LEGENDRE AND ISOTROPIC SUBMANIFOLDS IN CONTACT AND RIEMANNIAN GEOMETRY WITH MODERN DEVELOPMENTS

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BY

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DECLARAION

I do hereby declare that this thesis entitled "**Legendre and Isotropic Submanifolds in Contact and Riemannian Geometry with Modern Developments**" has been done by me under the supervision of Dr. Md. Showkat Ali, Professor and Chairman, Department of Applied Mathematics, University of Dhaka, Dhaka-1000. I do further declare that no portion of the work considered in this thesis has been submitted in support of an application for another degree or qualification of this or any other University or Institute of learning either in home or abroad.

October, 2020 **(A. K. M. Nazimuddin)**

CERTIFICATE

This is to certify that the research work presented in this thesis entitled "**LEGENDRE AND ISOTROPIC SUBMANIFOLDS IN CONTACT AND RIEMANNIAN GEOMETRY WITH MODERN DEVELOPMENTS**" submitted by **A. K. M. NAZIMUDDIN,** Reg. No: 09, Session: 2015-2016 of University of Dhaka, Bangladesh, in partial fulfillment of the requirement for the PhD Degree in Applied Mathematics of this University, is absolutely based upon his own work and that neither this thesis nor any part of it has been submitted for any degree/ diploma of any academic award anywhere before.This thesis is carried out by the author under the supervision of Dr. Md. Showkat Ali, Professor and Chairman, Department of Applied Mathematics, University of Dhaka, Bangladesh.

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Author

Dedicated to

My Parents, Wife and Son

CONTENTS

ABSTRACT

The aim of this thesis is to study interconnection among various branches of differential geometry and its application to dynamical systems. At first, we discuss about various branches of differential geometry and study interconnection among these branches. Then some recent development on the application of differential geometry to dynamical system, called Brusselator model, is studied. Organization of the thesis is as follows.

Chapter-1 provides some background materials on which the rest of the thesis is based. In this chapter basic definitions and theorems of real and complex manifolds are provided. This chapter is mainly a review.

In Chapter-2, a brief review on connections on manifolds and Riemannian manifolds is first of all provided. Given a connection on a manifold we can define geodesic, Riemannian curvature tensor, Ricci tensor, Ricci scalar on it. It should be mentioned here that, hand calculations become extremely tedious to evaluate components of connection, Riemannian curvature tensor, Ricci tensor etc. on higher dimensional manifolds. In this chapter we have developed some computer codes for computing these components. Using computer techniques, the components of connection, Riemannian curvature tensor, Ricci tensor etc. can be computed easily. The work is original.

In Chapter-3, interconnections among manifolds with symplectic structure are reviewed. This chapter is mainly a review. But there are some original calculations also. In this chapter we have studied connections of symplectic geometry with the contact geometry, Riemannian geometry and Kähler geometry using existing theorems.

In Chapter-4, a review on symplectic geometry and contact geometry with complex manifold is provided. Here we have developed a special comparison between complex symplectic geometry and complex contact geometry.

Chapter-5 is mainly a review on Kodaira, Legendre and isotropic moduli spaces. However, there are some original calculations also. Here we have studied the existence and stability of Kodaira and Legendre moduli spaces and also the existence, completeness and maximality of isotropic moduli spaces. Also, interconnection among Kodaira, Legendre and isotropic moduli spaces is established in this chapter.

Chapter-6 is original. It provides the main result. Here we analyze two slow-fast dynamical systems named Brusselator model and Lorenz-Haken model through differential geometry. First, we investigate the temporal and spatiotemporal Brusselator model, respectively and find periodic traveling wave solutions. As a result, we obtain a spot pattern of the model. Then, we investigate the Lorenz-Haken model. Next, we apply an old strategy called the Geometric Singular Perturbation Theory and another newly developed strategy that reflects the applications of differential geometry in the slow-fast dynamical system called the flow curvature method to the two models named as temporal Brusselator model and Lorenz-Haken model. According to the Flow Curvature Method, we determine the curvature of the trajectory curve analytically called flow curvature manifold by estimating the solution or trajectory curve of the dynamical system as a curve in Euclidean space. Since this manifold comprises the time derivatives of the velocity vector field and hence it receives knowledge about the dynamics of the corresponding system. In Model 1 named Brusselator model where we consider the temporal Brusselator model as a two dimensional slow-fast dynamical system. According to the Flow Curvature Method, we determine the flow curvature manifold which directly provides the slow invariant manifold where the Darboux invariance theorem is then used to show the invariance of the slow manifold. On the other hand, since the temporal Brusselator model has no singular approximation and hence, Geometric Singular Perturbation Theory fails to provide the slow invariant manifold associated with temporal Brusselator model. After that, we describe the effect of growth and curvature with surface deformation on pattern formation of the spatiotemporal Brusselator model. In Model 2 named Lorenz-Haken model, we consider as a three dimensional slow-fast dynamical system. By using Flow curvature method, we determined the flow curvature manifold which directly provides the third order approximation of the slow manifold where the

Darboux invariance theorem is then used to show the invariance of the slow manifold. Then, we analyze the stability of the fixed point of the L-H model using the flow curvature manifold. On the other hand, since L-H model has singular approximation and it can be considered as a singularly perturbed system. Hence, by using Geometric Singular Perturbation Theory we determine the order by order approximation in the small multiplicative parameter of the slow manifold where the Fenichel's invariance theorem is used to show the invariance of the slow manifold. After that, we compare the two geometric methods applied to the two slow-fast dynamical systems and highlight the significant results.

Finally, some concluding remarks and scope for future work in this direction are given in Chapter-7.

LIST OF CHAPTER WISE PUBLICATIONS

1. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2016) "**Connections with Symplectic Structures**", *American Journal of Computational Mathematics,* 6(4), 313-319. (Chapter-3)

2. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2019) **"Periodic Pattern Formation Analysis Numerically in a Chemical Reaction-Diffusion System"**, *International Journal of Mathematical Sciences and Computing,* 5(3), 17-26. (Chapter-6)

3. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2019) **"Pattern Formation in the Brusselator Model Using Numerical Bifurcation Analysis"**, *Punjab University Journal of Mathematics,* 51(11), 31-39. (Chapter-6)

4. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2019) **"Riemannian Geometry and Modern Developments"**, *GANIT: Journal of Bangladesh Mathematical Society*, 39, 71-85. (Chapter-2)

5. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2019) **"Symplectic and Contact Geometry with Complex Manifolds"**, *GANIT: Journal of Bangladesh Mathematical Society*, 39, 119-126. (Chapter-4)

6. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2020) **"Application of the Flow Curvature Method in Lorenz-Haken Model"**, *International Journal of Mathematical Sciences and Computing,* 6(1), 33-48. (Chapter-6)

7. **A. K. M. Nazimuddin** and Md. Showkat Ali, (2020) **"Slow Invariant Manifold of Brusselator Model"**, *International Journal of Mathematical Sciences and Computing,* 6(2), 79-87. (Chapter-6)

CHAPTER 1

MANIFOLDS

1.1 Real Manifolds

A manifold of dimension n is a topological space which locally resembles n-dimensional Euclidean space. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to $\mathbb{R}^n$. Lines and circles are one-dimensional manifolds. Two-dimensional manifolds are also called surfaces. Although a manifold locally resembles Euclidean space, but globally a manifold may have complicated structures. For example, the surface of the sphere is not a Euclidean space, but in a region it can be charted by means of geographic maps: map projections of the region into the Euclidean plane. When a region appears in two neighbouring maps, the two representations do not coincide exactly and a transformation is needed to pass from one to the other, called a transition map or transition function.

Definition 1.1. Let M be a topological space and $U \subseteq M$ an open set. Let $V \subseteq \mathbb{R}^n$ be open. A homeomorphism $\varphi: U \to V$, where $\varphi(u) = (x_1(u), \cdots, x_n(u))$ is called a coordinate system on U, and the functions $x_1(u), \cdots, x_n(u)$ are the coordinate functions. Also, φ^{-1} is a inverse map that is parameterization of U.

Definition 1.2. A pair $(U, \varphi)$of a topological manifold M is an open subset U of M called the domain of the chart, together with a homeomorphism $\varphi: U \to V$ of U onto an open set V in $\mathbb{R}^\text{n}$. Roughly speaking, a chart of M, is an open subset U in M with each point in U labeled by n numbers.

Definition 1.3. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are said to be compatible if

1. (i) $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ open
   (ii) $\varphi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ open

2. (i) $\phi_\alpha o \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a C^∞ diffeomorphism.

(ii) $\phi_\beta o \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a C^∞ diffeomorphism.

Definition 1.4. An atlas of class C^k on a topological manifold M is a set $\{(U_\alpha, \phi_\alpha), \alpha \in I\}$ of each chart, such that

(i) the domain U_α covers M i.e.; $M = \bigcup_{\alpha \in I} U_\alpha$.

(ii) the homeomorphism ϕ_α satisfy the following compatibility conditions: the maps

$$\varphi_\alpha o \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
$$\varphi_\beta o \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

must be of class C^k.

These homeomorphisms are the transition maps or coordinate transformations.

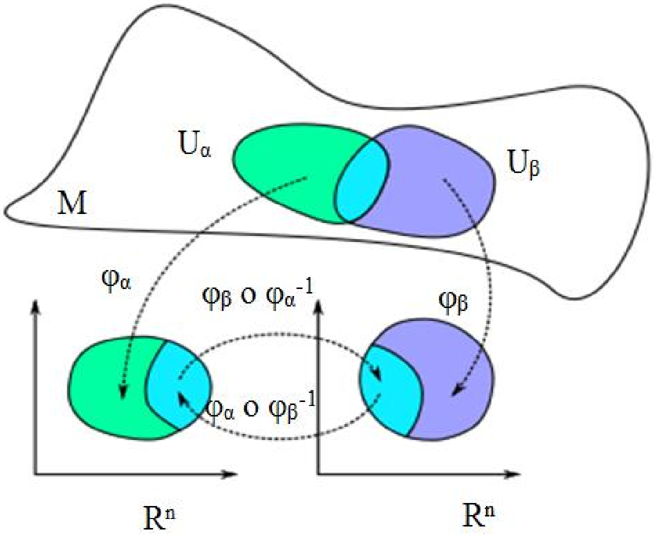


Figure 1.1. Transition maps

Definition 1.5. An C^∞ - atlas on M is a collection $A = \{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ of C^∞ chart which cover M and are C^∞ - compatible.

Definition 1.6. A second countable, Hausdorff topological space M is an n-dimensional topological manifold if it admits an atlas $\left\{ \left(U_\alpha, \phi_\alpha \right), \alpha \in I \right\}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 1.7. A topological manifold is said to be a differential or smooth manifold if all transition maps are C^∞ diffeomorphisms, that is, all partial derivatives exist and are continuous. Also, two smooth atlases are equivalent if their union is a smooth atlas.

Definition 1.8. A function $f : M \to \mathbb{R}$ from an n-dimensional manifold M to the reals is differentiable if and only if $f \circ x^{-1}$ is differentiable for any local chart $x : U \to \mathbb{R}^m$.

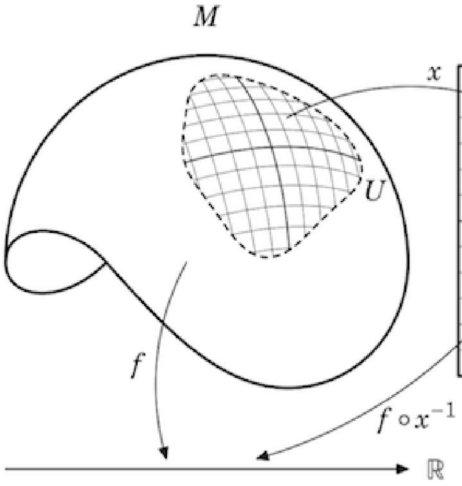


Figure 1.2. Differentiable function on a differentiable manifold

Definition 1.9. A map $f : X \to Y$ is called a homeomorphism if and only if f is bijection (hence $f^{-1} : Y \to X$ exist) both f and f^{-1} are continuous.

Definition 1.10. A diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is an injective (one - one) such that, both f and its inverse f^{-1} are C^∞ functions. But, not necessary the domain of f will be the whole at $\mathbb{R}^n$.

Definition 1.11. In vector calculus, the Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function.

Consider a map $f : U\to \mathbb{R}^m$ which is a class of c^k. That is, every function $f^i(x^1,\cdots,x^n)$, $i = \overline{1,m}$ is differentiable up to k-th order. Then the matrix

$$(\frac{\partial f}{\partial x})\Big|_{x_0} = \left\|\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{bmatrix}\right\|_{x_0\in \mathrm{U}}$$

is called the Jacobian matrix of the map f at the point $x_0 \in U$.

1.1.1 Tangent Vector Space on Manifolds

Definition 1.12. Tangent vector at x is $v = \left[(\alpha, x)\right]$. The set of all tangent vectors at x denoted by T_xM and it is called the tangent space of M at x.

Definition 1.13. Let M be a manifold and $x\in M$. Then for all $x \in M$, T_xM is defined as tangent space. The tangent bundle TM is defined by

$$TM \coloneqq \cup_{x\in M} T_xM$$

Definition 1.14. The cotangent bundle T^*M is the dual to the tangent bundle TM in the sense that each tangent space has a dual cotangent space as a vector space. The cotangent bundle T^*M is a smooth manifold itself, whose dimension is $2n$.

Definition 1.15. A fibre bundle is a space which locally looks like a product of two spaces but may possess a different global structure. Tangent and cotangent bundles are special cases of a fibre bundle. Every fiber bundle consists of a continuous surjective map: $\pi : E \to B$, where small regions in the total space E look like small regions in the product space $B \times F$. Here B is called the base space while F is the fiber space.

Definition 1.16. A jet bundle is a generalization of both the tangent bundle and the cotangent bundle. The Jet bundle is a certain construction which makes a new smooth fiber bundle out of a given smooth fiber bundle. It makes it possible to write differential equations on sections of a fiber bundle in an invariant form.

Definition 1.17. A (smooth) real vector bundle V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional real vector spaces which is locally trivial. More formally, a real vector bundle is a triple (M, V, π), where M and $V$are smooth manifolds and $\pi : V \to M$ is a smooth map. For each $m \in M$, the fiber $V_m \equiv \pi^{-1}(m)$ of V over m is a real k-dimensional vector space.

Definition 1.18. The projection map of a manifold M is defined by

$$\pi : TM \to M$$

given by

$$\pi(v) = x \text{ if } v \in T_x M.$$

Let (U, φ) be any chart containing $x \in M$. Define the map $\bar{\varphi}: T_x M \to \mathbb{R}^m$, where

$$T_x M \in [(\alpha, x)] = v \Rightarrow \bar{\varphi}(v) = \bar{\varphi}[(\alpha, x)] := (\varphi \circ \alpha)'(o).$$

Theorem 1.1. For the structure of tangent space $T_x M$, the map $\bar{\phi}$ is well- defined one-one, onto map.

Proof. (i) $\bar{\varphi}$ is well-defined

Indeed, if $[(\beta, x)] \in [(\alpha, x)]$ is another representative of $[(\alpha, x)]$ i.e. $[(\beta, x)] = [(\alpha, x)]$ then,

$$\bar{\varphi}[(\beta, x)] := (\varphi \circ \beta)'(o).$$

But $(\phi \circ \beta)'(o) = (\phi \circ \alpha)'(o)$; [By definition]

So, we have,

$$\overline{\varphi}[\beta,x]=\overline{\varphi}[(\alpha,x)].$$

$\therefore \overline{\phi}$ - is well defined.

(ii) $\bar{\varphi}$ is One–One

We assume that $\left[(\alpha,x)\right]$ and $\left[(\beta,x)\right]$ are two tangent vectors

such that, $\overline{\varphi}[(\alpha,x)]=\overline{\varphi}[(\beta,x)]$

Then by definition,

$$(\varphi\circ\alpha)'(o)=(\varphi\circ\beta)'(o)$$

Hence,

$$\alpha\tilde{x}\beta \Rightarrow [(\alpha,x)]=[(\beta,x)].$$

So, $\bar{\varphi}$ is one – one.

(iii) $\bar{\varphi}$ is Onto

Let $\bar{h} \in \mathbb{R}^m$. Then we have to show that these exists $\left[(\alpha,x)\right]\in T_xM$ such that $\overline{\phi}\left[(\alpha,x)\right]=\vec{h}$. If there is some α like this, then we have,

$$(\varphi \circ \alpha)'(o) = \vec{h} \tag{1.1}$$

So, we look for some α satisfying (1.1).

Take the line - $\beta(t) \coloneqq t\bar{h} + \varphi(x)$

where $\phi(x)$ is constant and set $\alpha(t) = \phi^{-1}\left(t.\vec{h}+\phi(x)\right)=\phi^{-1}\left(\beta(t)\right)=(\phi^{-1}o\beta)(t)$, for same β, restricted in a small interval of $\mathbb{R}$ say, J so that

$$\alpha(J)\subseteq U$$

For such an α we have, $\overline{\phi}\left[(\alpha,x)\right] =\left(\phi o\alpha\right)' t$

$$
\begin{aligned}
&= \left(\varphi o \varphi^{-1} o \beta\right)' t \\
&= \beta'(t) \\
&= \left(t\vec{h} + \varphi(x)\right)' \\
&= \frac{d}{dt}\left[t\vec{h} + \varphi(x)\right] \\
&= \vec{h}
\end{aligned}
$$

So, $\bar{\varphi}$ is onto.

Hence $\bar{\varphi}$ is a well defined, one-one and onto. □

Theorem 1.2. The projection map $\pi : TM \to M$ is C^∞ map.

Proof . Let us prove that, π is C^∞ at a tangent vector $u \in T_x M \subset TM$. By definition, we should find a chart of TM and a chart of M such that, the corresponding local representation of π is smooth. For this purpose, if $u \in T_x M$, we choose the chart $\left(U, \phi\right) \in A$ with $x \in U$ and there corresponding chart $\pi^{-1}\left((U), \varphi\right)$ of TM.

Clearly, $u \in \pi^{-1}(U)$. Thus, we can form the local representation

$$\varphi \circ \pi \circ \phi^{-1} : \phi\left(\pi^{-1}(U)\right) \to \varphi(U).$$

But $$\varphi\left(\pi^{-1}(U)\right) = \varphi(U) \times \mathbb{R}^m$$

So, $$\varphi \circ \pi \circ \varphi^{-1} \colon \varphi(U) \times \mathbb{R}^m \to \varphi(U) \subseteq \mathbb{R}^m$$

Now, we check that for every $(a, \vec{h}) \in \varphi(U) \times \mathbb{R}^m$.

$$
\begin{aligned}
(\varphi \circ \pi \circ \phi^{-1})(a, \vec{h}) \quad &= (\phi o \pi)\left(\phi^{-1}(a, \vec{h})\right) \\
&= (\phi o \pi)(v), where \phi^{-1}(a, h) = v \in T_{\phi^{-1}(a)} M
\end{aligned}
$$

$$= \varphi(x)$$
$$= \varphi(\pi(v)).$$
$$= \varphi(\varphi^{-1}(a))$$
$$= a.$$

i.e., $$(\varphi \circ \pi \circ \varphi^{-1})(a, \vec{h}) = a = Pr_1(a, \vec{h}), \forall (a, \vec{h}) \in \varphi(U) \times \mathbb{R}^m$$

or, $$(\varphi \circ \pi \circ \varphi^{-1}) = Pr_1 : \varphi(U) \times \mathbb{R}^m \to \varphi(U).$$

Therefore, the local representation is now the map,

$$(\varphi \circ \pi \circ \varphi^{-1}) = Pr_1 : \varphi(U) \times \mathbb{R}^m \to \varphi(U)$$

which is smooth map at every point of $\varphi(U) \times \mathbb{R}^m$.

Hence, the local representation is the smooth map at $\varphi(\pi^{-1}(U))$. Therefore, since the local representation is C^∞ at $\varphi\left(\pi^{-1}(U)\right)$. so, π is C^∞ at U. The same thing will be true for every $u \in TM$. Hence, π is C^∞ (smooth map) for all $u \in TM$.

Thus completes the proof of the theorem. □

Definition 1.19. Let $f : M \to N$ be a C^∞ map (smooth). The tangent map of f at x or differential of f at some $x \in M$, denoted by,

$T_x f = dxf = f_{*,x}$

is the map.

$T_x f : T_x M \to T_{f(x)} N$

And defined by $$T_x f\left(\left[(\alpha, x\right]\right) := \left[\left(fo\alpha, f(x)\right)\right] \in T_{f(x)} N.$$

1.1.2 Vector Fields on Manifolds

A vector field X on a manifold M is a cross-section in TM. Thus a vector field X assigns to every point $x \in M$, a tangent vector $X(x)$ such that the map $M \to TM$ so obtained is smooth. The vector fields on M form a module which will be denoted by $\mathfrak{X}(M)$.

Definition 1.20. A C^∞-vector field on M is a C^∞ map

$$X: M \to TM$$

such that

$$\pi o X = id_M \text{ , where } id_M = \text{identity of } M$$

Definition 1.21. If (U, φ) is a chart at $x \in U$ with coordinates $(x_1, x_2, \dots, x_m)$, then we have a basis $(\frac{\partial}{\partial x_i}\Big|_x)_{i=1,2,\dots,m}$ of $T_x M$. Thus

$$X(x) = \sum_{i=1}^{m} \lambda_i \left.\frac{\partial}{\partial x_i}\right|_x \quad \text{; where } x \in U \subseteq M$$

and

$$X(y) = \sum_{i=1}^{m} \mu_i \left.\frac{\partial}{\partial x_i}\right|_y \quad \text{; where } y \in U \subseteq M$$

In general,

$$X(x) = \sum_{i=1}^{m} f_i(x) \left.\frac{\partial}{\partial x_i}\right|_x \quad \text{; where } f_i \in U \subset M \to \mathbb{R}$$

Instead of f_i, we write X_i. Hence $X(x) = \sum_{i=1}^{m} X_i(x) \left.\frac{\partial}{\partial x_i}\right|_x$

We call X_i the coordinates of the vector field X with respect to the chart (U, φ).

Definition 1.22. Let M be an n-dimensional smooth manifold with domain U, O_M be the set of smooth functions. A smooth vector field on M is a map $X: O_M \to O_M$ such that,

(i) $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$

(ii) $X(fg) = X(f)g + fX(g)$; $\forall\, f, g \in O_M$ and $\alpha, \beta \in \mathbb{R}$

The set of all smooth vector fields on M is a vector space denoted by $\Gamma(TM)$.

Theorem 1.3. A vector field X is smooth if and only if its coordinates X_i's are smooth for all charts of M.

Proof. Assume that X is smooth. Take any chart (U, φ) with coordinates $(x_1, x_2, \dots, x_m)$.Then $X|_U$ is again smooth. Since X is smooth, if we take the charts (U, φ) of M and $(\pi^{-1}(U), \varphi)$ of TM , then the corresponding local representation

$$\varphi \ o \ X \ o \ \varphi^{-1} : \varphi(U) \to \varphi\big(\pi^{-1}(U)\big) = \varphi(U) \times \mathbb{R}^m$$

is smooth (as the local representation is defined, because $X(U) \subseteq \pi^{-1}(U)$).

Now for every $a \in \varphi(U)$, we have

$$\begin{aligned}
(\varphi \ o \ X \ o \ \varphi^{-1})(a) &= \varphi(X(\varphi^{-1}(a))) \\
&= \varphi\big(X(x)\big) \qquad [\because \varphi^{-1}(a) = x] \\
&= \varphi(X_x) \\
&= \Big(\varphi\big(\pi(X_x)\big), \bar{\phi}(X_x)\Big) \quad if \ X_x \in TM \\
&= (\varphi(x), \bar{\phi}(X_x)) \\
&= \Big(a, \bar{\phi}\big(X(x)\big)\Big) \quad [\because \varphi^{-1}(a) = x \Rightarrow \varphi(x) = a] \\
&= \Big(a, \bar{\phi}\big(X|_U(x)\big)\Big) \\
&= \left(a, \bar{\phi}\left(\textstyle\sum_{i=1}^m X_i \frac{\partial}{\partial x_i}\right)(x)\right) \quad [\because X|_U = \textstyle\sum_{i=1}^m X_i \frac{\partial}{\partial x_i}] \\
&= \left(a, \bar{\phi}\left(\textstyle\sum_{i=1}^m X_i \frac{\partial}{\partial x_i}\Big|_x\right)\right) \\
&= \left(a, \textstyle\sum_{i=1}^m X_i(x) \ \bar{\phi}(\frac{\partial}{\partial x_i}\Big|_x)\right) \quad [\because \bar{\phi} \text{ is a linear map}] \\
&= (a, \textstyle\sum_{i=1}^m X_i(x) \ e_i) \qquad [\because e_i = \bar{\phi}(\frac{\partial}{\partial x_i}\Big|_x)] \\
&= (a, \ (X_1(x), \ X_2(x), \dots, X_m(x))) \\
&= (a, \ (X_1(\ \varphi^{-1}(a)), \ X_2(\ \varphi^{-1}(a)), \dots, X_m(\ \varphi^{-1}(a)))) \\
&= (a, \ ((X_1 o \ \varphi^{-1})(a), \ (X_2 o \ \varphi^{-1})(a), \dots, (X_m o \ \varphi^{-1})(a)) \\
&= (id_{\varphi(U)}(a), \ ((X_1 o \ \varphi^{-1}), \dots, (X_m o \ \varphi^{-1}))(a))
\end{aligned}$$

$$= (id_{\varphi(U)}, (X_1 o\, \varphi^{-1}, \dots, X_m o\, \varphi^{-1})(a)$$

Therefore $(\varphi\ o\ X\ o\ \varphi^{-1}) = (id_{\varphi(U)}, (X_1 o\, \varphi^{-1}, \dots, X_m o\, \varphi^{-1}))$

Hence the smoothness of $\varphi\ o\ X\ o\ \varphi^{-1}$ implies the smoothness of each $X_i o\ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}$.
Since φ^{-1} is diffeomorphism, so $X_i \colon U \to \mathbb{R}$ is smooth (because $X_i = X_i o\ \varphi^{-1} o\ \varphi$ is a composition functions). The same thing is true for any other chart, say (V, ψ) and the corresponding coordinates $X(y_i) = Y_j$.
Hence, a vector field X is smooth implies its coordinates X_i are smooth for all charts of M.
Conversely, we can show shat the coordinates X_i of a vector field X are smooth for all charts of M implies the vector field X is smooth.

This completes the proof of the theorem. □

Definition 1.23. The set of C^∞ -vector fields on manifold M is defined by $\mathfrak{X}(M)$. Then we have

(i) For $X, Y \in \mathfrak{X}(M) \in x(M)$ defined $X + Y$ by

$$(X + Y)\,(x) := X(x) + Y(x) = X_x + Y_y \text{ for all } x \in M$$

(ii) For $\lambda \in \mathbb{R}$ and $X \in \mathfrak{X}(M)$,we define $\lambda . X$ by

$$(\lambda . X)(x) := \lambda . X(x) = \lambda . X_x$$

(iii) For $f \in C^\infty(M, \mathbb{R})$ and $X \in \mathfrak{X}(M)$we define $f.X$ by

$$(f.X)(x) := f(x).X(x) = f(x).X_x \text{ for all } x \in M$$

The set of vector fields $\mathfrak{X}(M)$ together with property (i) and (ii) is called vector space on manifold M and the set of vector fields $\mathfrak{X}(M)$ together with property (i) and (iii) is called module on manifold M.

Definition 1.24. Let M be a smooth n-dimensional manifold, O_M be the set of smooth functions and $\Gamma(TM)$ be the vector space of smooth vector fields. There is a well-defined bilinear map called Lie-brackets or Commutator

$$[\ ,] : (TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$(X,Y) \mapsto [X,Y]$$

Given by

$$[X,Y] : \ O_M \to O_M$$

$$f \mapsto [X,Y]f := X(Y(f)) - Y(X(f))$$

1.1.3 Riemannian Manifolds

Definition 1.25. Let M be a smooth manifold. A Riemannian metric g on M assigns to any smooth vector fields X and Y on M a smooth function $g(X,Y)$, where

(i) $g(X_1 + X_2, Y) = g(X_1, Y) + g(X_2, Y)$,

(ii) $g(X, Y_1 + Y_2) = g(X, Y_1) + g(X, Y_2)$,

(iii) $g(fX, Y) = f\,g(X,Y) = g(X, fY)$,

(iv) $g(X,Y) = g(Y,X)$

for all smooth real-valued functions f and vector fields X, X_1, X_2, Y, Y_1, Y_2 and

$$g(X,X) > 0 \text{ wherever } X \neq 0.$$

Definition 1.26. A Riemannian manifold (M,g) consists of a smooth manifold M together with a (smooth) Riemannian metric g on M.

Definition 1.27. A C^∞ connection ∇ on a manifold M is a mapping

$\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ denoted by $\nabla: (X,Y) \to \nabla_X Y$ which has the linearity properties:

For all $f, g \in C^\infty(M)$ and $X, X', Y, Y' \in \mathfrak{X}(M)$, we have

(i) $\nabla_{fX+gX'} Y = f(\nabla_X Y) + g(\nabla_{X'} Y)$,

(ii) $\nabla_X(fY + gY') = f\nabla_X Y + g\nabla_X Y' + (Xf)Y + (Xg)Y'$

Now the asymmetry in the rules of first and second vector fields X and Y; ∇ is $C^\infty(M)$ linear in X but not in Y. However if f is a constant function, then $Xf = 0$; thus ∇ is linear in both variables.

They do for M imbedded in Euclidean space. In addition, we have in these special case two further properties:

(iii) $[X, Y] = \nabla_X Y - \nabla_Y X$ (symmetry), and

(iv) $X(Y, Y') = (\nabla_X Y, Y') + (Y, \nabla_X Y')$.

Definition 1.28. A C^∞ connection which also has properties (iii) and (iv) is called a Riemannian connection.

Definition 1.29. A moduli space is a geometric space (usually a scheme or an algebraic stack) whose points represent algebro-geometric objects of some fixed kind, or isomorphism classes of such objects.

1.2 Preliminaries on $\mathbb{C}^n$

Definition 1.30. Let $f: \mathbb{C}^n \to \mathbb{C}$, $U \subseteq \mathbb{C}^n$ open with $\in U$, and let $z = (z_1, \dots, z_n)$ be the coordinates in $\mathbb{C}^n$. f is holomorphic in $a = (a_1, \dots, a_n) \in U$ if f has a convergent power series expansion:

$$f(z) = \sum_{k_1,\dots,k_n=0}^{+\infty} a_{k_1,\dots,k_n}(z_1 - a_1)^{k_1} \dots (z_n - a_n)^{k_n}$$

This means, in particular, that f is holomorphic in each variable. Moreover, we define

$$\mathcal{O}_{\mathbb{C}^n}(U) \coloneqq \{f: U \to \mathbb{C} | f \text{ is holomorphic}\}$$

A map $F = (F_1, \dots, F_m): U \to \mathbb{C}^m$ is holomorphic if each F_j is holomorphic.

Let f be a holomorphic function. One can write $f(z) = g(z) + ih(z)$, with $g, h: U \to \mathbb{R}$ smooth. The condition for f to be holomorphic on U is equivalent to the Cauchy-Riemann conditions:

$$\frac{\partial g}{\partial x_j}(\mathrm{a}) = \frac{\partial h}{\partial y_j}(\mathrm{a}) \text{ and } \frac{\partial g}{\partial y_j}(\mathrm{a}) = -\frac{\partial h}{\partial x_j}(\mathrm{a})$$

For $j = 1, \dots, n$, where $z_j = x_j + iy_j$.

Definition 1.31. Let $V \subseteq \mathbb{C}^n$ be open, let $F = (F_1, \dots, F_m): V \to \mathbb{C}^m$ be a holomorphic map. The complex Jacobian matrix of F is

$$J_{\mathbb{C}}F := \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_n} \end{pmatrix} = \left(\frac{\partial F_j}{\partial z_k}\right)$$

Now let $F_j = G_j + iH_j$ with $G_j, H_j: V \to \mathbb{R}$ smooth $\mathbb{R}$-valued functions. Let $\tilde{F}: V \to \mathbb{R}^{2n}$ defined as $\tilde{F} = (G_1(z), \dots, G_m(z), H_1(z), \dots, H_m(z))$. The real Jacobian matrix of F is

$$J_{\mathbb{R}}F := J_{\mathbb{R}}\tilde{F} = \begin{pmatrix} \frac{\partial G_j}{\partial x_k} & \frac{\partial G_j}{\partial y_k} \\ \frac{\partial H_j}{\partial x_k} & \frac{\partial H_j}{\partial y_k} \end{pmatrix}$$

Remark: If $F: \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic, then $\frac{\partial G_j}{\partial x_k} = \frac{\partial H_j}{\partial y_k}, \frac{\partial G_j}{\partial y_k} = -\frac{\partial H_j}{\partial x_k}$ that means

$$J_{\mathbb{R}}F = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ with } A = \frac{\partial G_j}{\partial x_k}, B = \frac{\partial H_j}{\partial y_k}$$

Moreover,

$$\frac{\partial F_j}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)(G_j + iH_j) = \frac{1}{2}\left(\frac{\partial G_j}{\partial x_k} + i\frac{\partial H_j}{\partial y_k} + i\left(\frac{\partial H_j}{\partial x_k} - i\frac{\partial G_j}{\partial y_k}\right)\right)$$

$$= \frac{\partial G_j}{\partial x_k} + i\frac{\partial H_j}{\partial y_k} = A_{jk} + iB_{jk}$$

$$\Rightarrow J_{\mathbb{C}}F = A + iB$$

Lemma 1.4. Let $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in M_{2n}(\mathbb{R})$, and let $J = \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}$. Then

$$JMJ^{-1} = M \Leftrightarrow P = S, Q = -R \Leftrightarrow M = \begin{pmatrix} P & -R \\ R & P \end{pmatrix}$$

Proof.

$$JMJ^{-1} = M \Leftrightarrow JM = MJ \Leftrightarrow \begin{pmatrix} -R & -S \\ P & Q \end{pmatrix} = \begin{pmatrix} Q & -P \\ S & -R \end{pmatrix}$$

□

Combining the above lemma and the previous remark, we can characterize a holomorphic function $F: \mathbb{C}^n \to \mathbb{C}^n$ analyzing its real Jacobian matrix:

Proposition 1.5. A function $F: \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic if and only if $J(J_{\mathbb{R}}F)J^{-1} = J_{\mathbb{R}}F$, with $J = \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}$.

Remark: It is worth to notice that J is the matrix representing the multiplication by i from $\mathbb{C}^n$ to itself. Thus, one can also state: a function $F: \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic if and only if its real Jacobian matrix is self-conjugate under the conjugation action of the multiplication by i (or simply, its real Jacobian matrix commutes with J) [75].

Proposition 1.6. Let $F: \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic. Then $\det(J_{\mathbb{R}}F) \geq 0$.

Proof. Consider the matrix N defined as

$$N = \begin{pmatrix} Id_n & i \cdot Id_n \\ Id_n & -i \cdot Id_n \end{pmatrix} \in M_{2n}(\mathbb{C}), N^{-1} = \frac{1}{2}\begin{pmatrix} Id_n & Id_n \\ -i \cdot Id_n & i \cdot Id_n \end{pmatrix}$$

Notice that,

$$
\begin{aligned}
N\,J_{\mathbb{R}}F\,N^{-1} &= \frac{1}{2}\begin{pmatrix} Id_n & i\cdot Id_n \\ Id_n & -i\cdot Id_n \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} Id_n & Id_n \\ -i\cdot Id_n & i\cdot Id_n \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} A+iB & -B+iA \\ A-iB & -B-iA \end{pmatrix}\begin{pmatrix} Id_n & Id_n \\ -i\cdot Id_n & i\cdot Id_n \end{pmatrix} \\
&= \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \\
&= \begin{pmatrix} J_{\mathbb{C}}F & 0 \\ 0 & \overline{J_{\mathbb{C}}F} \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\det(J_{\mathbb{R}}F) &= \det(N)\det(N^{-1})\det(J_{\mathbb{R}}F) = \det(N\,J_{\mathbb{R}}F\,N^{-1}) \\
&= \det(J_{\mathbb{C}}F)\det(\overline{J_{\mathbb{C}}F}) = \det(J_{\mathbb{C}}F)\overline{\det(J_{\mathbb{C}}F)} = |\det(J_{\mathbb{C}}F)|^2 \geq 0. \qquad \square
\end{aligned}
$$

A holomorphic function in one variable is a conformal mapping from $\mathbb{R}^2$ to itself, that is, it preserves orientations of angles. The latter proposition shows that, when dealing with a holomorphic function of several variables, the "orientation preserving" property translates to a strict condition on the determinant of the real Jacobian of the function. As we will see in the next section, this condition is related in some sensewith the notion of orientation (it will imply the orientability of complex manifolds, seen as differentiable manifolds).

Theorem 1.7. (Maximum Principle). Let $g: V \to \mathbb{C}$ be holomorphic, $V \subseteq \mathbb{C}$ open, connected. Assume there is a $v \in V$ such that $|g(v)| \geq |g(z)|\ \forall\, z \in V$ ($|g|$takes its maximum on). Then g is constant, so $g(z) = g(v)\ \forall\, z \in V$.

This fundamental result about one-variable holomorphic function has many consequences in complex analysis; we will only use it once in the following section to see that a holomorphic function on a compact complex manifold is nothing but a constant function. There exists also a "holomorphic version" of the Dini theorem (local invertibility of maps with invertible Jacobian):

Proposition 1.8. Let $V \subseteq \mathbb{C}$ open, $F: \mathbb{C}^n \to \mathbb{C}^n$ holomorphic. Assume that $J_{\mathbb{C}}F$ has rank n in $a \in V$ (i.e. $J_{\mathbb{C}}F(a)$ has non-zero determinant). Then there is a neighborhood W of a and a holomorphic inverse $G: F(W) \to W$ such that $F \circ G = id_{F(W)}, G \circ F = id_W$.

Proof. As $\det(J_{\mathbb{R}}F)$ has rank $2n$, $\det(J_{\mathbb{R}}F) = |\det(J_{\mathbb{C}}F)|^2 \neq 0$. So, by the Dini theorem there exist a neighborhood $W$of a such that it is possible to find an inverse G for the map F regarded as a map from $\mathbb{R}^{2n}$to $\mathbb{R}^{2n}$. We are going to show that $G$is already the map we need, that is, G is holomorphic; or equivalently, $J(J_{\mathbb{R}}G)J^{-1} = J_{\mathbb{R}}G$, J as in lemma1.1.

We know that $G \circ F = id_W \Rightarrow J_{\mathbb{R}}G \cdot J_{\mathbb{R}}F = Id$; moreover, since F is holomorphic $J(J_{\mathbb{R}}F)J^{-1} = J_{\mathbb{R}}F$. Then

$$J(J_{\mathbb{R}}G)^{-1}J^{-1} = J_{\mathbb{R}}G^{-1} \Rightarrow J(J_{\mathbb{R}}G)J^{-1} = (J(J_{\mathbb{R}}G)^{-1}J^{-1})^{-1} = (J_{\mathbb{R}}G^{-1})^{-1} = J_{\mathbb{R}}G. \quad \square$$

Definition 1.32. A function F is *biholomorphic* on $W \subseteq \mathbb{C}^n$ if there exists a holomorphic inverse $G: F(W) \to W$.

1.2.1 Basic Theory of Complex Manifolds

A complex manifold is a topological manifold equipped with an atlas of charts onto open disks in $\mathbb{C}^n$, such that the transition maps are biholomorphic. Consequently, each complex manifold is a real differentiable manifold. Moreover, since biholomorphic maps are orientation-preserving, a complex manifold is canonically oriented (not just orientable).

The theories of real (differentiable) manifolds and complex manifolds are substantially different, the main reason being that holomorphic functions are much more rigid than smooth (i.e. $\mathbb{C}^\infty$) functions. For example, there exists no holomorphic function on a compact complex manifold, apart from the trivial case of constant functions.The Whitney embedding theorem tells us that any real manifold can be (diffeomorphically) embedded in $\mathbb{R}^N$, while most complex manifolds do notadmit any holomorphic embedding into $\mathbb{C}^N$ (nor in $\mathbb{P}^N$, in the compact case).The classification of complex manifolds is more complicated than that of real manifolds. For example, a given topological manifold X admits only finitely many differentiable structures if dim $X \neq 4$, while a given complex manifold often admits uncountably many complex structures. As a matter of fact, the set of all complex structures (up to equivalence) on a given complex manifold, forms itself a continuous space, and in fact can be given the structure of a complex algebraic variety, called moduli space.

1.2.1.1 Complex Charts and Atlases

Let X be a topological manifold of dimension $2n$, that is, X is a Hausdorff topological space such that each point of X admits an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^{2n}$. Such a homeomorphism $x : U \to V$ is called coordinate neighborhood. In this section, we do not require $X$to be second countable (as it happened for differentiable manifolds).

Definition 1.33. A local complex chart (U, z) of X is an open subset $U \subseteq X$ and an homeomorphism $z : U \to V := z(U) \subset \mathbb{C}^n (\equiv \mathbb{R}^{2n})$.

Two local complex charts (U_α, z_α) , (U_β, z_β) are compatible if the map $f_{\beta\alpha} := z_\beta \circ {z_\alpha}^{-1}: z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic. The map $f_{\beta\alpha}$ is called transition function or coordinate change. (We note that $f_{\alpha\beta}$ is holomorphic, too).

Definition 1.34. A holomorphic atlas (or complex analytical atlas) of X is a collection $A = \{(U_\alpha, z_\alpha)\}$ of local complex charts, such that $X = \cup_\alpha U_\alpha$ and such that all transition functions $f_{\alpha\beta}$ are biholomorphic, for each α, β. (In this way, each pair of charts is compatible).

A complex analytic structure on X is a maximal holomorphic atlas $A = \{(U_\alpha, z_\alpha)\}_{\alpha \in I}$. Maximal means: if (U, z) is a local complex chart and (U, z) is compatible with (U_α, z_α) $\forall\ \alpha \in I$, then$(U, z) \in A$.

A complex (analytic) manifold is a topological manifold together with a complex analytic structure.

Remark: A holomorphic atlas $B = \{(U_\beta, z_\beta)\}_{\beta \in J}$determines a (unique) maximal atlas A with $B \subset A$ and hence it determines a complex manifold.

Given a complex manifold X, we can think about X without its complex structure, that is: if $dim_{\mathbb{C}}X = n$, Xdefines a differentiable manifold X_0 with $dim_{\mathbb{R}}X_0 = 2n$, wherea complex chart (U, z) gives rise to a real chart $(U, \tilde{z})$ via the identification

$$z = (z_1, \dots, z_n) \leftrightarrow \tilde{z} = (x_1, \dots, x_n, y_1, \dots, y_n) z_j = x_j + iy_j, x_j, y_j: U \to \mathbb{R}$$

One can easily check that if(U_α, z_α), (U_β, z_β) are compatible then $(U_\alpha, \tilde{z}_\alpha)$, $(U_\beta, \tilde{z}_\beta)$ are compatible too.

Proposition 1.9. Consider a complex manifold X as a differentiable manifold $X_0$with the coordinates inherited from the complex structure on X. Then X_0 is orientable.

Proof. Any transition map $F \coloneqq z_\beta \circ {z_\alpha}^{-1}: \mathbb{C}^n \to \mathbb{C}^n$ on $X$is holomorphic, and so is the inverse. As we've seen at the previous section, $\det(J_{\mathbb{R}}F) = |\det(J_{\mathbb{C}}F)|^2 > 0$ (it is not zero since Fhas an inverse). It is easy to show that $J_{\mathbb{R}}F$ is nothing else that the Jacobian matrix of the transition map $\tilde{F}$ on X_0. Then, each transition map of X_0 has Jacobian with positive determinant, i.e, X_0 is equipped with a positive atlas and is positively oriented. □

A simple consequence of this proposition is: not every differentiable manifold X_0 can be the underlying differentiable manifold of a complex manifold X.

1.2.1.2 Holomorphic Functions

Definition 1.35. Let $U \subseteq X$ be open, $f: \mathbb{C}^n \to \mathbb{C}$ be a function. Then f is *holomorphic* on U if, taken (U_α, z_α) such that $\cap U_\alpha \neq \emptyset$, the function

$$f \circ {z_\alpha}^{-1}: z_\alpha(U_\alpha \cap U) \to \mathbb{C}$$

is holomorphic. This definition does not depend on the choice of the coordinate (U_α, z_α). In addition, we define

$$\mathcal{O}_X(U) := \{f: U \to \mathbb{C} | f \text{is holomorphic}\}$$

Remark: Let $(U, z = (z_1, \dots, z_n))$ be a local complex chart on X. Let $a \in U$ with $z(a) = 0$, and let $f: U \to \mathbb{C}$ be holomorphic, Then

$$(f \circ {z_\alpha}^{-1})(u) = \sum_{k_1,\dots,k_n=0}^{+\infty} a_{k_1,\dots,k_n} {u_1}^{k_1} \dots {u_n}^{k_n}$$

where $x \in U, z(x) = u$. This means that,

$$f(x) = (f \circ {z_\alpha}^{-1})(z(x)) = \sum_{k_1,\dots,k_n=0}^{+\infty} a_{k_1,\dots,k_n} {z_1}^{k_1}(x) \dots {z_n}^{k_n}(x)$$

$$\Rightarrow f = \sum_{k_1,\dots,k_n=0}^{+\infty} a_{k_1,\dots,k_n} {z_1}^{k_1} \dots {z_n}^{k_n}$$

Definition 1.36. A map $\varphi: X^n \to Y^m$ between complex manifolds is holomorphic if

$$w_\beta \circ \varphi \circ {z_\alpha}^{-1}: z_\alpha(U_\alpha \cap \varphi^{-1}(V_\beta)) \to \mathbb{C}^m$$

is holomorphic for all charts (U_α, z_α) of X, (V_β, V_β) of Y. It is sufficient to verify that the above map is holomorphic for any (U_α, z_α), (V_β, V_β) in one atlas of X, Y respectively.

Example: The projection map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{p}^n$is holomorphic. To check this, we use the atlases $\{\mathbb{C}^{n+1} - \{0\}, \mathrm{id}_{\mathbb{C}^{n+1} - \{0\}}\}$for $\mathbb{C}^{n+1} - \{0\}$and $\{(U_j, z_j)\}_{j=1,\dots,n}$defined on $\mathbb{p}^n$as in the example of the previous section. We will check the definition only for$j = 0$.

$$\left(z_0 \circ \pi \circ \mathrm{id}_{\mathbb{C}^{n+1} - \{0\}}\right)(u_0, \dots, u_n) = z_0(u_0: \dots : u_n) = \left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0}\right)$$

This map is holomorphic on $\pi^{-1}(U_0)$.

Proposition 1.10. Let $\varphi: X^n \to Y^m$ be a holomorphic map between complex manifolds. Let $(U, z), (V, w)$ be local complex charts of X, Y respectively such that $(U) \subseteq V$. The map $F := w \circ \varphi \circ z^{-1}$ is holomorphic; assume that $J_{\mathbb{C}}F(a)$ has constant rank $k \,\forall a \in U$ (i.e., with the usual terminology, φ has constant rank on U). Then for any $a \in U$ there exists a neighborhood W of a, local complex charts $(U', z'), (V', w')$ with $a \in U' \subseteq W$ such that $\varphi(U') \subseteq V', z'(a) = 0, w'\big(\varphi(a)\big) = 0$ and $F' := w' \circ \varphi \circ (z')^{-1}: (u_1, \dots, u_n) \to (u_1, \dots, u_k, 0, \dots, 0)$.

Proof. Similar to the proof for differentiable manifolds, using Proposition 1.8, too.

Theorem 1.11. Let X be a (connected) compact complex manifold, let $f: X \to \mathbb{C}$ be a holomorphic function. Then f is constant.

Proof. $|f|: X \to \mathbb{R}$ is a continuous function, X is compact $\Rightarrow \{|f|: x \in X\}$ is compact, hence bounded. Thus, there is an $x_0 \in X$ such that $|f(x_0)| = M$ is maximal. Let$a = f(x_0) \in \mathbb{C}$. Obviously, $f^{-1}(a)$ is closed in X (it is a pre-image of a point); if we are able to show that $f^{-1}(a)$ is open, too, then $f^{-1}(a) = X$, that implies $f(x) = a \,\forall\, x \in X$.

Let $x \in f^{-1}(a)$, (U, z) be a chart with $z(x) = 0$. Then $F : f \circ z^{-1} : z(U) \in \mathbb{C}$ is holomorphic on the open subset $z(U) \subseteq \mathbb{C}^n; F\,(0) = f(x) = a$ and $|f|$ has a maximum in $z = 0$. Let $\epsilon > 0$ such that $B_\varepsilon := \{y \in \mathbb{C}^n: \|y\| < \varepsilon\} \subseteq z(U)$. For $y \in B_\varepsilon$, the function $g(t) := F\,(ty)$ is holomorphic on $\{t \in \mathbb{C}: \|ty\| < \varepsilon\}$ and $|g|$ takes its maximum in $t = 0$. By the "maximum principle" (Theorem 1.4), g is constant $\Rightarrow a = g(0) = g(1) = F\,(y)$, that means $f(y) = a \,\forall\, y \in B_\varepsilon$.

Hence $f \equiv a$ on $z^{-1}(B_\varepsilon)$, an open subset of X containing x. Hence $f^{-1}(a)$ is open. □

This result is somewhat surprising and disappointing: the condition of compactness for X, which usually makes life a lot easier when dealing with a manifold, does not allow us to consider holomorphic functions on X, since all of them are constant.

1.2.2 Complex Submanifolds

In this section we present a short treatment of the local theory of submanifolds of complex manifolds and the relation with the analytical theory of functions of several complex variables.

First, we must describe how complex manifolds are to be regarded from a differential geometric point of view. To eliminate confusion, it is desirable to eliminate complex numbers themselves from the definition, to regard complex manifolds from a "real" point of view.

Suppose that M is a manifold of dimension $2n$. Consider a coordinate chart from an open subset of M to an open subset of $\mathbb{R}^{2n}$. Now, $2n$ -dimensional Euclidean space is just $\mathbb{C}^n$, the space of n complex variables. We say that M has a complex manifold structure if an atlas of coordinate charts can be chosen so that the transition maps between two charts are defined by complex analytic functions. We say that a map between two manifolds with such structures is complex-analytic (or holomorphic) if, when referred $\mathbb{C}^n$ to by the coordinate charts, it is defined by complex analytic functions. Two such structures on the same manifold can be regarded as essentially the same if the identity map is holomorphic. It is important to realize, however, that a given manifold may have many different complex manifold structures and that a manifold need not admit any complex manifold structure. For example, the $2n$-dimensional spheres, for $n \neq 1$or 3, do not admit any. It is not known whether the six-dimensional sphere can admit one. Our first aim is to make this remark clearer by exhibiting a complex structure as a geometric structure defined by a tensor field on the manifold, just as, say, a Riemannian metric is a structure defined by a tensorfield. As a first step in this direction, we describe how the complex analytic structure on $\mathbb{C}^n$ itself is determined by a tensor field.

Consider $\mathbb{R}^{2n}$ or the space of variables (x_i, y_i) , with $1 \leq i, j, \dots \leq n$. Putting $z_i = x_i + \sqrt{-1}\, y_i$ gives the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ that we have in mind; that is, the coordinates of $\mathbb{R}^{2n}$ are considered as the real and imaginary parts of the complex variables of $\mathbb{C}^n$. Suppose $F = f + \sqrt{-1}\ g$ is a complex valued function on $\mathbb{R}^{2n}$ that is holomorphic.

The holomorphic conditions can be described by the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial x_i} = \frac{\partial g}{\partial y_i} \text{ and } \frac{\partial f}{\partial y_i} = -\frac{\partial g}{\partial x_i}$$

Define F($\mathbb{R}^{2n}$) linear map, that is, a tensor field, $J: V(\mathbb{R}^{2n}) \to V(\mathbb{R}^{2n})$ by setting

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i};\ J\left(\frac{\partial}{\partial x_j}\right) = -\frac{\partial}{\partial x_j} \text{ for } 1 \leq i \leq n$$

Then the Cauchy-Riemann equation becomes

$$X(f) = J(X)\ (g) \text{ for all } X \in V(\mathbb{R}^{2n}) \tag{1.2}$$

Note also that

$$J(JX) = -X \text{ for all } X \in V(\mathbb{R}^{2n}) \tag{1.3}$$

We can now characterize complex-analytic maps $\varphi: \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ by means of the J −tensor, namely for each point $p \in \mathbb{R}^{2n}$,

$$\text{each tangent vector } v \text{ to } p = \ \varphi_*(J(v)) = J\varphi_*(v) \tag{1.4}$$

To prove this, note that to prove φ is holomorphic. It suffices to show that $\varphi^*(F)$ is holomorphic for every holomorphic function F on $\mathbb{R}^{2n}$. However, the characterization of the Cauchy-Riemann equations as (1.2) makes it obvious that (1.3) is this condition.

Equation (1.4) tells us that a given complex structure on a manifold M defines a tensor field $J: V(M) \to V(M)$, with $J^2 = -$(identity). For if the J-tensor on $\mathbb{R}^{2n}$ is carried over to M by a coordinate chart, then (1.4) implies that J is actually independent of the coordinate chart associated with the complex structure.

Now, not every tensor field $J: V(M) \to V(M)$ with $J^2 = -$ (identity) arises in this way from a complex structure: Certain integrability conditions must be satisfied (M is said to carry on almost complex structure if it merely has such a tensor. The 6-sphere, for example, has such a tensor, which is not integrable). Such conditions are given by [40], and take the form

$$[X, Y] + J[JX, Y] = J[X, JY] + [JX, Y] = 0 \quad \text{for } X, Y \in \ V(M) \tag{1.5}$$

The key point is that the left-hand side, as a function of $X \in \ Y,$ is $F(M)$-bilinear; hence, defines a genuine tensor field. The verification of this is straight forward.

Having done this, notice that to prove it is zero, it suffices to show that it is zero for a basis of vector fields, for example, the basis$((\partial/\partial x_i),(\partial/\partial x_j))$,which is obvious. Then we can carry over (1.5) to a manifold with a complex structure. It turns out that, conversely, a J-tensor satisfying (1.3) and (1.5) arises in this way from a complex structure. If the data are real-analytic, this is not hard to prove. If the data are given as only $\mathbb{C}^{\infty}$it is considerably more difficult to prove but is true.

At any rate, we shall take our beginning point that a complex structure is defined on a manifold M by a J-tensor satisfying (1.3) and (1.5). Our main concern in this chapter is with the properties of submanifolds of $M.$ First we must consider those submanifolds that themselves are complex manifolds. Let N be a complex-analytic manifold and let $\varphi: N \to M$ be a submanifold map that is also complex-analytic. We shall call this acomplex submanifold of M. From the characterization of holomorphic maps in terms of the J-tensor, we see that$J\left(\varphi_*(N_p)\right) \subset \varphi_*(N_p)$ from now onlet us suppress the explicit notation for the submanifold map. Then the condition for a complex submanifold becomes

$$J(N_p) = N_p \qquad \text{for all } p \in N \tag{1.6}$$

We now have:

Theorem 1.12. A submanifold N of M is a complex submanifold if and only if (1.6) is satisfied.

Proof. We have already seen that (1.6) is necessary. To prove it is sufficient, notice that (1.6) implies that N itself causes a J-tensor obtained by restricting J to N. That the integrability conditions are satisfied, if for the J-tensor restricted to N, is a consequence of the fact that $[X, Y]$ is tangent to N if $X$and Y are vector fields of M that are tangent to N. □

Turn now to consideration of a submanifold N of arbitrary dimension. We want to find a method for describing the “maximal” complex submanifold of M that is contained in. Now if v is a tangent vector of N that is tangent to such a complex submanifold, then $J(v)$ is also tangent to N. Let us call a tangent vector to N with this property a holomorphic tangent vector. For $p \in N$, let H_p be the subspace of N_p consisting of all holomorphic tangent vectors; that is,

$$H_p = \{v \in N_p; J(v) \in N_p\} \tag{1.7}$$

Similarly, define H as the following subspace of $V(M)$:

$$H = \{X \in V(M) \colon X(p) \in H_p \text{for all} p \in N\} \tag{1.8}$$

Now there is a possibility of "singularities" in the field $p \to H_p$ of tangent subspaces; that is, $p \to \dim H_p$ is not necessarily constant on N. However, we shall not consider this sort of pathology here; then, also, for $p \in N$,

$$H_p = \{X(p) \colon X \in H\} \tag{1.9}$$

In order that this be so, we must have $[X, Y] \in H$ for $X, Y \in H$. We can construct a tensor field that "measures" the extent to which this is true: For $X, Y \in H$, set

$$L(X, Y) = J[JX, Y] \text{ projected into } V(M)/H \tag{1.10}$$

Now, we can verify that $L(,)$ has a tensorial behavior as a function of X and Y (although the term in the right-hand side of (1.10) does not have a tensorial behavior before it is projected).

$$L(fX, Y) = J[J(fX), Y] = J[fJ(X), Y] = J(g[JX, Y] - Y(f)JX) = fJ[JX, Y] + Y(f)X$$

which is equal to $fL(X, Y)$ when the right-hand side is projected mod$H$. Hence L passes to the quotient with respect to the restriction mapping $H \to H_p$ and we get, for each $p \in N$, a bilinear mapping (which we again denote by $L(,)$) of $H_p \times H_p \to {M_p}/{N_p}$. This field of bilinear mappings iscalled the Levi form of N. Explicitly, then,

for $X, Y \in H, p \in N$,

$$L\big(X(p),Y(p)\big) = J[JX,Y](p) \text{ projected into } {}^{M_p}\!\big/_{N_p}. \qquad (1.11)$$

Lemma 1.13. The Levi form is symmetric.

Proof. This follows from the integrability condition (1.5):

$$J[JX,Y] - J[JY,X] = [Y,X] + [JY,JX].$$

The right-hand side projects into zero when projected mod N_p .The left-hand side, though, is $L\big(X(p),Y(p)\big) - L(Y(p),X(p))$. □

Let us examine now the consequences of the Levi form vanishing identically.

Theorem 1.14. If the Levi form vanishes, then the field $p \rightarrow H_p$ of tangent subspaces of $T(N)$ is completely integrable. The maximal integral manifolds of this field then define a foliation of *N* by maximal complex submanifolds. In particular, if N is a hypersurface of M (that is, if $dim\, M = dim\, N + l$), then these complex submanifolds of *N* are hypersurfaces in *N;* hence *N* may be considered locally as a one-parameter family of complex-analytic hypersurfaces of *M.* Conversely, if a real hypersurface of *M* has this geometric property, then its Levi form vanishes.

Proof. To prove integrability of $p \rightarrow H_p$ we must show that $[H,H] \subset H$. If $X,Y \in H$, $L(X,Y) = 0$ if and only if $J[J\,X,Y]$ is tangent to *N,* hence if $[JX,Y]$ also belongs to H. This condition is obviously equivalent to $[H,H] \subset H$. That the maximal integral submanifolds of the field $H_p \rightarrow H_p$ are complex analytic submanifolds of M, since $J(H_p) = H_p$ and the tangent space to the maximal integral submanifolds is precisely H_p. The converse is obvious. □

Theorem 1.15. The hyperplanoids that are real-analytic are locally, precisely the hypersurfaces that can be written as $f = 0$, where f is the real part of a holomorphic function $f + \surd(-1)\, g = F$.

Proof. First notice that a hypersurface determined by $f = 0$ can also be written locally as the locus determined by,

$\sqrt{-1}\,F - t$ =0, where *t* is a real variable;

that is, the hypersurface is composed of a one-parameter family of complex analytic hypersurfaces.

Conversely, suppose that A is a complex manifold of one complex dimension less than M , and that $\varphi: A \times R \to M$ is a real-analytic submanifold mapping such that, for fixed t, the mapping $p \to \varphi(p, y)$of $A \to M$ is holomorphic.

We can suppose without loss in generality that M is $\mathbb{C}^n$ itself, and that A is $\mathbb{C}^{n-1}$. Since φ is real-analytic, we can extend φ to a mapping of $\mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^n$ by extending t to be a complex variable. The condition that φ be a submanifold map requires that this extended map of $\mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^n$ have nonzero Jacobian. Then, by the implicit function theorem, there is (always, locally, of course) an inverse holomorphic map $\mathbb{C}^n \to \mathbb{C}^{n-1} \times \mathbb{C}$. Following this map by the projection $\mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}$, we obtain a holomorphic function F on $\mathbb{C}^n$, that is, on M . The image of *N* in M is characterized by the condition that *F* take real values on*;* that is, *N* is obtained by setting the real part of $\sqrt{-1}\,F$ equal to zero. □

Theorem 1.16. Suppose that, in addition to the complex structure, M has an affine connection ∇ with zero torsion tensor such that the covariant derivative of the J-tensor defining the complex structure is zero. Let *N* be a submanifold of M, let $S(\,,)$ be its second fundamental form with respect to the affine connection, and let $L(\,,)$ be its Levi form with respect to the complex structure. Then

$$L(u, v) = S(u, v) + S(Ju, Jv) \text{ for } u, v \in H_p, p \in N. \tag{1.12}$$

Proof. The condition that the covariant derivative of the J-tensor be zero is explicitly

$$\nabla_X J(Y) = J\,\nabla_X Y \qquad \text{for } X, Y \in V(M)$$

The torsion-free condition is $\nabla_X Y - \nabla_X X = [X, Y]$. Then, for $X, Y \in H$,

$$J[JX, Y] = J\left(\nabla_{JX} Y - \nabla_Y JX\right) = \nabla_{JX} JY + \nabla_Y X$$

Taking the value of both sides at $p \in N$and projecting modN_p , gives (1.12). ☐

Theorem 1.17. Let N be a real hypersurface of a complex manifold. Suppose that the Levi form of N is nonzero at each point of N. Let f be a function on M that is the real part of a holomorphic function. Then the derivatives of f at points of N in direction normal to N are determined by derivatives of f in directions tangential to N.

Proof. Let $N \in H$. By hypotheses, for each $p \in N$ we can choose X so that $L(X\,(p\,), X\,(p\,))$ is not tangent to. Then $J[JX, X]$ is not tangent to N at p ; hence, also in a certain neighborhood of p . Then any vector field Z in a neighborhood of p can, after multiplication by a factor, be written as $J[JX, X] + Y$, where Y is tangent to N. Suppose $f + \sqrt{-1}\, g$ is holomorphic on M ; that is, $f$and g satisfy (1.2). Then

$$\begin{aligned} Z(f) = J[JX, X](f) + Y(f) &= -[JX, X](g) + Y(f) \\ &= X(JX)(g) - (JX)(X)(g) + Y(f) \\ &= X(X)(f) + (JX)(JX)(f) + Y(f) \end{aligned}$$

The left-hand side involves a derivative of f in a normal direction to N, while the right-hand side involves derivatives that are in direction tangent to N.The argument can be iterated to show that all normal derivatives can be so expressed. ☐

CHAPTER 2

RIEMANNIAN GEOMETRY AND MODERN DEVELOPMENTS

The beautiful subject initiated by Riemannin the 19th century on Riemann surfaces had deep influence on the development of complex geometry in the 20th century. While Hodge provided the fundamental structure relating complex analysis with topology via Hodge groups, Kodaira provided fundamental methods to construct holomorphic sections of bundles. With the works of Chern classes and Hirzebruch-Riemann-Roch formula, the works of Hodge and Kodaira have been developed to be most powerful tools in understanding Kähler geometry. The modern development has been emphasizing the use of non-linear elliptic equations, relating the concept of Kähler-Einstein metrics and Hermitian Yang-Mills equations to various fundamental concepts of stability introduced to study moduli spaces. In this chapter, we describe connections on manifolds and Riemannian manifolds. Given a connection on a manifold we can define geodesic, Riemannian curvature tensor, Ricci tensor, Ricci scalar on it. It should be mentioned here that, hand calculations become extremely tedious to evaluate components of connection, Riemannian curvature tensor, Ricci tensor etc. on higher dimensional manifolds. In this chapter we have developed some computer codes for computing these components. Using computer techniques, the components of connection, Riemannian curvature tensor, Ricci tensor etc. can be computed easily.

The original work of this chapter exists in subsection 2.2.4 under section 2.2 where the remaining part of this chapter indicates the brief review work.

2.1 The Work of Riemann

Riemann was one of the founders of complex analysis, along with Cauchy. Riemann pioneered several directions in the subject of holomorphic functions:

1. The idea of using differential equations and variational principle. The majorwork here is the Cauchy–Riemann equation, and the creation of Dirichlet principle to solve the boundary value problem for harmonic functions.

2. He gave the proof of the Riemann mapping theorem for simply connected domains. This theory of uniformization theorems has been extremely influential. There are methods based on various approaches, including methods ofpartial differential equations, hypergeometric functions and algebraic geometry. A natural generalization is to understand the moduli space of Riemann surfaces where Riemann made an important contribution by showing that itis a complex variety with dimension $3g - 3$.

3. The idea of using geometry to understand multivalued holomorphic functions, where he looked at the largest domain that a multivalued holomorphic function can define. He created the concept of Riemann surfaces, where hestudied their topology and their moduli space. In fact, he introduced the concept of connectivity of space by cutting Riemann surface into pieces. The concept of Betti number was introduced by him for spaces in arbitrary dimension. The idea of understanding analytic problems through topology orgeometry has far-reaching consequences. It influenced the later works ofPoincaré, Picard, Lefstchetz, Hodge and others. Important examples of Riemann's research are to use monodromy groups to study analytic functions. Such study has deep influence on the development of discrete groups in the 20th century. The Riemann–Hilbert problem was inspired by this and up tonow, is still an important subject in geometry and analysis. The study of ramified covering and the Riemann–Hurwitz formula gave an efficient technique in algebraic geometry and number theory.

4. The discovery of Riemann–Roch formula over algebraic curve. The generalizations by Kodaira, Hirzebruch, Grothendieck, Atiyah–Singer have led to tremendous progress in mathematics in the 20th century.

5. His study of period integrals related to Abel–Jacobi map and the hypergeometric equations:

$$z(1-z)y'' + [c-(a+b+c)z]y' - aby = 0$$

6. The study of Riemann bilinear relations, the Riemann forms and the theta functions. During his study of the periods of Riemann surfaces, he found that the period matrix must satisfy period relations with a suitable invertible skew symmetric integral matrix which is called Riemann matrix later. Riemann realized that the period relations give necessary and sufficient condition for the existence of non-degenerate Abelian functions.

First of all, we should say that Riemann was the mathematician that brought us a new concept of space that was not perceived by any mathematician before him. I believe that was the reason that Gauss was so touched by his famous address on the foundations of geometry in 1854. It is surprised that Riemann had rather liberal view about what geometry is supposed to be. His guiding principle was nature itself. [86]

The theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally. This takes us into the realm of another science–physics.

He thinks a deep understanding of geometry should be based on concepts of physics. And this is indeed the case as we experienced in the past century, especially in the past 50 years development of geometry. Although he was the one who introduced the concept of Riemann surface, which is the largest domain that a multivalued holomorphic function lives in, the precise modern conceptwas developed much later through the efforts of Klein, Poincaré and others. While Felix Klein [50] already used atlas to describe Riemann surface, it has to wait until Hermann Weyl [98] who first gave the modern rigorous definition of Riemann surface, in terms of coordinate charts. It was rather strange that a formal introduction of the concept of complex manifold was quite a bit later. Historically, generalization of one complex variable to several complex variables began by the study of functions on domains in $\mathbb{C}^n$. There were fundamental works of Levi, Oka, and Bergman. The natural generalization of the concept of two-dimensional surfaces to higher dimensional manifolds was done by O. Veblen and J.H.C. Whitehead in 1931–32. H. Whitney (1936) clarified the concept by proving that differentiable manifolds can be embedded into Euclidean space. However, it was only in 1932 at the International Congress of Mathematicians in Zurich, did Carathéodory study "four-dimensional Riemann surface" for its own sake. In 1944, Teichmüller mentioned "complex e analytische Mannigfaltigkeit" in his work on "*Veränderliche Riemannsce Flächen*". Chern was perhaps the first to use the English name "complex manifold" in his work [13].The general abstract concept of almost complex structure was introduced by Ehresmann and Hopf in the 1940s. In 1948, Hopf [41] proved that the spheres S^4 and S^8 cannot admit almost complex structures.The concept of Kähler geometry was introduced by Kähler [46] in 1933 where he demanded the Kähler form (which was first constructed by E. Cartan) to have a Kähler potential. Kähler had already observed special properties of such metric. He knew that the Ricci tensor associated to the metric tensor $g_{i\bar{j}}$ can be written rather simply as

$$R_{k\bar{l}} = \frac{\partial^2}{\partial z_k \partial \bar{z}_l}(\log det\, g_{i\bar{j}})$$

which gave a globally defined closed form on the manifold. He knew that it defines a topological invariant for the geometry. It defines a cohomology class independent of the metric. It was found later that, after normalization, it represents the first Chern class of the manifold. The simplicity of the Ricci form allows Kähler to define the concept of Kähler–Einstein metricand he wrote down the equation locally in terms of the Kähler potential. He gave examples of the Kähler metric of the ball. Slightly afterwards, Hodge developed Hodge theory, without knowing thework of Kähler, based on the induced metric from projective space to the algebraic manifolds. He studied the theory of harmonic forms with special attention to algebraic manifolds. The (p, q) decomposition of the differential forms have tremendous influence on the global understanding of Kähler manifolds. A very important observation is that the Hodge Laplacian commutes with the projection operator to the (p, q)-forms and hence the (p, q) decomposition descends to the de-Rham cohomology. The theory was soon generalized to cohomology with twisted coefficients. A very important cohomology with twisted coefficient is cohomology with coefficient in the tangent bundle or cotangent bundle, and their exterior powers. For the first cohomology with coefficient in tangent bundle, Kodaira and Spencer developed the fundamental theory of deformation of geometric structures, which gave far reaching generalization of the works of Riemann, Klein, Teichmüller and others on parametrization of complex structures over Riemann surfaces. They realize that the first cohomology with coefficient on tangent bundle, denoted by $H^1(T)$, parametrize the complex structure infinitesimally and that the second cohomology with coefficient on tangent bundle, denoted by $H^2(T)$, gives rise to obstruction to the deformation. The last statement was made very precisely by Kurinishi using Harmonic theory of Hodge–Kodaira. It describes the singular structure of the moduli space locally. Kodaira–Spencer studied how elements in $H^1(T)$ acts on other cohomology, which leads to study of variation of Hodge structures. The Hodge groups can be grouped in an appropriate way to form a natural filtration of the natural de-Rham group. The Kodaira–Spencer map plays a very important role in understanding the deformation of such filtrations. Cohomology with coefficient of cotangent bundle or wedge product of cotangent bundle gives to Hodge $(p. q)$-forms. The duality of tangent bundle and cotangent bundle gives rise to something called mirror symmetry studied extensively in the last thirty years in relation to the theory of Calabi–Yau manifolds. A very important tool in complex geometry was the introduction of Chern classes to complex bundles over a manifold and the representation of such classes by curvature of the bundle. When Chern introduced the concept of Chern classes, he was influenced by the works of Pontryagin classes. In the course of defining Chern classes by de-Rham forms given by symmetric polynomial of the curvature form, Chern defined the Chern connection for holomorphic bundles. He also proved that Chern classes of holomorphic bundles are represented by algebraic cycles on algebraic manifolds. This has been the major evidence of the Hodge conjecture: That every (p, p) class can be represented by algebraic cycles. Chern proved that three different ways to define Chern classes are equivalent. In particular, he proved they are integral classes. Weil explained how they are related to Lie algebra invariant polynomials. Weil remarked that the integrality of Chern classes should play a role in quantum theory. Chern–Weil theory forms a bridge between topology, geometry, and mathematical physics.

Kodaira was the first major mathematician who developed Hodge theory of harmonic forms right after its announcement by Hodge and he generalized the theory of harmonic forms to manifolds with boundaries, where various boundary conditions must be imposed. Perhaps his most important work was his deep understanding that the Bochner argument in Riemannian geometry can be used to prove a vanishing theorem for cohomology classes under curvature condition of the manifold. He realized that the natural place for such vanishing theorem is to deal with cohomology with coefficient on bundle or sheaf. The vanishing theorem of Kodaira says that for positive line bundle L on a compact complex manifold M:

$$H^q(M, K_M \otimes L) = 0$$

for $q > 0$. Coupled with the following theorem of Serre duality:

$$H^q(M, E) \cong H^{n-q}(M, K \otimes E^*)$$

Kodaira vanishing theorem implies that the Euler characteristic of cohomology with coefficients in a holomorphic vector bundle E with $E \otimes K^*$ positive, is simply the dimension of the group of holomorphic sections of E.

2.2 Computational Method on n-Dimensional Riemannian Manifolds

The purpose of this section is to discuss an implementation method on n-dimensional Riemannian manifolds using a computer technique. A Riemannian manifold is a differentiable manifold in which each tangent space is equipped with an inner product

$\langle\cdot,\cdot\rangle$ in a manner which varies smoothly from point to point. All differentiable manifolds (of constant dimension) can be given the structure of a Riemannian manifold. Geodesics plays an important role in many applications, especially in nuclear physics, image processing. Ovidiu Calin and Vittorio Mangione [10] considered the Heisenberg manifold structure to provide a qualitative characterization for geodesics under non-holonomic constraints. Our implementation approach can successfully well illustrate the important parameters such as Christoffel coefficients that are required in the determination of tensors. This symbol appears in many calculations in Geometry where we use non-Cartesian coordinates. In n-dimensions it has a total of n^3 components. Thus, whereas it is easy to compute this symbol in 2 or 3 dimensions, it becomes highly tedious to evaluate components of the Christoffel symbols in higher dimensions, but it is quite an easy task to deal with such situations if one can use algebraic computations for this purpose. However, it is not always possible to have the ready-made routines available that can be used in situations like this. Thus, it is of great use if one can write small routines to algebraically compute such expressions. Nevertheless, these routines can be written only when one has a reasonable knowledge of algebraic programming at the back of one's mind [8, 96].

2.2.1 Riemannian Metrics and Levi-Civita Connection

Let M be a smooth manifold. A bilinear symmetric positive-definite form

$$g_p : T_pM \times T_pM \rightarrow \mathbb{R}$$

defined for every $p \in M$ and smoothly depending on p is called a Riemannian metric on M. Positive-definite means that $g_p(v,v) > 0$ for every $v = 0, v \in T_pM$. Smoothly depending on p means that for every pair X_p, Y_p of $\mathbb{C}^\infty$ smooth vector fields on M the expression $g_p(X_p, Y_p)$ defines a $\mathbb{C}^\infty$-smooth function of $p \in M$.

Alternatively, consider a coordinate neighbourhood on M containing p and let x^i, $i = 1,\dots, dimM$ be the local coordinates. Then any two tangent vectors $u,v \in T_pM$ may be written as $u = u^i\left(\partial/\partial x^i\right)_p$, $v = v^i\left(\partial/\partial x^i\right)_p$ and $g_p(u,v) = g_{ij}(p)u^iv^j$ where the functions $g_{ij}(p) = g(\left(\partial/\partial x^i\right)_p, \left(\partial/\partial x^i\right)_p)$ express the coefficients of the metric g in local co-ordinates. One often uses the following notation for a metric in local coordinates $g = g_{ij}dx^i dx^j$. The bilinear form (metric) g will be smooth if and only if the local coefficients $g_{ij} = g_{ij}(x)$ are smooth functions of local coordinates x^i on each coordinate neighbourhood.

Theorem 2.1. Any smooth manifold M can be given a Riemannian metric. [28]

Definition 2.1. A connection on a manifold M is a connection on its tangent bundle TM. A choice of local coordinates x on M determines a choice of local trivialization of TM (using the basis vector fields $\frac{\partial}{\partial x^i}$ on coordinate patches). The transition function $\emptyset$ for two trivializations of TM is given by the Jacobi matrices of the corresponding change of coordinates $\left(\emptyset^i_{i'}\right) = \left(\frac{\partial x^i}{\partial x^{i'}}\right)$.

Let Γ^i_{jk} be the coefficients (Christoffel symbols) of a connection on M in local coordinates x^i. For any other choice $x^{i'}$ of local coordinates the transition law on the overlap becomes

$$\Gamma^i_{jk} = \Gamma^{i'}_{j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}$$

One can see from the above formula that if Γ^i_{jk} are the coefficients of a connection on M then Γ^i_{jk} also are the coefficients of some well-defined connection on M (in general, this would be a different connection). The difference $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ is called the torsion of a connection (Γ^i_{jk}). The transformation law for T^i_{jk} is

$$T^i_{jk} = T^{i'}_{j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k},$$

thus, the torsion of a connection is a well-defined anti-symmetric bilinear map sending a pair of vector fields X, Y to a vector field $T(X, Y) = T^i_{jk} X^j Y^k$ on M.

Theorem 2.2. On any Riemannian manifold (M, g) there exists a unique connection D such that

(1) $d(g(X,Y))(Z) = g(D_Z X,Y) + g(X,D_Z Y)$ for any vector fields X,Y,Z on M; and

(2) the connection D is symmetric, where D is called the Levi–Civita connection of the metric g.

The condition (1) in the above theorem is sometimes written more neatly as

$$dg(X,Y) = g(DX,Y) + g(X,DY).$$

2.2.2 Geodesics on a Riemannian Manifold

Let E → M be a vector bundle endowed with a connection (Γ^i_{jk}). A parameterized smooth curve on the base M may be written in local coordinates by $x^i(t)$. A lift of this curve to E is locally expressed as $(x^i(t), a^j(t))$ using local trivialization of the bundle E to define coordinates a^j along the fibres. A tangent vector $(\dot{x}(t), \dot{a}(t)) \in T_{(x^i(t),a^j(t))}E$ to a lifted curve will be horizontal at every t precisely when $a(t)$ satisfies a linear ordinary differential equation

$$\dot{a}^i + \Gamma^i_{jk}(x)\, a^j \dot{x}^k = 0$$

Where $i,j = 1, \ldots,$ rank $E, k = 1, \ldots, dim\ B$. Now if $E = TM$ then there is also a canonical lift of any smooth curve $\gamma(t)$ on the base, as $\dot{\gamma}(t) \in T_{\gamma(t)}M$.

Definition 2.2. A curve $\gamma(t)$ on a Riemannian manifold M is called a geodesic if $\dot{\gamma}(t)$ at every t is horizontal with respect to the Levi–Civita connection. The condition for a path in M to be a geodesic may be written explicitly in local coordinates as

$$\ddot{x}^i + \Gamma^i_{jk}(x)\, \dot{x}^j \dot{x}^k = 0$$

a non-linear second-order ordinary differential equation for a path $x(t) = (x^i(t))$ (here $i,j,k = 1,\ldots, dim\ M$). By the basic existence and uniqueness theorem from the theory of ordinary differential equations, it follows that for any choice of the initial conditions $x(0) = p, \dot{x}(t) = a$ there is a unique solution path $x(t)$ defined for $|t| < \varepsilon$ for some positive ε. Thus, for any $p \in M$ and $a \in T_p M$ there is a uniquely determined (at least for any small $|t|$) geodesic with this initial data (i.e. 'coming out of p in the direction a').

Proposition 2.3. If $\gamma(t)$ is a geodesic on(M, g) then $|\dot{\gamma}(t)|g = $ constant.

2.2.3 Curvature of a Riemannian Manifold

Let g be a metric on a manifold M. The (full) Riemann curvature $R = R(g)$ of g is, by definition, the curvature of the Levi–Civita connection of g. Thus $R \in \Omega^2_M(End\ (TM))$, locally a matrix of differential 2 -forms $R = \frac{1}{2}(R^i_{j,kl}\, dx^l \wedge dx^k)$, $i, j, k, l = 1 \ldots n = dim\, M$. The coefficients $(R^i_{j,kl})$ form the Riemann curvature tensor of (M, g). Given two vector fields X, Y, one can form an endomorphism field $R(X, Y) \in \Gamma(End(TM))$; its matrix inlocal coordinates is $R(X, Y)^i_j = R^i_{j,kl}X^kY^k$ (as usual $X = X^k\partial_k,\ Y = Y^l\partial_l$). Denote $R_k = R(\partial k, \partial l) \in End(T_p M)$ (here p is any point in the coordinate neighbourhood). In local coordinates a connection (covariant derivative) may be written as$d + A$, with $A = \Gamma^i_{jk}dx^k = A_k dx^k$. We write $D_k = D_{\frac{\partial}{\partial x_k}} = \frac{\partial}{\partial x_k} + A_k$. The definition of the curvature form of a connection yields an expression in local coordinates

$$R^i_{j,kl} = \left(D_l D_k \frac{\partial}{\partial x^j} - D_k D_l \frac{\partial}{\partial x^j}\right)^i, \text{or } R_{kl} = -[D_k, D_l]$$

considering the coefficient at $dx^l \wedge dx^k$. Now $D_X = X^k D_k$. So we have

$$\begin{aligned}
-[D_X, D_Y] &= -[X^k D_k, X^l D_l] \\
&= -X^k(\partial_k Y^l)D_l - X^k Y^l D_k D_l + Y^k(\partial_k X^l)D_l + X^k Y^l D_l D_k \\
&= X^k Y^l R_{kl} - [X, Y]^l D_l
\end{aligned}$$

We have thus proved.

Proposition 2.4 (i) $R_{ij,kl} = -R_{ij,lk} = R_{ji,lk}$ (ii) $R^i_{j,kl} + R^i_{k,lj} + R^i_{l,jk} = 0$ (iii) $R_{ij,kl} = R_{kl,ij}$.

Proof. (i) The first equality is clear. For the second equality, one has, from the definition of the Levi–Civita connection, $\frac{\partial g_{kl}}{\partial x^i} = g\left(D_i \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) + g\left(\frac{\partial}{\partial x^k}, D_i \frac{\partial}{\partial x^l}\right)$ and further

$$\frac{\partial^2 g_{kl}}{\partial x^j \partial x^i} =$$

$$g\left(D_j D_i \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) + g\left(D_i \frac{\partial}{\partial x^k}, D_j \frac{\partial}{\partial x^l}\right) + g\left(D_j \frac{\partial}{\partial x^k}, D_i \frac{\partial}{\partial x^l}\right) + g\left(\frac{\partial}{\partial x^k}, D_j D_i \frac{\partial}{\partial x^l}\right)$$

The right-hand side of the above expression is symmetric in i, j as$\frac{\partial^2 g_{kl}}{\partial x^j \partial x^i} = \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}$.The anti-symmetric part of the right-hand side (which must be zero) equals $R_{ij,kl} + R_{ji,kl}$.

(ii) Firstly,$\left(D_k \frac{\partial}{\partial x^j}\right)^i = \Gamma^i_{jk} = \left(D_j \frac{\partial}{\partial x^k}\right)^i$, by the symmetric property of the Levi–Civita. The claim now follows by straight forward computation.

(iii) Multiplying (ii) by g_{iq} gives $R_{ij,kl} + R_{ik,lj} + R_{il,jk} = 0$. Similarly $R_{jk,li} + R_{jl,ik} + R_{ji,kl} = 0$ $R_{kl,ij} + R_{ki,jl} + R_{kj,li} = 0$ and $R_{li,jk} + R_{lj,ki} + R_{lk,ij} = 0$.

Adding up the four identities and making cancellations using (i) (the 'octahedron trick') gives the required result. □

There are natural ways to extract "simpler" quantities (i.e. with less components) from the Riemann curvature tensor.

Definition 2.3. The Ricci curvature of a metric g at a point $p \in M, Ric_P = Ric(g)_p$, is thetrace of the endomorphism $v \to R_p(x, v)y$ of T_pM depending on a pair of tangent vectors $x, y \in T_pM.$ Thus in local coordinates $Ric(p)$ is expressed as a matrix $Ric = (Ric_{ij}), Ric_{ij} = \sum_q R^q_{i,jq}$.That is, the Ricci curvature at p is a bilinear form on T_pM. A consequence of Proposition 2.5(iii) is that this bilinear form is symmetric, $Ric_{ij} = Ric_{ji}$.

Definition 2.4. The scalar curvature of a metric g at a point $p \in M, s = scal(g)_p$ is asmooth function on M obtained by taking the trace of the bilinear form Ric_{ij} with respect to the metric g.

If local coordinates are chosen so that $g_{ij}(p) = \delta ij$, then the latter definition means that $s(p) = \sum_i R_{ii}(p) = \sum_{i,j} R_{ij,ji}(p)$. For a general g_{ij}, the formula may be writtenas $s = \sum_i g^{ij} Ric_{ij}$, where g^{ij} is the induced inner product on the cotangent space with respect to the dual basis, algebraically (g^{ij}) is the inverse matrix of (g_{ij}).

2.2.4 Computer Code

In the current section, we have presented our developed computer codes with an example. We have developed this codes by using a mathematical programming language MATLAB [99].

Example 2.1. Consider the metric for the three-sphere in coordinates $x^{\mu} = (\psi, \theta, \varphi)$ is given by [11]

$$ds^2 = \mathrm{d}\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\varphi^2)$$

MATLAB Code 1: (Calculating the Christoffel symbols of thefirst kind)

```
function [p]=christoffels1(i,j,k,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
result=diff(metric(j,k),coord(i))+diff(metric(i,k),coord(j
))-diff(metric(i,j),coord(k));
p=(1/2)*result;
end
```

If we run the above code for a particular input, then we will get a corresponding output. The followings are non-vanishing components and all other components are zero.

Input: `christoffels1(1,2,2)`

Output:`cos(shi)*sin(shi)`

Input:`christoffels1(1,3,3)`

Output:`cos(shi)*sin(shi)*sin(theta)^2`

Input:`christoffels1(2,1,2)`

Output:`cos(shi)*sin(shi)`

Input:`christoffels1(2,2,1)`

Output:`-cos(shi)*sin(shi)`

Input: `christoffels1(2,3,3)`

Output:`cos(theta)*sin(shi)^2*sin(theta)`

Input:`christoffels1(3,1,3)`

Output:`cos(shi)*sin(shi)*sin(theta)^2`

Input:`christoffels1(3,2,3)`

Output:`cos(theta)*sin(shi)^2*sin(theta)`

Input: `christoffels1(3,3,1)`

Output:`-cos(shi)*sin(shi)*sin(theta)^2`

Input: `christoffels1(3,3,2)`

Output: `-cos(theta)*sin(shi)^2*sin(theta)`

MATLAB Code 2: (Calculating the Christoffel symbols of the second kind)

```
function [e]=christoffels2(l,i,j,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
e=0;
```

```
for k=1:3
e=e+((1/2)*sum(inversemetric(l,k)*(diff(metric(j,k),coord(
i))+diff(metric(i,k),coord(j))-
diff(metric(i,j),coord(k)))));
end
end
```

If we run the above code for a particular input, then we will get a corresponding output. The followings are non-vanishing components and all other components are zero.

Input: `christoffels2(1,2,2)`

Output:`-sin(shi)*cos(shi)`

Input:`christoffels2(1,3,3)`

Output:`-sin(shi)*cos(shi)*sin(theta)^2`

Input:`christoffels2(2,1,2)`

Output:`cos(shi)/sin(shi)`

Input:`christoffels2(2,2,1)`

Output:`cos(shi)/sin(shi)`

Input: `christoffels2(2,3,3)`

Output:`-cos(theta)*sin(theta)`

Input:`christoffels2(3,1,3)`

Output:`cos(shi)/sin(shi)`

Input:`christoffels2(3,2,3)`

Output:`cos(theta)/sin(theta)`

Input: `christoffels2(3,3,1)`

Output:`cos(shi)/sin(shi)`

Input: `christoffels2(3,3,2)`

Output: `cos(theta)/sin(theta)`

MATLAB Code 3: (Calculating the geodesic)

```
function [final]=geodesic(l)
symsshithetaphiderivative(shi)derivative(theta)derivative(
phi);
d=[derivative(shi) derivative(theta) derivative(phi)];
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
s=0;
fori=1:3
for j=1:3
            q=0;
            p=1;

for k=1:3
q=q+((1/2)*sum(inversemetric(l,k)*(diff(metric(j,k),coord(
i))+diff(metric(i,k),coord(j))-
diff(metric(i,j),coord(k)))));
end
p=p*q*d(i)*d(j);
s=s+p;
end
end
    final=s*(-1);
disp('derivative of');
disp(d(l));
end
```

If we run the above code for a particular input, then we will get a corresponding output.

Input: `geodesic(1)`

Output:
```
derivative of
derivative(shi)=
cos(shi)*sin(shi)*derivative(phi)^2*sin(theta)^2 +
cos(shi)*sin(shi)*derivative(theta)^2
```

Input: `geodesic(2)`

Output:
```
derivative of
derivative(theta)=
derivative(phi)^2*cos(theta)*sin(theta) -
(2*cos(shi)*derivative(shi)*derivative(theta))/sin(shi)
```

Input: `geodesic(3)`

Output:
```
derivative of
derivative(phi)=
- (2*derivative(phi)*cos(shi)*derivative(shi))/sin(shi) -
(2*derivative(phi)*cos(theta)*derivative(theta))/sin(theta
)
```

MATLAB Code 4: (Calculating the Riemann Christoffel tensor)

```
function [a]=reichris(l,i,j,r1,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
q=0;
for s=1:3
p=diff(christoffels2(l,i,r1,shi,theta,phi),coord(j))-
diff(christoffels2(l,i,j,shi,theta,phi),coord(r1));
q=q+christoffels2(l,s,j,shi,theta,phi)*christoffels2(s,i,r
1,shi,theta,phi)-
christoffels2(l,s,r1,shi,theta,phi)*christoffels2(s,i,j,sh
i,theta,phi);
a=p+q;
end
```

```
function [e]=christoffels2(l,i,j,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
e=0;
for k=1:3
e=e+((1/2)*sum(inversemetric(l,k)*(diff(metric(j,k),coord(
i))+diff(metric(i,k),coord(j))-
diff(metric(i,j),coord(k)))));
end
end

end
```

If we run the above code for a particular input, then we will get a corresponding output. The followings are nonvanishing components and all other components are zero or are related via symmetries.

Input: `reichris(1,2,1,2)`

Output:`sin(shi)^2`

Input: `reichris(1,3,1,3)`

Output:`sin(shi)^2*sin(theta)^2`

Input: `reichris(2,3,2,3)`

Output:`sin(shi)^2*sin(theta)^2`

MATLAB Code 5: (Calculating the Ricci tensor)

```
function [f]=ricci(i,r1,shi,theta,phi)
symsshithetaphi ;
f=0;
```

```
for j=1:3
f=f+reichris(j,i,j,r1,shi,theta,phi);
end
function [a]=reichris(l,i,j,r1,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
q=0;
for s=1:3
p=diff(christoffels2(l,i,r1,shi,theta,phi),coord(j))-
diff(christoffels2(l,i,j,shi,theta,phi),coord(r1));
q=q+christoffels2(l,s,j,shi,theta,phi)*christoffels2(s,i,r
1,shi,theta,phi)-
christoffels2(l,s,r1,shi,theta,phi)*christoffels2(s,i,j,sh
i,theta,phi);
a=p+q;
end

function [e]=christoffels2(l,i,j,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
e=0;
for k=1:3
e=e+((1/2)*sum(inversemetric(l,k)*(diff(metric(j,k),coord(
i))+diff(metric(i,k),coord(j))-
diff(metric(i,j),coord(k)))));
end
end
end
end
```

If we run the above code for a particular input, then we will get a corresponding output. The followings are nonvanishing components and all other components are zero.

Input: `ricci(1,1)`
Output:`2`
Input: `ricci(2,2)`
Output:`2sin(shi)^2`
Input: `ricci(3,3)`
Output:`2sin(shi)^2*sin(theta)^2`

MATLAB Code 6: (Calculating the scalar curvature tensor)

```
function [c]=scalar(shi,theta,phi)
symsshithetaphi;
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
c=0;
fori=1:3
for r1=1:3
c=c+(inversemetric(i,r1)*ricci(i,r1,shi,theta,phi));
end
end
function [f]=ricci(i,r1,shi,theta,phi)
symsshithetaphi ;
f=0;
for j=1:3
f=f+reichris(j,i,j,r1,shi,theta,phi);
end
function [a]=reichris(l,i,j,r1,shi,theta,phi)
symsshithetaphi;
```

```
coord=[shi theta phi];
q=0;
for s=1:3
p=diff(christoffels2(l,i,r1,shi,theta,phi),coord(j))-
diff(christoffels2(l,i,j,shi,theta,phi),coord(r1));
q=q+christoffels2(l,s,j,shi,theta,phi)*christoffels2(s,i,r
1,shi,theta,phi)-
christoffels2(l,s,r1,shi,theta,phi)*christoffels2(s,i,j,sh
i,theta,phi);
a=p+q;
end

function [e]=christoffels2(l,i,j,shi,theta,phi)
symsshithetaphi;
coord=[shi theta phi];
metric=[1 0 0;0 (sin(shi))^2 0;0 0
(sin(shi)*sin(theta))^2];
inversemetric=inv(metric);
e=0;
for k=1:3
e=e+((1/2)*sum(inversemetric(l,k)*(diff(metric(j,k),coord(
i))+diff(metric(i,k),coord(j))-
diff(metric(i,j),coord(k)))));
end
end
end
end
end
```

If we run the above code, then we will get the following output.

Input: `scalar`

Output:`6`

2.3 Three-Dimensional Metrics as Deformations of a Constant Curvature Metric

It is known, since an old result by Riemann, that a n-dimensional metric has $f = n(n-1)/2$ degrees of freedom, that is, it is locally equivalent to the giving of f functions. As this feature is related to some particular choices of local charts, which are obviously non-geometric objects, it seems to be generically a not covariant property.

According to it, a two-dimensional metric has $f = 1$ degrees of freedom.In this case, however, a stronger result holds, as it is well known [18], namely: any two-dimensional metric g is locally conformally flat, $g = \varphi\eta$, φ being the conformal deformation factor and η the flat metric.

Contrarily to what the above Riemann's general result suggests, the two dimensional case is intrinsic and covariant, i.e. it only needs the knowledge of the metric g and only involves tensor quantities, specifically, the sole degree of freedom is represented by a scalar, the conformal deformation factor φ. The question thus arises of, whether or not, for $n > 2$ there exist similar intrinsic and covariant local relations between an arbitrary metric g, on the one hand, and the corresponding flat one η together with a set of f covariant quantities on the other.

To our knowledge, no result of this type has been published. Indeed, the known results concerning the diagonalization of any three-dimensional metric do not belong to this type. As a matter of fact, besides the $f = 3$ scalars and the (more or less implicit) flat metric, these results also involve a particular orthogonal triad of vector fields. Also, in the context of the General Theory of Relativity, such a n-dimensional relation has been proposed by one of us, but unfortunately it remains for the moment only a mere conjecture [15].

In this section we shall answer affirmatively the three-dimensional case. This dimension is the solution to the equation $f = n$, so that one is tempted to take (the components of) a vector field as the covariant set (of $f = 3$ quantities).

On the other hand, the result being deliberately local, it would seem that the essentials of the flat metric in this matter is its minimal freedom, i.e. the maximal dimension of its isometry group, so that it should be possible to substitute it by a prescribed constant curvature metric. We shall see that both assumptions work.

In fact, this section is devoted to proving the following result:

Theorem 2.5. Any three-dimensional metric g may be locally obtained from a constant curvature metric, h, by a deformation like

$$g = \sigma\, h + \varepsilon\, s \otimes s \tag{2.1}$$

where σ and s are respectively a scalar and a one-form, the sign $\varepsilon = \pm 1$ and a functional relation between σ and the Riemannian norm of s can be arbitrarily prescribed.

This result should be interesting in geometrical as well as in physical situations. In geometry, perhaps one of the first questions to be answered is the following: In two dimensions it is known that the gauge of the conformal factor σ or, equivalently, the set of flat metric tensors conformal to a given metric is given by the solutions of the Laplacian, $\Delta\sigma = 0$ [80].

In classical physics, the above theorem should be useful in (finite) deformation theory of materials; equation (2.1) may be considered as an ideal universal deformation law, allowing, from an unconstrained or not initial state (described in material coordinates by the tensor h), to reach any other deformation state (described in the same coordinates by the tensor g). This ideal universal law allows to associate, to every deformation state of a material, a vector field s among those of the gauge class of the flat metric.

In general relativity, any vacuum space-time is locally equivalent to its Cauchy data, $\{g, K\}$, g being the spatial metric and K is the extrinsic curvature of the initial instant. These data have to verify the constraint equations, a set of four equations for which many years ago Lichnerowicz showed [63] that to every arbitrarily given metric g' it corresponds a unique solution $\{g, K\}$ such that $g = \sigma g'$. This beautiful result is however useless for precise physical situations because, g being initially unknown, one does not see how to choose the good starting metric g' , which has to give g by conformity. Such an objection may be eliminated using (2.1) in the constraint equations. Our theorem also allows to translate notions such as asymptotic flatness or spatial singularity in terms of the differential 1-form $s$over a flat metric h.

2.3.1 Flat Deformation of a Given Metric

Instead of proving theorem 2.5 as stated in the introduction, we shall prove the following equivalent result:

Theorem 2.6. Let (M, g) be a Riemannian 3-manifold. There locally exist a function φ and a differential 1-form μ such that the tensor

$$g' \coloneqq \varphi g - \varepsilon \mu \otimes \mu \tag{2.2}$$

(with $\varepsilon = \pm 1$) is also a Riemannian metric with constant curvature. Besides, an arbitrary relation between φ and $|\mu|^2 = g^{ij} \mu_i \mu_j$ can be imposed in advance. The equivalence between both theorems follows immediately on substituting

$$h = g', \quad \sigma = \varphi^{-1}, \sigma = \varphi^{-\frac{1}{2}} \mu$$

into equation (2.1). The present formulation (2.2) stresses that we seek to derive $g'$from a given g. The proof is based on the comparison of the Riemannian geometries respectively defined by g and g'.

We start by considering the Riemannian connections Δ and Δ' . In an arbitrary frame $\{e_i\}_{i=1,2,3}$ the expression (2.2) reads:

$$g'_{ij} \coloneqq \varphi g_{ij} - V_{ij} \text{with} \;\; V_{ij} \coloneqq \mu_i \mu_j \tag{2.3}$$

We shall consider the difference tensor:

$$B_{ki}^{j} \coloneqq \gamma'^{j}_{ki} - \gamma_{ki}^{j} \tag{2.4}$$

which is symmetric:

$$B_{ki}^{j} \coloneqq B_{ik}^{j} \tag{2.5}$$

because both connections are torsion free.

Now, since $\Delta_k\, g_{ij} = \Delta'_k\, g'_{ij} = 0$ and taking (2.5) into account, we easily obtain that:

$$B_{ik}^{j} = \frac{1}{2}[\varphi_k\, g_{ir} + \varphi_i\, g_{kr} - \varphi_r\, g_{ik} - \Delta_k\, M_{ir} - \Delta_i\, M_{kr} + \Delta_r\, M_{ik}]h'^{rj} \tag{2.6}$$

where

$$h'^{rj} \coloneqq \varphi^{-1}\left(g^{rj} + \frac{1}{\varphi - m_0} M^{rj}\right), \text{ with } m_0 \coloneqq g^{ij} M_{ij} = \varepsilon|\mu|^2 \tag{2.7}$$

is the inverse metric for g'_{ij}.

For the sake of illustration, we shall consider an example of 3-dimensional Riemannian manifolds and locally deform them into flat metrics, in the sense stated in Theorem 2.6.

Example 2.2. (Schwarzschild Space)
The title is a shortening for the space 3-manifold for Schwarzschild coordinates in Schwarzschild space-time. The metric is:

$$\tilde{g} = k^{-1}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 sin^2\theta\, d\varphi \otimes d\varphi \tag{2.8}$$

with $k = 1 - \frac{2m}{r}$, in the region $r > 2m$ (otherwise the metric is not Riemannian). This metric can be deformed into a flat metric in several ways. Among others: **(A)** Choosing $s = \sqrt{k^{-1} - 1}\, dr$, we readily obtain:

$$\tilde{g} = g' + s \otimes s$$

Where $g' = dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 sin^2\theta\, d\varphi \otimes d\varphi$ is flat.

(B) It is well known that changing *r* into the coordinate

$$R = \frac{1}{2}\left(r\sqrt{k} + r - m\right), \qquad r = R\left(1 + \frac{m}{2R}\right)^2$$

the metric becomes:$\tilde{g} = \sigma g'$, where

$$\sigma = \left(1 + \frac{m}{2R}\right)^4, \qquad g' \coloneqq dR \otimes dR + R^2\, d\theta \otimes d\theta + R^2 sin^2\theta\, d\varphi \otimes d\varphi$$

is a flat metric.

We have shown that, locally, any Riemannian 3-dimensional metric g can be deformed along a direction $s$into a metric σhthat is conformal to a metric of constant curvature, as stated in theorem 2.5.The direction s is not uniquely determined by the metric g and the decomposition (2.1) can be achieved in an infinite number of ways. Determining more precisely the class of σ and s which deform a given g into a constant curvature metric h will be the object of future work. Specially the case where both g and h, are flat.

CHAPTER 3

CONNECTIONS WITH SYMPLECTIC STRUCTURES

Symplectic geometry originated in Hamiltonian dynamics. Symplectic geometry is the study of symplectic structures. These are certain topological structures, but these can only exist on even dimensional manifolds. Since symplectic structures are purely topological structures, so they do not depend on any metric structure of the underlying space. In the earlier work, Nazimuddin and Rifat (2014) developed a comparison between symplectic and Riemannian geometry [78]. After summarizes the basic definitions, examples and facts concerning symplectic geometry this chapter will proceed to discuss the connections of symplectic geometry with the contact geometry, Riemannian geometry and Kähler geometry.

This chapter is mainly a review. But there are some original calculations also. The original part of this chapter is to make several connections with the symplectic geometry which exists in section 3.4, section 3.5 and section 3.6.

3.1 Basic Concepts with Examples

Let M be an even dimensional smooth closed manifold, that is a compact smooth manifold without boundary. A symplectic structure ω on M is a closed ($d\omega = 0$), non-degenerate ($\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$) smooth 2-form. The nondegeneracy condition is equivalent to the fact that ω induces an isomorphism. In symplectic geometry, conformal changes to ω (i.e., multiplying by g) would usually force $d(g\omega) \neq 0$.

Example 3.1. The standard symplectic structure on $\mathbb{R}^{2n}$ is given by

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

where $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ are the coordinates of $\mathbb{R}^{2n}$ .It is clear that ω_0 is closed.

Example 3.2. All manifolds are not symplectic. For instance, S^4 is not. If ω_0 is a symplectic form on S^4, then ω_0 is exact, since the second homology class of S^4 vanishes [69]. In other words, since ω_0 is a closed 2-form $\omega_0 = d\alpha_0$, for some 1-form α_0 and $d(\omega_0 \wedge \alpha_0) = \omega_0 \wedge \omega_0$. Since $\omega_0 \wedge \omega_0$ is a volume form on S^4 , Stokes theorem implies that

$$\int_{S^4} \omega_0 \wedge \omega_0 = \int_{\delta S^4} \omega_0 \wedge \alpha_0 \neq 0$$

Since S^4 has no boundary, the last integral vanishes and ω_0 can have no symplectic form.

3.2 Local Theory

The natural equivalence between symplectic structures is symplectomorphism. Two symplectic structures ω_1 and ω_2 on manifolds M_1 and M_2 , respectively, are symplectomorphic if there exists a diffeomorphism $\varphi : M_1 \to M_2$ satisfying $\varphi^*(\omega_1) = \omega_2$. All symplectic structures are locally symplectomorphic. In consequence, there are no local invariants in symplectic geometry according to the following theorems. In particular case, we have Darboux's theorem which states that, all symplectic structures on a $2n$ dimensional manifold is locally symplectomorphic to the standard structure on $\mathbb{R}^{2n}$.

Theorem 3.1. (Darboux's theorem) Let M be a manifold of dimension $2n$with a closed non-degenerate 2-form ω_0. For any point p on a symplectic manifold, there exists a chart U with local coordinates $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$, such that on U

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Thus, locally all symplectic structures are symplectomorphic to example 3.1.

Theorem 3.2. (Weinstein's Theorem) If a submanifold $L$of a symplectic manifold (M, ω), then there exists a neighborhood of L which is symplectomorphic to a neighborhood of the zero section in the cotangent bundle T^*L.

Furthermore, symplectic structures are "local in time". That is symplectic deformations of symplectic structures do not produce new symplectic structures.

Theorem 3.3. (Moser's theorem) Let M be a closed manifold and $\omega_t, t \in [0, 1]$ is a family of cohomologous symplectic forms on M then there is an isotopy φ_t with $\varphi_0 = id$ such that ${\varphi_t}^*(\omega_t) = \omega_0$ for all t.

In particular, on a symplectic manifold all deformations of symplectic structures come from diffeomorphisms of the underlying manifold. The theorem is not true if the symplectic structures do not agree off of a compact set.

3.3 Existence and Classification

If a symplectic vector bundle is a pair (E, ω) over a smooth manifold M of rank $2n$, where $E \rightarrow M$ is a real vector bundle, then ω_q (skew-symmetric and non-degenerate) is a symplectic form on each fiber E_q, depending smoothly on q. Each of the following two characteristics is equivalent to the existence of a symplectic structure (a) the existence of a reduction of the structure group of $E$from general linear group $GL(2n)$ to symplectic group $Sp(2n, \mathbb{R})$ and (b) the existence of an (almost) complex structure on E: $J \in End(E)$ such that $J^2 = -Id$.

Now we discuss some recent results on the existence of symplectic structures on both open and closed manifolds. The existence problem of symplectic structures on even dimensional closed manifolds is quite difficult. However, Gromov has shown that symplectic structures on open manifolds obey an h-principle rule. As the existence problem of symplectic structures is based on a differential equation, but it can be reduced to a differential inequality and then solved by the h-principle.

Theorem 3.4. (Gromov's Theorem) Every $2n$ dimensional manifold M with almost symplectic structure is homotopic through almost symplectic structures to a symplectic structure, if M is open.

If the manifolds are closed, then the existence problem is much more subtle. Often there are no h-principle rules. The following result was obtained using Seiberg–Witten theory:

Theorem 3.5. (Taubes Theorem) The connected sum of an odd number of copies of $\mathbb{CP}^2$ does not admit a symplectic structure (even though it admits an almost symplectic structure and a cohomology class $\beta \in H^2(M)$ such that $\beta^2 \neq 0$).

In higher dimensions the uniqueness problem for symplectic forms on closed manifolds does not reduce to topological obstruction theory. There is often a dramatic difference between the space of non-degenerate two-forms and the space of symplectic forms [70].

3.4 Connections with Contact Geometry

The even dimensional analogue theory to contact geometry is symplectic geometry. In general, contact manifolds come naturally as boundaries of symplectic manifolds. Also, a contact manifold by symplectic means by looking at its symplectization [19, 42].

Consider (X, ω) be a symplectic manifold. A vector field v satisfying

$$L_v\omega = \omega$$

where $L_v\omega$ is the Lie derivative of ω in the direction of v, is called a symplectic dilation. A compact hypersurface M in $(X, \omega)$is said to have contact type if there exists a symplectic dilation v in a neighborhood of M that is transverse to$M$. Given a hypersurface M in (X, ω) the characteristic line field LM in the tangent bundle of M is the symplectic complement of TM in TX. (Since M is codimension one it is coisotropic and thus the symplectic complement lies in TM and is one dimensional.)

Theorem 3.6. Let M be a compact hypersurface in a symplectic manifold (X, ω) and denote the inclusion map $i : M \to X$. Then M has contact type if and only if there exists a 1-form α on M such that $d\alpha = i^*\omega$ and the form α is never zero on the characteristic line field.

If M is a hypersurface of contact type, then the 1-form α is obtained by contracting the symplectic dilation v into the symplectic form: $\alpha = l_v\omega$. It is easy to verify the 1-form α is a contact from on M. Thus, a hypersurface of contact type in a symplectic manifold inherits a co-oriented contact structure.

Given a co-orientable contact manifold (M, ξ) its symplectization *Symp*$(M, \xi) = (X, \omega)$ is constructed as follows. The manifold $X = M \times (0, \infty)$ and given a global contact form α for ξ the symplectic form is $\omega = d(t\alpha)$, where t is the coordinate on $\mathbb{R}$.

Example 3.3. The symplectization of the standard contact structure on the unit cotangent bundle is the standard symplectic structure on the complement of the zero section in the cotangent bundle.

The symplectization is independent of the choice of contact from α. To see this fix a co-orientation for ξ and note the manifold X can be identified (in may ways) with the subbundle of T^*M whose fiber over $x \in M$ is $\{\beta \in T_x^*M : \beta(\xi_x) = 0$ and $\beta > 0$ on vectors positively transverse to $\xi_x\}$ and restricting $d\lambda$ the this subspace yields a symplectic form ω, where λ is the Liouville form on T^*M. A choice of contact form α fixes an identification of X with the subbundle of T^*M under which $d(t\alpha)$ is taken to $d\lambda$.

The vector field $v = \frac{\partial}{\partial t}$ on (X, ω) is a symplectic dilation that is transverse to $M \times \{1\} \subset X$. Clearly, $l_v\omega|_{M\times\{1\}} = \alpha$. Thus, we see that any co-orientable contact manifold can be realized as a hypersurface of contact type in a symplectic manifold. In summary we have the following theorem.

Theorem 3.7. If (M, ξ) is a co-oriented contact manifold, then there is a symplectic manifold $Symp\ (M, \xi)$ in which M sits as a hypersurface of contact type. Moreover, any contact form α for ξ gives an embedding of M into $Symp\ (M, \xi)$ that realizes M as a hypersurface of contact type.

We also note that all the hypersurfaces of contact type in (X, ω) look locally, in X, like a contact manifold sitting inside its symplectification.

Theorem 3.8. Given a compact hypersurface M of contact type in a symplectic manifold $(X, \omega)$with the symplectic dilation given by v there is a neighborhood of M in X symplectomorphic to a neighborhood of $M \times \{1\}$ in $\mathrm{Symp}\,(M, \xi)$ where the symplectization is identified with $M \times (0, \infty)$using the contact form $\alpha\ = l_v\omega|_M$ and $\xi\ =\ ker\alpha.$

The following proposition shows how symplectic structures can be generated from contact structures.

Proposition 3.9. [71] Let α be a contact structure on a 3-manifold. Then $d(e^{\theta}\alpha)$ is a symplectic form on the 4-dimensional manifold $M \times \mathbb{R}$, where θ is the coordinate on ℝ. (Here α is written as a form on $M \times \mathbb{R}$).

Proof. We have $\omega_0 =\ d(e^{\theta}\alpha) = e^{\theta}(d\theta \wedge \alpha + d\alpha)$. Thus,

$$\omega_0 \wedge \omega_0 = e^{2\theta}(2d\theta \wedge \alpha \wedge d\alpha + d\alpha \wedge d\alpha)$$

Since $\alpha \wedge d\alpha$ is never zero and since $d\alpha \wedge d\alpha$does not contain differentials of θ, the claim follows. □

There are also other relations between contact and symplectic geometry [20].

3.5 Connections with Riemannian Geometry

The differentiable structure of a smooth manifold M gives rise to a canonical symplectic form on its cotangent bundle T^*M. Giving a Riemannian metric g on M is equivalent to prescribing its unit cosphere bundle ${S_g}^*M \subset T^*M$ and the restriction of the canonical 1-form from T^*M gives S^*M the structure of a contact manifold.

The following examples of known results are closely related to Riemannian and symplectic aspects of geometry.

(a) A submanifold L of a symplectic manifold (M, ω) is called lagrangian if $\omega = 0$ on TL.

(i) Endow complex projective space $\mathbb{CP}^n$ with the usual Kähler metric and the usual Kähler form. The volume of submanifolds is taken with respect to this Riemannian metric. According to a result of Givental–Kleiner–Oh, the standard $\mathbb{RP}^n$ in $\mathbb{CP}^n$ has minimal volume among all its Hamiltonian deformations [78]. A partial result for the Clifford torus in $\mathbb{CP}^n$ can be found in [27]. The torus $S^1 \times S^1 \subset S^2 \times S^2$ formed by the equators is also volume minimizing among its Hamiltonian deformations [42]. If L is a closed Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$ there exists according to [94] a constant C depending on L such that

$$Vol(\varphi_H(L)) \geq C \text{ for all Hamiltonian deformations of } L.$$

(ii) The mean curvature form of a Lagrangian submanifold Lin a Kähler-Einstein manifold can be expressed through symplectic invariants of L [12].

(b) To estimate the first eigenvalue of the Laplacian operator on functions for certain Riemannian manifolds, symplectic methods can be used [84].

(c) Consider a bounded domain $U \subset \mathbb{R}^{2n}$ with smooth boundary. There exists a periodic billiard trajectory on $\overline{U}$ of length l with

$$l^n \leq C_n vol(U)$$

where C_n is an explicit constant depending only on n [20].

(d) Also Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

is satisfied as a consequence of the closure of the symplectic form, $d\omega = 0$.

3.6 Connections with Kähler Geometry

Kähler manifolds are the remarkable class of symplectic manifolds. M. Gromov [29] observed that some of the tools used in the Kähler context can be used for the study of symplectic manifolds. One part of his wondering work has grown into which is now called Gromov–Witten theory [72]. All Kähler manifolds are symplectic, since the Kähler form is closed and non-degenerate. For instance, the complex projective space $\mathbb{CP}^n$ is Kähler so that this space is also symplectic. But the converse need not be true, but we have the following theorem:

Theorem 3.10. A structure (M, ω, J) on a smooth manifold X is a Kähler structure if ω is a symplectic form, J is a complex structure, g is a Riemannian metric such that $g(X, Y) = \omega(X, JY)$.

Many techniques and constructions from complex geometry are most useful in symplectic geometry. For instance, there is a symplectic version of blowing-up, which is closely related to the symplectic packing problem [73, 74], also Donaldson's construction of symplectic submanifolds [17].

Also, any complex surface admits a Kähler structure if and only if the first Betti number is even [9]. There are many symplectic 4-manifolds with even $b1$ (or $b1 = 0$) admitting no Kähler structure [31]. For a minimal Kähler surface we have the following theorem.

Theorem 3.11. Let (X, J) be a minimal Kähler surface. Then inside the symplectic cone, the Kähler cone can be enlarged across any of its open face determined by an irreducible curve with negative self-intersection. In fact, if the curve is not a rational curve with odd self-intersection, then the reflection of the Kähler cone along the corresponding face is in the symplectic cone.

In addition, for a minimal surface of general type, the canonical class K_J is shown to be in the symplectic cone in [14, 88].

CHAPTER 4

SYMPLECTIC AND CONTACT GEOMETRY WITH COMPLEX MANIFOLDS

In this chapter we discuss about almost complex structures and complex structures on Riemannian manifolds, symplectic manifolds and contact manifolds. We have also shown a special comparison between complex symplectic geometry and complex contact geometry. Finally, we investigate the existence of a complex submanifold of positive dimension in $\mathbb{C}^n$ that intersects a real submanifold along two absolutely and real analytic submanifolds.

The first example of compact symplectic manifold with no K$\ddot{a}$hler structure is provided in [92]. Thurston's example had already been discovered as a complex manifold, by Kodaira during his work on the classification of compact complex surfaces [50]. On the other side, the thought of complex contact manifold was discovered as an end result of the works of Kobayashi and Boothby [56, 57, 58] in late 1950s and the early 1960s. Then in 1965, Wolf [97] studied homogeneous complex contact manifolds. Ishihara and Konishi [44, 45] delivered a notion of normality for complex contact structures. In this development however, the notion of normality looks too robust due to the fact it precludes the complicated Heisenberg group as one of the canonical examples, even though it does include complex projective spaces as odd complex dimension as one would expect. Then B. Korkmaz [47, 48, 49] provide a new situation for the normality.

In this chapter, we study on symplectic geometry and contact geometry with complex manifold. Here we have developed a special comparison between complex symplectic geometry and complex contact geometry. This chapter is mainly a review. But the original part of this chapter is to develop a special comparison using some special characteristic which exists in section 4.3.

4.1 Complex Symplectic Manifolds

Let (M,ω) be a complex-symplectic manifold with $dim_{\mathbb{C}}M = 2n$ and complex structure J_0. Then ω is a closed, holomorphic 2-form with $\omega^n \neq 0$. Let $\omega = \omega_1 + i\omega_2$, where ω_1 and ω_2 are real 2-forms. Since ω is closed, so are ω_1 and ω_2. Also, ω being holomorphic means that $\omega(X + iJ_0X,*) = 0$ as a1-form on $T^{\mathbb{C}}M$. It is easy to see then that

$$\omega_2(X,Y) = -\omega_1(J_0X,Y) = -\omega_1(X,J_0Y)$$

for any real vectors X and Y. Now, we may use the complex version of Darboux's theorem to find local holomorphic functions $(z_1,\dots,z_n,w_1,\dots,w_n)$ such that $\omega = dz_1 \wedge dw_1 +\cdots + dz_n \wedge dw_n$. If we derive real coordinates $z_j = x_j + iy_j$, $w_j = s_j + it_j$, then

$$\begin{aligned}\omega_1 &= dx_1 \wedge ds_1 - dy_1 \wedge dt_1 +\cdots + dx_n \wedge ds_n - dy_n \wedge dt_n\\ \omega_2 &= dx_1 \wedge dt_1 - dy_1 \wedge ds_1 +\cdots + dx_n \wedge dt_n - dy_n \wedge ds_n\end{aligned}$$

from which we see that $\omega_1^{2n} \neq 0$ and $\omega_2^{2n} \neq 0$. Thus, we have two distinct symplectic structures on M. For now, we will assume that each represents an integral class in cohomology.

Gromov proved that an open almost complex manifold M always carries a compatible symplectic structure [30]. For compact manifolds existence of an almost complex structure does not imply existence of a symplectic structure and the simplest additional necessary condition is the existence of a closed 2-form ω such that its powers ω^j are cohomologically for $j = 1,\cdots,N$: $[\omega]^j \neq 0$ in $H^{2j}(M)$.

A complex manifold $M$is called a Kähler manifold if it carries a Hermitian metric $h_{ij}dz^i d\bar{z}^j$ such that the form $\omega = h_{ij}dz^i d\bar{z}^j$ is closed. This form is symplectic and therefore any Kähler manifold carries a natural symplectic structure.

The simplest examples of Kähler manifolds are algebraic manifolds which are complex submanifolds of the complex projective spaces. For such manifold a Kähler structure is given by the metric induced from the Fubiny–Study metric by the embedding. Denote by ($\mathbb{CP}^n$, ω_{FS}) the complex projective space $\mathbb{CP}^n$ with a K$\ddot{a}$hler form ω_{FS} induced by the Fubiny–Study metric. These symplectic manifolds serve as universal symplectic manifolds in the following sense.

Proposition 4.1. [90] Let (M, ω) be a compact symplectic manifold of dimension $2n$ such that the form ω is integer, i.e. $[\omega] \in H^2(M;\ \mathbb{Z}) \subset H^2(M;\ \mathbb{R})$. Then there exists an embedding

$$f : M \to \mathbb{CP}^{2n+1}$$

such that $f * \omega_{FS} = \omega$.

4.1.1 Complex symplectic structure on T$^*\mathcal{T}$ (S)

It is a basic fact that if M is any complex manifold (in particular when $M = \mathcal{T}\ (S)$), the total space of its holomorphic cotangent bundle T^*M is equipped with a canonical complex symplectic structure.

The canonical 1-form ξ is the holomorphic (1, 0)-form on T^*M defined at a point $\phi \in T^*M$ by $\xi_\phi := \pi^*\phi$, where $\pi : T^*M \to M$ is the canonical projection and ϕ is seen as a complex covector on M in the right-hand side of the equality. The canonical complex symplectic form on T^*M is then simply defined by $\omega_{\text{can}} = d\xi$. If (z_k) is a system of holomorphic coordinates on M so that an arbitrary (1,0)-form has an expression of the form $\alpha = \sum w_k dz_k$, then (z_k, w_k) is a system of holomorphic coordinates on T^*M for which $\xi = \sum w_k dz_k$and $\omega_{\text{can}} = \sum dw_k \wedge dz_k$ .The canonical 1-form satisfies the following reproducing property. If α is any (1,0)-form on M, it is in particular a map $M \to T^*M$ and as such it can be used to pull back differential forms from T^*M to M. It is then not hard to show that $\alpha^*\xi = \alpha$ and as a consequence $\alpha^*\omega_{\ can} = d\alpha$.

4.2 Complex Contact Manifolds

Let X be a complex manifold and TX its holomorphic tangent bundle. The complex manifold X is called contact if there is a complex-codimension one holomorphic sub-bundle D of TX which is maximally non-integrable, i.e. the tensor

$$D \times D \rightarrow T\,X/D$$
$$(v, w) \longmapsto [v, w]\ mod\ D$$

is non-degenerate for every point of X.

Let $L := TX/D$ be the quotient line bundle and $\theta : TX \rightarrow L$ the tautological projection, so that we have the short exact sequence

$$0 \rightarrow D \rightarrow TX \rightarrow L \rightarrow 0.$$

The projection θ can be thought of as a 1-form with values in the line bundle $L, \theta \in \Gamma(X, \Omega^1(L))$, with $ker(\theta) = D$. The sub-bundle D must have even rank $2n$ and, therefore, the manifold X has odd complex dimension $2n + 1 \geq 3$. Moreover, the non-degeneracy condition implies

$$\theta \wedge (d\theta)^n \in \Gamma(X, \Omega^{2n+1}(L^{n+1}))$$

is nowhere zero. This provides an isomorphism of the anti-canonical line bundle [57, 60] of X and L^{n+1}. Since $L = TX/D$, there is a C^∞ isomorphism

$$TX \cong D \oplus L,$$

so that

$$c(X) = c(D) \cdot c(L).$$

There is also the following isomorphism

$$D \cong D^* \otimes L$$

By means of the splitting principle we can write the Chern classes in terms of formal roots

$$c(D) = (1 + y_1)(1 + y_2) \cdots (1 + y_{2n}),$$

and

$$c(L) = (1 + y_{2n+1}),$$

so that

$$c_1(X) = (n + 1)y_{2n+1}.$$

4.2.1 General facts about global complex contact structures

We will now review some facts about global complex contact structures and their corresponding vertical sub-bundles. Throughout this section, we assume that $\eta = u - iv$ is a global holomorphic contact form (u and v are real 1-forms with $v = u \circ J$) and that V is the subbundle of TP defined as the span of $\{U, V = -JU\}$ where

$$u(U) = 1, v(U) = 0, \iota(U)du = 0,$$
$$u(V) = 0, v(V) = 1, \iota(V)dv = 0.$$

Theorem 4.2. If P is a complex contact manifold with a global holomorphic contact form $\eta = u - iv$ and corresponding vertical subbundle $V = span\{U, V = -JU\}$ given by

$$u(U) = 1, v(U) = 0, \iota(U)du = 0,$$
$$u(V) = 0, v(V) = 1, \iota(V)dv = 0.$$

Then

1. U and JU are infinitesimal automorphisms of J, i.e., $\mathcal{L}_U J = \mathcal{L}_{JU} J = 0$.
2. $[U, JU] = 0$, so that V is a foliation of TP.
3. $\mathcal{L}_U u = \mathcal{L}_{JU} u = \mathcal{L}_U v = \mathcal{L}_{JU} v = 0$.
4. $\mathcal{L}_U (du) = \mathcal{L}_{JU} (du) = \mathcal{L}_U (dv) = \mathcal{L}_{JU} (dv) = 0$.

Proof. If we use the complex Darboux Theorem to derive holomorphic coordinates $(z_1, \dots, z_{2n+1})$ such that

$$\eta = dz_1 - z_2 dz_3 - \cdots - z_{2n} dz_{2n+1},$$

then we see immediately that $\frac{1}{2}(U - iJU) = \partial/\partial z_1$. In other words, both $U = \partial/\partial x_1$ and $JU = \partial/\partial y_1$ are infinitesimal automorphisms of J. So, $\mathcal{L}_U J = \mathcal{L}_{JU} J = 0$. In particular, $[U, JU] = J[U, U] = 0$, i.e., V is a foliation. Also, note that, on each vertical leaf, we have a hermitian metric given by

$$g' = u \otimes u + v \otimes v,$$

i.e., U and JU are taken to be orthonormal vector fields. By assumption, $d\eta$ is a holomorphic 2-form on P. In particular,

$$dv(X,Y) = du(JX,Y) = du(X,JY)$$

for any vectors X,Y on P .we also have $dv(U,X) = du(U,JX) = 0$ and similarly, $du(V,X) = 0$ for any $X \in TP$. Thus, if $X \in H$, then

$$v([U,X]) = -2dv(U,X) = 0, u([U,X]) = -2du(U,X) = 0.$$

So, $[U,X] \in \mathcal{H}$. Similarly, $[V,X] \in \mathcal{H}$. Furthermore, for any $z \in P$, there is an open subset Y of P such that the space $\frac{Y}{V}$, the space of maximal vertical leaves on Y given the quotient topology, is an open manifold and $P_Y : Y \to \frac{Y}{V}$ is a submersion. Then, for any basic vector field X on Y, i.e., X is horizontal and$(P_Y)_* X$ is a well-defined vector field on $\frac{Y}{V}$, we have

$$(P_Y)_*([U,X]) = (P_Y)_*([V,X]) = 0.$$

So, $[U,X]$ and $[V,X]$ are also vertical. Thus, $[U,X] = [V,X] = 0$.

If X is any horizontal vector and we extend X to be a local basic vector field on P, then

$$\mathcal{L}_U u(X) = -u([U,X]) = 0.$$

Hence, $\mathcal{L}_U u = 0$. Similarly, we have

$$\mathcal{L}_U v = 0 = \mathcal{L}_{JU} v = \mathcal{L}_{JU} u.$$

Using this same argument, we have

$$\mathcal{L}_U(du) = \mathcal{L}_{JU}(du) = \mathcal{L}_U(dv) = \mathcal{L}_{JU}(dv) = 0.$$

This completes the proof. □

4.3 Comparison between Complex Symplectic Geometry and Complex Contact Geometry

Complex Symplectic Geometry	**ComplexContact Geometry**
1. Complex Symplectic Manifold The complex manifold X of complex dimension $2n$ is called symplectic if it has	**1. Complex Contact Manifold** The complex manifold X of complex dimension $2n+1$is called contact if

a holomorphic symplectic 2-form ω is closed with $\omega^n \neq 0$. Let $$\omega = \omega_1 + i\,\omega_2$$ These two closed forms ω_1 and ω_2 are real symplectic forms and define the structure of a complex symplectic manifold on X.	there is a complex co-dimension one holomorphic sub-bundle D of TX which is maximally non-integrable, i.e. the tensor $$D \times D \to T\,X/D$$ $$(v,w) \longmapsto [v,w]\; mod\; D$$ is non-degenerate for every point of X.
2. Examples $(\boldsymbol{i})$ Kodaira- Thurston manifold represents a complex symplectic manifold. Let g be the Lie algebra of $G$and let g*be its dual. We identify tensors on gand g* with left-invariant objects on G. It is easy to check that g has a basis $< X_1,\ X_2,\ X_3,\ X_4 >$in which the only non-zero bracket is $[X_1,\ X_2] = -X_3$. Let $< x_1,\ x_2,\ x_3,\ x_4 >$be the dual basis of g*. The only non-zero differential on g* is computed to be $dx_3 = x_1 \wedge x_2$ .The element $\omega = x_1 \wedge x_4 + x_2 \wedge x_3$is closed and non-degenerate. $(\boldsymbol{ii})$ Consider the holomorphic Lie groups $\mathbb{C}^3 \cong H_{\mathbb{C}} = \left\{\begin{pmatrix} 1 & z_2 & z_1 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_1,\ z_2,\ z_3 \in M_{3\times 3}(\mathbb{C})\right\}$ and $\mathbb{C}^2 = \begin{pmatrix} z_1 \\ z_1 \end{pmatrix}$. Then the map $\pi: \mathbb{C}^3 \to \mathbb{C}^2$ is a surjective holomorphic Lie group homomorphism. The holomorphic 2-	**2. Examples** $(\boldsymbol{i})$The odd-dimensional complex projective space $\mathbb{P}^{2n+1}$ is a complex contact manifold. Any 2-homogeneous symplectic form ω on $\mathbb{C}^{2n+2}$defines a contact form on $\mathbb{P}^{2n+1}$. $(\boldsymbol{ii})$ Complex Heisenberg group $H_{\mathbb{C}}$ represents a complex contact manifold, where $$\mathbb{C}^3 \cong H_{\mathbb{C}} = \left\{\begin{pmatrix} 1 & z_2 & z_1 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_1,\ z_2,\ z_3 \in M_{3\times 3}(\mathbb{C})\right\}$$ The complex contact structure of this manifold is given by the left invariant 1-

form $\omega = -dz_1 \wedge dz_2$ on $\mathbb{C}^2$ is a left-invariant complex symplectic form.	form $\eta = dz_1 - z_2 dz_3$ and $\eta \wedge d\eta \neq 0$
3. Equivalence Let U_X (resp. U_Y) be an open subset of T^*X (resp. T^*Y) and let $\varphi : U_X \simeq U_Y$ be a symplectic isomorphism. Then, locally on U_X , there exists a τ -preserving contact isomorphism $$\psi :\ \rho^{-1}(U_X) \simeq \rho^{-1}\ (U_Y)$$ making the diagram below commutative $$\begin{tikzcd} T^*X \supset U_X \arrow[r, "\varphi"] & U_Y \subset T^*Y \\ \dot{P}^*(X \times \mathbb{C}) \supset \rho^{-1}(U_X) \arrow[u, "\rho"] \arrow[r, "\psi"] & \rho^{-1}(U_Y) \subset \dot{P}^*(Y \times \mathbb{C}) \arrow[u, "\rho"'] \end{tikzcd}$$	**3. Equivalence** Let V_X (resp.V_Y) be an open subset of $\dot{P}^*(X \times \mathbb{C})$ (resp. of $\dot{P}^*(Y \times \mathbb{C})$) and let $\psi : V_X \simeq V_Y$ be a contact isomorphism. Then, we say that ψ is a τ -preserving contact isomorphism if it lifts as a homogeneous symplectic isomorphism $$\tilde{\psi}: \gamma^{-1}(V_X) \simeq \gamma^{-1}\ (V_Y)$$ making the diagram below commutative $$\begin{tikzcd} \dot{P}^*(X \times \mathbb{C}) \supset V_X \arrow[r, "\psi"] & V_Y \subset \dot{P}^*(Y \times \mathbb{C}) \\ T^*X \times \dot{T}^*\mathbb{C} \supset \gamma^{-1}(V_X) \arrow[u, "\gamma"] \arrow[r, "\tilde{\psi}"] & \gamma^{-1}(V_Y) \subset T^*Y \times \dot{T}^*\mathbb{C} \arrow[u, "\gamma"'] \end{tikzcd}$$
4. Quantization-deformation modules Let X be a complex symplectic manifold. There exists canonically a K-Abelian stack $mod(\mathcal{W}^{\sqrt{v}}, X)$ on X such that if $U \subset X$ is an open subset isomorphic by a contact transformation φ to an open subset $U_X \subset T^*X$, then $mod(\mathcal{W}^{\sqrt{v}}, X)\|_U$ is equivalent by φ to the stack $mod(\mathcal{W}_X^{\sqrt{v}} \| U_X)$.	**4. Quantization-deformation modules** Let Y be a complex contact manifold. There exists canonically a $\mathbb{C}$ -Abelian stack $mod(\varepsilon^{\sqrt{v}}, Y)$ on Y such that if $V \subset Y$ is an open subset isomorphic by a contact transformation ψ to an open subset $V_Y \subset P^*Y$, then $mod(\varepsilon^{\sqrt{v}}, Y)\|_V$ is equivalent by ψ to the stack $mod(\varepsilon_Y^{\sqrt{v}} \| V_Y)$.

5. Local Characteristic	**5. Local Characteristic**
Let X be a complex symplectic manifold. Now Darboux's theorem implies that, the local model of X is an open subset of the cotangent bundle T^*M with $M = \mathbb{C}^{\frac{1}{2}\dim X}$.	Let Y be a complex contact manifold. Now Darboux's theorem implies that, the local model of Y is an open subset of the projective cotangent bundle P^*M with $M = \mathbb{C}^{\frac{1}{2}(\dim Y+1)}$.

4.4 Existence of complex submanifolds

We are interested in complex submanifolds S in $\mathbb{C}^{2n}$ that intersect the real submanifold M at the origin. Recall that M has real dimension $2n$. Generically, the origin is an isolated intersection point if dim $S = n$. Let us consider the situation when the intersection has dimension n. Without further restrictions, there are many such complex submanifolds; for instance, we can take a n-dimensional totally real and real analytic submanifold S_1 of M. We then let S be the complexification of S_1. To ensure the uniqueness or finiteness of the complex submanifolds, we therefore introduce the following.

Definition 4.1. Let M be a formal real submanifold of dimension 2n in $\mathbb{C}^{2\mathrm{n}}$. We say that a formal complex submanifold S is attached to M if $S \cap M$ contains at least two germs of totally real and formal submanifolds S_1, S_2 of dimension n that intersect transversally at the origin and S has dimension n. Such a pair $\{S_1, S_2\}$ are called a pair of asymptotic formal submanifolds of M.

We first derive the results at the formal level. We then apply the results of [83, 89]. The proof of the co-existence of convergent and divergent attached submanifolds will rely on a theorem of Pöschel on stable invariant submanifolds and Siegel's small divisor technique.We now describe the formal results. When $n = 1$, a non-resonant hyperbolic M admits a unique attached formal holomorphic curve [59].

When $n > 1$, new situations arise. First, we show that there are obstructions to attach formal submanifolds. However, the formal obstructions disappear when M admits the maximum number of deck transformations and $M$is non-resonant.We will consider a real submanifold M which is a higher order perturbation of a non resonant product quadrics. By adapting the proof of Klingenberg to the manifold $M,$ we will show the existence of a unique attached formal submanifold for a prescribed non-resonance condition. We also show that the complexification of S in $M'$is a pair of invariant formal submanifolds S'_1, S'_2 of σ. Furthermore, S is convergent if and only if S'_1 is convergent.

We now can prove the following theorem.

Theorem 4.3. Let M be a real analytic submanifold in $\mathbb{C}^{2n}$ without elliptic components. Assume that in (ξ, η) coordinates, $D\sigma(0)$ is diagonal and has distinct eigenvalues $\boldsymbol{\mu_1, \mu_2, \dots, \mu_n, \mu_1^{-1}, \mu_2^{-1}, \dots, \mu_n^{-1}}$**.** Let $v = v_\varepsilon$, then M admits a unique pair of formal asymptotic submanifold $\{S^\varepsilon{}_1, S^\varepsilon{}_2\}$ such that the complexification of $S^\varepsilon{}_1$ in $M'$is an invariant formal submanifold H_ε of σ that is tangent to

$$\cap_{\varepsilon_j=1} \{\eta_j = 0\} \cap \cap_{\varepsilon_j=-1} \{\xi_j = 0\}.$$

Furthermore, the complexification of $S^\varepsilon{}_2$ equals $\tau_1 H_\varepsilon$.

Proof. Let $S_i = S^\varepsilon{}_i$. We will follow Klingenberg's approach for $n = 1$, byusing the deck transformations. Suppose that S is an attached formal complex submanifold which intersects with M at two totally real formal submanifolds S_1, S_2. We first embed $S_1 \cup S_2$ into M' as M is embedded into M'. Let S'_i be the complexification of S_i in M'. Since ρ fixes S_i pointwise, then $\rho S_i = S_i$.

We want to show that $\tau_1(S'_1) = S'_2$; thus S'_i is invariant under σ. We can see that $S'_i$is defined by

$$\overline{\rho}_i(z') = w'$$

On S'_1 ,we have $L(z', w') + E(z', w') = -f(z')$. The latter defines a complex submanifold of dimension n. Thus, it must be S'_1. On M',

$$\left(L_j(z',w') + E_j(z',w')\right)^2 = z_{p+j}$$

are invariant by τ_1. Thus each $L_j(z',w') + E_j(z',w')$ is either invariant or skew-invariant by τ_1. Computing the linear part, we conclude that they are all skew-invariant by τ_1. Hence $\tau_1(S'_1)$ is defined by $L_j(z',w') + E_j(z',w') = f(z')$, which is the defining equations for S'_2. We must identify the tangent space of S'_1 at the origin.

Finally, if S'_1 is convergent, then $\overline{\rho_1}$ is convergent. Hence S'_1, the fixed-point set of ρ_1, is convergent. □

CHAPTER 5

EXISTENCE AND STABILITY OF LEGENDRE AND ISOTROPIC SUBMANIFOLDS IN CONTACT MANIFOLDS

Without references to differential geometry or twistor theory a solution of a certain moduli problem was solved by Kodaira [52] in 1962. Kodaira's initial data is a pair, $X \hookrightarrow Y$ consisting of a compact complex submanifold X of a complex manifold Y. He showed that if the normal bundle $N_{X|Y}$ of the initial submanifolds $X \hookrightarrow Y$ is such that $H^1(X, N_{X|Y}) = 0$, then the moduli set M has two properties: first, it is a manifold with dim $M = H^0(X, N_{X|Y})$; second, a tangent vector at any point$t \in M$ can be realized canonically as a global section of the normal bundle N_t of the associated submanifold $X_t \hookrightarrow Y$, i.e., there is a canonical isomorphism $k_t : T_tM \rightarrow H^0(X_t, N_t)$, called the *Kodaira map*. The manifold (parameter space) M is called the Kodaira moduli space. In [68], Merkulov proved the completeness and maximality of moduli spaces as well as the stability of compact Legendre submanifolds in complex contact manifolds. A completeness and maximality of moduli spaces of compact complex isotropic submanifolds in complex contact manifold are studied in [1]. This result generalizes the result of Merkulov [68] on Legendre submanifolds.

This is mainly a review on Kodaira, Legendre, and isotropic moduli spaces. However, there are some original calculations also. The original work of this chapter is to establish an interconnection among Kodaira, Legendre, and isotropic moduli spaces which exist in section 5.5.

5.1 Kodaira Moduli Spaces

In this part we recall some useful facts about relative deformation theory of compact complex submanifolds of complex manifolds. [55]

Let Y and M be complex manifolds and let $\pi_1 : Y \times M \to Y$ and $\pi_2 : Y \times M \to M$ be two natural projections. An analytic family of compact submanifolds of the complex manifold Y with the moduli space M is a complex submanifold $F \hookrightarrow Y \times M$ such that the restriction of the projection π_2 on F is a proper regular holomorphic map [76] (regularity means that the rank of the differential of $\nu \equiv \pi_2 |_F : F \to M$ is equal to dim M at every point). Thus, the family F has double fibration structure [67]

$$Y \overset{\mu}{\leftarrow} F \overset{\nu}{\rightarrow} M,$$

where $\mu = \pi_1 |_F$. For each $t \in M$ we say that the $F \hookrightarrow Y \times M$ compact complex submanifolds $X_t := \mu \circ \nu^{-1}(t) \hookrightarrow Y$ belong to the family. If is an analytic family of compact submanifolds, then, for any $t \in M$, there is a natural linear map,

$$k_t : T_t M \to H^0(X_t, N_{X_t|Y}),$$

from the tangent space at t to the vector space of global holomorphic sections of the normal bundle $N_{X_t|Y} = TY|_{X_t} / TX_t$ to the submanifold $X_t \hookrightarrow Y$.

An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds is called *complete* if the Kodaira map k_t is an isomorphism at each point t in the moduli space M. It is called *maximal* if for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ of compact complex submanifolds such that $\mu \circ \nu^{-1}(t) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t})$ for some points $t \in M$ and $\tilde{t} \in \tilde{M}$, there is a neighborhood $\tilde{U} \subset \tilde{M}$ of the point $\tilde{t}$ and a holomorphic map $f : \tilde{U} \to M$ such that $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}') = \mu \circ \nu^{-1}(f(\tilde{t}'))$ for every $\tilde{t}' \in \tilde{U}$.

Theorem 5.1.[55] If $X \hookrightarrow Y$ is a compact complex submanifold in complex manifold Y with normal bundle $N_{X|Y}$ such that $H^1(X, N_{X|Y}) = 0$, then X belongs to the complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex submanifolds with the moduli space M being a $H^0(X, N_{X|Y})$ -dimensional complex manifold. This moduli space is called Kodaira moduli space.

5.2 Complex Contact Manifolds

Definition 5.1. A complex contact manifold is pair (Y, D) consisting of a $(2n+1)-$ dimensional complex manifold Y and a rank $2n$-holomorphic subbundle $D \subset TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{gathered}\phi : D \times D \to TY / D \\ (v, w) \mapsto [v, w] \operatorname{mod} D\end{gathered}$$

is non-degenerate. Define the contact line bundle $L := TY/D$, on Y by the exact sequence

$$0 \to D^{2n} \to TY^{2n+1} \overset{\theta}{\to} L \to 0,$$

where θ is the tautological projection and $D = \ker\theta$. It may easily be verified that the maximal non-degeneracy of the distribution D is equivalent to the fact that the above defined "twisted" 1-form satisfies the condition

$$\theta \wedge (d\theta)^n \neq 0.$$

Definition 5.2. A compact complex n-dimensional submanifold X of the complex contact manifold Y is called *Legendre Submanifold* if $TX \subset D$. The normal bundle $N_{X|Y}$ of any Legendre submanifold $X \hookrightarrow Y$ is isomorphic to $J^1 L_X$ [61] where $L_X = L|_X$, and, therefore, fits into the exact sequence

$$0 \to \Omega^1 X \otimes L_X \to N_{X|Y} \overset{pr}{\to} L_X \to 0.$$

Definition 5.3. A compact complex p-dimensional submanifold $X^p \hookrightarrow Y^{2n+1}$ of a complex contact manifold Y^{2n+1} is called *isotropic* if $TX \subset D|X$.

An isotropic submanifold of possible maximal dimension n is called a Legendre submanifold.

Definition 5.4. The bundle S_X is defined to be the kernel of the canonical projection

$$p : N_{X|Y} \to J^1 L_X ,$$

i.e., it is defined by the exact sequence

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0.$$

5.3 Legendre Moduli Spaces

5.3.1 Existence of Legendre Moduli Spaces

Let Y be a complex contact manifold. An analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact submanifolds of Y [54] is called an *analytic family of compact Legendre submanifolds* if, for any point $t \in M$, the corresponding subset $X_t := \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is a Legendre submanifolds. The parameter space M is called a Legendre moduli space.

Let $F \hookrightarrow Y \times M$ be a family of compact Legendre submanifolds. If $F \hookrightarrow Y \times M$ is an analytic family of compact complex Legendre submanifolds, it is also an analytic family of complex submanifolds in the sense of Kodaira and thus, for each $t \in M$, there is a linear map

$$k_t : T_t M \to H^0(X_t, N_{X_t|Y}).$$

Definition 5.5. The analytic family $F \hookrightarrow Y \times M$of compact Legendre submanifolds is *complete* at a point $t \in M$ if the composition

$$s_t : T_t M \overset{kt}{\to} H^0(X_t, N_{X_t|Y}) \overset{pr}{\to} H^0(X_t, L_{X_t})$$

provides an isomorphism between the tangent space to M at the point t and the vector space of global sections of the contact line bundle over X_t. The analytic family $F \hookrightarrow Y \times M$ is called complete if it is complete at each point of the moduli space M.

Lemma 5.2. [1] If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre submanifolds is complete at a point $t_0 \in M$, then there is an open neighbourhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.

Definition 5.6. An analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre submanifolds is *maximal* at a point $t_0 \in M$, if for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ of compact complex Legendre submanifolds such that $\mu \circ \nu^{-1}(t_0) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}_0)$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists neighbourhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f : \tilde{U} \to M$ such that $f(\tilde{t}_0) = t_0$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}_0') = \mu \circ \nu^{-1}(f(\tilde{t}'))$ for each $\tilde{t}' \in \tilde{U}$. The family $F \hookrightarrow Y \times M$ is called maximal if it is maximal at each point t in the moduli space M.

Lemma 5.3. [1] If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre submanifolds is complete at a point $t_0 \in M$, then it is maximal at the point t_0.

The map $s_t : T_t M \to H^0(X_t, L_{X_t})$ studies by the Lemma 5.2 and Lemma 5.3 will also play a fundamental role in our study of the rich geometric structure induced canonically on moduli spaces of complete and maximal analytic families of compact Legendre submanifolds described by the following theorem.

Theorem 5.4. [65] Let X be a compact complex Legendre submanifold of a complex contact manifold Y with contact line bundle L. If $H^1(X, L_X) = 0$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre submanifolds containing X with Legendre moduli space M, is a $H^0(X, L_X)$ - dimensional complex manifold.

This theorem is proved by working in local coordinates adapted to the contact structure and expanding the defining functions of nearby compact Legendre submanifolds in terms of local coordinates on the moduli space M. This is much in the spirit of the original proof of Kodaira's theorem of the existence, completeness and maximality of compact submanifolds of complex manifolds. The essential difference from the Kodaira case is that the infinite sequence of obstructions to agreements on overlaps of formal power series is situated now in $H^1(X, L_X)$ rather than in $H^1(X, N_{X|Y})$.

5.3.2 Stability of Legendre Moduli Spaces

A *family of complex contact manifolds* is by definition a quadruple $(\mathcal{Y}, S, \mathfrak{D}, \varpi)$, consisting of complex manifolds $\mathcal{Y}$ and $\mathcal{S}$, a holomorphic submersion $\varpi: \mathcal{Y} \hookrightarrow \mathcal{S}$ and a maximally non-integrable codomain 1 vector subbundle, $\mathfrak{D} \subset \ker \varpi_*$, of the bundle of ϖ -vertical tangent vectors. Therefore, each fibre $Y_s = \pi^{-1}(s)$, $s \in S$, is a complex contact manifold with contact line bundle L_s isomorphic to $(\ker \varpi_* / \mathfrak{D})\big|_{Y_s}$. The manifold S is often called a *parameter space*. For any $s \in S$ there is a canonical linear map

$$\rho_s : T_s S \to H^1(Y_s, L_s).$$

According to Kodaira [53], if (Y, L) is a compact complex contact manifold with $H^2(Y, L) = 0$, then there exists a complete analytic family $(\mathcal{Y}, S, \mathfrak{D}, \varpi)$ of contact manifolds such that

(i) each fibre Y_s is compact,

(ii) $Y = Y_{s_0}$ for some $s_0 \in S$, and

(iii) the map $\rho_s : T_s S \to H^1(Y_s, L_s)$ is an isomorphism for each $s \in M$.

In the present section it is more suitable to call a family of complex contact manifolds $(\mathcal{Y}, S, \mathfrak{D}, \varpi)$ simply *a complex contact fibre manifold* and denote by $\mathcal{Y}$ .Then a submanifold $x \hookrightarrow \mathcal{Y}$ is called a *complex Legendre fibre submanifold* if the restriction of ϖ to x defines a holomorphic submersion $\omega: x \to S$ whose fibres $X_s = \omega^{-1}(s)$ are complex Legendre submanifolds of Y_s . If all these fibres are compact, then x is called a *complex Legendre fibre submanifold with compact fibres*.

Definition 5.7. A compact complex Legendre submanifold X of a complex contact manifold Y is called *stable* if for any complex fibre manifold $\mathcal{Y}$ such that $\varpi^{-1}(s_0)=Y$ for some point $s_0 \in S$, there exists a neighborhood U of s_0 in S and a complex Legendre fibre submanifold $x \subset \mathcal{Y}|_U$ with compact fibres such that $x \cap Y = X$.

Let $X \hookrightarrow Y$ be a compact complex Legendre submanifold and $\mathcal{Y}$ a complex contact fibre manifold such that $\varpi^{-1}(s_0)=Y$ for some point $s_0 \in S$. Then the normal bundle $N_{X|\mathcal{Y}}$ of $X \hookrightarrow \mathcal{Y}$ fits into the exact sequence

$$0 \rightarrow N_{X|Y} \xrightarrow{i} N_{X|\mathcal{Y}} \rightarrow \mathbb{C}^p \otimes \mathcal{O}_X \rightarrow 0$$

where $N_{X|Y}$ is the normal bundle of $X \hookrightarrow Y$ and $p = \dim S$. Therefore, the quotient bundle $\mathcal{N} = N_{X|\mathcal{Y}} / i(\Omega^1 X \otimes L_X)$ has the extension structure

$$0 \rightarrow L_X \rightarrow \mathcal{N} \rightarrow \mathbb{C}^p \otimes \mathcal{O}_X \rightarrow 0$$

Theorem 5.5. Let $X \hookrightarrow Y$ be a compact complex Legednre submanifold and $\mathcal{Y}$ a complex contact fibre manifold such that $\varpi^{-1}(s_0)=Y$ for some point $s_0 \in S$. If $H^1(X, L_X)=0,$ then there exists an analytic family of compact complex submanifolds $\{X_t \hookrightarrow \mathcal{Y}|\ t \in M\ \}$ such that each X_t is a Legendre submanifold of Y_s for some $s \in S$ and such that there is a canonical isomorphism $T_t M \rightarrow H^0(X_t,\ \mathcal{N}_t)$ for all $t \in M$.

The proof is omitted since it is based on a rather straightforward generalization of the arguments used in the proof of the existence Theorem [53].

Theorem 5.6. [68] Let X be a complex compact Legendre submanifold of a complex contact manifold (Y,L). If $H^1(X,L_X)=0$, then X is a stable Legendre submanifold of Y.

This generalizes the result in Kodaira's stable submanifold. There are strong indications in [66] that the Legendre moduli spaces we studied in this section will play a pivotal role in the twistor theory of G-structures with restricted invariant torsion.

5.4 Isotropic Moduli Spaces

5.4.1 Families of Complex Isotropic Submanifolds

Let Y be a complex contact manifold. An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds of the complex manifold Y is called an *analytic family of isotropic submanifolds*, if for any $t \in M$, the corresponding subset $X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is an isotropic submanifold. Use is made of the symbol $\{X_t \hookrightarrow Y \mid t \in M\}$ to denote an analytic family of isotropic submanifolds.

Let $X = X_{t_0}$ for some $t_0 \in M$. If $X^p \hookrightarrow Y^{2n+1}$ is an isotropic submanifold, then each point in X has a neighbourhood U in Y such that the contact structure in a suitable trivialization of L over U is

$$\theta = d\omega^0 + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}} d\omega^{\bar{\bar{a}}} + \sum_{a=1}^{p} \omega^a dz^a$$

and X in U is given by

$$\omega^0 = \omega^a = \omega^{\bar{a}} = \omega^{\bar{\bar{a}}} = 0.$$

There exists an adopted coordinate covering $\{U_i\}$ of a tubular neighbourhood of X inside Y. As a consequence one can always choose local coordinate functions $(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, z_i^a)$, in U_i where $\bar{a}, \bar{\bar{a}} = 1, \cdots, n$ and $a=1,\cdots,p$ such that the contact structure in U_i is represented by

$$\theta_i = d\omega_i{}^0 + \sum_{\ddot{a}=p+1}^{n} \underbrace{\omega_i{}^{\bar{a}} d\omega_i{}^{\bar{\bar{a}}}}_{(n-p)-terms} + \sum_{a=1}^{p} \underbrace{\omega_i{}^a dz_i{}^a}_{p-terms}$$

with $U_i \cap X$ given by

$$\omega_i{}^0 = \omega_i{}^a = \omega_i{}^{\bar{a}} = \omega_i{}^{\bar{\bar{a}}} = 0,$$

and

$$\theta_i\,|_{Ui\cap Uj} = A_{ij}\theta_j\,|_{Ui\cap Uj}$$

for some nowhere vanishing holomorphic functions $A_{ij}(\omega_j, z_j)$. They satisfy the condition

$$A_{ik} = A_{ij}A_{jk}$$

on every triple intersection $U_i \cap U_j \cap U_k$. Clearly, $\{A_{ij}\}$are gluing functions of the contact line bundle L.

On the intersection $U_i \cap U_j$, the coordinates $\omega_i{}^A := (\omega_i{}^0, \omega_i{}^a, \omega_i{}^{\bar{a}}, \omega_i{}^{\bar{\bar{a}}})$ and z_i^a are holomorphic functions of $\omega_j{}^B := (\omega_j{}^0, \omega_j{}^b, \omega_j{}^{\bar{b}} \omega_j{}^{\bar{\bar{b}}})$ and z_j^b,

$$\begin{cases} \omega_i{}^0 = f_{ij}{}^0(\omega_j{}^B, z_j{}^b) \\ \omega_i{}^a = f_{ij}{}^a(\omega_j{}^B, z_j{}^b) \\ \omega_i{}^{\bar{a}} = f_{ij}{}^{\bar{a}}(\omega_j{}^B, z_j{}^b) \\ \omega_i{}^{\bar{\bar{a}}} = f_{ij}{}^{\bar{\bar{a}}}(\omega_j{}^B, z_j{}^b) \\ z_i{}^a = g_{ij}{}^a(\omega_j{}^B, z_j{}^b) \end{cases}$$

$$\Leftrightarrow \begin{cases} \omega_i{}^A = f_{ij}{}^A(\omega_j{}^B, z_j{}^b) \\ z_i{}^a = g_{ij}{}^a(\omega_j{}^B, z_j{}^b) \end{cases}$$

with$f_{ij}^A(0, z_j^b) = 0$, where $A = 0, a, \bar{a}, \bar{\bar{a}}$

For any point t in a sufficient small coordinate neighbourhood $M_0 \subset M$ of t_0 with coordinate functions $t^{\alpha}, \alpha = 1, \dots, m = dim\ M$, the associated isotropic submanifold $X_t = \mu \circ \nu^{-1}(t)$ is expressed in the domain U_i by equations of the form [3]

$$\omega_i^{\ A} = \phi_i^{\ A}(z_i^{\ a}, t^{\alpha}),\ \alpha = 0, a, \bar{a}, \bar{\bar{a}}\ .$$

Lemma 5.7. X_t is isotropic if and only if

$$\phi_i^{\ a}(z_i,t) = -\frac{\partial \phi_i^{\ 0}(z_i,t)}{\partial z_i^{\ a}} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\ \bar{b}}(z_i,t)\frac{\partial \phi_i^{\ \bar{\bar{b}}}(z_i,t)}{\partial z_i^{\ a}}$$

holds.

Proof. Let $X^p \hookrightarrow Y^{2n+1}$be an isotropic submanifold in complex contact manifold Y. An arbitrary X_t, deformation of X inside Y , is given by

$$\begin{cases} \omega_i^{\ 0} = \phi_i^{\ 0}(z_i,t) \\ \omega_i^{\ a} = \phi_i^{\ a}(z_i,t) \\ \omega_i^{\ \bar{a}} = \phi_i^{\ \bar{a}}(z_i,t) \\ \omega_i^{\ \bar{\bar{a}}} = \phi_i^{\ \bar{\bar{a}}}(z_i,t) \end{cases}$$

$$\Rightarrow \omega_i^{\ A} = \phi_i^{\ A}(z_i,t)$$

Then, $\left\{\frac{\partial \varphi_i^A}{\partial t}|_0\right\}$ is a global section of $N_{X|Y}$.

X_t is isotropic if and only if $\theta_i = d\omega_i^{\ 0} + \omega_i^{\ \bar{a}} d\omega_i^{\ \bar{\bar{a}}} + \omega_i^{\ a} dz_i^{\ a}$ vanishes on X_t. Then

$$0 = \theta_i | X_t = d\varphi_i^{\ 0}(z_i,t) + \varphi_i^{\ \bar{a}}(z_i,t) d\varphi_i^{\ \bar{\bar{a}}}(z_i,t) + \varphi_i^{\ a}(z_i,t) dz_i^{\ a}$$

$$= \frac{\partial \varphi_i^{\ 0}(z_i,t)}{\partial z_i^{\ a}} dz_i^{\ a} + \varphi_i^{\ \bar{a}}(z_i,t) \frac{\partial \varphi_i^{\ \bar{\bar{a}}}}{\partial z_i^{\ b}} dz_i^{\ b} + \varphi_i^{\ a}(z_i,t) dz_i^{\ a}$$

$$= [\varphi_i^{\ a}(z_i,t) + \frac{\partial \varphi_i^{\ 0}(z_i,t)}{\partial z_i^{\ a}} + \sum_{\bar{b}=p+1}^{n} \varphi_i^{\ \bar{b}}(z_i,t) \frac{\partial \varphi_i^{\ \bar{\bar{b}}}(z_i,t)}{\partial z_i^{\ a}}] dz_i^{\ a}$$

Thus, we obtain

$$\phi_i^{\,a}(z_i,t) = -\frac{\partial \phi_i^{\,0}(z_i,t)}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\,\bar{b}}(z_i,t)\frac{\partial \phi_i^{\,\bar{\bar{b}}}(z_i,t)}{\partial z_i^{\,a}}$$

$\square$

5.4.2 Completeness and Maximality of Isotropic Moduli Spaces

Let Y be a complex contact manifold and let $F \hookrightarrow Y \times M$ be an analytic family of compact complex isotropic submanifolds. The latter is also an analytic family of compact complex submanifolds in the sense of Kodaira and thus, for each $t \in M$, there is a canonical linear map

$$k_t : T_tM \to H^0(X_t, N_{X_t|Y})$$

The exact sequence

$$0 \to S_{X_t} \to N_{X_t|Y} \to J^1L_{X_t} \to 0,$$

has an expansion as follows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \Omega^1X_t \otimes L_{X_t} \arrow[d] & & \\
0 \arrow[r] & S_{X_t} \arrow[r] & N_{X_t|Y} \arrow[r] \arrow[d] & J^1L_{X_t} \arrow[r] & 0 \\
 & & L_{X_t} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Hence, there is a canonical map represented by a dashed arrow,

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathrm{H}^0(X_t, \Omega^1X_t \otimes L_{X_t}) \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^0(X_t, S_{X_t}) \arrow[r] & \mathrm{H}^0(X_t, N_{X_t|Y}) \arrow[r] \arrow[dr, dashed] & \mathrm{H}^0(X_t, J^1L_{X_t}) \arrow[r] \arrow[d] & 0 \\
 & & & \mathrm{H}^0(X_t, L_{X_t}) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Thus, there is a canonical sequence of linear spaces,

$$0 \to \mathrm{H}^0(X_t, S_{X_t}) \to \mathrm{H}^0(X_t, N_{X_t|Y}) \to \mathrm{H}^0(X_t, L_{X_t}) \to 0$$

which is not exact in general.

Definition 5.8. The analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is complete at a point $t \in M$ if the Kodaira map k_t makes the induced sequence

$$0 \to \mathrm{H}^0(X_t, S_{X_t}) \to k_t(T_t M) \to \mathrm{H}^0(X_t, L_{X_t}) \to 0$$

exact. The analytic family $F \hookrightarrow Y \times M$ is complete if it is complete at each point of the moduli space.

Theorem 5.8. [1] If an analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is complete at a point $t_0 \in M$, then there is an open neighbourhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.

Definition 5.9. An analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is *maximal* at a point $t_0 \in M$, if for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ of compact complex isotropic submanifolds such that $\mu \circ \nu^{-1}(t_0) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}_0)$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists neighbourhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f : \tilde{U} \to M$ such that $f(\tilde{t}_0) = t_0$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}'_0) = \mu \circ \nu^{-1}(f(\tilde{t}'))$ for each $\tilde{t}' \in \tilde{U}$. The family $F \hookrightarrow Y \times M$ is called *maximal* if it is maximal at each point t in the moduli space M.

5.4.3 Existence Theorem of Isotropic Moduli Spaces

Theorem 5.9. [2] If $X \hookrightarrow Y$ is a compact complex isotropic submanifold in a complex contact manifold, then its normal bundle $N_{X|Y}$ fits into an extension

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0,$$

If $H^1(X, L_X) = H^1(X, S_X) = 0$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of isotropic submanifolds such that

(i) $X_{t_0} = X$ for some $t_0 \in M$;
(ii) the moduli space M is smooth;
(iii) dim $M = H^0(X, L_X) + H^0(X, S_X)$;
(iv) the tangent space $T_t M, t \in M$, fits into the extension

$$0 \to H^0(X_t, S_{X_t}) \to k_t(T_t M) \to H^0\left(X_t, L_{X_t}\right) \to 0.$$

5.4.4 Stability of Isotropic Moduli Spaces

Theorem 5.10. Let (Y, D) be complex contact manifold and $X \subset Y$ be a isotropic submanifold of Y with contact line bundle L. Then there is a following exact sequence

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0$$

Proof. Consider a particular 1-form θ that represents the contact structure. Let, for $p \in X, Z \in T_p X$ be a vector in the normal bundle and $Q \in T_p Y$. Then there are two equations

$$f(p) = \theta(Q), d\theta(Z, Q) = Z(f)|_p$$

which uniquely determines the 1-jet on X at p of a function f. Consider rescaling $\theta \to g\theta$ where g is a function on Y. If we set $\hat{\theta} = g\theta$ and $\hat{f} = gf$, then we have

$$\hat{\theta}(Q) = g\theta(Q) = gf(p) = \hat{f}|_p$$

$$d\hat{\theta}(Z, Q) = (dg \wedge \theta)(Z, Q) + gd\theta(Z, Q)$$

$$= dg(Z)\theta(Q) - dg(Q)\theta(Z) + gZ(f)|_p$$

$$= Z(g)f(p) - 0 + gZ(f)|_p$$

$$= Z(gf)|_p$$

$$= Z(\hat{f})|_p$$

Since $T_pX \subseteq T_pX^{\perp} \subset D$ then $Z \in D$ so that $\theta(Z) = 0$. Therefore this elementary calculation shows that the above two conditions are satisfied by gf and so we conclude that we have defined a map $N_{X|Y} \to J^1L_X$. Furthermore, it is clear that the kernel is $TX^{\perp}/TX$.

This completes the proof . □

Theorem 5.11. If an analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is complete at a point $t_0 \in M$, then it is maximal at the point t_0.

Proof. Let $\tilde{F} \hookrightarrow Y \times \tilde{M}$ be any analytic family of compact complex isotropic submanifolds such that $X_t = \mu \circ \nu^{-1}(t_0) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}_0)$ for some point $\tilde{t}_0 \in \tilde{M}$. Let $\{U_i\}$ be a covering of Y by coordinate charts with coordinate functions $\left(\omega_i^A, z_i^a\right)$ such that

$$d\omega^0{}_i + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}} d\omega^{\bar{\bar{a}}} \,|\, U_i \cap U_j + \sum_{a=1}^{p} \omega^a{}_i dz^a{}_i \,|\, U_i \cap U_j$$

$$= A_{ij}(d\omega^0{}_i + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}} d\omega^{\bar{\bar{a}}} \,|U_i \cap U_j + \sum_{a=1}^{p} \omega^a{}_i dz^a{}_i \,|U_i \cap U_j)$$

for some non-vanishing holomorphic functions A_{ij} , and the isotropic submanifold X_t is given in each intersection $X_t \cap U_i$ by equations $\omega_i^A = 0$. Define

$$\phi_i{}^A(z_i,t) = \begin{bmatrix} \phi_i{}^0(z_i,t) \\ \phi_i{}^a(z_i,t) \\ \phi_i{}^{\bar{a}}(z_i,t) \\ \phi_i{}^{\bar{\bar{a}}}(z_i,t) \end{bmatrix} \qquad \text{and} \qquad \bar{\phi}_i{}^A(z_i,t) = \begin{bmatrix} \bar{\phi}_i{}^0(z_i,\tilde{t}) \\ \bar{\phi}_i{}^a(z_i,\tilde{t}) \\ \bar{\phi}_i{}^{\bar{a}}(z_i,\tilde{t}) \\ \bar{\phi}_i{}^{\bar{\bar{a}}}(z_i,\tilde{t}) \end{bmatrix}$$

Then, for sufficiently small neighbourhoods U and $\tilde{U}$ of points $t_0 \in M$ and $\tilde{t}_0 \in \tilde{M}$, the submanifolds $\nu^{-1}(U) \hookrightarrow Y \times U$ and $\tilde{\nu}^{-1}(\tilde{U}) \hookrightarrow Y \times \tilde{U}$ are given respectively by equations

$$\phi_i^{\,a}(z_i,t) = -\frac{\partial \phi_i^{\,0}(z_i,t)}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\,\bar{b}}(z_i,t)\frac{\partial \phi_i^{\,\bar{\bar{b}}}(z_i,t)}{\partial z_i^{\,a}},$$

and

$$\tilde{\phi}_i^{\,a}(z_i,\tilde{t}) = -\frac{\partial \tilde{\phi}_i^{\,0}(z_i,\tilde{t})}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \tilde{\phi}_i^{\,\bar{b}}(z_i,\tilde{t})\frac{\partial \tilde{\phi}_i^{\,\bar{\bar{b}}}(z_i,\tilde{t})}{\partial z_i^{\,a}},$$

where $t = (t^1, \ldots, t^m)$, $m = \dim M$, and $\tilde{t} = (\tilde{t}^1, \ldots, \tilde{t}^l), l = \dim \tilde{M}$, are coordinates on U and $\tilde{U}$ respectively, and $\varphi_i^A(z_i,t)$ and $\tilde{\varphi}_i^A(z_i,\tilde{t})$ are some holomorphic functions. We may assume without loss of generality that coordinate functions $t^1, \ldots, t^m$ vanish at $t_0 \in U$, while coordinate functions $\tilde{t} = (\tilde{t}^1, \ldots, \tilde{t}^l)$ vanish at $\tilde{t}_0 \in \tilde{U}$.

To prove this Theorem 5.11, we have to construct a holomorphic map $f : \tilde{U} \to U$ such that $f(\tilde{t}_0) = t_0$ and

$$\tilde{\phi}_i^{\,A}(z_i,t) = \phi_i^{\,A}(z_i, f(\tilde{t})) \tag{5.1}$$

for all $\tilde{t}$ in some sufficiently small neighbourhood of $\tilde{t}_0$. Let us first prove the existence of a unique formal power series $f(\tilde{t})$ satisfying this equation. For this purpose, we introduce the following notations. If $P(s)$ is a power series in variables $s = (s^1, \ldots, s^k)$ we write

$$P(s) = P_0(s) + P_1(s) + \ldots + P_q(s) + \ldots$$

where each term $P_q(s)$ is a homogeneous polynomial of degree q in $s^1, \ldots, s^k$, and denote it by $P^{[q]}(s)$ the polynomial

$$P^{[q]}(s) = P_0(s) + P_1(s) + \ldots + P_q(s)$$

If $Q(s)$ is another power series in s, we write $P(s) \equiv_q Q(s)$ if $P^{[q]}(s) = Q^{[q]}(s)$. Let us look for a solution of equation (5.1) in the form of a formal power series

$$f(\tilde{t}) = f_1(\tilde{t}) + f_2(\tilde{t}) + \ldots + f_q(\tilde{t}) + \ldots$$

Then equations (5.1) are reduced to the system of congruencies

$$\tilde{\phi}_i^{\,A}(z_i, \tilde{t}) \equiv_q \phi_i^{\,A}(z_i, f^{[q]}(\tilde{t})),\ i \in I,\ q = 1,2,\ldots \tag{5.2}$$

First, we shall construct polynomials $f^{[q]}(\tilde{t})$ by induction on q. Let

$$\phi_i^{\,A}(z_i, t) = \phi_{i|1}^{\,A}(z_i, t) + \phi_{i|2}^{\,A}(z_i, t) + \ldots$$

be the power series expansion of $\varphi_i^{\,A}(z_i, t)$ in $t^1, \ldots, t^m$. By hypothesis, the family $F \hookrightarrow Y \times M$ is complete at $t_0 \in M$. According to Theorem 5.10, and definition 5.8, the sequence

$$0 \to \mathrm{H}^0(X, S_X) \to k_{t_0}\left(T_{t_0}M\right) \to \mathrm{H}^0(X, L_X) \to 0,$$

is exact. On the other hand, if there exists a sheaf of Abelian groups $\tilde{N}_{X|Y}$, which fits into there exact sequence,

$$0 \to \mathrm{H}^0(X, S_X) \to \mathrm{H}^0\left(X, \tilde{N}_{X|Y}\right) \to \mathrm{H}^0(X, L_X) \to 0.$$

Moreover, the Kodaira map k_0 maps exactly $T_{t_0}M$ to the space of the global sections of $\tilde{N}_{X|Y}$. Thus, we have an isomorphism

$$k_{t_0} : T_{t_0}M \to \mathrm{H}^0\left(X_t, \tilde{N}_{X|Y}\right)$$

According to the local coordinate description of the map k_{t_0}, given in the proof of Theorem 5.10, this means that the collection of 0 -cocycles $\left\{\frac{\partial \varphi_{i|1}{}^{A}(z_i,t)}{\partial t^{\alpha}}\right\}$, $\alpha = 1, \dots, m$, represents a basis of the vector space $\mathrm{H}^0\left(X, \tilde{N}_{X|Y}\right)$. Since

$$\tilde{\phi}_{i|1}{}^{A}(z_i,\tilde{t}) = \sum_{\gamma=1}^{l} \frac{\partial \tilde{\phi}_i{}^{A}(z_i,t)}{\partial \tilde{t}^{\gamma}}|_{\tilde{t}=0}\ \tilde{t}^{\gamma}$$
,

and each 1 – cochain$\left\{\frac{\partial \tilde{\varphi}_i{}^{A}(z_i,\tilde{t})}{\partial \tilde{t}^{\gamma}}|_{\tilde{t}=0}\right\}$,$\gamma = 1, \cdots, l$ represents a global section of $\tilde{N}_{X|Y}$ over X, we conclude that the collection $\left\{\tilde{\varphi}_{i|1}{}^{A}(z_i,\tilde{t})\right\}$ may be interpreted as a homogeneous polynomial of degree 1 in $\tilde{t}$ with coefficients in $\mathrm{H}^0\left(X, \tilde{N}_{X|Y}\right)$. Therefore, we can decompose

$$\tilde{\phi}_{i|1}{}^{A}(z_i,\tilde{t}) = \sum_{\alpha=1}^{m} f_1^{\alpha}(\tilde{t}) \frac{\partial \phi_{i|1}{}^{A}(z_i,t)}{\partial t^{\alpha}}$$
,

where coefficients $f_1(\tilde{t})$ are linear vector-valued functions of $\tilde{t}^1,...,\tilde{t}^l$. Thus, we have,

$$\tilde{\phi}_{i|1}{}^{A}(z_i,\tilde{t}) = \phi_{i|1}{}^{A}(z_i, f_1(\tilde{t})),\ i \in I,$$

which means that the functions $f_1(\tilde{t})$ satisfy the congruence (5.2).

Assume the polynomials $f^{[q]}(\tilde{t})$ satisfying (5.2)are already constructed. Define a homogeneous polynomial $\omega_i^{A}(z_i,\tilde{t})$of degree $q+1$ in $\tilde{t}$ by the congruence

$$\omega_i^{A}(z_i,\tilde{t}) \equiv_{q+1} \tilde{\phi}_i^{A}(z_i,\tilde{t}) - \phi_i^{A}(z_i, f^{[q]}(\tilde{t})).$$

From the obvious equalities

$$\phi_i^{A}(g_{ij}(\phi_j^{B}(z_j,t),z_j),t) = f_{ij}^{A}(\phi_j^{B}(z_j,t),z_j),$$

$$\widetilde{\phi}_i^{\,A}(g_{ij}(\widetilde{\phi}_j^{\,B}(z_j,t),z_j),t) = f_{ij}^{\,A}(\widetilde{\phi}_j^{\,B}(z_j,t),z_j),$$

where

$$\phi_i^{\,a}(z_i,t) = -\frac{\partial \phi_i^{\,0}(z_i,t)}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\,\bar{b}}(z_i,t)\frac{\partial \phi_i^{\,\bar{\bar{b}}}(z_i,t)}{\partial z_i^{\,a}},$$

and

$$\widetilde{\phi}_i^{\,a}(z_i,t) = -\frac{\partial \widetilde{\phi}_i^{\,0}(z_i,t)}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \widetilde{\phi}_i^{\,\bar{b}}(z_i,t)\frac{\partial \widetilde{\phi}_i^{\,\bar{\bar{b}}}(z_i,t)}{\partial z_i^{\,a}},$$

we find

$$\omega_i^{\,A}(z_i,\widetilde{t}\,)\,|\,z_i = g_{ij}(0,z_j) \equiv_{q+1} \omega_i^{\,A}(z_i,\widetilde{t}\,)\,|\,z_i = g_{ij}(\widetilde{\phi}_j^{\,B}(z_j,\widetilde{t}\,),z_j)$$

$$\equiv_{q+1} [\widetilde{\phi}_i^{\,A}(z_i,\widetilde{t}\,) - \phi_i^{\,A}(z_i,f^{[q]}(\widetilde{t}\,))]\,|\,z_i = g_{ij}(\widetilde{\phi}_j^{\,B}(z_j,\widetilde{t}\,),z_j)$$

$$\equiv_{q+1} [\widetilde{\phi}_i^{\,A}(z_i,\widetilde{t}\,)\,|\,z_i = g_{ij}(\widetilde{\phi}_j^{\,B}(z_j,\widetilde{t}\,),z_j) - \phi_i^{\,A}(z_i,f^{[q]}(\widetilde{t}\,))]\,|\,z_i = g_{ij}(\phi_j^{\,B}(z_j,f^{[q]}(\widetilde{t}\,)),z_j)$$

$$\equiv_{q+1} f_{ij}^{\,A}(\tilde{\varphi}_j^{\,B}(z_j,\tilde{t}),z_j) - f_{ij}^{\,A}(\varphi_j^{\,B}(z_j,f^{[q]}(\tilde{t})),z_j) \equiv_{q+1} \frac{\partial f_{ij}^{\,A}}{\partial \omega_j^{\,B}}\Big|_{\omega_j=0}\, \omega_j^{\,B}(z_j,t).$$

The latter congruence means that the collection $\left\{\omega_i^A(z_i,\tilde{t})\right\}$ is a homogeneous polynomial of degree $q+1$ in $\tilde{t} = (\tilde{t}^1,\ldots,\tilde{t}^l)$ with coefficients in $H^0\left(X,\, \tilde{N}_{X|Y}\right)$. Let us now show that $\left\{\omega_i^A(z_i,\tilde{t})\right\}$ takes values in $\tilde{N}_{X|Y}$ in fact. For this, we have to show that

$$\omega_i^{\,a}(z_i,\widetilde{t}\,) = -\frac{\partial \omega_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}}.$$

We have by definition,

$$\omega_i^{\,0}(z_i,\widetilde{t}\,) \equiv_{q+1} \widetilde{\phi}_i^{\,0}(z_i,\widetilde{t}\,) - \phi_i^{\,0}(z_i, f^{[q]}(\widetilde{t}\,)) \tag{5. 3}$$

and

$$\omega_i^{\,a}(z_i,\widetilde{t}\,) \equiv_{q+1} \widetilde{\phi}_i^{\,a}(z_i,\widetilde{t}\,) - \phi_i^{\,a}(z_i, f^{[q]}(\widetilde{t}\,)) \tag{5.4}$$

Differentiate (5.3) with respect to z_i^a, we get

$$\frac{\partial \omega_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}} = \frac{\partial \widetilde{\phi}_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}} - \frac{\partial \phi_i^{\,0}(z_i, f^{[q]}(\widetilde{t}\,))}{\partial z_i^{\,a}} \tag{5.5}$$

Equation (5.4) implies,

$$\begin{aligned} \omega_i^{\,a}(z_i,\widetilde{t}\,) = &-\frac{\partial \widetilde{\phi}_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}} - \sum_{\bar{b}=p+1}^{n} \widetilde{\phi}_i^{\,\bar{b}}(z_i,\widetilde{t}\,)\frac{\partial \widetilde{\phi}_i^{\,\bar{\bar{b}}}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}} + \frac{\partial \phi_i^{\,0}(z_i, f^{[q]}(\widetilde{t}\,))}{\partial z_i^{\,a}} \\ &+ \sum_{\bar{b}=p+1}^{n} \phi_i^{\,\bar{b}}(z_i,t)\frac{\partial \phi_i^{\,\bar{\bar{b}}}(z_i, f^{[q]}(\widetilde{t}\,))}{\partial z_i^{\,a}} \end{aligned} \tag{5.6}$$

As $\tilde{\varphi}_i^{\bar{a}}(z_i,\tilde{t}) =_q \varphi_i^{\bar{a}}(z_i,t)$, $\tilde{\varphi}_i^{\bar{\bar{a}}}(z_i,\tilde{t}) =_q \varphi_i^{\bar{\bar{a}}}(z_i,t)$ and degree $\varphi_i^{\bar{a}}, \varphi_i^{\bar{\bar{a}}} \geq 1$, degree $\tilde{\varphi}_i^{\bar{a}}, \tilde{\varphi}_i^{\bar{\bar{a}}} \geq 1$ so the second and fourth terms of equation (5.6) cancel out by induction assumption, and we obtain

$$\omega_i^{\,a}(z_i,\widetilde{t}\,) = -\left(\frac{\partial \widetilde{\phi}_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}} - \frac{\partial \phi_i^{\,0}(z_i, f^{[q]}(\widetilde{t}\,))}{\partial z_i^{\,a}}\right).$$

Hence,

$$\omega_i^{\,a}(z_i,\widetilde{t}\,) = -\frac{\partial \omega_i^{\,0}(z_i,\widetilde{t}\,)}{\partial z_i^{\,a}}.$$

Therefore, $\left\{\omega_i^{\,A}(z_i,\tilde{t})\right\}$ represents a global sections of bundle $\tilde{N}_{X|Y}$ so that we can decompose again over the basis section $\left\{\varphi_{i|1}^{\;\;A}(z_i,t)\right\}$,

$$\omega_i^{\ A}(z_i,\tilde{t}) = \sum_{\alpha=1}^{m} f_{q+1}^{\ \ \alpha}(\tilde{t}) \frac{\partial \phi_{i|1}^{\ \ A}(z_i,t)}{\partial t^{\alpha}},$$

where coefficients $f_{q+1}(\tilde{t})$ are vector-valued homogeneous polynomials of degree $q+1$ in $\tilde{t}^1,\ldots,\tilde{t}^l$. Defining

$$f^{[q+1]}(\tilde{t}) = f^{[q]}(\tilde{t}) + f_{q+1}(\tilde{t}),$$

we obtain,

$$\tilde{\phi}_i^{\ A}(z_i,\tilde{t}) \equiv_{q+1} \phi_i^{\ A}(z_i, f^{[q]}(\tilde{t})) + \omega_i^{\ A}(z_i,\tilde{t})$$

$$\equiv_{q+1} \phi_i^{\ A}(z_i, f^{[q+1]}(\tilde{t})).$$

This completes our inductive construction of the polynomials $f^{[q]}(\tilde{t})$ satisfying equations (5.2). The convergence of the resulting formal power series

$$f(\tilde{t}) = f_1(\tilde{t}) + f_2(\tilde{t}) + \ldots + f_q(\tilde{t}) + \ldots$$

for all $\tilde{t}$ in some open neighbourhood of the origin in $\mathbb{C}^1$ follows from estimates obtained by Kodaira in [54], which carry over verbatim to our case. This fact completes the proof. $\square$

5.5 Interconnections among Isotropic, Legendre and Kodaira Moduli Spaces

If $X \hookrightarrow Y$ is a complex submanifold, there is an exact sequence of vector bundles

$$0 \to N^* \to \Omega^1 Y|_X \to \Omega^1 X \to 0,$$

which induces a natural embedding $\mathbb{P}(N^*) \to \mathbb{P}(\Omega^1 Y)$ of total spaces of the associated projectivized bundles. The manifold $\hat{Y} = \mathbb{P}(\Omega^1 Y)$ carries a natural contact structure such that the constructed embedding $\hat{X} = \mathbb{P}(N^{*}) \;\to\; \hat{Y}$ is an isotropic as well as Legendre one [3]. Indeed, the contact distribution $D \subset T\hat{Y}$ at each point $\hat{y} \in \hat{Y}$ consists of those tangent vectors $V_{\hat{y}} \in T_{\hat{y}}\hat{Y}$ which satisfy the equation $< \hat{y}, \tau_{*}(V_{\hat{y}}) > = 0$, where $\tau : \hat{Y} \to Y$ is a natural projection and angular brackets denote the paring of 1-forms and vectors at $\tau(\hat{y}) \in \hat{Y}$. Since the submanifold $\hat{X} \subset \hat{Y}$ consists precisely of those projective classes of 1-forms in $\Omega^{1}Y|_{X}$ which vanish when restricted on TX, we conclude that $T\hat{X} \subset D|_{\hat{X}}$.

CHAPTER 6

MODERN DEVELOPMENTS IN DIFFERENTIAL GEOMETRY APPLIED TO DYNAMICAL SYSTEM

In this chapter we discuss about a slow-fast dynamical system called Brusselator model through differential geometry. Differential geometry based new developed approach called the flow curvature method is considered to analyze the temporal Brusselator model. According to this method, the trajectory curve or flow of any dynamical system of dimension n considers as a curve in Euclidean space of dimension n. Then the flow curvature or the curvature of the trajectory curve may be computed analytically. The set of points where the flow curvature is null or empty defines the flow curvature manifold. This manifold connected with the dynamical system of any dimension n directly describes the analytical equation of the slow invariant manifold incorporated with the same dynamical system. We apply the flow curvature method for the first time on the two-dimensional Brusselator model to describe the main characteristics of this dynamical system. Also, we discuss about the pattern formation phenomena of the spatiotemporal Brusselator model through differential geometry.

This chapter is original and it provides the main result.

6.1 Preliminaries of Dynamical System and Differential Geometry

6.1.1 Dynamic System

We consider a system of differential equations defined on a compact E of $\mathbb{R}$ by:

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}) \tag{6.1}$$

where

$$\vec{X} = \left[x_1, x_2, ..., x_n\right]^t \in E \subset \ \mathbb{R}^n$$

and

$$\vec{F}\left(\vec{X}\right) = \left[f_1\left(\vec{X}\right), f_2\left(\vec{X}\right), ..., f_n\left(\vec{X}\right)\right]^t \in E \subset \mathbb{R}^n$$

Here, $\vec{F}\left(\vec{X}\right)$ is the velocity vector field whose components f_i are C^∞ continuous functions in E with values in $\mathbb{R}$. Since each component of the speed vector field does not depend here on time. So, the system (6.1) is autonomous.

6.1.2 Kinematic Vector Functions

According to the mechanics formalism, the integral curve defined by the vector function $\overrightarrow{X}(t)$ of a dynamical system is considered as the coordinates of a moving point M at the instant t, then three following kinematics variables are attached to this point which represents the trajectory curve of M:

$\vec{X} \rightarrow$ parametric representation of orbit,

$\vec{V} \rightarrow$ instantaneous velocity vector,

$\vec{\gamma} \rightarrow$ instantaneous acceleration vector.

Definition 6.1. Since the vector function $\overrightarrow{X}(t)$ of the scalar variable t represents the trajectory of the point M, the total differential of $\overrightarrow{X}(t)$ is the vector function $\vec{V}(t)$ of the scalar variable t which represents the instant velocity vector of the point M at the moment t. Mathematically, this can be represented by the following formula:

$$\vec{V}(t) = \frac{d\vec{X}}{dt} = \vec{F}\left(\vec{X}\right) \tag{6.2}$$

The instantaneous velocity vector $\vec{V}(t)$ is tangent in every point to the trajectory.

Definition 6.2. Since the vector function $\vec{V}(t)$ of the scalar variable t represents the velocity vector of the point M, the total differential of $\vec{V}(t)$ is the vector function $\vec{\gamma}(t)$ of the scalar variable t which represents the instantaneous acceleration vector of the point M at the instant t. Mathematically, we can write

$$\vec{\gamma}(t) = \frac{d\vec{V}}{dt} \tag{6.3}$$

The components f_i of the velocity vector are assumed to be continuous, of class C^∞ on E and with values in $\mathbb{R}$, it is possible to calculate the total differential of velocity vector field defined by (6.1). We can write

$$\frac{d\vec{V}}{dt} = \frac{d\vec{F}}{d\vec{X}} \frac{d\vec{X}}{dt}$$

Here $\frac{d\vec{F}}{d\vec{X}}$ represents the Jacobian functional matrix J of the system (6.1) and considering equations (6.2) and (6.3), we obtain the following relationship whose role is very important:

$$\vec{\gamma} = J\vec{V}$$

Using the S-Frenet marker [22], i.e., a movable marker constructed from the trajectory curve $\vec{X}(t)$ oriented in the direction of the movement of the current point M, it is possible to define $\vec{\tau}$, the unit vector tangent to the trajectory curve in M, $\vec{n}$, the normal vector, i.e., the main normal in M directed inwards of the concavity of the curve and $\vec{\beta}$, the binormal unit vector at the trajectory curveso that the trihedron $\left(\vec{\tau}, \vec{n}, \vec{\beta}\right)$ is direct (figure 6.1).

Definition 6.3. The osculating plane is the plane which passes through a fixed point $\vec{X}^*$ of the dynamical system and parallel to the unit tangent and normal vectors of a tangent curve.

Definition 6.4. The curvature, which expresses the rate of change of the tangent to the trajectory curve which is defined by:

$$\kappa = \frac{1}{\Re} = \frac{\left\| \vec{\gamma} \wedge \vec{V} \right\|}{\left\| \vec{V} \right\|^3}$$

where $\Re$ represents the radius of curvature.

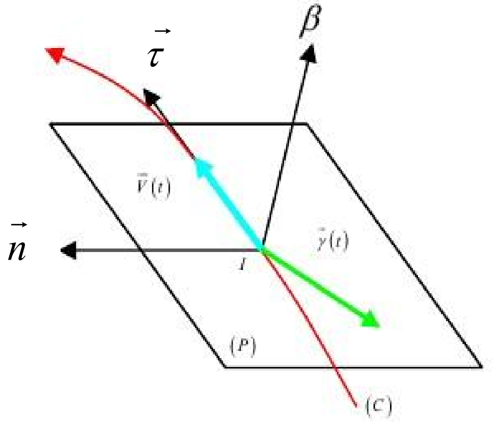


Figure 6.1. S-Frenet Frame and Osculating Plane

Definition 6.5. A manifold $M \subset \mathbb{R}^n$ is defined as a set of points in satisfying a system of m scalar equations:

$$\phi\left(\vec{X}\right) = 0$$

where ϕ: $\mathbb{R}^n \to \mathbb{R}^m$ for $m < n$ with $\vec{X} = \left[x_1, x_2, ..., x_n\right]^t \in E \subset \mathbb{R}^n$. The manifold M is differentiable if ϕ is differentiable.

6.1.3 Slow-Fast Dynamical Systems

Dynamical system (6.1) comprising small multiplicative parameters in one or several components of its velocity vector field may be defined in a compact E included in $\mathbb{R}$ by:

$$\begin{cases} \vec{x}' = \vec{f}(\vec{x}, \vec{z}, \varepsilon) \\ \vec{z}' = \varepsilon\, \vec{g}(\vec{x}, \vec{z}, \varepsilon) \end{cases} \tag{6.4}$$

where $\vec{x} \in \mathbb{R}^m, \vec{z} \in \mathbb{R}^n, \varepsilon \in \mathbb{R}^+$ and the prime denotes differentiation with respect to the independent variable t. The functions $\vec{f}$ and $\vec{g}$ are assumed to be C^∞ functions of $\vec{x},\ \vec{z}$ and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$. When $0 < \varepsilon \ll 1$, i.e. is a small positive number, variable $\vec{x}$ is called fast variable, and $\vec{z}$ is called slow variable. Reformulating system (6.4) in terms of the rescaled variable $\tau\ = \varepsilon t$, we obtain the singularly perturbed systems:

$$\begin{cases} \varepsilon\, \dot{\vec{x}} = \vec{f}(\vec{x}, \vec{z}, \varepsilon) \\ \dot{\vec{z}} = \vec{g}(\vec{x}, \vec{z}, \varepsilon) \end{cases} \tag{6.5}$$

Dots $(\cdot)$ represent the derivatives with respect to the new independent variable τ. The independent variables t and τ are referred to the fast and slow times, respectively, and (6.4) and (6.5) are called fast and slow system, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled singular perturbation problems when $\varepsilon \ll 1$, i.e. is a small positive parameter.

A non-singularly perturbed dynamical system (6.1) defined in a compact E included in $\mathbb{R}$ may be considered as slow-fast if its functional Jacobian matrix has at least one "fast" eigenvalue, i.e. with the largest absolute value of the real part over a huge domain of the phase space.

6.2 Dynamical System Analysis

6.2.1 <u>Model 1</u>: Brusselator Model

The Brusselator system describes the following chemical reactions [85]

$$\begin{gathered} A \to U\,, B + U \to V + D, \\ 2D + V \to 3U\,, U \to E \end{gathered}$$

Since it is important to consider at least a cubic nonlinearity in the rate equations, so the non-dimensional form of the Brusselator model (spatiotemporal) is as follows:

$$\frac{\partial u}{\partial t} = D_u \Delta u + a - (b+1)u + u^2 v,$$
$$\frac{\partial v}{\partial t} = D_v \Delta v + bu - u^2 v. \tag{6.6}$$

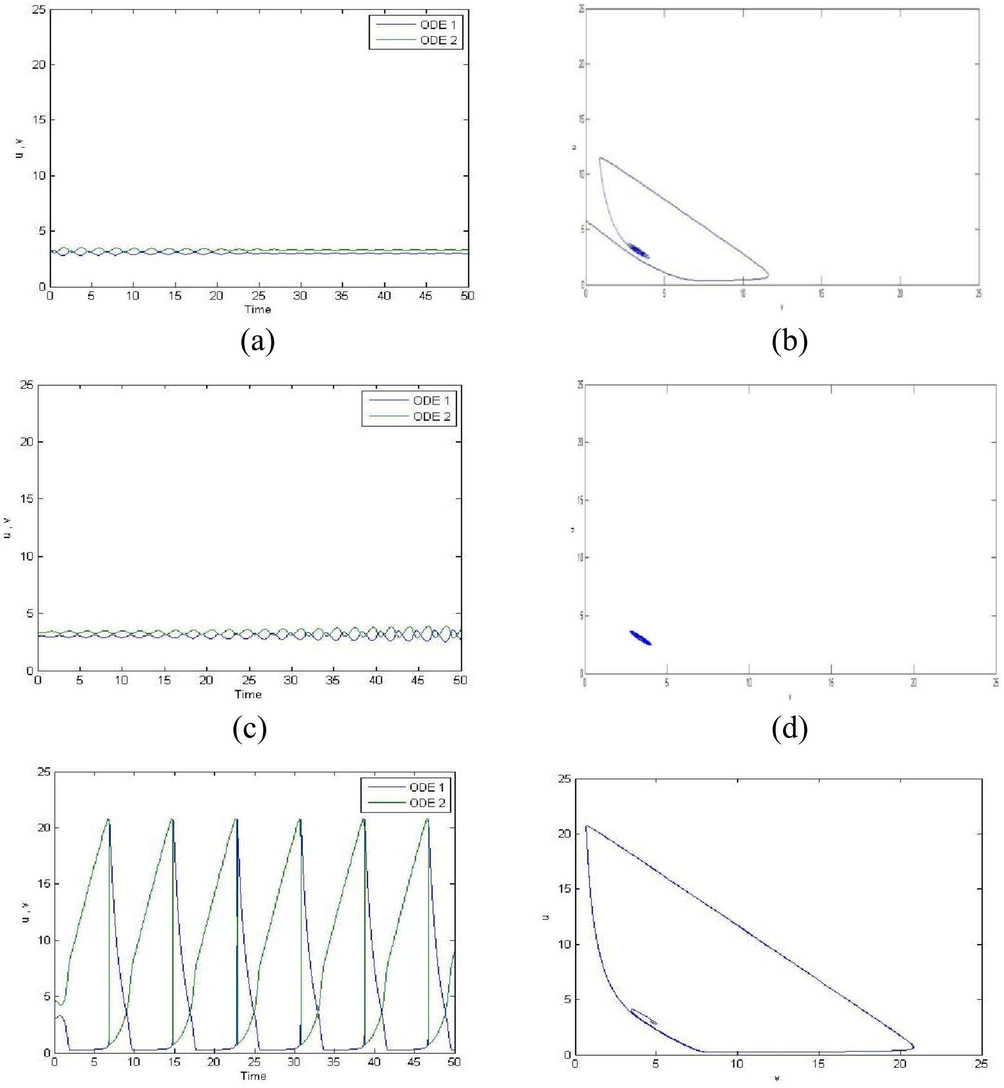
where u and v are the dimensionless concentration called activator and inhibitor and a and b are the kinetic parameters. The equilibrium point for the system (6.6) is $(a, b/a)$.

6.2.1.1 Existence of Limit Cycles (Local Dynamics Analysis)

In this part, we establish the existence of limit cycle solutions of the local dynamics of the model (6.6). Local dynamics of the model (6.6) without diffusion term can be represented by the following system of ordinary differential equations (ODEs).

$$\frac{du}{dt} = a - (b+1)u + u^2 v,$$
$$\frac{dv}{dt} = bu - u^2 v. \tag{6.7}$$

A periodic solution (u, v) is a periodic orbit or limit cycle of the system of ODEs (6.7). Linear stability analysis shows that when $b < 1 + a^2$ then the equilibrium point $(a, b/a)$ is stable and all other non-equilibrium solutions of (6.7) approach to the unstable limit cycle. Also, when $b > 1 + a^2$ then the equilibrium point $(a, b/a)$ is unstable and all other non-equilibrium solutions of (6.7) approach to the stable limit cycle. Hence, we get an equation for the Hopf bifurcation points and which is $b = 1 + a^2$. Now, we investigate the system of ODEs (6.7) numerically to verify the linear stability analysis results.



(a) (b)

(c) (d)

(e) (f)

Figure 6.2. An illustration of the local dynamics of the model (6.6) which is represented by (6.7).

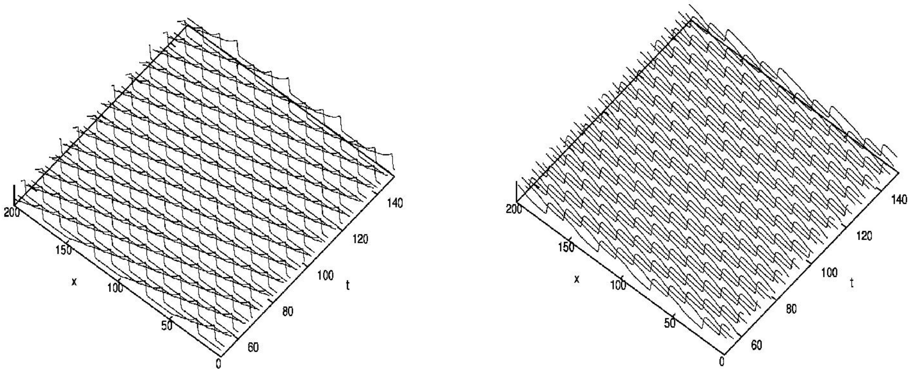
In fig. 6.2, we plot activator and inhibitor densities u and v with respect to time as well as u versus v where we take $a = 3.0$. In this case, the kinetics have a Hopf bifurcation at $b = 10.0$ and we get a stable limit cycle solution for greater values of $b = 10.0$. Fig. 6.2(a) and fig. 6.2(b) represents the solutions of (6.7) when $a = 3.0$ and $b = 9.9$, where we get the stable equilibrium point $(3, 3.3)$ and shows the existence of the unstable limit cycle. Fig. 6.2(c) and fig. 6.2(d) represents the solutions of (6.7) when $a = 3.0$ and $b = 10.1$ where we get the unstable equilibrium point $(3, 3.37)$ shows the existence of the stable limit cycle. Fig. 6.2(e) and fig. 6.2(f) represents the solutions of (6.7) when $a = 3.0$ and $b = 14$, where we get the unstable equilibrium point $(3, 4.67)$ and shows the existence of the stable limit cycle. From the stable limit cycle solutions of (6.7), we see that the cycles are of low amplitude for *b* close to the Hopf bifurcation value $b = 10.0$ and they increase in amplitude as *b* is increased.

6.2.1.2 Existence of Periodic Solutions in the One-Dimensional Space through Direct Partial Differential Equation Simulation

In this subsection, we perform the direct Partial Differential Equation (PDE) numerical simulations of (6.6) in one dimensional space. Fig. 6.3 shows the qualitative behavior of the periodic solutions of (6.6) with the periodic patterns. We use table 6.1 for the numerical simulations.

Table 6.1. Typical parameter values of (6.6) for the numerical computations.

Parameters	a	b	D_u	D_v
Values	3.0	14.0	3.0	10.0



(a) (b)

Figure 6.3. Space-time plot via direct PDE simulation. (a) Solutions of activator, u (b) Solutions of inhibitor, v.

Fig. 6.3(a) and Fig. 6.3(b) shows that the PTW solutions of the activator, u and the inhibitor, v respectively. We apply an implicit scheme with periodic boundary conditions over the domain $\left[0, D_x\right]$. Here D_x represents the system size which is defined by $D_x = n \times p$ where n is the number of pulses and p is the spatial period. We consider, $dx = 0.09$ as space size and $dt = 0.01$ as time stepon 2181 grid elements. Here, we consider a small perturbation of the steady state solution as the initial data and continue our simulation process for a long time until we get a stable pattern. In this simulation, we take $D_x = 200$ as the system size with four pulses that means, the spatial period is $p = 50$ and also we take $50 \le t \le 150$ as the time range for the solutions of (6.6). Finally, we obtain periodic pattern solutions of the activator, u as well as the inhibitor, v. Hence, we get a good agreement between the results obtained from this subsection and the result from the subsection 6.2.1.1.

6.2.1.3 Periodic Patten Formation in the Two-Dimensional Spaces through Direct PDE Simulation

In this subsection, we use alternating direction implicit (ADI) method with Neumann boundary conditions to perform a series of direct PDE numerical simulations of (6.6) in two dimensions. Numerical simulation is performed on the spatiotemporal grid (x_i, y_j) with $x_i = i\Delta x,\ i = 0, \cdots, N_x$ and $y_j = j\Delta y,\ j = 0, \cdots, N_y$ where $\Delta x = \Delta y$ for a uniform mesh grid and time $t_n = n\Delta t,\ n = 0,1,2,3\cdots$, where Δt is the time step. Therefore, the space steps in the x-direction and in the y-direction are as follows:

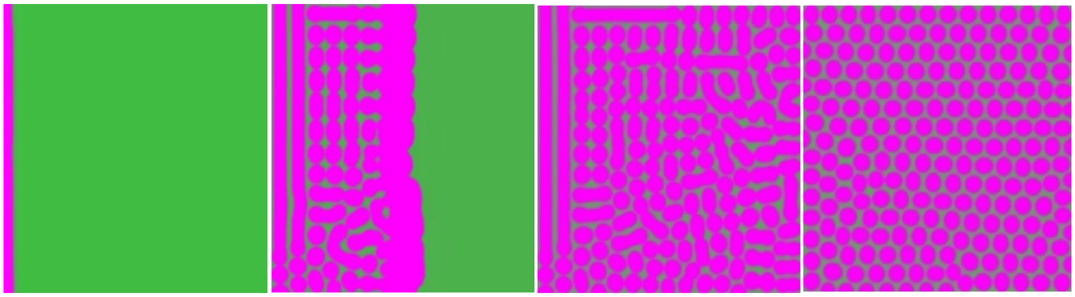
$$\Delta x = \frac{L_x}{N_x},\ \Delta y = \frac{L_y}{N_y},\ N_x, N_y \in \mathbb{Z}, \tag{6.8}$$

where $0 < x < L_x$ and $0 < y < L_y$ is used as the domain in the (x, y) parameter plane. In (6.6), we represent the grid approximations by $U_{i,j}^n \approx u(x_i, y_j, t_n)$ and $V_{i,j}^n \approx v(x_i, y_j, t_n)$ .Therefore, the full discrete grid approximation of $U_{i,j}^n$ is as follows:

$$\frac{U_{i,j}^{n+1/2} - U_{i,j}^{n}}{\Delta t/2} = D_u \frac{U_{i-1,j}^{n+1/2} - 2U_{i,j}^{n+1/2} + U_{i+1,j}^{n+1/2}}{\Delta x^2} + D_u \frac{U_{i,j-1}^{n} - 2U_{i,j}^{n} + U_{i,j+1}^{n}}{\Delta y^2} + f(U_{i,j}^{n}, V_{i,j}^{n}) \tag{6.9}$$

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{\Delta t/2} = D_u \frac{U_{i-1,j}^{n+1/2} - 2U_{i,j}^{n+1/2} + U_{i+1,j}^{n+1/2}}{\Delta x^2} + D_u \frac{U_{i,j-1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j+1}^{n+1}}{\Delta y^2} + f(U_{i,j}^{n+1/2}, V_{i,j}^{n+1/2}) \tag{6.10}$$

Equation (6.9) indicates the first half of the total time step and (6.10) indicates the rest half of the total time step. The central difference operator is defined as $\delta_x U_{i,j}^{k} = U_{i+1/2,j}^{k} - U_{i-1/2,j}^{k}$ and a similar formula can be defined for δ_y. Equialently, we can define the approximation equations for $V_{i,j}^{n}$.



(a) (b) (c) (d)

Figure 6.4. Pattern evolution as a function of time. (a) at $t = 0$ (b) at $t = 47$ (c) at $t = 122$ (d) at $t = 2000$.

In this simulation, we use $\Delta x = \Delta y = 0.5$ as space step and $\Delta t = 0.04$ as time step on a grid of 220×220 elements and eventually, we get a periodic spot pattern. Again, it was checked that the decreasing values of step size did not lead to any changes in the results. We continue our numerical simulations until they are in a stationary or until they have behavior that the characteristics results do not seem anymore. Fig. 6.4 shows the dynamics of a periodic pattern of (6.6) as a function of time. Here, we consider a small perturbation of the steady state solution as an initial guess and continue our simulation process for a long time until we get a periodic pattern. We use the parameter values of (6.6) as mentioned in tab.6.1. Fig. 6.4 (a) shows the initial data at time $t = 0$. Fig. 6.4 (b), fig. 6.4 (c) shows the development process of the spot pattern at time $t = 47$ and $t = 122$ respectively. Finally, we get a periodic spot pattern at time $t = 2000$ which shows in fig. 6.4 (d). Hence, we get a good agreement among the results obtained from this subsection and the results from the subsection 6.2.1.1 and 6.2.1.2.

6.2.2 Model 2: Lorenz-Haken Model

In [39], Haken introduced an optical model. Since the Haken model is similar to the Lorenz model, hence the system is called Lorenz- Haken (L-H) model. The slow-fast nonlinear system of equations in three variables for the standard L-H model is given by:

$$\begin{aligned} \frac{dE}{dt} &= \bar{k}(P-E), \\ \frac{dP}{dt} &= nE-P, \\ \frac{dn}{dt} &= \bar{\gamma}(B-n-EP). \end{aligned} \tag{6.11}$$

In the laser system (6.11), the real amplitude of the electromagnetic field is denoted by E, the polarization of the cavity medium is denoted by P and n is the inversion of the state within the two levels of the development due to the pumping. Also, $\bar{k}$ and $\bar{\gamma}$ are the relaxation rate parameters and B is the pump parameter. If we consider x, y, z in place of E, P, n respectively and also consider μ, δ in place of $\bar{k}, \bar{\gamma}$, then equation (6.11) can be written as the following system of non-linear ordinary differential equations.

$$\begin{aligned} \frac{dx}{dt} &= \mu(y-x), \\ \frac{dy}{dt} &= zx-y, \\ \frac{dz}{dt} &= \delta(B-z-xy). \end{aligned} \tag{6.12}$$

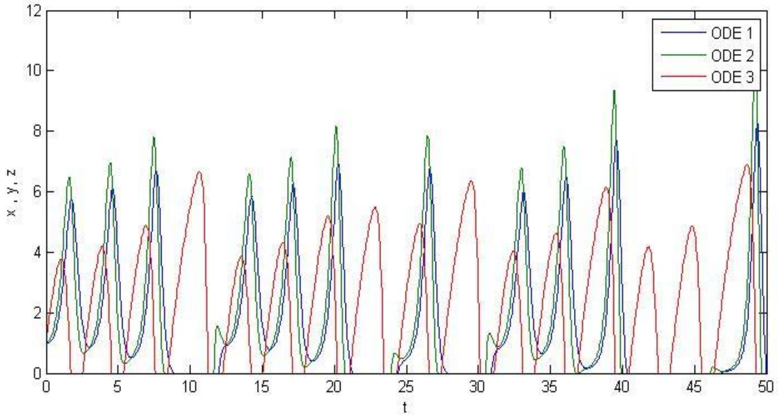


Figure 6.5. Numerical simulation of the model (6.12)

In fig. 6.5, we plot the state variables x, y and z with respect to time t. We use the parameter values of (6.12) as mentioned in Table 6.2.

Table 6.2. Typical parameter values of (6.12) for the numerical computation.

Parameters	μ	δ	B
Values	4.0	0.4	12.0

6.3 Dynamical System Analysis through Differential Geometry

6.3.1 Flow Curvature Method

Singularly perturbed systems can have invariant manifolds where the trajectories of the flow move slowly and these slow manifolds are invariant with respect to the flow [5, 62, 91]. Several methods have been developed to find out the analytical slow manifold equations of the singularly perturbed systems. In [16, 23, 24, 25, 26, 81, 82, 95], introduces the geometric singular perturbation technique to establish the existence of the slow manifold equation along with the local invariance of the slow manifold for the singularly perturbed system. In the case of non-singularly perturbed system this technique fails to provide the slow manifold.

The flow curvature method [32, 33, 34, 35, 36] is the new method in recent publications for computing the analytical implicit equation of the slow manifold. This method can be applied to any autonomous or non-autonomous dynamical systems in n-dimensions whether it is singularly perturbed or not. Recent applications of the flow curvature method of the singularly perturbed systems are FitzHugh-Nagumo model, Van der pol model, Chua's model, etc and the applications of the flow curvature method of the non-singularly perturbed systems are Lorenz model, Rikitake model, etc. [37] used the flow curvature method to construct the slow invariant manifold of the heartbeat model. In [38], author developed the slow invariant manifold analytical implicit equation of the generalized Lorenz-Krishnamurthy model and conservative generalized Lorenz-Krishnamurthy model. The most important feature of this method is that, the flow curvature manifold directly gives us the analytical equation of the slow manifold. Without using any asymptotic expansions, this method allows us to find the flow curvature manifold and hence slow invariant manifold equation. To the best of our knowledge, this method is the best to find the analytical equation of the slow invariant manifold for any dimensional dynamical system.

Now, we briefly discuss the flow curvature method in terms of differential geometry. This method uses the properties of curvatures of trajectory curve or flow of the dynamical system. Using this method, one can define the flow curvature manifold corresponding to the dynamical system. Any n-dimensional dynamical system can have the $(n-1)$ dimensional flow curvature manifold that means flow curvature manifold contains the information about the flow with highest curvature.

6.3.1.1 Analytical Implicit Equation of the Slow Manifold of the Dynamical System

Invariant manifold implies a very significant role to explain the stability as well as dynamical behavior of a system, especially for a slow-fast dynamical system. Although geometric perturbation technique is well known to find the analytical equation of slow manifold, the main difference between geometric perturbation technique and the flow curvature method is that it neither uses asymptotic expansions nor eigenvectors. Another difference is that this method can be used for any dynamical system which may or may not singularly perturbed.

Proposition 6.1. The set of points where the curvature of the flow of the model (6.1) vanishes represented by the following flow curvature manifold equation of the dynamical system.

$$\phi(\vec{X}) = \det\left(\overrightarrow{\dot{X}}, \overrightarrow{\ddot{X}}\right) = 0$$

Proof. See [33, 34]

Note that for any n-dimensional dynamical system, maximum $(n-1)^{th}$ flow curvature is possible.

Proposition 6.2. The flow curvature manifold of the dynamical system (6.1) directly provides its implicit analytical equation of the slow manifold.

Proof. See [33, 34]

6.3.1.2 Darboux Invariance Theorem

According to [64, 87], the concept of the invariant manifold is first introduced by G. Darboux (1878, p. 71). We consider the trajectories of the dynamical system (6.1) is represented by a motion of a point in a two dimensional space and the coordinates of the point is $\overrightarrow{X} = (u, v)$ and the velocity vector of this point is $\vec{V} = (\dot{u}, \dot{v})$.

Proposition 6.3. Consider $\phi(\vec{X}) = \det\left(\overrightarrow{X}, \overrightarrow{\dot{X}}\right) = 0$ is a slow manifold of the dynamical system (6.1) where ϕ is a first time continuously differentiable function, then this manifold is invariant with respect to the flow of (6.1) if there exist a first time continuously differentiable function called cofactor $C(\overrightarrow{X})$ which satisfies the following equation:

$$\mathcal{L}_{\vec{V}}\phi(\vec{X}) = C(\vec{X})\phi(\vec{X}),$$

with the Lie derivative defined as the following:

$$\mathcal{L}_{\vec{V}}\phi(\vec{X}) = \vec{V}\cdot\vec{\nabla}\phi = d\phi/dt$$

Proof. See [33, 34]

6.3.1.3 The Osculating Plane Equation

Definition 6.5. The osculating plane is the plane which passes through a fixed point $\vec{X}^*$ of the dynamical system and parallel to the unit tangent and normal vectors to a trajectory curve. The Osculating plane can be defined using for a dynamical system (6.1) as the following

$$P(\vec{X}) = \left(\vec{X} - \vec{X}^*\right)\cdot\dot{\vec{X}} = 0$$

Theorem 6.4. The Flow curvature manifold $\phi(\vec{X})$ of the three-dimensional dynamical system (6.1) merges with its Lie derivative $\mathcal{L}_{\vec{V}}\phi(\vec{X})$ and with its osculating plane $P(\vec{X})$ in the vicinity of the fixed point $\vec{X}^*$.

Proof. See [34]

6.3.1.4 Stability Analysis of the Fixed Points

Definition 6.6. The fixed points $\vec{X}^*$ of any dynamical system may also be fixed points of the flow curvature manifold if the following two equations are satisfied:

$$\phi(\vec{X}^*) = 0$$
$$\vec{\nabla}\phi(\vec{X}^*) = 0$$

Definition 6.7. The Hessian of a function $\phi(\vec{X})$ at the point $\vec{X}$ is denoted by $H_{\phi(\vec{X})}$ and defined by

$$\mathrm{H}_{\phi(\vec{X})} = \begin{vmatrix} \dfrac{\partial^2 \phi}{\partial x^2} & \dfrac{\partial^2 \phi}{\partial x \partial y} \\ \dfrac{\partial^2 \phi}{\partial y \partial x} & \dfrac{\partial^2 \phi}{\partial y^2} \end{vmatrix}$$

Theorem 6.5. (a) If $\mathrm{H}_{\phi(\vec{X}^*)} \leq 0$ then both eigenvalues are real and the fixed point $\vec{X}^*$ is a saddle or a node.

(b) If $\mathrm{H}_{\phi(\vec{X}^*)} > 0$ then both eigenvalues are complex conjugated and the fixed point $\vec{X}^*$ is a focus.

Proof. See [34]

6.3.2 Geometric Singular Perturbation Theory

Geometric Singular Perturbation Theory is based on the following assumptions and theorem stated by Nils Fenichel in the middle of the seventies.

6.3.2.1 Assumptions

(A1) Functions $\vec{f}$ and $\vec{g}$ are C^∞ functions in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and I is an open interval containing $\varepsilon = 0$.

(A2) There exists a set M_0 that is contained in $\{(\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = 0\}$ such that M_0 is a compact manifold with boundary and M_0 is given by the graph of a C^1 function $\vec{x} = \vec{X}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^n$ is a compact, simply connected domain and the boundary of D is an $(n-1)$ dimensional C^∞ submanifold. Finally, the set D is overflowing invariant with respect to (6.5) when $\varepsilon = 0$.

(A3) M_0 is normally hyperbolic relative to the reduced fast system and in particular it is required for all points $\vec{p} \in M_0$, that there are k (resp.l) eigenvalues of $D_{\vec{x}}\, \vec{f}(\vec{p}, 0)$ with positive (resp. negative) real parts bounded away from zero, where $k + l \ = m$.

6.3.2.2 Fenichel Persistence Theory for Singularly Perturbed Systems

For compact manifolds with boundary, Fenichel's persistence theory states that, provided the hypotheses (A1)–(A3) are satisfied, the system (6.4) has a slow (or center) manifold and this slow manifold has fast stable and unstable manifolds.

Theorem 6.6. Let system (6.4) satisfying the conditions (A1)$-$(A3). If $\varepsilon > 0$ is sufficiently small, then there exists a function $\vec{X}(\vec{z}, \varepsilon)$ defined on D such that the manifold $M_\varepsilon = \{(\vec{x}, \vec{z}) : \vec{x} = \vec{X}(\vec{z}, \varepsilon)\}$ is locally invariant under (6.4). Moreover, $\vec{X}(\vec{z}, \varepsilon)$ is C^r for any $r < +\infty$ and M_ε is $C^r O(\varepsilon)$ close to M_0. In addition, there exist perturbed local stable and unstable manifolds of M_ε. They are unions of invariant families of stable and unstable fibers of dimensions l and k, respectively, and they are $C^r O(\varepsilon)$ close for all $r < +\infty$, to their counterparts.

Proof. See [23-26]

6.3.2.3 Invariance

Generally, Fenichel theory enables to turn the problem for explicitly finding functions $\vec{x} = \vec{X}(\vec{z}, \varepsilon)$ whose graphs are locally slow invariant manifolds M_ε of system (6.4) into regular perturbation problem. Invariance of the manifold M_ε implies that $\vec{X}(\vec{z}, \varepsilon)$ satisfies:

$$\varepsilon\, D_{\vec{z}}\ \vec{X}(\vec{z}, \varepsilon)\ \vec{g}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) = \vec{f}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) \tag{6.13}$$

Then, plugging the perturbation expansion:

$$\vec{X}(\vec{z}, \varepsilon) = \vec{X}_0(\vec{z}) + \varepsilon\, \vec{X}_1(\vec{z}) + O(\varepsilon^2)$$

into (6.13) enables to solve order by order for $\vec{X}(\vec{z}, \varepsilon)$.

The Taylor series expansion for $\vec{f}\left(\vec{X}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right)$ up to terms of order two in ε leads at order ε^0 to

$$\vec{f}\left(\vec{X}_0(\vec{z}), \vec{z}, 0\right) = \vec{0}$$

which defines $\vec{X}_0(\vec{z})$ due to the invertibility of $D_{\vec{x}}\,\vec{f}$ and the implicit function theorem.
At order ε^1 we have:

$$D_{\vec{z}}\ \vec{X}_0(\vec{z})\ \vec{g}\left(\vec{X}_0(\vec{z}), \vec{z}, 0\right) = D_{\vec{x}}\,\vec{f}\left(\vec{X}_0(\vec{z}), \vec{z}, 0\right) \vec{X}_1(\vec{z}) + \frac{\partial \vec{f}}{\partial \varepsilon}\left(\vec{X}_0(\vec{z}), \vec{z}, 0\right)$$

which yields $\vec{X}_1(\vec{z})$ and so forth.

So, regular perturbation theory enables to build locally slow invariant manifolds M_ε. But for high-dimensional singularly perturbed systems slow invariant manifold analytical equation determination leads to tedious calculations.

6.3.3 Geometric Analysis of Dynamical Systems

In this section, we analyze two dynamical systems using two geometric methods. In subsection 6.3.3.1, we analyze a two dimensional non-singularly perturbed dynamical system called Brusselator model using two geometric methods named as flow curvature method and geometric singular perturbation theory as well as we provide the effect of growth and curvature with surface deformation on pattern formation of the Brusselator model. In subsection 6.3.3.2, we analyze a three dimensional singularly perturbed dynamical system called Lorenz-Haken model using two geometric methods named as flow curvature method and geometric singular perturbation theory.

6.3.3.1 Geometric Analysis of Model 1

6.3.3.1.1 Analysis Using the Flow Curvature Method

We consider an activator-inhibitor Brusselator model which represents an autocatalytic oscillating chemical reaction. In paper [6], authors discussed the asymptotic behaviour of the solutions of the Brusselator model numerically. In paper [7], author studied various types of pattern formation of the Brusselator model arising in chemical reactions with the numerical investigation. We study the slow invariant manifold of the Brusselator model for the first time and this study advances the field from the previous related work. So, our goal of this section is to apply the flow curvature method on the two-dimensional chemical system called Brusselator model to find the analytical implicit equation of the slow invariant manifold. The invariance of the slow manifold of the Brusselator model is then proved by using the Darboux theory. To simulate the Brusselator model, we use MATHEMATICA as a software tool.

According to the flow curvature method, the trajectory curves of any dynamical system which may or may not singularly perturbed considered the curves in the Euclidean space. We consider the system model (6.7) as the slow-fast dynamical system.

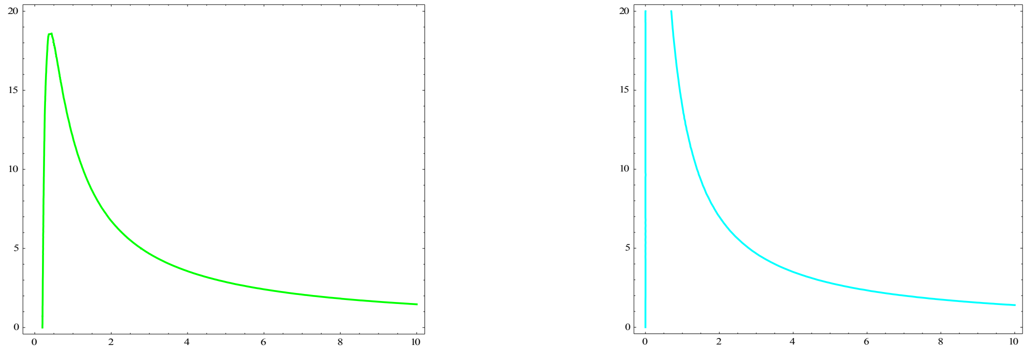
We use the parameter values of (6.7) as mentioned in tab. 6.1 and for the numerical simulation, we consider the range of the state variables connected with the dynamical system (6.7) as the following

$$[u_{\min}, u_{\max}] = [-.1, 10];$$
$$[v_{\min}, v_{\max}] = [0, 20];$$

By putting the right hand side parts of the dynamical system (6.7) equal to zero, that is,

$$\begin{aligned} & a-(b+1)u+u^2v=0, \\ & bu-u^2v=0, \end{aligned} \tag{6.14}$$

We obtain two following graphs for the two null-clines of the system (6.7).



(a) (b)

Figure 6.6. Nullclines of the model (6.7). (a) first equation of the system (6.11), (b) second equation of the system (6.11).

Thus, we get the following fixed point by solving the system (6.11)

$$u_1 = 3; \; v_1 = \frac{14}{3};$$

We use explicit Runge-Kutta method to solve the model (6.7) numerically where we use $(u_0, v_0) = (1,1)$ as an initial point. Fig. 6.7 shows the phase diagram along with the two nullclines represented by (6.7) where t ranges from 80 to 100. Also, the purple point in the fig. 6.7 indicates the fixed point of the model (6.7).

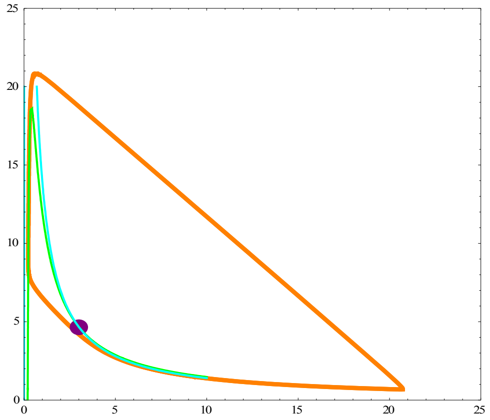


Figure 6.7. Phase plot analysis for the model (6.7) along with the two nullclines and single fixed point obtained from the same model.

Now, in order to calculate the flow curvature manifold of the model (6.7) using the flow curvature method, we need the velocity and acceleration because of our 2-dimensional dynamical model. The velocity vector field of the model (6.7) can be represented by the following way.

$$\vec{\mathrm{V}} = \left\{3 - 15u + u^2 v, 14u - u^2 v\right\}$$

The Jacobian matrix corresponding to the model (6.7) may be written as

$$J = \left\{\left\{-15 + 2uv, u^2\right\}, \left\{14 - 2uv, -u^2\right\}\right\}$$

Now we get the acceleration vector by using the formula $\vec{A} = J\vec{V}$ and hence, we obtain

$$\vec{\gamma} = \left\{u^2(14u - u^2 v) + (-15 + 2uv)(3 - 15u + u^2 y), -u^2(14u - u^2 v) + (14 - 2uv)(3 - 15u + u^2 v)\right\}$$

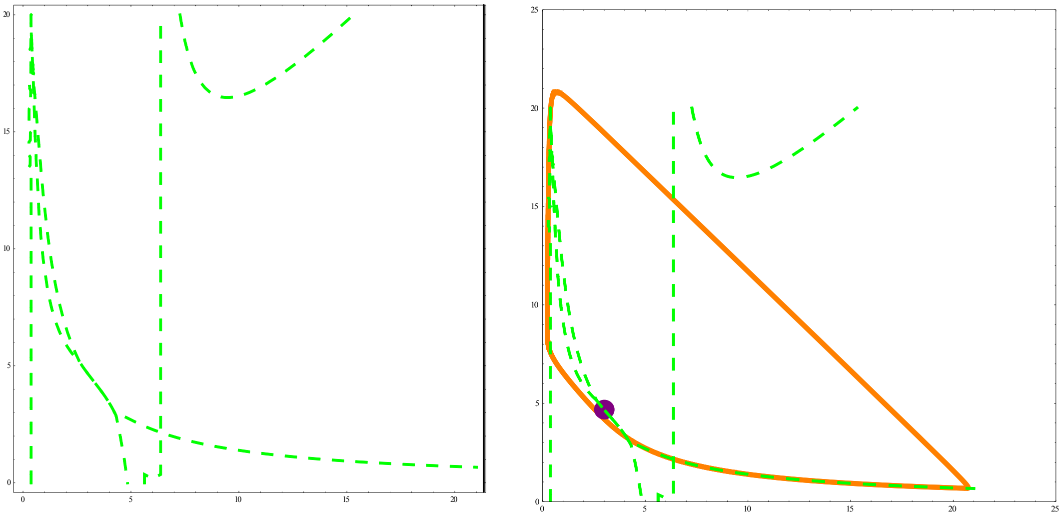
Then, we find the slow manifold function of the model (6.7) as the following

$$\phi(u,v) = 126 - 630u - 42u^3 + 14u^4 - 18uv + 135u^2 v - 15u^3 v + 3u^4 v - u^5 v - 6u^3 v^2 + u^4 v^2$$

Now the analytical implicit equation of the slow manifold of the model (6.7) can be written as

$$\phi(u,v) = 0 \tag{6.15}$$

In fig. 6.8 (a) shows the graphical representation of the analytical implicit equation of the slow manifold represented by the equation (6.15) and fig. 6.8 (b) represents the slow manifold as well as phase space diagram in the same graph.



(a) (b)

Figure 6.8. (a) Graphical representation of the slow manifold analytical equation of the model (6.7) using flow curvature method, (b) Graphical representation of the slow manifold analytical equation along with the phase diagram represented by (6.7).

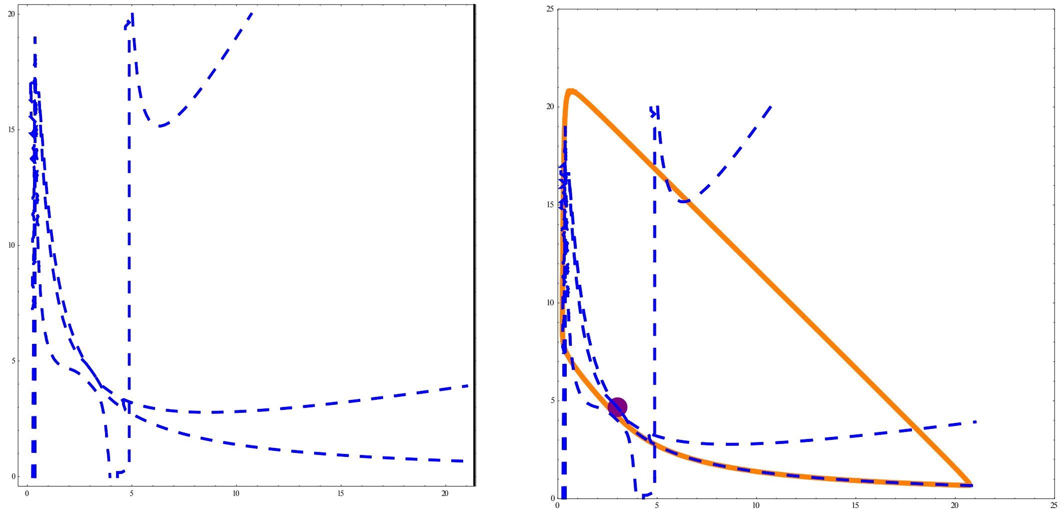
The Lie derivative of the slow manifold function is then evaluated as the following by using the Darboux invariance theory to establish the flow curvature invariance of the equation (6.15). We first find the normal vector of the flow curvature manifold and we get

$$\vec{\nabla}\phi=\begin{Bmatrix}-630-126u^2+56u^3-18v+270uv-45u^2v+12u^3v-5u^4v-18u^2v^2+4u^3v^2, \\ -18u+135u^2-15u^3+3u^4-u^5-12u^3v+2u^4v\end{Bmatrix}$$

Now according to proposition 6.3, we compute Lie derivative of the slow manifold as follows

$$\mathcal{L}_{\vec{V}}\phi=-1890+9450u-630u^2+3948u^3-1050u^4+42u^5-14u^6-54v+ \\ 1080uv-4815u^2v+729u^3v-624u^4v+174u^5v-3u^6v+u^7v-72u^2v^2+ \\ 552u^3v^2-105u^4v^2+24u^5v^2-7u^6v^2-18u^4v^3+4u^5v^3$$

In fig. 6.9 (a) shows the graphical representation of the equation $\mathcal{L}_{\vec{V}}\phi = 0$ that means the graphical representation of the flow curvature invariance manifold where the rate of change of $\psi(u,v)$ is equal to zero and fig. 6.9 (b) shows the combined graph of the invariance manifold and phase space plot.



(a) (b)

Figure 6.9. (a) Graphical representation of the invariance equation of the slow manifold analytical equation of the model (6.7) according to the Darboux theorem (b) Graphical representation of the invariance equation of the slow manifold analytical equation along with the phase diagram represented by (6.7).

The osculating plane equation for the system (6.7) is obtained as follows:

$$P(\vec{X}) = -14 + 3v + 1/3 u (336 - 45v + u (-42 + v (-23 + 3u + 3v)))$$

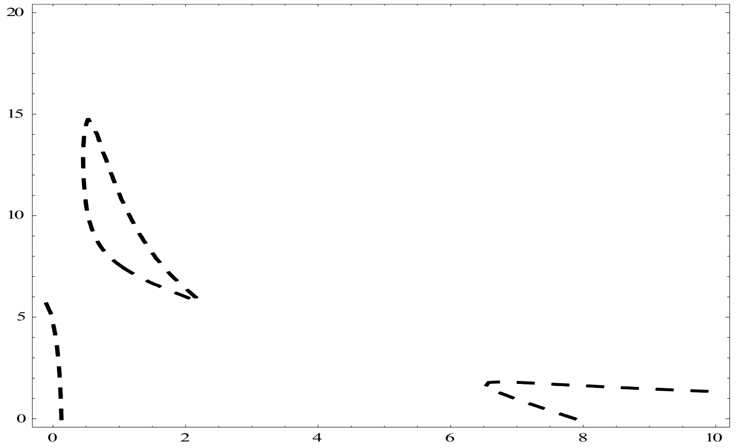


Figure 6.10. Graphical Representation of the Osculating Plane equation.

To perform the stability analysis of the fixed-point using flow curvature manifold of the Brusselator model, we need to calculate the Hessian of flow curvature manifold and we get

$$\mathrm{H}_{\phi(\vec{X})} = \begin{vmatrix} 270y-20x^3y+12x^2(14+3y+y^2)-18x(14+5y+2y^2) & -18+270x-5x^4+4x^3(3+2y)-9x^2(5+4y) \\ -18+270x-5x^4+4x^3(3+2y)-9x^2(5+4y) & 2(-6+x)x^3 \end{vmatrix}$$

and

$$\mathrm{H}_{\phi(\vec{X}^*)} = 1620$$

Since, Hessian is positive, so the fixed point (u_1, v_1) is a focus.

6.3.3.1.2 Analysis Using the Geometric Singular Perturbation Theory

Geometric Singular Perturbation Theory is entirely devoted to singularly perturbed system and provides their slow invariant manifold according to Fenichel's theorem. The Brusselator model has nosingular approximation, it has been numerically checked that its functional jacobian matrix possesses at least a largest absolute value of the real part over a huge domain of the phase space. So, it can be considered as a slow fast dynamical system but not as a singularly perturbed system. Thus, Geometric Singular Perturbation Theory can not provide the slow invariant manifold associated with Brusselator model.

6.3.3.1.3 The Effect of Growth and Curvature with Surface Deformation on Pattern Formation of the Brusselator model

Since the seminal paper by Turing [93], reaction-diffusion models have been proposed to account for pattern formation in a wide variety of biological situations. The simplest version of the model consists of two coupled non-linear reaction-diffusion equations describing the spatiotemporal evolution of the concentration of two substances (termed morphogens by Turing). Turing showed that for conditions under which the reaction kinetics admitted a linearly stable spatially uniform steady state, it was possible for diffusion to cause instability, leading to spatially varying profiles in morphogen concentration. These are the Turing patterns and they arise from the so-called diffusion-driven instability. It has been shown that these models exhibit a variety of spatial patterns consistent with those observed in a number of biological systems. From a theoretical viewpoint, the hypothesis that spatial patterns in early development arise via a Turing instability has been criticized for a number of reasons. Murray [77] found that changes in spatial scale can produce dramatic changes in the patterns exhibited by the Turing model. In [4], the effect of a growing domain is incorporated by choosing a time-dependent scaling factor. This brief review shows that understanding the effects of growth and geometry on Turing patterns is currently an issue of importance. The main purpose of the section is to provide a general framework for the study of pattern formation using reaction diffusion equations in which the effects of both growth and geometry are taken into account.

Consider a domain which grows in one dimension and also consider the parameter $s \in [0,1]$ (the spatial parameter) and define the mapping ψ_t, such that for every time $t \geq 0$,

$$\psi_t : [0,1] \to \mathbb{R}^3,\ \psi_t(s) \equiv X(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$$

where $X(s,t)$ represents a curve in space parameterized by s, for each time t. This curve can be used to represent a one-dimensional spatial domain which grows in time. It will be convenient to introduce at this stage the arc length as a function of s and t,

$$\sigma(s,t) = \int_0^s |X_s(s',t)| ds'$$

For two-dimensional growth we assume that for every time $t \geq 0$, there is a surface M parametrized by $(r,s) \in \Omega \subset \mathbb{R}^2$ that models the shape and size of the growing domain (the organism). Hence, there is a mapping

$$X(r,s,t) \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3, \qquad \psi_t(r,s) \equiv X(r,s,t) = \begin{pmatrix} x(r,s,t) \\ y(r,s,t) \\ z(r,s,t) \end{pmatrix}$$

that defines a two-dimensional surface embedded in $\mathbb{R}^3$.

We suppose that the evolution of the studied surface M is driven by the morphogens u and v, where u is the inhibitor and v is the activator. In mathematical terms we have that

$$\frac{\partial M}{\partial t} = M_t = h(u,v)N$$

where N is the normal to the surface and h a function of the two morphogens. The simplest case for h that we have adopted in the following is a linear function of one morphogen:

$$h(u,v)N = kvN$$

where K is a parameter in $\mathbb{R}$ and also the normal vector is given by

$$N(r,s,t)=X_r \times X_s \neq 0,$$

Since we assume surface is regular for each t. We also have the expression for the metric on the surface M_t, which is given by

$$dl^2 = dx^2 + dy^2 + dz^2$$

By using Riemannian metric for two-dimensional space, we can rewrite the above equation as

$$dl^2 = g_{ij} dx_i dx_j$$

where

$$x_1 = r, x_2 = s \quad \text{and} \quad g_{ij} = X_{x_i} \cdot X_{x_j}, i, j = 1, 2.$$

We denote

$$E = |X_r|^2 = g_{11}, E = X_r \cdot X_s = g_{12}, G = |X_s|^2 = g_{22}$$

Here we are assuming that the parametrization (r, s) is such that it defines an orthogonal system on the surface M_t, that is,

$$X_r \cdot X_s = 0$$

for each time t. Hence the matrix G of coefficients of the first fundamental form is

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} |X_r|^2 & 0 \\ 0 & |X_s|^2 \end{pmatrix}$$

with inverse

$$G^{-1} = \frac{1}{g_{11}g_{22}}\begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} = \begin{pmatrix} |X_s|^2 & 0 \\ 0 & |X_r|^2 \end{pmatrix}$$

Now, let ϕ be the morphogen concentration of a substance on the surface M_t . $\phi = \phi(X,t)$ is the number of molecules per unit area at time t, and $X \in M_t$. Consider a region $\Omega(t)$ on the surface, where diffusion takes place, and assume $\Omega(t) = \psi_t(\Omega_0)$ for some open, bounded domain $\Omega_0 \in \mathbb{R}^2$, with $\partial\Omega_0$ smooth. Then the diffusion process for ϕ on $\Omega(t)$ is given by

$$\frac{d}{dt}\int_{\Omega(t)} \phi(X,t)dS_X = D \oint_{\partial\Omega(t)} \nabla\phi \cdot ndl$$

where $\partial\Omega(t)$ is a regular curve on the surface and n is the unit vector normal to the curve, which lies on the tangent plane.

Since the surface on which evolve the morphogens is modified with time, we have to adapt the system of equations (6.6) to take into account the geometric changes. The problem of reaction-diffusion on growing domains has been well-studied in the past years. It leads generally to add convective and dilution terms to $\partial_t u$ ($\partial_t v$ respectively) that can be combined in $div(au)$ where

$$a(X,t) = \frac{dX}{dt} = X_t$$

represents the flow velocity of the growing surface.

If the surface M_t is parameterized by $X(r,s,t)$ then the reaction-diffusion System (6.6) can be rewritten as

$$\frac{\partial u}{\partial t}+u\frac{\partial \ln g}{\partial t}=D_u\Delta_{M_t}u+a-\left(b+1\right)u+u^2v,$$

$$\frac{\partial v}{\partial t}+v\frac{\partial \ln g}{\partial t}=D_v\Delta_{M_t}v+bu-u^2v.$$

where Δ_{M_t} is the Laplace-Beltrami operator. It is well known that differential geometry provides a convenient basis for describing the behavior of a shape. In differential geometry, the Laplace operator can be generalized to operate on functions defined on surfaces in Euclidean space and more generally on Riemannian manifolds. This general operator is Laplace–Beltrami operator. Like the Laplacian, the Laplace–Beltrami operator is defined as the divergence of the gradient, and is a linear operator taking functions into functions.

Now, g can be defined as the following

$$g=\left|g_{ij}\right|=\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$$

and also, we can define

$$g^{ij}=\frac{cofactor(g)}{g}$$

where g^{ij}, g_{ij} are the fundamental metric tensors associated to the Riemannian manifold M.

6.3.3.2 Geometric Analysis of Model 2

6.3.3.2.1 Analysis Using the Flow Curvature Method

According to the flow curvature method, the trajectory curves of any dynamical system which may or may not singularly perturbed considered the curves in the Euclidean space. We consider the system model (6.12) as the slow-fast dynamical system.

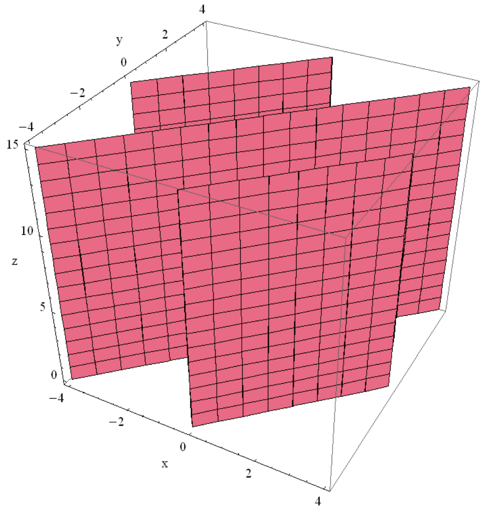
We use the parameter values of (6.12) as mentioned in Table 6.2 and for the numerical simulation, we consider the range of the state variables connected with the dynamical system (6.12) as the following

$$[\mathrm{x}_{\min}, \mathrm{x}_{\max}] = [-4, 4];$$
$$[\mathrm{y}_{\min}, \mathrm{y}_{\max}] = [-4, 4];$$
$$[\mathrm{z}_{\min}, \mathrm{z}_{\max}] = [0, 15];$$

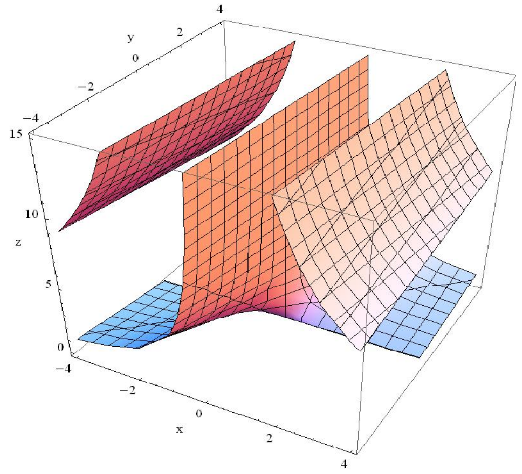
By putting the right hand side parts of the dynamical system (6.12) equal to zero, that is,

$$\begin{aligned} &\mu(y-x)=0, \\ &yzx-y=0, \\ &\delta(B-z-xy)=0, \end{aligned} \qquad (6.16)$$

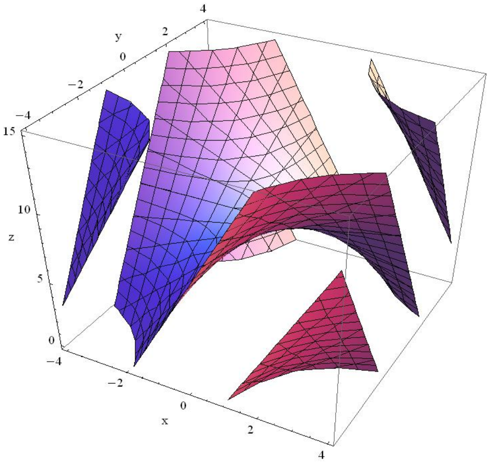
We obtain three following graphs for the three null-clines of the system (6.12).



(a)



(b)



(c)

Figure 6.11. Nullclines of the model (6.12). (a) first equation of the system (6.16), (b) second equation of the system (6.16) and (c) third equation of the system (6.16).

Thus, we get the following three fixed points by solving the system (6.16)

x_1 = -3.3166247903554; y_1= -3.3166247903554; z_1 = 1;
x_2 = 0; y_2 = 0; z_2= 12;
x_3 = 3.31662479103554; y_3 = 3.3166247903554; z_3 = 1;

We use explicit Runge-Kutta method to solve the model (6.12) numerically where we use $(x_0, y_0, z_0) = (1,1,1)$ as an initial point. Fig. 6.12 shows the phase diagram represented by (6.12) where t ranges from 500 to 1000. Also, the three green points in the fig. 6.12 indicate the fixed points of the model (6.12).

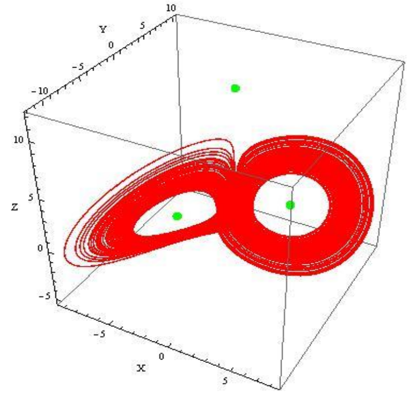


Figure 6.12. Phase plot analysis for the model (6.12) along with the three fixed points obtained from the same model.

Now, in order to calculate the flow curvature manifold of the model (6.12) using the flow curvature method, we need the velocity, acceleration and over-acceleration (Jerk) because of our 3-dimensional dynamical model. The velocity vector field of the model (6.12) can be represented by the following way.

$$\vec{V}=\{4\ (-x + y),\ -y + xz,\ 0.4\ (12 - xy - z)\}$$

The Jacobian matrix corresponding to the model (6.12) may be written as

$$J=\begin{pmatrix} -4 & 4 & 0 \\ z & -1 & x \\ -0.4y & -0.4x & -0.4 \end{pmatrix}$$

Now we get the acceleration vector by using the formula $\vec{A}=J\vec{V}$ and hence, we obtain

$$\vec{A}=4(4x-5y+xz),-0.4(-12x-2.5y+x^2y+13.5xz-10yz),-1.6(1.2-1.35xy+y^2-0.1z+0.25x^2z)$$

Then, the over-acceleration or jerk is calculated according to the formula

$$\overrightarrow{A^*}=J\vec{A}+\text{Total Differential}\,(J)\vec{V}$$

and we get the result as the following.

$$\begin{aligned}\overrightarrow{A^*}&=-1.6(-12x-2.5y+x^2y+13.5xz-10yz),3.2(-x+y)(12-xy-z)+4z(4x-5y+xz)-\\ &1.6x(1.2-1.35xy+y^2-0.1z+0.25x^2z)+0.4(-12x-2.5y+x^2y+13.5xz-10yz),-\\ &1.6y(4x-5y+xz)+0.64(1.2-1.35xy+y^2-0.1z+0.25x^2z)+0.16x(-12x-2.5y+x^2y+13.5xz-10yz)\end{aligned}$$

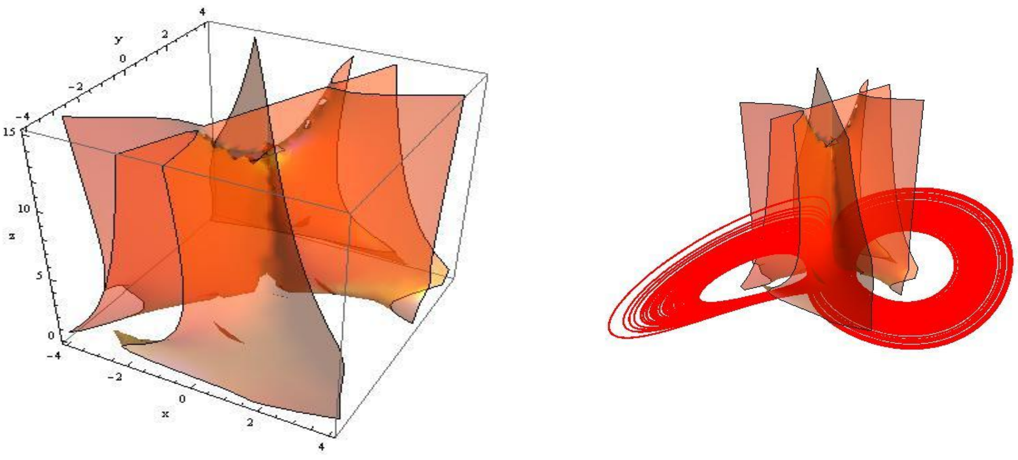
After that, we find the slow manifold function of the model (6.12) as the following

$$\psi(x,y,z)=0.256\begin{pmatrix}-8208\,x^2+144\,x^4+18702\,x\,y+1218\,x^3y-24\,x^5y-13896\,y^2-2226\,x^2y^2+\\ 18\,x^4y^2+\,x^6y^2+764\,x\,y^3-110\,x^3y^3+720\,y^4+192\,x^2y^4-120\,x\,y^5+\\ 3186\,x^2z-282\,x^4z-4462.5\,x\,y\,z+672\,x^3y\,z-1.5\,x^5y\,z+4110\,y^2z-\\ 718\,x^2y^2z-20\,x^4y^2z-24\,x\,y^3z+30\,x^3y^3z+160\,y^4z-808.5\,x^2z^2+45\,x^4z^2+\\ 2.5\,x^6z^2+1142\,x\,y\,z^2-260\,x^3y\,z^2-1446\,y^2z^2+425\,x^2y^2z^2-300\,x\,y^3z^2+\\ 350\,x^2z^3-50\,x^4z^3-75\,x\,y\,z^3+75\,x^3y\,z^3+100.\,y^2z^3-25\,x^2z^4\end{pmatrix}$$

Now the analytical implicit equation of the slow manifold of the model (6.12) can be written as

$$\psi(x,y,z)=0 \tag{6.17}$$

In Fig.6.13 (a) shows the graphical representation of the analytical implicit equation of the slow manifold represented by the equation (6.17) and Fig.6.13 (b) represents the slow manifold as well as phase space diagram in the same graph.



(a) (b)

Figure 6.13. (a) Graphical representation of the slow manifold of the model (6.12) using flow curvature method, (b) Graphical representation of the slow manifold along with the phase diagram represented by (6.12).

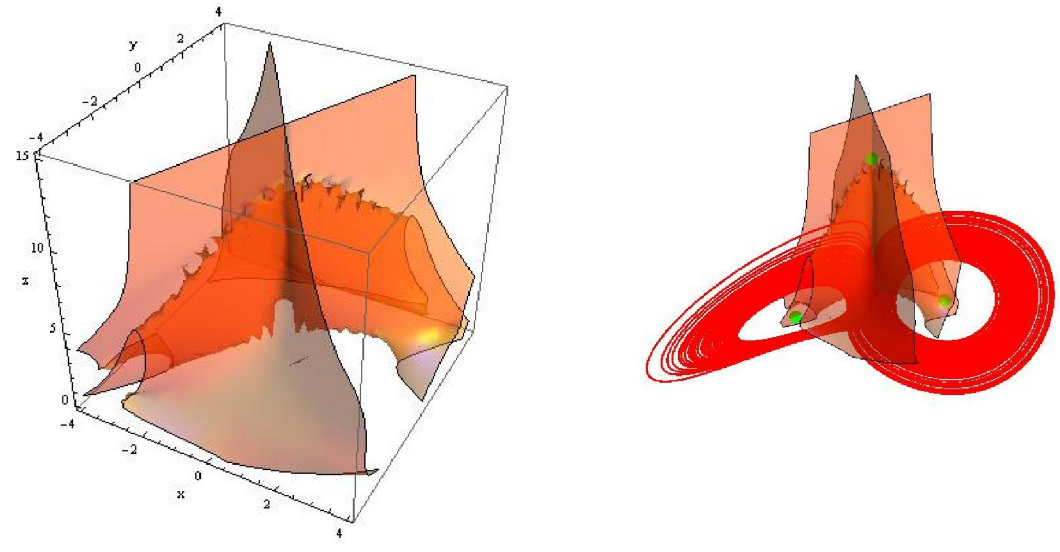
The Lie derivative of the slow manifold function is then evaluated as the following by using the Darboux invariance theory to establish the flow curvature invariance of the equation (6.17). We first find the normal vector of the flow curvature manifold and we get

$$\begin{aligned}
&\vec{\nabla}\psi=\{0.256(-16416.x+576.x^3+18702.y+3654.x^2y-120.x^4y-4452.xy^2+\\
&72.x^3y^2+6.x^5y^2+764.y^3-330.x^2y^3+384.xy^4-120.y^5+6372.xz-1128.x^3z-\\
&4462.5yz+2016.x^2yz-7.5x^4yz-1436.xy^2z-80.x^3y^2z-24.y^3z+90.x^2y^3z-\\
&1617.xz^2+180.x^3z^2+15.x^5z^2+1142.yz^2-780.x^2yz^2+850.xy^2z^2-300.y^3z^2+\\
&700.xz^3-200.x^3z^3-75.yz^3+225.x^2yz^3-50.xz^4),0.256(18702.x+1218.x^3-\\
&24.x^5-27792.y-4452.x^2y+36.x^4y+2.x^6y+2292.xy^2-330.x^3y^2+2880.y^3+\\
&768.x^2y^3-600.xy^4-4462.5xz+672.x^3z-1.5x^5z+8220.yz-1436.x^2yz-40.x^4yz-\\
&72.xy^2z+90.x^3y^2z+640.y^3z+1142.xz^2-260.x^3z^2-2892.yz^2+850.x^2yz^2-\\
&900.xy^2z^2-75.xz^3+75.x^3z^3+200.yz^3),0.256(3186.x^2-282.x^4-4462.5xy+\\
&672.x^3y-1.5x^5y+4110.y^2-718.x^2y^2-20.x^4y^2-24.xy^3+30.x^3y^3+160.y^4-\\
&1617.x^2z+90.x^4z+5.x^6z+2284.xyz-520.x^3yz-2892.y^2z+850.x^2y^2z-600.xy^3z+\\
&1050.x^2z^2-150.x^4z^2-225xyz^2+225x^3yz^2+300y^2z^2-100x^2z^3)\}
\end{aligned}$$

Now according to proposition 6.3, we compute Lie derivative of the slow manifold as follows

$$\mathcal{L}_{\vec{V}}\psi=-6.5024\left(\begin{array}{l}
-3187.28\,x^2+144.\,x^4+7110\,x\,y+455.858\,x^3y-24\,x^5y-4816.06\,y^2-1386.4\,x^2y^2+\\
46.0157\,x^4y^2+\,x^6y^2+980.913\,x\,y^3-93.2756\,x^3y^3-1.25984\,x^5y^3-37.1654\,y^4+\\
142.299\,x^2y^4+0.472441\,x^4y^4-100.472\,x\,y^5+18.8976\,y^6+622.913\,x^2z-247.039\,x^4z-\\
1289.63\,x\,y\,z+780.236\,x^3y\,z-1.26378\,x^5y\,z+1637.62\,y^2z-826.362\,x^2y^2z-\\
8.50394\,x^4y^2z+173.606\,x\,y^3z+13.937\,x^3y^3z+31.4961\,y^4z-4.37096\times10^{-15}x^2y^4z-\\
302.846\,x^2z^2+31.6535\,x^4z^2+2.5\,x^6z^2+234.315\,x\,y\,z^2-139.055\,x^3y\,z^2-3.14961\,x^5y\,z^2-\\
395.937\,y^2z^2+302.835\,x^2y^2z^2-246.457\,x\,y^3z^2+47.2441\,y^4z^2+100.709\,x^2z^3-23.622\,x^4z^3-\\
14.685\,x\,y\,z^3+38.3858\,x^3y\,z^3+24.4094\,y^2z^3-6.49606\,x^2z^4-2.95276\,x^4z^4-1.09274\times10^{-15}x\,y\,z^4
\end{array}\right)$$

In Fig.6.14 (a) shows the graphical representation of the equation $\mathcal{L}_{\vec{V}}\psi=0$ that means the graphical representation of the flow curvature invariance manifold where the rate of change of $\psi(x,y,z)$ is equal to zero and Fig.6.14 (b) shows the combined graph of the invariance manifold and phase space plot.



(a) (b)

Figure 6.14. (a) Graphical representation of the invariance equation of the slow manifold of the model (6.12) according to the Darboux theorem (b) Graphical representation of the invariance equation of the slow manifold along with the phase diagram represented by (6.12).

Now, we calculate the osculating plane equation for the fixed point x_1 = -3.3166247903554, y_1= -3.3166247903554, z_1 = 1 . We find the first osculating plane expression as the following

$$P_1(\vec{X}) = 6.4\begin{pmatrix} 23.8797x - 0.6x^2 - 47.2619y + 5.475xy + 3.15079x^2y + 0.35x^3y - 11.3y^2 - \\ 4.56036xy^2 + 0.325x^2y^2 - 0.0829156x^3y^2 - 0.025x^4y^2 + 4.14578y^3 - \\ 0.85xy^3 + 1.y^4 + 20.3972xz + 0.175x^2z - 0.829156x^3z - 6.01138yz + \\ 6.1375xyz + 0.124373x^2yz + 0.0375x^3yz - 3.85y^2z + \\ 1.15069\times10^{-16}xy^2z - 0.25x^2y^2z - 1.8656xz^2 + 1.1875x^2z^2 - \\ 0.207289x^3z^2 - 0.0625x^4z^2 + 0.829156yz^2 - 2.125xyz^2 + 2.5y^2z^2 - 0.625x^2z^3 \end{pmatrix}$$

Now the graphical representation of the osculating plane equation $P_1(\vec{X}) = 0$ can be shown as follows:

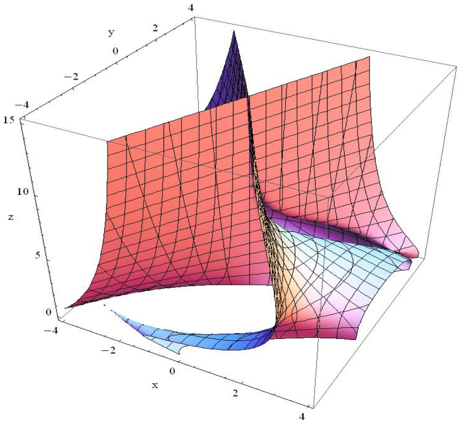


Figure 6.15. Graphical representation of the osculating plane equation corresponding to the fixed point (x_1, y_1, z_1).

Then we calculate the osculating plane equation for the fixed point $x_2 = 0, y_2 = 0, z_2 = 12$. We find the second osculating plane expression as the following

$$P_2(\vec{X}) = 6.4\begin{pmatrix} 32.4x^2 - 48.15xy - 2.4x^3y + 16.2y^2 + 3.075x^2y^2 - 0.025x^4y^2 - 0.85xy^3 + 1.y^4 - \\ 9.45x^2z + 29.5125xyz + 0.0375x^3yz - 31.35y^2z - 0.25x^2y^2z + 8.0625x^2z^2 - \\ 0.0625x^4z^2 - 2.125xyz^2 + 2.5y^2z^2 - 0.625x^2z^3 \end{pmatrix}$$

Now the graphical representation of the osculating plane equation $P_2(\vec{X}) = 0$ can be shown as follows:

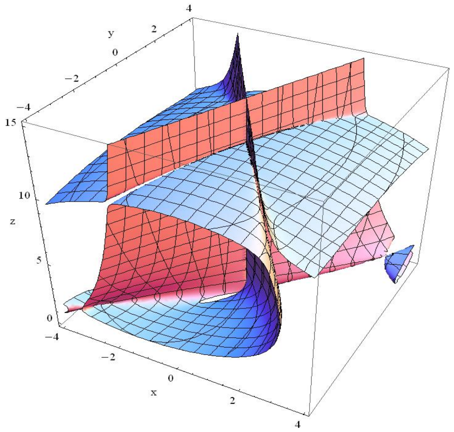


Figure 6.16. Graphical representation of the osculating plane equation corresponding to the fixed point (x_2, y_2, z_2).

Similarly, we calculate the osculating plane equation for the fixed point $x_3 = 3.31662479103554,\ y_3 = 3.3166247903554,\ z_3 = 1$. We find the third osculating plane expression as the following

$$P_3(\vec{X}) = 6.4\left(\begin{array}{l} -23.8797x - 0.6x^2 + 47.2619y + 5.475xy - 3.15079x^2y + 0.35x^3y - 11.3y^2 + 4.56036xy^2 + \\ 0.325x^2y^2 + 0.0829156x^3y^2 - 0.025x^4y^2 - 4.14578y^3 - 0.85xy^3 + 1.y^4 - 20.3972xz + \\ 0.175x^2z + 0.829156x^3z + 6.01138yz + 6.1375xyz - 0.124373x^2yz + 0.0375x^3yz - 3.85y^2z - \\ 1.15069\times10^{-16}xy^2z - 0.25x^2y^2z + 1.8656xz^2 + 1.1875x^2z^2 + 0.207289x^3z^2 - 0.0625x^4z^2 - \\ 0.829156yz^2 - 2.125xyz^2 + 2.5y^2z^2 - 0.625x^2z^3 \end{array}\right)$$

Now the graphical representation of the osculating plane equation $P_3(\vec{X}) = 0$ can be shown as follows:

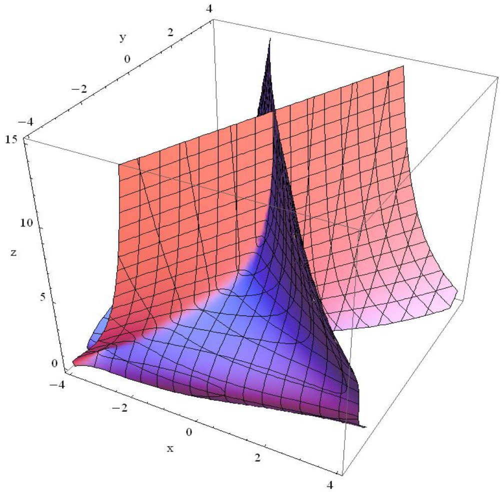


Figure 6.17. Graphical representation of the osculating plane equation corresponding to the fixed point (x_3, y_3, z_3).

Observation shows that, Fig.6.13 (a) merges with Fig.6.14 (a), Fig.6.15, Fig.6.16 and Fig.6.17 in the vicinity of the three fixed points. That means theorem 6.4 holds.

To perform the stability analysis of the fixed-points using flow curvature manifold of the L-H model, we need to calculate the Hessian of flow curvature manifold.

The following expression represents the first row vector of the Hessian matrix of the flow curvature manifold at the point $\vec{X} = (x, y, z)^t$.

$$\begin{gathered}
\{-4202.5+98.304y^4+x^3y(-122.88-7.68z)+1631.23z-413.952z^2+179.2z^3-12.8z^4+ \\
x^4(7.68y^2+19.2z^2)+y^2(-1139.71-367.616z+217.6z^2)+x^2(442.368+y^2(55.296-61.44z)- \\
866.304z+138.24z^2-153.6z^3)+xy(1870.85+1032.19z-399.36z^2+115.2z^3+y^2(-168.96+46.08z)), \\
4787.71+3.072x^5y-153.6y^4+x^3y(36.864-40.96z)+x^4(-30.72-1.92z)-1142.4z+292.352z^2- \\
19.2z^3+y^2(586.752-18.432z-230.4z^2)+xy(-2279.42+393.216y^2-735.232z+435.2z^2)+ \\
x^2(935.424+516.096z-199.68z^2+57.6z^3+y^2(-253.44+69.12z)),-1.92x^4y+7.68x^5z+ \\
x^3(-288.768-20.48y^2+92.16z-153.6z^2)+y(-1142.4+y^2(-6.144-153.6z)+584.704z-57.6z^2)+ \\
x^2y(516.096+23.04y^2-399.36z+172.8z^2)+x(1631.23-827.904z+537.6z^2-51.2z^3+y^2(-367.616+435.2z))\}
\end{gathered}$$

The following expression represents the second row vector of the Hessian matrix of the flow curvature manifold at the point $\vec{X}$.

$$\begin{gathered}
\{4787.71+3.072x^5y-153.6y^4+x^3y(36.864-40.96z)+x^4(-30.72-1.92z)-1142.4z+ \\
292.352z^2-19.2z^3+y^2(586.752-18.432z-230.4z^2)+xy(-2279.42+393.216y^2- \\
735.232z+435.2z^2)+x^2(935.424+516.096z-199.68z^2+57.6z^3+y^2(-253.44+69.12z)), \\
-7114.75+0.512x^6+x^4(9.216-10.24z)+2104.32z-740.352z^2+51.2z^3+x^3y(-168.96+46.08z)+ \\
y^2(2211.84+491.52z)+xy(1173.5-614.4y^2-36.864z-460.8z^2)+x^2(-1139.71+589.824y^2- \\
367.616z+217.6z^2),-0.384x^5-10.24x^4y+x^2y(-367.616+435.2z)+x(-1142.4+y^2(-18.432-460.8z)+ \\
584.704z-57.6z^2)+x^3(172.032+23.04y^2-133.12z+57.6z^2)+y(2104.32+163.84y^2-1480.7z+153.6z^2)\}
\end{gathered}$$

The following expression represents the third row vector of the Hessian matrix of the flow curvature manifold at the point $\vec{X}$.

$$\begin{gathered}
\{-1.92x^4y+7.68x^5z+x^3(-288.768-20.48y^2+92.16z-153.6z^2)+y(-1142.4+y^2(-6.144-153.6z)+ \\
584.704z-57.6z^2)+x^2y(516.096+23.04y^2-399.36z+172.8z^2)+x(1631.23-827.904z+537.6z^2- \\
51.2z^3+y^2(-367.616+435.2z)),-0.384x^5-10.24x^4y+x^2y(-367.616+435.2z)+x(-1142.4+ \\
y^2(-18.432-460.8z)+584.704z-57.6z^2)+x^3(172.032+23.04y^2-133.12z+57.6z^2)+ \\
y(2104.32+163.84y^2-1480.7z+153.6z^2),1.28x^6+xy(584.704-153.6y^2-115.2z)+ \\
x^4(23.04-76.8z)+x^3y(-133.12+115.2z)+y^2(-740.352+153.6z)+ \\
x^2(-413.952+217.6y^2+537.6z-76.8z^2)\}
\end{gathered}$$

By combining these three row vectors we find the complete Hessian matrix. The determinant of this Hessian matrix is denoted by $H_{\psi(\vec{X})}$. The relative Hessian of the flow curvature manifold $\psi(\vec{X})$ can be defined as follows:

$$\hat{H}_{\psi(\vec{X})} = \frac{H_{\psi(\vec{X})}}{\psi(\vec{X})}$$

We now calculate the relative Hessian at the point $(x_1 + \varepsilon, y_1 + \varepsilon, z_1 + \varepsilon)$ and get the following expression

$$(3.64235\times10^{-33} + \varepsilon(-3.71533\times10^{-18} + \varepsilon(-0.000552965 + \varepsilon(-1.53476\times10^{11} + \varepsilon(5.30899\times10^{11} + \\ \varepsilon(-8.87961\times10^{11} + \varepsilon(9.18391\times10^{11} + \varepsilon(-5.94088\times10^{11} + \varepsilon(2.21072\times10^{11} + \varepsilon(-2.68341\times10^{10} + \\ \varepsilon(-1.64071\times10^{10} + \varepsilon(1.03109\times10^{10} + \varepsilon(-2.97524\times10^{9} + \varepsilon(5.26477\times10^{8} + \varepsilon(-5.95009\times10^{7} + \\ \varepsilon(4.16963\times10^{6} + \varepsilon(-167288. + (3408.12 - 27.5251\varepsilon)\varepsilon))))))))))))))))) / (8.31538\times10^{-}12 + \\ \varepsilon(7.79567\times10^{-}12 + \varepsilon(-2.18279\times10^{-11} + \varepsilon(1844.25 + \varepsilon(-1388.47 + \varepsilon(288.525 + \varepsilon(13.4831 + \\ \varepsilon(-10.795 + 1.\varepsilon))))))))$$

By considering $\varepsilon \to 0$, we get the positive value of the above expression and which is 4.38026×10^{-22}. Now according to theorem 6.5, the fixed point (x_1, y_1, z_1) is a saddle-node. Then we calculate the relative Hessian at the point $(x_2 + \varepsilon, y_2 + \varepsilon, z_2 + \varepsilon)$ and get the following expression

$$(0. + \varepsilon^2(0.00224639 + \varepsilon(-2.59169\times10^{12} + \varepsilon(-3.29468\times10^{12} + \varepsilon(-1.62118\times10^{12} + \varepsilon(9.43401\times10^{12} + \\ \varepsilon(4.36915\times10^{12} + \varepsilon(4.53163\times10^{12} + \varepsilon(2.3954\times10^{12} + \varepsilon(6.03654\times10^{11} + \varepsilon(7.94268\times10^{10} + \\ \varepsilon(4.72054\times10^{9} + \varepsilon(-7.13001\times10^{7} + \varepsilon(-2.74377\times10^{7} + \varepsilon(-1.22338\times10^{6} + \varepsilon(12093.5 + \\ (1462.76 - 27.5251\varepsilon)\varepsilon)))))))))))))))) / (\varepsilon^2(-1.9749\times10^{-11} + \varepsilon(-9904.71 + \varepsilon(5526.71 + \\ \varepsilon(2178. + \varepsilon(343.714 + \varepsilon(26.7143 + 1.\varepsilon)))))))$$

By considering $\varepsilon \to 0$, we get the negative value of the above expression and which is -1.13747×10^{8}. Now according to theorem 6.5, the fixed point (x_2, y_2, z_2) is a saddle-focus or center. Similarly, we calculate the relative Hessian at the point $(x_3 + \varepsilon, y_3 + \varepsilon, z_3 + \varepsilon)$ and get the following expression

$$
\begin{aligned}
&(3.15053\times10^{-33}+\varepsilon(3.55972\times10^{-18}+\varepsilon(0.000208321+\varepsilon(-4.44938\times10^{11}+\varepsilon(-1.79251\times10^{12}+\\
&\varepsilon(-2.97502\times10^{12}+\varepsilon(-2.69281\times10^{12}+\varepsilon(-1.51324\times10^{12}+\varepsilon(-5.66076\times10^{11}+\varepsilon(-1.47005\times10^{11}+\\
&\varepsilon(-2.74232\times10^{10}+\varepsilon(-3.83535\times10^{9}+\varepsilon(-4.22314\times10^{8}+\varepsilon(-3.57007\times10^{7}+\varepsilon(-1.58142*10^{6}+\\
&\varepsilon(86197.8+\varepsilon(15605.6+(382.487-27.5251\varepsilon)\varepsilon)))))))))))))))))/(8.31538*10^{-}12+\varepsilon(2.91038*10^{-11}+\\
&\varepsilon(2.39067*10^{-}11+\varepsilon(5346.61+\varepsilon(5115.9+\varepsilon(1914.62+\varepsilon(356.517+\varepsilon(32.795+1.\varepsilon))))))))
\end{aligned}
$$

By considering $\varepsilon \to 0$, we get the positive value of the above expression and which is 3.7888×10^{-22}. Now according to theorem 6.5, the fixed point (x_3, y_3, z_3) is a saddle-node.

6.3.3.2.2 Analysis Using the Geometric Singular Perturbation Theory

In this part, we derive the slow manifold equation of the L-H system by using geometric singular perturbation theory.

Taking $\varepsilon = \frac{1}{\mu}$ then we can treat the L-H system (6.12) as slow-fast autonomous system. Therefore we can analyze it and can obtain the slow manifold equation by using geometric singular perturbation method. Now, the equation (6.12) can be rewritten as follows:

$$
\begin{aligned}
\varepsilon\frac{dx}{dt} &= f(x,y,z) = (y-x),\\
\frac{dy}{dt} &= g(x,y,z) = zx-y,\\
\frac{dz}{dt} &= h(x,y,z) = \delta(B-z-xy).
\end{aligned}
\tag{6.18}
$$

where $\boldsymbol{x}$ is the fast variable, $\boldsymbol{y}$ and $\boldsymbol{z}$ are slow variables. We use the parameter values of (6.18) as mentioned in Table 6.2.

L-H model (6.18) which is checking Fenichel's assumptions (A1)−(A3), the singular approximation M_0 is contained in $\{(x,y,z): f(x,y,z) = 0\}$ such that M_0 is a compact manifold with boundary given by the graph of the C^1function: $x = X_0(y,z) = y$.

So, the problem is to find a function $x = X(y, z, \varepsilon)$ whose graph is locally slow invariant manifold M_ε of the L-H system. Let's pose:

$$X(y, z, \varepsilon) = X_0(y, z) + \varepsilon\, X_1(y, z) + \varepsilon^2\, X_2(y, z) + O(\varepsilon^3) \qquad (6.19)$$

As previously stated in section 6.3.2.3,

At order ε^0:

$$\frac{\partial X_0}{\partial y} f(X_0(y, z), y, z) = 0 \Leftrightarrow x = X_0(y, z) = y$$

which defines the singular approximation $X_0(y, z) = y$ due to the invertibility of $\frac{\partial f}{\partial x}$ and the implicit function theorem.

At order ε^1:

$$X_1(y, z)\frac{\partial X_0}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial X_1}{\partial y} f(X_0(y, z), y, z) + \frac{\partial X_0}{\partial z} h(X_0(y, z), y, z) = g(X_0(y, z), y, z)$$

Since according to implicit function theorem $f(X_0(y, z), y, z) = 0$ we have:

$$X_1(y, z) = \frac{g(X_0(y, z), y, z) - \frac{\partial X_0}{\partial z} h(X_0(y, z), y, z)}{\frac{\partial X_0}{\partial y}\frac{\partial f}{\partial x}}$$

$$\Leftrightarrow X_1(y, z) = y - yz$$

At order ε^2:

$$X_2(y, z)\frac{\partial X_0}{\partial y}\frac{\partial f}{\partial x} + \frac{1}{2} X_1{}^2(y, z)\frac{\partial X_0}{\partial y}\frac{\partial^2 f}{\partial x^2} + X_1(y, z)\frac{\partial X_1}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial X_2}{\partial y} f(X_0(y, z), y, z) +$$
$$X_1(y, z)\frac{\partial X_0}{\partial z}\frac{\partial h}{\partial x} + \frac{\partial X_1}{\partial z} h(X_0(y, z), y, z) = X_1(y, z)\frac{\partial g}{\partial x}$$

Since according to implicit function theorem $f(X_0(y, z), y, z) = 0$ we have:

$$X_2(y,z) = \frac{X_1(y,z)\frac{\partial g}{\partial x} - \frac{1}{2}X_1{}^2(y,z)\frac{\partial X_0 \partial^2 f}{\partial y\ \partial x^2} - X_1(y,z)\frac{\partial X_1 \partial f}{\partial y\ \partial x} - X_1(y,z)\frac{\partial X_0 \partial h}{\partial z\ \partial x} - \frac{\partial X_1}{\partial z} h(X_0(y,z),y,z)}{\frac{\partial X_0 \partial f}{\partial y\ \partial x}}$$

$$\Leftrightarrow X_2(y,z) = 0.4\,\mathrm{y}^3 + \mathrm{y}\,(-5.8 + 1.4\,\mathrm{z})$$

and so on.

Now, from the equation (6.19) we can write the slow manifold equation associated with the L-H system as

$$X(y,z,\varepsilon) = y + \varepsilon\,(y - yz) + \varepsilon^2\,\{0.4\,\mathrm{y}^3 + \mathrm{y}\,(-5.8 + 1.4\,\mathrm{z})\,\} \qquad (6.20)$$

Equation (6.20) represents the second order approximation in ε of the slow manifold associated with the L-H model.

Now the graphical representation of the slow manifold equation (6.20) associated with the L-H model can be shown as follows:

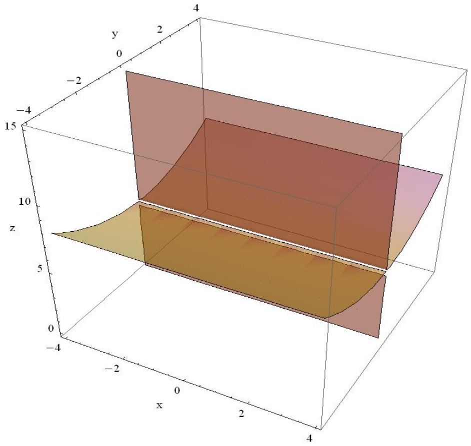


Figure 6.18. Graphical representation of the slow manifold determimned by the geometric singular perturbation theory

6.3.4 Comparison and Discussion

In this section, we compare the two geometric methods applied to the two slow-fast dynamical systems and highlight the significant results discussed in this chapter.

(1) Flow Curvature Method can be applied to any n-dimensional slow-fast dynamical systems not only singularly perturbed systems but also non-singularly perturbed systems. On the other hand, Geometric Singular Perturbation Theory can be applied to only n-dimensional singularly perturbed dynamical systems.

(2) Flow Curvature Method uses the local metrices properties of curvatures inherent to differential geometry and does not require the use of asymptotic expansions. On the other side, Geometric Singular Perturbation Theory uses regular asymptotic expansions.

(3) Using the Flow Curvature Method, we can form flow curvature manifold where the curvature of the flow directly provides the slow invariant manifold analytical equation determination of any high-dimensional slow-fast dynamical systems. On the contrary, using the Geometric Singular Perturbation Theory, the determination of the slow invariant manifold analytical equation turned into a regular perturbation problem and for dimension greater than three, slow manifold determination with the Geometric Singular Perturbation Theory leads to tedious calculations.

(4) In Flow Curvature Method, Darboux invariance theorem is used to show the invariance of the flow curvature manifold whereas that in the Geometric Singular Perturbation Theory, Fenichel's invariance theorem is used to show the invariance of the slow manifold.

(5) In this chapter, we use Model 1 named Brusselator model where we consider the temporal Brusselator model as a two dimensional slow-fast dynamical system. By using Flow curvature method, we determine the flow curvature manifold

which directly provides the slow invariant manifold where the Darboux invariance theorem is then used to show the invariance of the slow manifold. After that, we analyze the stability of the fixed point of the temporal Brusselator model using the flow curvature manifold. Besides, since the Brusselator model has no singular approximation and it can be considered as a slow fast dynamical system but not as a singularly perturbed system. Hence, Geometric Singular Perturbation Theory fails to provide the slow invariant manifold associated with temporal Brusselator model.

(6) In this chapter, we use another model Model 2 named Lorenz-Haken model as a three dimensional slow-fast dynamical system. By using Flow curvature method, we determine the flow curvature manifold which directly provides the third order approximation of the slow manifold where the Darboux invariance theorem is then used to show the invariance of the slow manifold. Then, we analyze the stability of the fixed point of the L-H model using the flow curvature manifold. Furthermore, since L-H model has singular approximation and it can be considered as a singularly perturbed system. Hence, by using Geometric Singular Perturbation Theory we determine the order by order approximation in the small multiplicative parameter of the slow manifold where the Fenichel's invariance theorem is then used to show the invariance of the slow manifold and the calculations of the higher order approximations are very tedious.

According to the above discussions comparing two geometric methods, we can conclude that the Flow Curvature Method is the best to find the analytical equation of the slow invariant manifold for any dimensional slow-fast dynamical system.

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

In this chapter, we summarize the significant results discussed in this thesis and we suggest some ideas for future work.

7.1 Conclusions

In the first part of this thesis, we discussed various branches of differential geometry. We developed some computer codes to compute several important components of Riemannian geometry. We developed a special comparison between symplectic and contact geometry with complex manifolds. We then reviewed Kodaira, Legendre and isotropic moduli spaces and established interconnection among Legendre, isotropic and Kodaira moduli spaces.

In the second part of this thesis, we applied an old strategy called the Geometric Singular Perturbation Theory and another newly developed strategy that reflects the applications of differential geometry in the slow-fast dynamical system called the flow curvature method to the two models named as temporal Brusselator model and Lorenz-Haken model. According to the Flow Curvature Method, we determined the curvature of the trajectory curve analytically called flow curvature manifold by estimating the solution or trajectory curve of the dynamical system as a curve in Euclidean space. Since this manifold comprises the time derivatives of the velocity vector field and hence it receives knowledge about the dynamics of the corresponding system. In Model 1 named Brusselator model where we considerd the temporal Brusselator model as a two dimensional slow-fast dynamical system. According to the Flow Curvature Method, we determined the flow curvature manifold which directly provides the slow invariant manifold where the Darboux invariance theorem is then used to show the invariance of the slow manifold. On the other hand, since the temporal Brusselator model has no singular approximation and hence, Geometric Singular Perturbation Theory fails to provide the slow invariant manifold associated with temporal Brusselator model. Finally, we described the effect of growth and curvature with surface deformation on pattern formation of the spatiotemporal Brusselator model. In Model 2 named Lorenz-Haken model, we considered as a three dimensional slow-fast dynamical system. By using Flow curvature method, we determined the flow curvature manifold which directly provides the third order approximation of the slow manifold where the Darboux invariance theorem is then used to show the invariance of the slow manifold. Then, we analyzed the stability of the fixed point of the L-H model using the flow curvature manifold. On the other hand, since L-H model has singular approximation and it can be considered as a singularly perturbed system. Hence, by using Geometric Singular Perturbation Theory we determined the order by order approximation in the small multiplicative parameter of the slow manifold where the Fenichel's invariance theorem is used to show the invariance of the slow manifold.

7.2 Future Extensions

We suggest the following ideas which are the extensions to our future work. The following ideas may be tested:

(a) We may further analyze any n-dimensional dynamical model through differential geometry.

(b) We may investigate the stationary periodic wave solutions of the dynamical system through differential geometry comparing with periodic traveling wave solutions of the dynamical system.

(c) We may analyze the whole reaction-diffusion model in terms of differential geometry analytically and numerically.

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