_n^m(v), \sigma_n^m(u)\right) \geq \frac{1}{m-1}$ ……...(4.4.7)

Now from equations (4.4.5), (4.4.6) and (4.4.7) and Lemma 4.4.1, we can say that P is a dense invariant uncountable 1-scrambled set of transitive points for σ_n.

Finally, for $u \in U$ and $v \in P$, $\lim_{p\to\infty} \sup d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = \frac{1}{m-1}$ and

$\lim_{p\to\infty} inf\ d\left(\sigma_n^p(u), \sigma_n^p(v)\right) = 0$. (Proved)

Theorem 4.4.7 $\left(\sum_m, \sigma_n\right)$ has chaotic dependence on initial conditions.

Proof: Let $p = (p_1 p_2 p_3 \ldots \ldots) \in \sum_m$ and $N(p)$ $\exists$ an open set U of $\sum_m$ such that

$p \in U \subseteq N(p)$.

As $p \in U$ so $\exists$ an open ball $B(p, r)$ with $r > 0$ such that $B(p, r) \subseteq U \subseteq N(p)$. At that time for $r > 0$, choose k such that $\frac{1}{m^k} < r$.

We now make out $q \in B(p, r) \subseteq U \subseteq N(p)$ such that the pair $(p, q) \in \sum_m^2$ is Li-Yorke.

Now after using the letters in $p = (p_1 p_2 p_3 \ldots \ldots) \in \sum_m$, we define the words $W(p, 3nk), W(p, 5nk),\ W(p, 7nk), \ldots \ldots$etc. as follows:

$W(p, 3nk) = (p^*_{3nk+1}p^*_{3nk+2} \cdots p^*_{4nk}p_{4nk+1}p_{4nk+} \quad \cdots p_{5nk}),$

$W(p, 5nk) = (p^*_{5nk+1}p^*_{5nk+2} \cdots p^*_{6nk}p_{6nk+1}p_{6nk+2} \ \cdots p_{7n} \ )$

$W(p, 7nk) = (p^*_{7nk+1}p^*_{7nk+2} \cdots p^*_{8nk}p_{8nk+1}p_{8nk+2} \cdots p_{9nk}) , \cdots$

and so on.

Now using the above-defined words, we form the point q as follows:

$q = \left(p_1p_2p_3 \dots p_{nk}(0^*)^{nk}(0)^{nk}W(p, 3nk)W(p, 5nk)W(p, 7nk)W(p, 9nk) \dots\right)$

where $(0^*)^{nk} = \underbrace{(0^*0^*0^* \dots 0^*)^n}_{k-terms}, (0)^{nk} = \underbrace{(000 \dots 0)^n}_{k-terms}$ and

$0^* = (m-1) - 0 = m - 1.$

From the construction of q it is evident that q agrees with p up to the k^{th} term. Hence using Theorem 3.3.1, we can write $d(p, q) \le \frac{1}{m^k} < r$ and hence

$q \in B(p,r) \subseteq U \subseteq N(p)$. Here, we observe that q contains infinitely many words of the type $W(p, (2n-1)k)$ containing *2k* letters each where $n \ge 2$.

Moreover

$$\sigma_n^{3k}(q) = (p^*_{3nk+1}p^*_{3nk+2} \cdots p^*_{4nk}p_{4nk+} \ p_{4nk+2} \cdots p_{5n} \ p^*_{5nk+} \ p^*_{5nk+2} \cdots)$$

$$\sigma_n^{4k}(q) = (p^*_{4nk+1}p^*_{4nk+} \ \cdots p_{5nk}p^*_{5nk+1}p^*_{5nk+2} \cdots p^*_{6nk}p_{6nk+1}p_{6nk+2} \cdots)$$

Therefore, $sup \ d\left(\sigma_n^k(p), \sigma_n^k(q)\right) \ge d(\sigma_n^{3k}(p), \sigma_n^{3k}(q))$ and so

$$\lim_{k\to\infty} sup \ d\left(\sigma_n^k(p), \sigma_n^k(q)\right) \ge \lim_{k\to\infty} d(\sigma_n^{3k}(p), \sigma_n^{3k}(q)) \ge \lim_{k\to\infty} \sum_{r=1}^{n} \frac{|p_{3nk+r} - p^*_{3nk+} \ |}{m^r}$$

$$\ge \lim_{k\to\infty} \left\{\frac{1}{m} + \frac{1}{m^2} + \cdots + \frac{1}{m^k}\right\}$$

$$= \frac{1}{m-1}$$

Again, $0 \le \lim_{n\to\infty} inf \ d\left(\sigma_n^k(p), \sigma_n^k(q)\right)$

$$\le \lim_{n\to\infty} d(\sigma_n^{4k}(p), \sigma_n^{4k}(q))$$

$$=\lim_{n\to\infty} d((p_{4nk+1}\cdots p_{5nk}p_{5nk+1}\cdots p_{6nk}p_{6nk+1}\cdots),$$

$$(p_{4nk+1}\cdots p_{5n}\ \ p^*_{5nk+1}p^*_{5nk+2}\cdots\cdots p^*_{6nk}p_{6nk+1}\cdots\cdots))$$

$$\leq \lim_{k\to\infty}\left\{\left(\frac{m-1}{m^{nk+1}}+\frac{m-1}{m^{nk+2}}+\cdots+\frac{m-1}{m^{nk}}\right)+\left(\frac{m-1}{m^{3nk+1}}+\frac{m-1}{m^{3nk+2}}+\cdots+\frac{m-1}{m^{4n}}\right)+\cdots\right\}$$

$$=\lim_{k\to\infty}\left\{\left(\frac{m-1}{m}+\frac{m-1}{m^2}+\cdots+\frac{m-1}{m^k}\right).\left(\frac{1}{m^k}+\frac{1}{m^{3k}}+\frac{1}{m^{5k}}+\cdots\right)\right\}$$

$$=\lim_{k\to\infty}\left\{\left(1-\frac{1}{m^k}\right).\frac{1}{m^k}\left(1+\frac{1}{m^{2k}}+\frac{1}{m^{4k}}+\frac{1}{m^{6k}}\cdots\cdots\right)\right\}$$

$$=\lim_{k\to\infty}\left\{\left(1-\frac{1}{m^k}\right).\frac{1}{m^k}.\frac{1}{1-\frac{1}{m^{3k}}}\right\}$$

$$=(1-0).\,0.\left(\frac{1}{1-0}\right)=0$$

Now, $0\leq \lim_{n\to\infty} inf\ d\left(\sigma_n^k(p),\sigma_n^k(q)\right)\leq 0 \Rightarrow \lim_{n\to\infty} inf\ d\left(\sigma_n^k(p),\sigma_n^k(q)\right)=0.$

So, it follows that

$\lim_{n\to\infty} sup\ d\left(\sigma_n^k(p),\sigma_n^k(q)\right)\geq \frac{1}{m-1}$ and $\lim_{n\to\infty} inf\ d\left(\sigma_n^k(p),\sigma_n^k(q)\right)=0.$

Hence, $(p,q)\in\sum_m^2$ with modulus $\delta=\frac{1}{m-1}>0$. Hence, $(\sum_m,\sigma_n)$ has chaotic dependence on initial conditions.

4.5 Summary and Conclusions

In this chapter, we try to establish some stronger chaotic features of σ_n. From example 4.4.1, we see that $g(x)$ is chaotic but not transitive (topologically). In theorem 4.4.3, we observed that σ_n on $\sum_m$ is generically δ-chaotic with $\delta=diam\left(\sum_m\right)=1$.

We know that Devaney chaos is stronger than Li-Yorke chaos, and already, we have shown σ_n satisfies all the requirements of Devaney's chaos, so we can say that it is also Li-Yorke chaos. At present, the shift map is used to chaotic model of a dynamical system, and we can now try to use σ_n instead of σ to the chaotic model of a dynamical system. So, we can use generalized shift map, σ_n as a new model for chaotic dynamical systems.

CHAPTER -5

CHAOTIC FEATURES OF THE COMPLEMENTED SHIFT MAP ON m-SYMBOL SPACE

5.1 Introduction

We have already established stronger chaotic features of σ_n in chapter 4. This chapter is devoted to studying the chaoticity and related properties of the complemented shift map σ^c. Few important chaotic properties of σ^c have been discussed. In the last part of this chapter, it is proved that σ^c is conjugate (topologically) to σ. Finally, we provide an example to show that σ^c is chaotic, and for that reason, we say that σ^c is an alternative chaotic model in m-symbol space.

In the previous chapter, we defined m-symbol sequence space, which is

$\sum_m = \{0,1,2, \dots \dots m-1\}^N = \{(u_i)_{i=1}^{\infty} : u_i \in \{0,1,2 \dots \dots m-1\}, m \in \mathbb{N}\}$,

where $m(\geq 2) \in \mathbb{N}$, under $d(u,v) = \sum_{k\geq 1} \frac{|u_k - v_k|}{m^k}$

for $u = (u_1 u_2 u_3 \dots \dots), v = (v_1 v_2 v_3 \dots \dots) \in \sum_m$.

Topological conjugacy can be used to make predictions of the behavior of a dynamical system up to comparing it with another dynamical system whose specific properties are known [5]. The Topological conjugacy feature has an essential role in studying the chaotic behavior of a map. With the help of this feature, we can explore the chaotic significance by comparing one map with another map. Topological conjugacy has such importance as it can protect many topological dynamical properties.

If we consider $s = (s_0 s_1 \dots \dots \dots)$ is any point of $\sum_m$. Then $\sigma^c : \sum_m \to \sum_m$ is defined by $\sigma^c(s) = ({s_1}^c {s_2}^c \dots \dots \dots \dots)$, where complement of s_i is ${s_i}^c$. Here, the map shifts the first element of a point and then changes all others into its complement.

5.2 Properties of the Complemented shift map

The dynamical behavior of σ^c is discussed in this section. Here we prove that σ^c is a continuous on $\sum_m$. Using Theorem 5.2.2 we proved that it is ω-chaotic.

Theorem 5.2.1: The dynamical systems $\left(\sum_m, \sigma^c\right)$ is continuous on on $\sum_m$.

Proof: We pick n so large that $\frac{1}{m^n} < \varepsilon$, where $\varepsilon > 0$. Consider $g = (g_0 g_1 g_2 \dots \dots \dots)$ and $e = (e_0 e_1 e_2 \dots \dots \dots)$ be any two points of $\sum_m$. Now we choose $\delta = \frac{1}{m^{n+1}}$.

Then $d(g, e) < \delta = \frac{1}{m^{n+1}}$

$\Rightarrow d\big((g_0 g_1 \dots g_{n+1} \dots), (e_0 e_1 \dots e_{n+1} \dots)\big) < \dfrac{1}{m^{n+1}}$

$\Rightarrow g_i = e_i$ for $i = 0,1,2, \dots \dots, n+1$ (using Lemma 3.2.1)

$\Rightarrow g_i^c = e_i^c$ for $i = 1,2, \dots \dots, n+1$

$\Rightarrow d\big((g_1^c \dots g_{n+1}^c \dots), (e_1^c \dots e_{n+1}^c \dots)\big) < \frac{1}{m^n}$

$\Rightarrow d\big(\sigma^c(g), \sigma^c(e)\big) < \frac{1}{m^n} < \varepsilon$

Hence $\left(\sum_m, \sigma^c\right)$ is a continuous on $\sum_m$.

Theorem 5.2.2: There exists an uncountable Ω of $\sum_m$ such that for arbitrary $g, e \in \Omega$ with $g \neq e$, i) $\omega_{\sigma^c}(g) \setminus \omega_{\sigma^c}(e)$ is uncountable, (ii) $\omega_{\sigma^c}(g) \cap \omega_{\sigma^c}(e) \neq \emptyset$, (iii) $\omega_{\sigma^c}(g) \cap Per(\sigma^c) \neq \emptyset$.

In addition,

(iv) either $\lim_{n\to\infty} inf\ d\big(\sigma^{cn}(g), \sigma^{cn}(e)\big) = 0$, or $\lim_{n\to\infty} sup\ d\big(\sigma^{cn}(g), \sigma^{cn}(e)\big) = 1$, for arbitrary $g, e \in \Omega$ with $g \neq e$, and (v) if p is any periodic point of σ^c, $\lim_{n\to\infty} sup\ d\big(\sigma^{cn}(p), \sigma^{cn}(g)\big) \geq \frac{1}{2}$, for any $g \in \Omega$.

Proof: Let S be a subset of $\sum_m$ with the following properties,

(i) for any two elements $\alpha, \beta \in S$, $\alpha_s \neq \beta_s$, at least for one s, where $\alpha = (\alpha_0 \alpha_1 \dots \dots)$ and $\beta = (\beta_0 \beta_1 \dots \dots)$

(ii) for any element $\gamma \in S$, there does not exist any positive integer m, such that $\sigma^{cm}(\gamma) = \gamma$.

Then clearly, the set S is uncountable. For any $\beta = (\beta_0 \beta_1 \dots \dots) \in S$, we know define $B(\beta, k, S) = (\beta_0 \beta_1 \dots \dots \beta_{k-1} (10)(1100) \dots \dots (1^{k!} 0^{k!}))$, where $k \geq 2$ is a fixed integer. The length of the finite string $(10)(1100) \dots \dots (1^{k!} 0^{k!})$ is always even whatever be the value of k. Hence the length of the string $B(\beta, k, S)$ is even if k is even and odd if k is odd. Further, let $T_\beta = ((0)^{10}(1)^{10}B(\beta,k,S)B(\beta,k+1,S)B(\beta,k+2,S)\dots\dots),\beta \in S$.

It is also noted that if the length of $B(\beta,k,S)$ is odd (or even) then the length of the following string $B(\beta,k+1,S)$ is even (or odd) and so on. So by our construction of T_β we get that lengths of the finite strings such as $B(\beta,n,S)$, $n \geq 2$ are alternatively even and odd in T_β. We now consider the set $\Omega = \{\sigma^{cp}(T_\beta): \beta \in S, p \geq 0\}$. Since the set S is uncountable, the set Ω is also uncountable.

Let $\sigma^{cp}(T_\beta)$ and $\sigma^{cq}(T_\gamma)$ be two arbitrary elements of Ω. Then by our construction $\beta_s \neq \gamma_s$ for some s.

Let $\varepsilon_1 > 0$ and $\frac{1}{m^{n_1}} < \varepsilon_1$, for sufficiently large n_1, such that $n_1 > s$. Then the set $A_1 = \{\delta: \delta = (\beta_0\beta_1\dots\dots\beta_s\dots\dots\beta_{n_1}\lambda_0\lambda_1\dots\dots)\}$, where $\lambda = (\lambda_0\lambda_1\dots\dots)$ is any point of $\sum_m$.

Similarly, let $\varepsilon_2 > 0$ and $\frac{1}{m^{n_2}} < \varepsilon_2$, for sufficiently large n_2, such that $n_2 > s$.

In the same way, we can write the set $B_1 = \{\rho: \rho = (\gamma_0\gamma_1\dots\dots\gamma_s\dots\dots\gamma_{n_1}\lambda_0\lambda_1\dots\dots)\}$, where $\lambda = (\lambda_0\lambda_1\dots\dots)$ is any point of $\sum_m$.

To prove (i), (ii) and (iii) now consider the following three cases:

Case I: Both p and q are even.

We now choose an even integer l_1 such that $\sigma^{cl_1}\sigma^{cp}(T_\beta)$ starts from β_0 and with $k > n_1 + 1$. Note that by our construction, we can get infinitely many such l_1's. So, $\sigma^{cl_1}\sigma^{cp}(T_\beta)$ and δ agree at least up to β_{n_1}, for all $\delta \in A_1$. So by the application of the Lemma 3.2.1 we get $d\left(\sigma^{cl_1}(\sigma^{cp}(T_\beta)),\delta\right) < \frac{1}{m^{n_1}} < \varepsilon_1$, for all $\delta \in A_1$. Hence A_1 is a set of limit points of the orbit of $\sigma^{cp}(T_\beta)$. Since $\lambda \in \sum_m$ is arbitrary, the set $A_1$is uncountable. So, we can say that $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right)$ contains at least an uncountable number of points.

We now similarly choose an even integer l_2 such that $\sigma^{cl_2}\sigma^{cp}(T_\gamma)$ starts from γ_0 and with $k > n_2 + 1$. By our construction, we can get infinitely many such l_2's. So,

$\sigma^{cl_2}\sigma^{cp}(T_\gamma)$ and ρ agree at least up to γ_{n_2}, $\forall\, \rho \in B_1$. So using Lemma 3.2.1 again, we have $d\left(\sigma^{cl_2}\left(\sigma^{cp}(T_\gamma)\right), \rho\right) < \frac{1}{m^{n_2}} < \varepsilon_2$, $\forall\, \rho \in B_1$. Hence B_1 is a set of limit points of the orbit of $\sigma^{cp}(T_\gamma)$. Since $\lambda \in \sum_m$ is arbitrary, the set B_1 is obviously uncountable. So we can say that $\omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right)$ contains at least an uncountable number of points. Since $\beta_s \neq \gamma_s$, $A_1 \cap B_1 = \emptyset$.

Hence $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \backslash \omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right)$ is always uncountable.

Now both $\sigma^{cp}(T_\beta)$ and $\sigma^{cp}(T_\gamma)$ contain infinitely many finite sequences of the type $(10)(1100) \dots \dots (1^{k!}0^{k!})$ in $B(\beta, k, S)$ (or $B(\gamma, k, S)$. By our construction k is continuously increasing in the next term $B(\beta, k+1, S)$ (or $B(\gamma, k+1, S)$ and the length of $B(\beta, k, S)$ (or $B(\gamma, k, S)$ is alternatively even or odd.

We now consider the point $t = (10)(1100) \dots \dots (1^{k!}0^{k!})(1^{k!+1}0^{k!+1})$. By the above argument and Lemma 3.2.1, we get that the point t is a limit point of the orbit of $\sigma^{cp}(T_\beta)$ and $\sigma^{cp}(T_\gamma)$ both.

Hence $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \cap \omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right) \neq \emptyset$.

Consider the point $g = \left(\beta_0\beta_1 \dots \dots \beta_{n_1} a_0 a_1 \dots \dots\right)$ of A_1, where $a = (a_0 a_1 \dots \dots)$ is a non-periodic point of σ^c. Then obviously, g is also a non-periodic point of σ^c.

Hence $g \notin Per(\sigma^c)$. But by our construction $g \in \omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right)$.

Hence, we get $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \backslash Per(\sigma^c) \neq \emptyset$.

This proves the requirements of (i), (ii), and (iii) if p and q are both even.

Case II: Both p and q are odd.

Now choose an odd integer l_3 (that is, $p + l_3$ even) such that $\sigma^{cl_3}\left(\sigma^{cp}(T_\beta)\right)$ starts from β_0 and with $k > n_1 + 1$. By our construction, again, we can get infinitely many such l_3's. So, $\sigma^{cl_3}\left(\sigma^{cp}(T_\beta)\right)$ and δ agree at least up to β_{n_1}, for all $\delta \in A_1$. Hence by Lemma 3.2.1 A_1 is an uncountable set of limit points of the orbit of $\sigma^{cp}(T_\beta)$. So we can say that $\omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right)$ contains at least an uncountable number of points.

Similarly, we choose an odd integer l_4 (that is, $q + l_4$ even) such that $\sigma^{cl_4}\left(\sigma^{cp}(T_\gamma)\right)$ starts from γ_0 and with $k > n_2 + 1$. Again we can get infinitely many such l_4's. So,

$\sigma^{cl_4}\left(\sigma^{cp}(T_\gamma)\right)$ and ρ agree at least up to γ_{n_2}, for all $\rho \in B_1$. Hence B_1 is an uncountable set of limit points of the orbit of $\sigma^{cp}(T_\gamma)$. So get that $\omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right)$ contains at least an uncountable number of points. Since $\beta_s \neq \gamma_s$, $A_1 \cap B_1 = \emptyset$.

Hence $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \backslash \omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right)$ is always uncountable.

Here also the point $t = (10)(1100)\dots\dots(1^{k!}0^{k!})(1^{k!+1}0^{k!+1})$ is a limit point of the orbit of $\sigma^{cp}(T_\beta)$ and $\sigma^{cp}(T_\gamma)$ both. Because we can always choose two odd integers i (for $\sigma^{cp}(T_\beta)$) and j (for $\sigma^{cp}(T_\gamma)$) such that an arbitrary neighborhood of t contains both the points $\sigma^{ci}(\sigma^{cp}(T_\beta))$ and $\sigma^{cj}\left(\sigma^{cp}(T_\gamma)\right)$.

Hence $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \cap \omega_{\sigma^c}\left(\sigma^{cp}(T_\gamma)\right) \neq \emptyset$.

In this case also the point $g = (\beta_0\beta_1 \dots\dots \beta_{n_1}a_0a_1 \dots\dots)$ as defined in Case 1 does not belong to $Per(\sigma^c)$, but by our construction $d \in \omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right)$. Hence we get, $\omega_{\sigma^c}\left(\sigma^{cp}(T_\beta)\right) \backslash Per(\sigma^c) \neq \emptyset$.

This proves the requirements of (i), (ii), and (iii) if p and q are both odd.

Case III: Exactly one of p and q is even.

When we take p is even and q is odd. Combining Case I and Case II above, we get that satisfies the conditions (i), (ii), and (iii), if exactly one of p and q is even.

We now prove the last part of this theorem. We consider the points $g = T_\beta$ and $e = \sigma^c(T_\gamma)$ of Ω. Then $\underset{n\to\infty}{Lt\,sup}\, d\left(\sigma^{cn}(g), \sigma^{cn}(e)\right) = 1$.

We now take the points $g = T_\beta$ and $e = T_\gamma$ of Ω.

Then $\underset{n\to\infty}{Lt\,inf}\, d\left(\sigma^{cn}(g), \sigma^{cn}(e)\right) = 0$.

Hence either $\underset{n\to\infty}{Lt\,inf}\, d\left(\sigma^{cn}(g), \sigma^{cn}(e)\right) = 0$ or $\underset{n\to\infty}{Lt\,sup}\, d\left(\sigma^{cn}(g), \sigma^{cn}(e)\right) = 1$, for arbitrary $g, e \in \Omega$ with $g \neq e$.

Lastly, if $p = (p_0p_1 \dots\dots)$ is any periodic point of σ^c we can always choose a positive integer m such that $\sigma^{cm}(p)$ and $\sigma^{cm}(g)$ are different in the first term of the sequence for any $g \in \Omega$.

Hence $\underset{n\to\infty}{Ltsup} d\left(\sigma^{cn}(g), \sigma^{cn}(e)\right) \geq \underset{n\to\infty}{Lt\,sup}\, d\left(\sigma^{cm}(p), \sigma^{cm}(g)\right) \geq \frac{1}{m}$.

So, we get that Ω satisfies the conditions (iv) and (v). Hence the theorem is proved.

Theorem 5.2.3: $\sigma^c: \sum_m \to \sum_m$ is ω- chaotic in $\sum_m$.

Proof: By construction of the set **Ω** in Theorem 5.2.2 and by (i), (ii) and (iii) of Theorem 5.2.2, we get that **Ω** is an ω-scrambled set for σ^c.

Hence $\sigma^c: \sum_m \to \sum_m$ is ω- chaotic on $\sum_m$.

5.3 Some Strong Chaotic Properties of σ^c

In this section, we present some essential chaotic properties of the complemented shift map.

Theorem 5.3.1 $\sigma^c: \sum_m \to \sum_m$ is transitive (topologically).

Proof: To prove σ^c is topologically transitive, we need to prove that for any two non-empty open sets, P and Q of $\sum_m$, $\exists\, n \in \mathbb{N}$ such that $\sigma^{cn}(P) \cap Q \neq \emptyset$.

Let $p = (p_1 p_2 p_3 \ldots \ldots) \in P$ and $q = (q_1 q_2 q_3 \ldots \ldots) \in Q$ be arbitrary. Now, $p \in P$, $q \in Q$. So, $\exists$ open balls $B(p, r_1) \subseteq P$ and $B(q, r_2) \subseteq Q$.

If $r = \min\{r_1, r_2\}$ then $B(p, r) \subseteq P$, $B(q, r) \subseteq Q$ and $\frac{1}{m^n} < r, n \in \mathbb{N}$.

Consider the point $W = (p_1 p_2 p_3 \ldots \ldots p_n q_1 q_2 q_3 \ldots \ldots) \in \sum_m$ which agrees with p up to the n^{th} term. Therefore, by Theorem 3.3.1, we have that

$d(p, W) \leq \frac{1}{m^n} < r \Rightarrow W \in B(p, r) \subseteq P$ and it follows that $\sigma^{cn}(W) \in \sigma^{cn}(P)$.

Again, $\sigma^{cn}(W) = (q_1 q_2 q_3 \ldots \ldots) = q \in Q, q = \sigma^{cn}(W) \in \sigma^{cn}(P)$

$$\Rightarrow q = \sigma^{cn}(W) \in \sigma^{cn}(P) \cap Q.$$

So we have $\sigma^{cn}(P) \cap Q \neq \emptyset$ and hence $\sigma^c: \sum_m \to \sum_m$ is transitive (topologically).

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Theorem 5.3.2: $\sigma^c: \sum_m \to \sum_m$ is topologically mixing.

Proof: Let $p = (p_1 p_2 p_3 \dots \dots) \in P$ and $q = (q_1 q_2 q_3 \dots \dots) \in Q$ be arbitrary. Then since $p \in P$, $q \in Q$ and P, Q are open sets in $\sum_m$, $\exists$ open balls $B(p, r_1) \subseteq P$ and $B(q, r_2) \subseteq Q$.

If $r = \min\{r_1, r_2\}$ then $B(p, r) \subseteq P$, $B(q, r) \subseteq Q$ and choose $k \in N$ such that $\frac{1}{m^k} < r$. Now we set up a sequence $\{w_n\}$ of points in $\sum_m$ with the help of *k*, *p*, and *q* such that

$w_1 = (p_1 p_2 p_3 p_4 \dots p_k q_1 q_2 q_3 \dots)$,

$w_2 = (p_1 p_2 p_3 p_4 \dots p_k a_1 q_1 q_2 q_3 \dots)$,

$w_3 = (p_1 p_2 p_3 p_4 \dots p_k a_1 a_2 q_1 q_2 q_3 q_4 \dots)$, … …

$w_i = (p_1 p_2 p_3 p_4 \dots p_k a_1 a_2 \dots \dots a_{i-1} q_1 q_2 q_3 q_4 \dots \dots)$, $i \geq 2$,

$a_i \in \{0,1,2, \dots \dots, m-1\}$.

Now, every w_i, $i \geq 2$ is constructed by using the finite word attained by taking first (*i*-1) consecutive symbols of $a = (a_1, a_2, a_3, \dots \dots a_{i-1}, \dots) \in \sum_m$ chosen arbitrarily. In particular, the first *k* letters of w_i, for each $i \geq 2$, is the finite word $p_{[1,k]} = (p_1 p_2 p_3 p_4 \dots p_k)$ taken from $p \in P$ and then the word $a_{[1,i-1]} = (a_1, a_2, a_3, \dots \dots, a_{i-1})$ taken from *a* and at last the sequence representing q, i.e.$w_i = (p_{[1,k]}, a_{[1,i-1]}, q)$. In this situation, we can also use *a* fixed letter from the alphabet set $\{0,1,2, \dots, m-1\}$ repeating it for (*i*-1) times rather than using $a_{[1,i-1]}$.

Now, by using the Theorem 3.3.1, we get, $d(p, w_i) \leq \frac{1}{m^k} < r$ [since p and w_i agree up to the *k*th digits], $\forall\, i \in N$. So, $w_i \in B(p, r) \subseteq P$ and hence

$\sigma^{c(k+i-1)}(w_i) \in \sigma^{c(k+i-1)}(B(p, r)) \subseteq \sigma^{c(k+i-1)}(P)$ for all $i \in N$.

Also $\sigma^{c(k+i-1)}(w_i) = (q_1 q_2 q_3 \dots \dots) \in Q$, $\sigma^{c(k+i-1)}(w_i) \in \sigma^{c(k+i-1)}(P)$ imply that $\sigma^{c(k+i-1)}(P) \cap Q \neq \emptyset$, $\forall\, i \geq 2$. So, $\sigma^{cn}(P) \cap Q \neq \emptyset$, for all $n \geq k$.

Hence, σ^c is topologically mixing.

Theorem 5.3.3: σ^c is generically δ-chaotic on $\sum_m$with $\delta = diam(\sum_m) = 1$.

Proof: In theorem 5.3.2, it is proved that σ^c is topologically mixing on $\sum_m$. We know that a topologically mixing map which is continuous on a compact metric space is also topologically weak mixing, so σ^c is topologically weak mixing. Using Proposition 3.2.1, it follows that σ^c is generically δ-chaotic on $\sum_m$ with

$\delta = diam(\sum_m) = 1$.

Theorem 5.3.4: $\sigma^c : \sum_m \to \sum_m$ has chaotic dependence on initial conditions.

Proof: Consider $g = (g_1 g_2 g_3 \dots \dots)$ be any point of $\sum_m$. Also, let P be an open neighborhood of g. Since P is open, so $\exists$ an open ball $Q \subset P$ with radius $\varepsilon > 0$. Choose a large positive integer n such that $\frac{1}{m^n} < \varepsilon$.

(i) Consider $u = (u_0 u_1 \dots \dots \dots u_i)$ and $v = (v_0 v_1 \dots \dots \dots v_m)$ be two finite sequences then $uv = u_0 u_1 \dots \dots \dots u_i v_0 v_1 \dots \dots \dots v_m$. Further, if we guess that $T_1 T_2 \dots \dots T_p$ are p finite sequences then $T_1 T_2 \dots \dots T_p$ can be defined in a similar way as above.

(ii) Let $W(u, 2n+2) = (u_{n+1} u_{n+2} \dots \dots \dots u_{2n+1} u_{2n+2}^c u_{2n+3}^c \dots \dots \dots u_{3n+2}^c)$,

$W(u, 2n+4) = (u_{3n+3} u_{3n+4} \dots \dots \dots u_{4n+4} u_{4n+5}^c u_{4n+6}^c \dots \dots \dots u_{5n+6}^c)$,

$W(u, 2n+6) = (u_{5n+7} u_{5n+8} \dots \dots u_{6n+9} u_{6n+10}^c u_{6n+11}^c \dots \dots \dots u_{7n+1}^c\ )$, and so on.

Note that for $k > 0, W(u, 2n+k)$is a finite string of length $(2n+k)$.

(iii) We take $t \in \sum_m$ such that,

$t = (u_0 u_1 \dots \dots \dots u_n\ W(u, 2n+2) W(u, 2n+4) W(u, 2n+6) \dots \dots \dots \dots \dots)$.

Using those three notations and Lemma 3.2.1 as above, we now prove the theorem. By making u and t agree up to u_n. So, $d(u, t) < \frac{1}{m^n} < \varepsilon$. Therefore $t \in Q \Rightarrow t \in P$.

Now consider the following two cases, which are favorable to prove this theorem.

Case I: We consider n is an odd integer.

Now $\sigma^{cn+1}(u) = (u_{n+1} u_{n+2} \dots \dots \dots u_{2n+1} \dots \dots \dots)$ and

$\sigma^{cn+}\ (t) = (u_{n+1} u_{n+2} \dots \dots \dots u_{2n+1} \dots \dots \dots)$.

Since t formation of infinitely many finite sequences of the type $W(u, 2n+k)$.

So we get,

$$\underset{n\to\infty}{Lt}\ inf\, d(\sigma^{cn}(u), \sigma^{cn}(t))$$

$$\leq \underset{n\to\infty}{Lt}\ d((u_{n+1} u_{n+2} \dots \dots \dots u_{2n+1} \dots \dots), (u_{n+1}^c u_{n+2}^c \dots \dots \dots u_{2n+1}^c \dots \dots)$$

$$\leq \underset{n\to\infty}{Lt} \left(\frac{0}{m} + \frac{0}{m^2} + \dots\dots\dots + \frac{0}{m^{n+1}}\right)$$
$$= 0$$

So, $\underset{n\to\infty}{Lt\ inf}\, d(\sigma^{cn}(u), \sigma^{cn}(t)) = 0$. ……………………………………………..….(5.3.1)

Similarly, $\sigma^{c2n+2}(u) = (u_{2n+2}u_{2n+3} \dots\dots\dots u_{3n+2} \dots\dots)$ and

$\sigma^{c2n+2}(t) = (u^c_{2n+2}u^c_{2n+3} \dots\dots\dots u^c_{3n+2} \dots\dots)$.

Hence,

$$\underset{n\to\infty}{Lt\ sup}\, d(\sigma^{cn}(u), \sigma^{cn}(t))$$
$$\geq \underset{n\to\infty}{Lt}\, d((u_{2n+2}u_{2n+3} \dots\dots\dots u_{3n+2} \dots\dots), (u^c_{2n+2}u^c_{2n+3} \dots\dots\dots u^c_{3n+2} \dots\dots))$$
$$\geq \underset{n\to\infty}{Lt} \left(\frac{1}{m} + \frac{1}{m^2} + \dots\dots\dots.. + \frac{1}{m^{n+1}}\right)$$
$$= \frac{1}{m-1} = \delta > 0$$

Hence $\underset{n\to\infty}{Lt\ sup}\, d\big(\sigma^{cn}(s), \sigma^{cn}(t)\big) \geq \delta$. …………………………………………(5.3.2)

Case II: Here n is an even integer.

Then $\sigma^{cn+1}(u) = (u^c_{n+1}u^c_{n+2} \dots\dots\dots u^c_{2n+1} \dots\dots)$ and

$\sigma^{cn+1}(t) = (u^c_{n+1}u^c_{n+2} \dots\dots\dots u^c_{2n+1} \dots\dots)$.

In this situation, also t consists of infinitely many finite sequences of the type $W(u, 2n + k)$.

So, we have

$$\underset{n\to\infty}{Lt\ inf}\, d(\sigma^{cn}(u), \sigma^{cn}(t))$$
$$\leq \underset{n\to\infty}{Lt}\, d((p^c_{n+1}p^c_{n+2} \dots\dots\dots p^c_{2n+1} \dots\dots), (p^c_{n+1}p^c_{n+2} \dots\dots\dots p^c_{2n+1} \dots\dots))$$
$$\leq \underset{n\to\infty}{Lt} \left(\frac{0}{m} + \frac{0}{m^2} + \dots\dots\dots.. + \frac{0}{m^{n+1}}\right)$$
$$= 0$$

Hence, $\underset{n\to\infty}{Lt\ inf}\, d(\sigma^{cn}(u), \sigma^{cn}(t)) = 0$.………………………….……………(5.3.3)

Similarly, $\sigma^{c2n+2}(u) = (u_{2n+2}u_{2n+3} \dots\dots\dots u_{3n+2} \dots\dots)$ and

$\sigma^{c2n+2}(t) = (u^c_{2n+2}u^c_{2n+3} \dots\dots\dots u^c_{3n+2} \dots\dots)$.

So we get

$$\underset{n\to\infty}{Lt\,sup}\; d(\sigma^{cn}(u), \sigma^{cn}(t))$$

$$\geq \underset{n\to\infty}{Lt}\; d((u_{2n+2}u_{2n+3}\dots\dots u_{3n+2}\dots\dots),(u^c_{2n+2}u^c_{2n+3}\dots\dots u^c_{3n+2}\dots\dots))$$

$$\geq \underset{n\to\infty}{Lt}\;(\tfrac{1}{m}+\tfrac{1}{m^2}+\;\dots\dots\dots+\tfrac{1}{m^{n+1}})$$

$$= \frac{1}{m-1} = \delta > 0$$

Hence,

$\underset{n\to\infty}{Lt\,sup}\; d\big(\sigma^{cn}(u), \sigma^{cn}(t)\big) \geq \delta$ ……………………………………………….(5.3.4)

By virtue of (5.3.1), (5.3.2), (5.3.3), (5.3.4) in the above two cases, we get

$\underset{n\to\infty}{Lt\,inf}\, d(\sigma^{cn}(p), \sigma^{cn}(t)) = 0$ and $\underset{n\to\infty}{Lt\,sup}\; d\big(\sigma^{cn}(s), \sigma^{cn}(t)\big) \geq \delta$…………..…...(5.3.5)

By virtue of (5.3.5), it is proved that the pair (u, t) is Li-Yorke. Hence $(\sum_m, \sigma^c)$ has chaotic dependence on initial conditions.

The following example establishes that if any continuous map has strong sensitive dependence on initial conditions, then it has sensitive dependence on initial conditions, but the converse is not always true.

Example 5.3.1 Let $M: [-1,1] \to [-1,1]$ be a map defined by

$$M(x) = \begin{cases} \frac{8}{7}x + \frac{8}{7}, & -1 \leq x \leq -\frac{1}{8} \\ -8x, & -\frac{1}{8} \leq x \leq 0 \\ x\,, & 0 \leq x \leq 1 \end{cases}$$

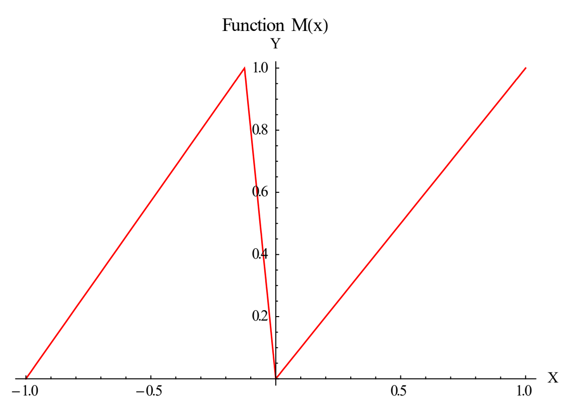

Figure 5.3.1: Function of $M(x)$ on the interval [-1,1]

The map defined above is continuous. It can be quickly shown that this map has sensitive dependence on initial conditions. We observe that the maximum distance between any two points of [-1,1] equals 2. If consider the point $-\frac{8}{15}$ and $U = (0,1)$. Then ∃ no point $y \in U$ such that $d\left(M^n(x), M^n(y)\right) = 2$, for any $n \geq 0$. Hence given map does not have strong sensitive dependence on initial conditions.

The following theorem shows that the complemented shift map is topologically conjugate to the shift map.

Theorem 5.3.5: σ^c and σ are conjugate topologically.

Proof: Consider a map $f: \sum_m \to \sum_m$ by $f(a) = (a_0 a_1^c a_2 a_3^c \dots \dots)$, where $a = (a_0 a_1 a_2 \dots \dots)$ is any point of $\sum_m$. At first, we prove the continuity of the function f.

Let $u = (u_0 u_1 u_2 \dots \dots)$ and $v = (v_0 v_1 v_2 \dots \dots)$ be any points of $\sum_m$, choose $\varepsilon_1 > 0$ be arbitrary and an even integer *n* so large that $\frac{1}{m^n} < \varepsilon_1$.

Then $d(u, v) < \delta_1 = \frac{1}{m^n}$

$$\Rightarrow d\left((u_0 u_1 u_2 \dots \dots u_n \dots), (v_0 v_1 v_2 \dots \dots v_n \dots)\right) < \frac{1}{m^n}$$

$\Rightarrow u_i = v_i$ for $i = 0,1,2, \dots \dots, n$ using Lemma 3.2.1

$$\Rightarrow u_i^c = v_i^c \text{ for } i = 1,2, \dots \dots, n$$

$$\Rightarrow d\left((u_0 u_1^c u_2 u_3^c \dots \dots u_n \dots), (v_0 v_1^c v_2 v_3^c \dots \dots v_n \dots)\right)$$

$$\Rightarrow d\left(f(u), f(v)\right) < \frac{1}{m^n} < \varepsilon_1$$

which proves that on $\sum_m$, our assuming map $f: \sum_m \to \sum_m$ is continuous. In the same way, we can show that the inverse of f is also continuous.

Next, we need to show that the map $f: \sum_m \to \sum_m$ is bijective. Consider $f(s) = f(t)$, then we get $(u_0 u_1^c u_2 u_3^c \dots \dots) = (v_0 v_1^c v_2 v_3^c \dots \dots)$.

Hence $u_m = v_m$, for $m = 0,2,4, \dots \dots$ and $u_m^c = v_m^c$, for $m = 1,3,5, \dots \dots$ So we get $u_m = v_m$ for al $m \geq 0$, that is $u = v$, which proves that $f: \sum_m \to \sum_m$ is injective.

To show that f is a homomorphism, we are only to show that f is surjective on $\sum_m$. Let $b = (b_0b_1b_2b_3 \dots \dots)$ be any point of $\sum_m$. After that $b^* = (b_0^c b_1 b_2^c b_3 \dots \dots)$ is a point of $\sum_m$, such that $g(b^*) = b$. Hence the mapping $f: \sum_m \to \sum_m$ is surjective. Hence we conclude that the map $f: \sum_m \to \sum_m$ is a homomorphism. Now we try to show that the map f is a conjugacy between σ and σ^c.

Let $u = (u_0u_1u_2 \dots \dots)$ be the point defined as above. Then $\sigma \circ f(s) = \sigma(u_0u_1^cu_2u_3^c \dots \dots) = (u_1^cu_2u_3^c \dots \dots)$.
On the other hand,
we get $f \circ \sigma^c(s) = g(s_1^cs_2^cs_3^c \dots \dots) = (s_1^cs_2s_3^c \dots \dots)$.

Hence $\sigma \circ f(s) = f \circ \sigma^c(s)$. So σ^c and σ are conjugate topologically.

The discussion of transitive maps on compact spaces is extensively motivated. It is essential to know if some iterate exists that is not transitive for a transitive map. From the following example, we see that G is not totally transitive, but it is transitive (topologically).

Example 5.3.2 Consider $G(x)$ be a continuous map defined by

$$G(x) = \begin{cases} 6x + \frac{1}{7}, & 0 \le x \le \frac{1}{7} \\ -6x + \frac{13}{7}, & \frac{1}{7} \le x \le \frac{2}{7} \\ \frac{1}{5} - \frac{1}{5}x, & \frac{2}{7} \le x \le 1 \end{cases}$$

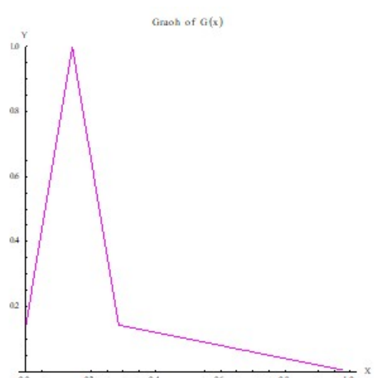

Figure 5.3.2 (a): Function of G(x) acting on [0,1]

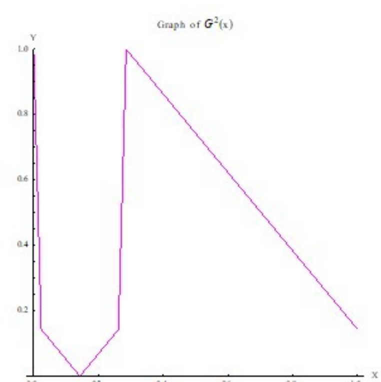

Figure 5.3.2 (b): Function of $G^2(x)$ acting on [0,1]

Since G is transitive, it has an orbit dense in [0,1], and hence, the interval is the unique invariant compact with a non-empty interior. It can be spontaneously shown G is transitive (topologically) on [0,1], and we can find under G^2, the subintervals $\left[0,\frac{2}{7}\right]$and $\left[\frac{2}{7},1\right]$ are invariant. But the above map is not transitive (topologically) on [0,1]. Hence $G(x)$ is not totally transitive.

Problem 5.3.1: Prove that $\sigma^c: \sum_m \to \sum_m$ is chaotic.

Solution: We have already proved that according to the definition of Devaney, it is chaotic, and we know that Devaney chaos implicates Li- Yorke. So σ^c is Li- Yorke chaotic. Now Γ_α is defined by,

$$\Gamma_\alpha = \alpha_0 01 \alpha_0\alpha_0\alpha_1\alpha_1 0011 \alpha_0\alpha_0\alpha_0\alpha_1\alpha_1\alpha_1\alpha_2\alpha_2\alpha_2 000111 \dots\dots\dots\dots,$$

$\forall\alpha = (\alpha_0\alpha_1\alpha_2 \dots\dots\dots)$ in $\sum_m$, we can prove it directly. So in the symbol $\sum_m$, σ^c is a strong chaotic map. Since the shift map is a chaotic model so now we can try to usage σ^c in the position of σ. Hence for chaotic dynamical systems, we accomplish that σ^c is a new model.

5.4 Summary and Conclusion

We have investigated in this work strong chaotic properties of σ^c. We have proved that σ^c is ω- chaotic in $\sum_m$ with some additional features. Since ω- chaos is equivalent to chaos in the sense of Devaney, σ^c is Devaney chaotic. Hence it is also Li-Yorke chaotic, Devaney chaos is stronger than Li-York chaos. It is also established that σ^c is generally δ-chaotic with $\delta = diam(\sum_m) = 1$. Section 5.3 of this chapter provides an example of a continuous function with strong sensitive dependence on initial conditions, but the converse is not always true. We have found from Theorem 5.3.5, σ and σ^c are conjugate toplogically. In example 5.2.3, we have shown that if any continuous map has an dense orbit, so that map is transitive on given interval. But that map is not totally transitive. So, we can say that not all transitive maps are totally transitive, and not all chaotic maps are totally transitive. Finally, we proved that σ^c is chaotic.

SUMMARY AND FUTURE WORK

The vision of this thesis is to investigate some chaotic features of the generalized shift map and the complemented shift map. On shift of finite type, the implication is true but not the case on the unit interval. The study of chaotic behavior on shift of finite type has been done extensively in various approaches. This thesis is divided in five chapters to facilitate the completion of the research work.

In the first chapter, dynamical systems and different types of chaos are briefly introduced. Various types of chaos as defined by mathematicians of the world are discussed. Different kind of chaotic maps and their chaotic behavior with changing values are also studied. Graphically, it is observed that the tent map is chaotic as the iterations increase.

Basic concepts related to symbolic dynamics are presented in the second chapter. The basic definition of shift map, Cantor set, Cantor's middle-thirds set, Itineraries are presented. Symbolic dynamics for quadratic, logistic, and Smale's Horseshoe are also explained. The action of the Smale's horseshoe which is based on the Baker's map, the Bernoulli shift map has many periodic orbits that are countably infinite and many non-periodic orbits that are uncountable. The dense orbit on $\sum_2$ is also shown. In the last section of this chapter, topological conjugacy for one-dimensional map is discussed. It is observed that the tent map, T_b and the logistic map, F_μ are conjugate. The doubling map conjugate to logistic map is also found.

Some important definitions which are helpful to prove chaotic properties are presented in the third chapter. Essential chaotic features of the shift map σ on $\sum_m$ is discussed in this chapter. Proximity theorem is established and based on this theorem, it is proved that shift map is topologically transitive, topologically mixing, and has sensitive dependence on initial conditions (SDIC). It is found that σ is δ-chaotic on $\sum_m$ with $\delta = diam(\sum_m) = 1$. Other important chaotic properties are proved, such as modified weakly chaotic dependence on initial conditions and chaotic dependence on initial conditions. The shift map is homeomorphism is proved. The fact that sequence space $\sum_m$ is a Cantor set is shown. It is found that the shift map and $f_m(u) = mu(mod\ 1)$ are topologically semi-conjugated. The shift map is exact Devaney chaotic (EDevC), and that it is conjugate to the quadratic map is proved.

In this chapter four, some stronger chaotic features of the generalized shift map, σ_n is established. From example 4.3.2, it is found that chaotic functions are not necessarily topologically transitive.

The chaoticity and related properties of the complemented shift map σ^c are studied in chapter five. A few essential chaotic properties of this map are discussed. That the complemented shift map, σ^c is generally δ-chaotic with $\delta = diam(\sum_m) = 1$ is established. It is proved that the shift map and complemented shift map are conjugate. Finally, it is found that the complemented shift map is chaotic.

In the future, we will discuss several important applications of symbolic dynamics in the field of dynamical systems, such as Markov Partitions, homoclinic orbits, and topological entropy etc. We will use the generalized shift map and the complemented shift map as another models for chaotic dynamical systems. We will attempt to establish some chaotic models in the field of "Population Dynamics" considering some contemporary and interesting phenomena in nature. We are interested in showing their chaotic behaviors as they play an important role in the analysis of dynamical systems.

Shift maps are proposed as a new framework to describe various geometric rearrangement problems that can be computed as a global optimization. Extending shift-map to use multiple source images, as described in shift map composition, can also be used for inpainting. Input images can include transformations of the original input image like rotation, scaling, etc. The generation of the binary sequence using a modified chaotic sine map will be studied. The possibilities of using the generated binary sequence in the radar pulse compression technique will be explored.

APPENDIX

Figure 1.5.1:

```
f[x_]:=2x+1/2/;0≤x≤1/2
f[x_]:=-2x+3/;1/2≤x≤1
f[x_]:=2-x/;1≤x≤2
```

Plot[f[x], {x, 0, 2}, AxesLabel → {"X", "Y"}, PlotLabel → "f(x)", PlotStyle → RGBColor[1, 0, 0], DisplayFunction → Identity

Figure 1.5.2:

```
recurrenceList[a_,x0_,n_:50]:=NestList[(a # (1-#))&,x0,n-
1]
Module[{fns,colors},colors={Red,Yellow,Green,Blue,Lighter
[Purple,0.6]};

GraphicsGrid@Partition[Table[fns=Table[recurrenceList[a,x
][[n+1]],{n,1,5}];
    Plot[fns,{x,0,1},PlotStyle→colors],{a,1,4}],2]]
```

Figure 1.5.3: (Using MATLAB)

```
clear
rmin = input('rmin = ')
rmax = input('rmax = ')
xo = input('x0 = ')
num = 100
num2 = num/2;

rmn = round(rmin*1000); %Matlab requires integer
subscripts
rmx = round(rmax*1000);
rct = 0; %rct is a counter for the number of r-
valuesinterated
for r = rmn:rmx
x(1)=xo; %set initial condition--Matlab requires
subscript > 0
rdec = r/1000; % converts back to decimal r
for n=2:num
x(n)=rdec*(x(n-1))*(1-(x(n-1)));
if n > num2
ir = n+(num-2)*rct; %ir is a counter for total
interations
itx(ir) = x(n); %after the first num2 iterations
rv(ir)=rdec;
end
end
```

```
rct=rct+1;
end

plot(rv,itx,'.')
axis([(rmin),ceil(rmax),0,1])
numitr = num2str(num);
title(['Bifurcation Diagram for the Logistic Map with
Interations = ',numitr])
```

Figure 1.5.4:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=1;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio->1, PlotStyle→RGBColor[1,0,0]]
```

Figure 1.5.5:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=2;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio->1, PlotStyle→RGBColor[1,0,0]]
```

Figure 1.5.6:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=3;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio->1, PlotStyle→RGBColor[1,0,0]]
```

Figure 1.5.7:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=4;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio->1, PlotStyle→RGBColor[1,0,0]]
```

Figure 1.5.8:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=5;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio->1, PlotStyle→RGBColor[1,0,0]]
```

Figure 1.5.9:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=6;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotStyle→RGBColor[1,0,0],
PlotRange-> {xmin,xmax},AspectRatio->1]
```

Figure 1.5.10:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=7;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotStyle→RGBColor[1,0,0],
PlotRange-> {xmin,xmax},AspectRatio->1]
```

Figure 1.5.11:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=8;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotStyle→RGBColor[1,0,0],
PlotRange-> {xmin,xmax},AspectRatio->1]
```

Figure 1.5.12:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=9;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotStyle→RGBColor[1,0,0],
PlotRange-> {xmin,xmax},AspectRatio->1]
```

Figure 1.5.13:

```
Clear[T,x];
T[x_]:=If[x<=.5,2x,2-2x];
xmin=0;
xmax=1;
NumberOfIterations=10;
Plot[{Nest[T,x,NumberOfIterations],x},{x,xmin,xmax},PlotStyle→RGBColor[1,0,0],
PlotRange-> {xmin,xmax},AspectRatio->1]
```

Figure: 2.5.8:

```
Clear[T,x];
T[x_]:=If[x≤.5,3x,3-3x];
xmin=0;
xmax=1.5;
NumberOfIterations=1;
Plot[Nest[T,x,NumberOfIterations],{x,xmin,xmax},PlotRange
→{xmin,xmax},AspectRatio→1, PlotLabel→"Cantor Tent
Map",AxesLabel→{"x","y"}, PlotStyle→RGBColor[1,0,0]]
```

Figure: 2.5.9:

```
Clear[T,x];
T[x_]:=If[x≤.5,3x,3-3x];
xmin=0;
xmax=1.5;
NumberOfIterations=2;
Plot[Nest[T,x,NumberOfIterations],{x,xmin,xmax},PlotRange
→{xmin,xmax},AspectRatio→1, PlotLabel→"Cantor Tent Map",
BaseStyle→{FontWeight→"Bold",
FointSize→14},PlotStyle→RGBColor[1,0,0],AxesLabel→{"x","y
"}]
```

Figure 2.5.13:

```
Clear[T, x]

T[x_]:=3x/;0≤x≤1/3
T[x_]:=3x-1/;1/3≤x≤2/3
T[x_]:=3x-2/;2/3≤x≤1

Plot[T[x], {x, 0,1}, AxesLabel → {"X", "Y"}, DisplayFunction → Identity, PlotStyle
        → RGBColor[1,0,0]]
```

Figure 3.3.1:

```
J[x_]:=11/10(x+1)/;-1≤x≤-1/11
J[x_]:=-11x/;-1/11≤x≤0
J[x_]:=x/;0≤x≤1
```

Plot[J[x], {x, −1,1}, AxesLabel → {"X", "Y"}, PlotStyle
→ RGBColor[1,0,1], DisplayFunction → Identity, PlotLabel
→ "Function J(x)"]

Figure 5.3.1:

```
M[x_]:=8/7x+8/7/;-1≤x≤-1/8
M[x_]:=-8x/;-1/8≤x≤0
M[x_]:=x/;0≤x≤1
```

Plot[M[x], {x, −1,1}, AxesLabel → {"X", "Y"}, PlotStyle
→ RGBColor[1,0,0], DisplayFunction → Identity, PlotLabel
→ "Function M(x)"]

Figure 5.3.2 (a):

```
Clear[G,x];
G[x_]:=6x+1/7/;0≤x≤1/7
G[x_]:=-6x+13/7/;1/7≤x≤2/7
G[x_]:=-1/5x+1/5/;2/7≤x≤1
 xmin=0;
 xmax=1;
 NumberOfIterations=1;
 Plot[{Nest[G,x,NumberOfIterations]},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio→Automatic, PlotLabel→"Graoh of G(x)",
AxesLabel→{"X","Y"},PlotStyle→RGBColor[1,0,1]]
```

Figure 5.3.2 (b):

```
Clear[G,x];
G[x_]:=6x+1/7/;0≤x≤1/7
G[x_]:=-6x+13/7/;1/7≤x≤2/7
G[x_]:=-1/5x+1/5/;2/7≤x≤1
 xmin=0;
 xmax=1;
 NumberOfIterations=2;
 Plot[{Nest[G,x,NumberOfIterations]},{x,xmin,xmax},PlotRange->
{xmin,xmax},AspectRatio→Automatic, PlotLabel→"Graoh of 2nd iteration of G(x)",
AxesLabel→{"X","Y"},PlotStyle→RGBColor[1,0,1]]
```

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List of Research Outputs During The Ph.D Study Period

It is declared that the following research articles have been published and presented papers in different conferences from this Ph.D. thesis of the Ph.D. Scholar Hena Rani Biswas.

List of Publications

- Biswas, H. R., and Md. Shahidul Islam, Stronger Chaotic features of the Complemented shift map on *m*-symbol space, International Journal of Statistics and Applied Mathematics, 6 (5): 34-41, (2021).
- Biswas, H. R., and Md. Shahidul Islam, Topological Conjugacy and Symbolic dynamics of the one-dimensional map, European Journal of Applied Sciences, 9 (5): 44-55, 2021.
- Biswas, H. R., and Md. Shahidul Islam, Chaotic features of the forward Shift Map on the Generalized *m*-Symbol Space, Journal of Applied Mathematics and Computation, 4 (3): 104-112, (2020).
- Biswas, H. R. and Monirul Islam, Shift Map and Cantor Set of Logistic Function, IOSR Journal of Mathematics, 16: 01-08, (2020).
- Biswas H.R., Investigation of chaoticity of the generalized shift map under a new definition of chaos and compare with shift map, Barisal University Journal Part 1, 4(2), 261-270, (2017).

Paper Presentation

- Presented the paper titled "Different types of maps routes to chaos in bi-infinite symbol space with strong chaotic features" at 22nd International Mathematics Conference held during 10-11 December 2021, organized by Bangladesh Mathematical Society (BMS) in Zoom Platform.
- Presented the paper titled "Investigation of Chaoticity of the shift map

and the generalized shift map on *m*-symbol space" at 1st INTERNATIONAL CONGRESS ON NATURAL SCIENCES(ICNAS-2021) held during 10-12 SEPTEMBER2021 organized by Atatürk University, ERZURUM,TURKEY in Zoom Platform.

- Achieved **Best Presenter Certificate** of the paper "Symbolic Dynamics for One dimensional Chaotic maps and its Applications" during the Technical Session VI, chaired by Prof. S. Ahmed, Rajiv Gandhi University, Arunachal Pradesh, India, on 1-3 September 2020, in ICAMST-2020 Organized by Department of Mathematics, Rajiv Gandhi University, Arunachal Pradesh, India.

Attendance to Courses And Research Schools

- Attended CIMPA School, "Functional Equations: Theory, Practice and Interactions", organized at the Institute of Mathematics, **Vietnam Academy of Science and Technology**, from April 12-23, 2021.

- Attended the CIMPA SCHOOL, "Numeration and Fractals", **Quezon city (Philippines)**, January 6-17, 2020.

- Achieved certificates for two courses from "**AARMS Summer School 2019**" held at the **University Prince Edward Island in Charlottetown, PE, Canada** during June 17 - July 12, 2019.

- Attended the CIMPA SCHOOL, "Dynamical Systems and Applications to biology", **Dhaka, Bangladesh,** 2019.

- Attended the CIMPA RESEARCH SCHOOL, "Dynamical Systems", **Kathmandu (Nepal)**, October 25 –November 6, 2018.