Studies on Eigenvalue Analysis for a Class of Differential Equations by the Methods of Weighted Residuals

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in partial fulfillment of the requirement for the award of the degree of

Doctor of Philosophy in Applied Mathematics

by

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This work is dedicated to the sacred memory of my late beloved mother

CANDIDATE'S DECLARATION

I hereby declare that this work which is being presented in this thesis entitled "Studies on Eigenvalue Analysis for a Class of Differential Equations by the Methods of Weighted Residuals" submitted in partial fulfillment of the requirement for the award of the degree of Ph.D in Applied Mathematics under the faculty of science of the University of Dhaka, Dhaka-1000, Bangladesh is an authentic record of my own work. It has not been submitted elsewhere (Universities or Institutions) for the award of any other degree.

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Date: September 30, 2018

CERTIFICATE

This is to certify that the thesis entitled "**Studies on Eigenvalue Analysis for a Class of Differential Equations by the Methods of Weighted Residuals**" submitted by Humaira Farzana, for the award of the degree of Doctor of Philosophy in Applied Mathematics, to the University of Dhaka, Bangladesh, is a piece of bona fide research work carried out by her under our supervision and guidance. To the best of our knowledge, this thesis has not been previously submitted to any other University or institution for the award of any degree/ diploma or any other similar title. We further certify that the work is worth submitting for the award of the said degree.

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Contents

Abstract

The problem of finding eigenvalues and eigenfunctions and studying their behaviour plays a crucial role in modern mathematics and engineering. The importance of the eigenvalue problem in applied mathematics (as well as in engineering and other areas) is that it arises on the way of solution of systems of linear ordinary differential equations with constant coefficients.

The investigations are of utmost importance for theoretical and applied mechanics, physics, physical chemistry, biophysics, mathematical economics, theory of systems and their optimization, theory of random process and many other branches of natural science. There are many applications of matrices in both engineering and science utilizing eigenvalues and sometimes, eigenvectors. Control theory, vibration analysis, electric circuits, advanced dynamics and quantum mechanics are just a few of the application areas.

Many researchers studied a large number of second and fourth order Sturm Liouville eigenvalue problems utilizing diversified numerical techniques. However, only few numerical methods are accessible in literature for higher order Sturm Liouville eigenvalue problems using some special techniques. In our thesis we formulate and compute the eigenvalues of general linear second order Sturm-Liouville problems. We also compute eigenvalues of higher even order linear problems (from fourth up to twelfth order) in one dimension. We utilize the technique of Galerkin weighted residual exploiting polynomial basis functions. In addition, we also calculate second, fourth and sixth order eigenvalue problems applying Weighted Residual Collocation and Spectral Collocation methods.

(i) The major steps in this thesis, dependent on the method of Galerkin weighted residual are:

Ascertaining a new method namely weighted residual method with various types of boundary conditions and minimizing the condition number as well as the cost of computations. Legendre and Bernstein piecewise polynomials over [0, 1] as

trial functions are used to compute the eigenvalues.

Rigorous matrix formulations for computing the eigenvalues of Sturm-Liouville eigenvalues problems are derived. Special care is taken when determining how the polynomials satisfy the corresponding homogeneous form of Dirichlet boundary conditions. For computation, the differential eigenvalue problems are reduced to algebraic system eigenvalues which gives fairly accurate eigenvalues of the problems.

(ii) The major steps of Weighted Residual Collocation method are the followings:

To approximate the solution of eigenvalue problems as a weighted sum of polynomials. To select at least the same number of collocation points as the unknown parameters and determine the co-efficient matrices /column vector at these collocation points. The residual is required to vanish point wisely at a set of pre-assigned points.

(ii) The major steps of Pseudo Spectral collocation method are:

To provide support functions for Spectral Collocation differentiation matrices corresponding to Chebyshev, Legendre interpolants.

The thesis entitled **"Studies on Eigenvalue Analysis for a Class of Differential Equations by the Methods of Weighted Residuals"** contains five chapters. Among them the first chapter is confined as **"Introduction"**. In this chapter, we discuss some mathematical preliminaries which are essential to study the problems examined in this thesis, such as a few definitions, theorems, prepositions, corollaries which are used in the subsequent chapters. We have also illustrated Bernstein and Legendre polynomials showing their properties and convergence criteria.

To accomplish the objectives specified in the previous section, the plan of the thesis comprises the following five chapters:

- Chapter 1 includes some important definitions which are related to our thesis and will be used in the subsequent chapters. This chapter also emphasizes in detail the properties of Bernstein and Legendre polynomials, some theorems on convergence of Bernstein polynomials. Objectives and scope of the thesis are also given in this chapter.

- Chapter 2 is dealt with the numerical computation of the second order SLEs where Bernstein and Legendre polynomials are utilized with three types of methods namely Galerkin WRM, Collocation and Spectral Collocation. Bernstein and Legendre polynomials are employed as basis functions for Galerkin WRM. For collocation and Spectral collocation method, Bernstein and Legendre-Lagrange polynomials are used respectively with grid points. We derive matrix Formulation I, Formulation II and Formulation III by applying the Galerkin method with three different types of boundary conditions for solving these problems. The basic polynomial approximation theorems and completeness of Sturm-Liouville are stated in this chapter. A wide range of examples is discussed in chapter 2 and our idea is extended for solving nonlinear differential equations with the case of Spectral Collocation scheme. The method is based on a direct discretization of the BVP as a nonlinear equation in the eigenvalue together with an iterative procedure.

- Chapter 3 is devoted to find the numerical computations of eigenvalues for fourth order BVPs. In this chapter the matrix formulations for solving linear SLEs are illustrated for two key boundary conditions out of various kinds of boundary conditions which are used in physical problems.

- The numerical calculations of sixth order BVPs by means of Formulation I and Formulation II for two different types of boundary conditions by the Galerkin method have been provided in Chapter 4. Stability and convergence criteria are demonstrated for Galerkin WRM in this Chapter. The Bernstein Collocation and Spectral Collocation method are formulated as a rigorous matrix form together with Clamped and Hinged type boundary conditions. This chapter is devoted to report a few numerical experiments separately which demonstrates the accuracy of the proposed numerical schemes for each class of method.

- In Chapter 5 we illustrate numerical computations of eigenvalue/ Rayleigh number eighth, tenth and twelfth order BVPs by the Galerkin. Precise matrix Formulation I and Formulation II for two different types of boundary conditions are developed. Numerical results of some Hydrodynamic BVPs are tabulated to compare the error with those developed so far. The obtained results prove that the offered Galerkin WRM is of high precision, competent and expedient. Conclusions are given for each chapter separately and references are given towards the last of the thesis.

List of chapter wise publications

1. Humaira Farzana, Md. Shafiqul Islam **(2015)** – "Computation of some second order Sturm-Liouville BVP using Chebyshev Legendre Collocation method", ***GANIT*** (Journal of Bangladesh Mathematical Society), **35**, 97 – 114.(Chapter 2)

2. Humaira Farzana, Md. Shafiqul Islam, Samir Kumar Bhowmik **(2015)**– "Computation of Eigenvalues of the Fourth Order Sturm-Liouville BVP by Galerkin Weighted Residual Method", ***British Journal of Mathematics and Computer Science***, **9** (1), 73 – 85.(Chapter 3)

3. Humaira Farzana, Md. Shafiqul Islam **(2015),** "Application of Galerkin Weighted Residual Method to 2nd, 3rd and 4th order Sturm-Liouville Problems", Journal of ***Mathematical Theory and Modeling,*** ISSN 2224-5804, **5, (2),**195-206. (Chapter 2-3)

4. Md. Shafiqul Islam, Humaira Farzana, Samir Kumar Bhowmik, **"Numerical Solutions of Sixth Order Eigenvalue Problems Using Galerkin Weighted Residual Method",** Differ Equ Dyn Syst, ***Springer,*** **25(2):** 187–205. DOI-10.1007/s12591-016-0323-9 (April 2017). (Chapter 4)

CHAPTER 1

Introduction

The concept of eigenvalue problems is rather important both in pure and applied mathematics in physical systems such as pendulums, vibrating and rotating shafts which are connected with eigenpairs of the system. For example, they describe the vibration modes of various systems, such as the vibrations of a string, the critical loads which a column supports before deformation or the energy eigenfunctions of a quantum mechanical oscillator, in which case the eigenvalues correspond to the resonant frequencies of vibration or energy levels. There have been great achievements in the studies of finite dimensional vibration systems and theory of wave processes in the past (Lyapunov, Poincare, Mandelstam, Timoshenko, Maxwell, Lord Kelvin, Sommerfield, Rayleigh, Helmholtz, Lord, Morse, etc.)

The discrete energy levels observed in atomic systems could be obtained as the eigenvalues of a differential operator which led Schrodinger to propose wave equation. In electrical engineering, eigenvalues /eigenvector analysis has a large role in the simulation of power systems where they determine frequency response of an amplifier or a reliability of a national power system. In aeronautical engineering, eigenvalue can be used to determine whether a flow over a wing is laminar or turbulent. In nuclear physics, random eigenvalues are used to model nuclear energy levels. In fluid mechanics, the linear stability of a plane Poiseuille flow and plane coquette flow depends upon the eigenvalues known as Reynolds numbers and Rayleigh numbers which confirm the presence of stability or instability. In quantum mechanics, quantities like energy, momentum, position etc. are represented by Hermitian operators on Hilbert space which is diagonalizable and eigenvalues are always real. Quantum mechanics are concerned with eigenvalues and eigenfunctions to differential operators/ equations.

For those class of eigenvalue problems, a well-developed theory and various codes for the computation of the numerical solution exist. However, our aim is not only to focus on Sturm-Liouville problems but to consider numerical methods, which are capable of solving a wide range of differential eigenvalue problems including higher order examples.

The resulting theory of the existence and asymptotic behavior of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space became known as Sturm–Liouville theory. This theory is important in applied mathematics, where Sturm-Liouville problems occur very commonly, particularly when dealing with linear partial differential equations that are separable.

If the interval is unbounded, or if the coefficients have singularities at the boundary points, this case, the spectrum no longer consists of eigenvalues alone and can contain a continuous component. There is still an associated eigen function expansion (similar to Fourier series versus Fourier transform). This is important in quantum mechanics, since the one-dimensional time-independent Schrödinger equation is a special case of a Sturm–Liouville equation with linear partial differential equations that are separable. Firstly, these equations can be solved as a Sturm–Liouville problem. Since there is no general analytical (exact) solution to Sturm–Liouville problems, we can assume we already have the solution to this problem, that is, we have the eigenfunctions and eigenvalues. Secondly, these equations can be analytically solved once the eigenvalues are known. In the recent years' numerical solution for Sturm-Liouville problems have been studied by many researchers exploiting various techniques and different algorithms have been applied to minimize the convergence rates.

Two standard approaches to the numerical approximation of eigenvalues of a boundary value problem can be distinguished as discretization and shooting. Discretization methods (such as finite differences and finite elements) involve substantial arithmetic and the storage of large matrices. Moreover, the accuracy quickly deteriorates for the higher eigenvalues. Shooting methods require less storage and arithmetic, but usually they do not determine the index of the eigenvalue. For Sturm-Liouville problems, these difficulties are avoided by the Prufer method, which is a shooting method based on oscillation. This Prufer-based shooting method has been implemented by Bailey *et al* (1978) in the SLEIGN code and its successor SLEIGN2 [Bailey *et al* (2001)] and by the NAG library code D02KDF. Popular algorithms known as piecewise constant midpoint methods, or Pruess methods which approximate the coefficient functions $p(x)$, $q(x)$ and $r(x)$ by piecewise-constant approximations, solving the problem analytically on the piecewise-constant intervals. This algorithmic theme was introduced in Canosa and Gomes (1970) and eventually developed into well analyzed packages [Marletta and Pryce (1992), Ledoux *et al* (2009), Ledoux and Daele (2010)]. This results in a set of problems which may each be solved analytically, again producing the most thorough analysis of general piecewise polynomial interpolants is due to Pruess, whose papers [Pruess (1973, 1975)] provide a wealth of convergence results approximations to a number of the lower eigenvalues.

Piecewise constant approximations are rather crude. The difficulties are reduced for piecewise linear approximation illustrated in Baily *et al* (2001) and Ledoux (2006-2007) because the bases can be expressed efficiently in terms of Airy functions.

Numerical methods for Sturm–Liouville eigen-problems that have been implemented in software include finite difference, finite element and spectral element discretization each of which leads to generalized algebraic eigen-problems. Here approximations to a number of the lower eigenvalues are available simultaneously.

If the eigenvalue problem is singular, the code BVPSUITE [Kitzhofer *et al* (2009)] for singular boundary value problems has been successfully applied. Singular problems are defined on an infinite domain and with singular endpoints which require a special numerical treatment. In these cases, an interval truncation procedure must be implemented. Different algorithms are implemented in the available Sturm-Liouville eigenvalue problems library codes to determine a truncated endpoint and appropriate boundary conditions to give a prescribed accuracy are illustrated in Pruess and Fulton (1993), Marletta and Pryce (1991). They also commonly arise from linear PDEs in several space dimensions when the equations are separable in some coordinate system, such as cylindrical or spherical coordinates. Some examples of these equations and their applications are the Bessel, Legendre, and Laguerre equations. Bessel equations arise when solving the Laplace and Helmholtz equations by separation of variables in cylindrical polar coordinates. Legendre equation arises in solving Laplace equations in spherical polar coordinates, and they give expressions for the spherical harmonic functions. While the Laguerre equation arises in solutions of three-dimensional Schrodinger equation with an inverse-square potential and in Gaussian integration. Moreover, the Galerkin WRM [Reedy (1993)] can be easily extended into two-dimensional problems, which are relatively difficult for various discretization methods.

1.1 Objectives and Scopes of the thesis.

The Sturm-Liouville systems arise from vibration problems in continuum mechanics such as the vibrations of a string. For example, if a string stretched tightly between two supports, located at $x=0$ and $x=l$, and subjected to a distributed vertical force of intensity $q(x)$ per unit length. If we assume that the string has linear density ρ (mass/unit length) and is rotating with uniform angular speed ω, such action generates a distributed inertia force (force / length) of magnitude $\rho\omega^2 u$ in a direction transverse to the string which displaces the string away from its initial rest configuration. Hence the problem of finding these deflection modes is mathematically equivalent to determining the static equilibrium position of a tightly stretched string subject to the distributed load $q(x)=\rho\omega^2 u$. Making the replacement in one dimensional Poisson equation, $\frac{d^2u}{dx^2}=-\frac{q(x)}{T}$ and introducing the parameter $\lambda=\frac{\rho\omega^2}{T}$ (T is tension). We can write the said equation in the form of an eigen-equation $\frac{d^2u}{dx^2}+\lambda u=0$, which is referred to as the one-dimensional Helmholtz equation.

The Schrodinger equation is a separable Partial Differential Equation (PDE), and in separating the PDE, Ordinary Differential equations (ODEs) can be generated which are of the form of Sturm-Liouville differential equations or presented by a second order BVP.

The process of solving certain linear evolution equations such as the heat or wave equations we are led in a very natural way to an eigenvalue problem for a second order linear differential operator with two boundary conditions but where no unique solution exists. Since many eigenvalue problems are of second order, for example Sturm Liouville problems, we also implemented a code for second order problems and paid special attention to the approximation of the boundary conditions in the singular case.

Fourth-order differential equations can model the bending of an elastic beam and, in this sense, we refer them as beam equations. They have received increased interest from several fields of science and engineering, either on bounded domains. The deformations of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by the fourth-order boundary value problem. Also, a traditionally important example of a fourth order BVP is the Orr Somerfield equation from the field of hydrodynamic stability.

The Euler Bernoulli theory is based on the assumption that plane cross-sections remain plane and perpendicular to the longitudinal axis after deformation. It is thus a special case of Timoshenko beam theory. In this theory the transverse deflection w of the beam is governed by the fourth order differential equation given by

$$\frac{d^2}{dx^2}\left(b\frac{d^2 w}{dx^2}\right) = f(x) \quad , \quad 0 < x < l$$

Where, $b = EI$ is the product of modulus of elasticity E and the moment of inertia I, f is the transversely distributed load.

Timoshenko Beam theory $M = EI\frac{d\psi}{dx}\ \frac{dw}{dx} - \psi(x) = -\frac{V(x)}{\kappa^2 GA}$

Euler-Bernoulli Theory, $M = EI\frac{d^2 w}{dx^2}\ \psi = \frac{dw}{dx}$

The vital piece of information required is the smallest eigenvalue which gives potentially the most visual structure of dynamical system is called critical buckling load. Fourth-order eigenvalue problems appear routinely in the linear stability analysis of 2-D incompressible flows. In many cases it is not possible to find analytically solutions to the eigenvalue problem in fluid mechanics, and therefore a suitable numerical method is required. For two dimensional flows the problem can often be reduced to a single fourth order equation for the amplitude $\psi(w)$ of the stream function $\psi(x,y,t)=\psi(w).exp[i\alpha(x-\omega t)]$.

The energy eigenfunctions of a quantum mechanical oscillator, in which case the eigenvalues correspond to the resonant frequencies of vibration or energy levels. The Schrodinger equation is a separable Partial Differential Equation (PDE), and in separating the PDE, Ordinary Differential equations (ODEs) can be generated which are of the form of Sturm-Liouville differential equations or presented by a second order BVP.

Mathematical model of astrophysics and free vibration analysis of ring structures give rise to many sixth-order boundary value problems. The thin convecting layers that are bounded by stable layers which are believed to surround A-type stars and dynamo action in some stars may be modeled by such equation. Moreover, when considering the instability setting in an infinite horizontal layer of fluid, which is heated from below and is subject to the action of rotation, we model the instability as ordinary convection and over stability by a sixth-order ordinary differential equation (ODE). In fluid mechanics the linear stability of a plane Poiseuille flow and plane Coquette flow depends upon the eigenvalues known as Reynolds numbers and Rayleigh numbers. Vibration characteristic of circular ring structure with constraints which has rectangular cross-sections of constant width and parabolic variable thickness is expressed by a sixth-order ordinary differential equation.

Electro hydrodynamics (EHD), magneto hydrodynamics (MHD) and Ferro hydrodynamics (FHD) are interested in an eigenvalue problem in EHD which implies the presence of electric forces. Electro hydrodynamic systems have important industrial application in the construction of devices using the electro viscous effect or charge entrainment, for instance EHD clutch development and EHD high voltage generators. The linear stability of their steady states typically leads to high order differential eigenvalue problems. Eighth order BVPs govern the physics of some hydrodynamic stability problems. These problems also arise in the study of astrophysics, hydrodynamics and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, applied mathematics, engineering and applied physics. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability sets in as over stability, it is modeled by an eighth order ordinary differential equation. Eighth order differential equations are also modeled while considering the motion of a cylindrical shell. Equations for the equilibrium in terms of displacement components for an orthotropic thin circular cylindrical shell subjected to a load that is not symmetric about the axis of the shell, which resulted in eighth order differential equations. Eighth order BVPs arise in the torsional vibration of uniform beam. Tenth and twelfth order equations arise when instability setting in as ordinary convection and due to acts of a uniform magnetic fields across the fluid in the same direction as gravity.

1.2 Sturm-Liouville Problems

A classical Sturm-Liouville equation, named after Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1882), is a real second-order linear differential equation of the form:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=\lambda r(x)u \qquad (1.1a)$$

where, $p(x)$, $q(x)$, $r(x)$ are all piecewise continuous functions and $p(x)$, $r(x)>0$ on the finite closed interval $[\gamma, \mu]$. In the regular Sturm-Liouville theory these boundary conditions have the form

$$\alpha_1 u(\gamma)+\beta_1 u'(\gamma)=0 \qquad (1.1b)$$

$$\alpha_2 u(\mu)+\beta_2 u'(\mu)=0 \qquad (1.1c)$$

$\alpha_1, \beta_1, \alpha_2, \beta_2$ are all real constants; α_1, β_1 are not both zero and α_2, β_2 are not both zero. Here, $u=u(x)$ that will also be required to satisfy appropriate boundary conditions. Finding the values of λ for which there exists a nontrivial (nonzero) solution u of (1.1a) satisfying the boundary conditions is part of the problem called the Sturm-Liouville problem. Such values of λ, when they exist, are called the eigenvalues of the boundary value problem defined by (1.1a) and the prescribed set of boundary conditions.

The nature of the boundary conditions depends on the classification of the endpoints as regular or singular.

The general solution of equation (1.1a) depends upon both x and the parameter λ .Thus, if u_1, u_2 constitute linearly independent solutions of (1.1a), we can write the general solution as

$$u = A_1 u_1(x,\lambda) + A_2 u_2(x,\lambda) \tag{1.2}$$

Orthogonal polynomials: An orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. A sequence of polynomials

$\{p_n(x)\}_{n=0}^{n=\infty}$ with degree deg $[p_n(x)] = n$ such that

$$\int_{\gamma}^{\mu} p_m(x) p_n(x) dx = 0 \text{ for } m \neq n \tag{1.3}$$

The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials.

Weight function: A function $\theta(x)$ used to normalize orthogonal functions is called weight function. The weight function $\theta(x)$ should be continuous and positive on (γ, μ) .

$$\int_{\gamma}^{\mu} p_m(x) p_n(x) \theta(x) dx = \delta_{mn} C_n \tag{1.4}$$

where,

$\theta(x)$ weighting function and δ_{mn} is the Kronecker delta

$$C_n = \int_{\gamma}^{\mu} \left[p_n(x) \right]^2 \theta(x) dx \tag{1.4a}$$

$$\delta_{mn} := \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{1.4b}$$

Self-adjoint differential operators: self-adjoint (differential) operators which play an important role in the spectral theory of linear operators and the existence of the eigenfunctions.

The domain of a differential operator L is the set of all $u \in L^2[\gamma, \mu]$.

Residual Function:

To obtain the residual function [Lewis and Ward (1991)], we first collect all the terms in the differential equation on the left-hand side. The exact solution will produce an answer which is identically zero for all values of x in the problem domain when substituted into the left-hand side. But an approximate solution will not produce an identically zero function but a function says, $R(x)$ which is called residual function.

1.3 The Galerkin weighted residual method

Let the approximate solution of equation (1.1a) be

$$\tilde{u}(x) \approx \tilde{u}(x) = \phi_0(x) + \sum_{i=1}^{n} c_i \phi_i(x) \tag{1.5}$$

where,

$\{\phi_k\}$ are trial or (basis) functions and c_i's are parameters.

i) The function $\phi_0(x)$ is chosen to satisfy the given boundary conditions of the problem.

ii) The functions $\phi_i(x)$, $i = 1, 2, 3, \ldots\ldots, n$ must each satisfy the corresponding homogenous form of the boundary conditions. In this method, we determine the n unknown parameters by selecting n weighting functions which are multiplied by unknown parameters.

Galerkin's method then involves determining these parameters by solving the n weighted residual equations given by

$$\left(R_n, \phi_j\right)_\theta := \int_\Omega R_n(x)\phi_j(x)\theta(x)dx = 0, \tag{1.6}$$

1.3.1 Modified Galerkin Method:

If we continue to use the Galerkin technique in conjunction with piecewise linear coordinate functions, then second derivative terms in the differential equation would make no contribution to the approximation leading to poor results. Hence it is desirable to use an alternative weighted residual technique which involves only first derivative terms. The new technique is obtainable using integration by parts from the standard Galerkin approach and is known as the modified Galerkin method [Lewis and Ward (1991)]. Also, in the modified Galerkin technique we shall demand of the trial solution still taken in the form

$$u(x) \approx \tilde{u}_n(x) = \phi_0(x) + \sum_{i=1}^{n-1} c_i \phi_i(x) \tag{1.7}$$

where $\phi_0(x)$ satisfies any essential boundary condition present and $\phi_i(x),\ i = 1, 2, 3, \ldots\ldots\ldots, n$

should satisfy the corresponding homogeneous form of any such essential boundary condition.

It is to note that boundary conditions are of two basic types, referred to as essential and suppressible. For second-order differential equations a boundary condition containing a derivative term is called suppressible; otherwise it is referred to as essential.

For example, we consider a second-order Sturm-Liouville eigenvalue problem:

$$-\frac{d}{dx}\left(\tilde{p}(x)\frac{du}{dx}\right) + \left[\lambda \tilde{r}(x) - \tilde{q}(x)\right]u = 0, \quad x \in (\gamma, \mu) \tag{1.8a}$$

$$\alpha_1 u(\gamma) + \beta_1 u'(\gamma) = 0 \tag{1.8b}$$

$$\alpha_2 u(\mu) + \beta_2 u'(\mu) = 0 \tag{1.8c}$$

$\alpha_1, \beta_1, \alpha_2, \beta_2$ are all real constants; Here if β_1, β_2 are both non-zero then both boundary conditions are suppressible. If $\beta_1 = 0$ and $\beta_2 \neq 0$ then the first condition is essential, the second is suppressible and so on.

Now using equation (1.5) into equation (1.6) we obtain weighted residual equations of the form

$$\int_{\gamma}^{\mu}\left[-\frac{d}{dx}\left(p(x)\frac{d\tilde{u}}{dx}\right)+q(x)\tilde{u}-\lambda r(x)\tilde{u}\right]\phi_j(x)\,dx=0 \tag{1.9}$$

$$\int_{\gamma}^{\mu}\left\{p(x)\frac{d\phi_0}{dx}\frac{d\phi_j}{dx}+q(x)\phi_0\phi_j-\lambda r(x)\phi_0\phi_j+\sum_{i=1}^{n}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i\phi_i\right]c_i\right\}dx$$

$$-p(\mu)\tilde{u}'(\mu)\phi_j(\mu)+p(\gamma)\tilde{u}'(\gamma)\phi_j(\gamma)=0 \tag{1.10}$$

Now we consider the boundary conditions

Case 1: ***Robin*** (mixed) boundary conditions ($\alpha_1 \neq 0, \alpha_2 \neq 0, \beta_1 \neq 0, \beta_2 \neq 0$)

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+\left[\lambda r(x)-q(x)\right]u=0,\ \ \gamma<x<\mu \tag{1.11a}$$

$$\tilde{u}'(\gamma)=-\frac{\alpha_1\tilde{u}(\gamma)}{\beta_1}\quad\text{and}\quad\tilde{u}'(\mu)=-\frac{\alpha_2\tilde{u}(\mu)}{\beta_2} \tag{1.11b}$$

We assume the trial solution in terms of polynomials, $\phi_j(x)$ as

$$\tilde{u}(x)=\phi_0(x)+\sum_{i=1}^{n}c_i\phi_i(x),\qquad n\geq 1 \tag{1.12}$$

where c_i's are unknown parameters. Let $\phi_0(x)=0$ is specified by the homogeneous boundary conditions.

Now the weighted residual equations corresponding to the equation (1.8a) given by

$$\int_{0}^{1}\left[-\frac{d}{dx}\left(p(x)\frac{d\tilde{u}}{dx}\right)+q(x)\tilde{u}-\lambda r(x)\tilde{u}\right]\phi_j(x)dx=0,\ \ j=1, 2, 3,\ldots.. \tag{1.13}$$

Again, from equation (1.7), we have

$$\tilde{u}(\gamma)=\sum_{i=0}^{n}c_i\phi_i(\gamma)\quad\text{and}\quad\tilde{u}(\mu)=\sum_{i=0}^{n}c_i\phi_i(\mu) \tag{1.14a}$$

$$\frac{du}{dx}=\sum_{i=0}^{n}c_i\frac{d\phi_i}{dx} \tag{1.14b}$$

Integrating each term of equation (1.13) by parts and using equations (1.14), we obtain the Galerkin weighted residual equations:

$$\sum_{i=0}^{n}\left[\int_{\gamma}^{\mu}\left[p(x)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}+q(x)\phi_i(x)\phi_j(x)-\lambda r(x)\phi_i(x)\phi_j(x)\right]dx - \frac{\alpha_2 p(\mu)\phi_i(\mu)\phi_j(\mu)}{\beta_2}+\frac{\alpha_1 p(\gamma)\phi_i(\gamma)\phi_j(\gamma)}{\beta_1}\right]c_i=0 \qquad (1.15)$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(F_{i,j}-\lambda E_{i,j}\right)c_i=0, \quad j=1, 2, 3,\ldots\ldots\ldots\ldots, n \qquad (1.16)$$

where,

$$F_{i,j}=\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx-\frac{\alpha_2 p(\mu)\phi_i(\mu)\phi_j(\mu)}{\beta_2}+\frac{\alpha_1 p(\gamma)\phi_i(\gamma)\phi_j(\gamma)}{\beta_1} \qquad (1.16a)$$

$$E_{i,j}=\int_{\gamma}^{\mu} r(x)\phi_i(x)\phi_j(x)dx, \quad i,j=1,2,3\ldots\ldots,n \qquad (1.16b)$$

Case 2: Dirichlet boundary conditions (*i.e.*, $\alpha_1 \neq 0$, $\beta_1 = 0$, $\alpha_2 \neq 0$, $\beta_2 = 0$)

Here the boundary terms vanish because the boundary conditions imply $\phi_j(\gamma)=0$ and $\phi_j(\mu)=0$

Hence

$$\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[\tilde{p}(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]c_i\right\}dx=\lambda\sum_{i=1}^{n-1}\int_{\gamma}^{\mu}\left[r(x)\phi_i(x)\phi_j(x)dx\right]c_i \qquad (1.17)$$

where,

$$F_{i,j}=\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx \qquad (1.17a)$$

$$E_{i,j}=\int_{\gamma}^{\mu} r(x)\phi_i(x)\phi_j(x)dx\ , \quad i,j=1,2,3\ldots\ldots,n \qquad (1.17b)$$

Case 3: Numann boundary conditions (i.e., $\alpha_1 = 0$, $\beta_1 \neq 0$, $\alpha_2 = 0$, $\beta_2 \neq 0$)

so that $u'(\gamma) = 0$ and $u'(\mu) = 0$

$$\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx+p(\gamma)\tilde{u}'(\gamma)\phi_j(\gamma)-p(\mu)\tilde{u}'(\mu)\phi_j(\mu)\right\}c_i\,dx$$

$$=\lambda\sum_{i=1}^{n-1}\int_{\gamma}^{\mu}\left[r(x)\phi_i(x)\phi_j(x)dx\right]c_i \quad (1.18)$$

where,

$$F_{i,j}=\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx \quad (1.18a)$$

$$E_{i,j}=\int_{\gamma}^{\mu}r(x)\phi_i(x)\phi_j(x)dx \quad (1.18b)$$

Case 4: Cauchy boundary conditions:

(i) when $\beta_1 \neq 0$, $\beta_2 = 0$

We obtain from equations (1.8b) and (1.8c)

$$u'(\gamma)=-\frac{\alpha_1 u(\gamma)}{\beta_1} \text{ and } \alpha_2 u(\mu)=0 \quad (1.19a)$$

It follows that $\phi_j(\mu)=0$ (1.19b)

$$\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]-p(\mu)\phi_i'(\mu)\phi_j(\mu)+p(\gamma)\phi_i'(\mu)\phi_j(\mu)\right\}c_i\,dx$$

$$=\lambda\sum_{i=1}^{n-1}\int_{\gamma}^{\mu}\left[r(x)\phi_i(x)\phi_j(x)dx\right]c_i \quad (1.20)$$

where,

$$F_{i,j}=\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx+\frac{\alpha_1 p(\gamma)\phi_i(\gamma)\phi_j(\gamma)}{\beta_1} \quad (1.20a)$$

$$E_{i,j}=\int_{\gamma}^{\mu}r(x)\phi_i(x)\phi_j(x)dx \quad (1.20b)$$

(ii) when $\beta_1 = 0, \ \beta_2 \neq 0$

We obtain from equation (1.8b) and (1.8c) that

$$u'(\mu) = -\frac{\alpha_2 u(\mu)}{\beta_2} \quad \text{and} \quad \alpha_2 u(\gamma) = 0 \tag{1.21a}$$

It follows that $\phi_j(\gamma) = 0$ (1.21b)

Equation (1.10) takes the form as:

$$\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + q(x)\phi(x)\phi_j(x)\right]dx - p(\mu)\phi_i'(\mu)\phi_j(\mu) + p(\gamma)\phi_i'(\mu)\phi_j(\mu)\right\}c_i$$

$$= \lambda\sum_{i=1}^{n-1}\int_{\gamma}^{\mu}\left[r(x)\phi_i(x)\phi_j(x)dx\right]c_i \tag{1.22}$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(F_{i,j} - \lambda E_{i,j}\right)c_i = 0, j=1, 2, 3, \ldots\ldots\ldots, n \tag{1.23}$$

$$F_{i,j} = \int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + q(x)\phi_i(x)\phi_j(x)\right]dx - \frac{\alpha_2 p(\mu)\phi_i(\mu)\phi_j(\mu)}{\beta_2} \tag{1.23a}$$

$$E_{i,j} = \int_{\gamma}^{\mu} r(x)\phi_i(x)\phi_j(x)dx \tag{1.23b}$$

Case 5: Periodic boundary Conditions: $u(\gamma) = u(\mu)$; $u'(\gamma) = u'(\mu)$

Equation (1.10) reduces to

$$\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + q(x)\phi_i(x)\phi_j(x)\right]dx - \phi_i'(\mu)\phi_j(\mu)\left[p(\mu) - p(\gamma)\right]\right\}c_i$$

$$= \lambda\sum_{i=0}^{n}\left\{\int_{\gamma}^{\mu}\left[r(x)\phi_i(x)\phi_j(x)\right]\right\}c_i dx \tag{1.24}$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(F_{i,j} - \lambda E_{i,j}\right)c_i = 0, \ j=1, 2, 3, \ldots\ldots\ldots, n \tag{1.25}$$

where,

$$F_{i,j}=\int_{\gamma}^{\mu}\left[p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}+q(x)\phi_i(x)\phi_j(x)\right]dx-\phi_i'(\mu)\phi_j(\mu)\left[p(\mu)-p(\gamma)\right] \quad (1.25a)$$

$$E_{i,j}=\int_{\gamma}^{\mu} r(x)\,\phi_i(x)\phi_j(x)\,dx \quad (1.25b)$$

Case 6: Boundary conditions for **semi-infinite domain**

We use polynomials as trial functions which are derived over the interval $0 < x < \mu$. We first convert SLE in $(1.8a)$ over the domain $|1, \infty|$ to an equivalent problem on $[0,1]$. This exertion is performed by placing $x = log\, t$ so that $\frac{dx}{dt}=\frac{1}{t}$;

$$\frac{d\tilde{u}}{dx}=\frac{d\tilde{u}}{dt}\frac{dt}{dx}=t\frac{d\tilde{u}}{dt};$$

$$\frac{d^2\tilde{u}}{dx^2}=\frac{d}{dx}\left(\frac{d\tilde{u}}{dx}\right)=\frac{d}{dt}\left(t\frac{d\tilde{u}}{dt}\right)\frac{dt}{dx}=t\left[t\frac{d^2\tilde{u}}{dt^2}+\frac{d\tilde{u}}{dt}\right]=t^2\frac{d^2\tilde{u}}{dt^2}+t\frac{d\tilde{u}}{dt} \quad (1.26)$$

Sturm-Liouviile problems in infinite range as $\mu \to \infty$. We can further increase the range of the interval and equation (1.9) gives

$$\int_{1}^{\mu}\left[-\frac{d}{dx}\left(p(t)\frac{d\tilde{u}}{dx}\right)+q(t)\tilde{u}-\lambda r(t)\tilde{u}\right]\phi_j(t)\,dt=0,\;\; 1<t<\infty \quad (1.26a)$$

We consider the interval the endpoint boundary conditions $u(\mu)=0$ as $\mu \to \infty$ in $0 < x < \mu$. Let us assume $u(1000)=0$ by taking $\mu=1000$.

$$\sum_{i=0}^{n}\left\{\int_{1}^{\mu}\left[p(t)\frac{d\phi_i}{dt}\frac{d\phi_j}{dt}+q(t)\phi_i(t)\phi_j(t)\right]dt-\sum_{i=0}^{n}\left[\lim_{\mu\to\infty}p(\mu)\tilde{u}'(\mu)\phi_j(\mu)-p(1)\tilde{u}'(1)\phi_j(1)\right]\right\}c_i$$

$$=\lambda\sum_{i=0}^{n}\int_{1}^{\mu}\left[r(t)\phi_i(t)\phi_j(t)\right]c_i\,dt \quad (1.27)$$

$\phi_j(0)=0$ and we assume $\lim_{\mu\to\infty}\phi_j(\mu)=0$.

$$\sum_{i=0}^{n}\left(F_{i,j}-\lambda E_{i,j}\right)c_i=0 \quad (1.28a)$$

where,

$$F_{i,j} = \int_{1}^{\mu} \left[p(t)\frac{d\phi_i}{dt}\frac{d\phi_j}{dt} + q(t)\phi_i(t)\phi_j(t) \right] dt \tag{1.28b}$$

$$E_{i,j} = \int_{1}^{\mu} r(t)\phi_i(t)\phi_j(t)\,dt \qquad i,j = 1,2,3 \ldots\ldots, n \tag{1.28c}$$

1.4 Basic properties and theorems on Sturm-Liouville problems

During time an extensive theory was developed for the regular boundary value problem (1.1a) - (1.1b), the so-called Sturm-Liouville theory. In this section we bring together those facts which seem especially relevant for the subject of this thesis. For a more elaborated study of the Sturm-Liouville theory we can refer to [Marletta and Pryce (1992), Baily *et al* (2001), Aukulenko and Nesterov (2006), Ledoux and Daele (2010), Ledoux *et al* (2009)].

Adjoint operator: Let A be a complex $n \times n$ square matrix viewed as a linear operator on C^n. Then for every $\phi, \psi \in C^n$, we have

$$\langle A\phi, \psi \rangle = (A\phi)^T \bar{\psi} = \phi^T A^T \bar{\psi} = \phi^T \overline{\bar{a}^T \psi} = \phi^T \overline{A^* \psi} = \langle \phi, A^* \psi \rangle \tag{1.29}$$

Thus, the conjugate transpose matrix $A^* = \overline{A}^T$ is the adjoint of A.

Self-adjoint operator: A linear operator L on inner product space V is called a self-adjoint operator if $L^* = L$.

$$\langle L\phi, \psi \rangle = \langle \phi, L\psi \rangle \quad \forall \phi, \psi \in V \tag{1.30}$$

Homogeneous, second-order, linear ordinary differential equation (with real-valued coefficients) can be written in self-adjoint form using a procedure illustrated below.

Here $I \in (\gamma, \mu)$ is a bounded or unbounded open interval of the real line R i.e., $-\infty \le \gamma < \mu \le \infty$, the coefficients $p(x)$, $q(x)$, $r(x) : (\gamma, \mu)$ into R, $\lambda \in C$, the complex field.

1.4.1 Existence, uniqueness and linearity

From the basic existence and uniqueness theorem for (linear) ordinary differential equations it follows that if, $p(x)$, $q(x)$, $r(x)$ are all piecewise continuous functions and $p(x)$, $r(x)>0$, then the Sturm-Liouville equation (1.1a) has a unique solution satisfying any given initial conditions

$u(\xi)=c_1$, $(pu')(\xi)=c_2$ at a point ξ of the interval .

Preposition: Suppose that (1.1a) is a Sturm-Liouville equation with $p(x)$, $q(x)$ and $r(x)$ continuous, and $p(x)>0$ for all $x\in[\gamma,\mu]$. Then the set of all functions $u(x)$ satisfying (1.1a) is a vector space of dimension two. In other words, there exist two linearly independent solutions given in equation (1.2) [Ledoux (2006-2007)].

Proof: The differential equation (1.1a) is equivalent to the non-autonomous linear system

$$\begin{cases} u'(x)=\dfrac{1}{p(x)}v(x) \\ v'(x)=[q(x)-\lambda r(x)]u(x) \end{cases} \tag{1.31}$$

Hence, by the basic existence and uniqueness theorem, there exists a unique solution of (1.1a) with initial values $u(\gamma)=1$, $p(\gamma)u'(\gamma)=0$. Similarly, there exists a unique solution of (1.1a) with initial values $u(\gamma)=0$, $p(\gamma)u'(\gamma)=1$.

Let us denote these solutions by $u_1(x)$ and $u_2(x)$. Moreover, if $u(x)$ is any solution of (1.1a), then

$$u(x)=u(\gamma)u_1(x)+p(\gamma)u'(\gamma)u_2(x) \tag{1.32}$$

To verify this we consider the function

$$\tilde{u}(x)=u(x)-u(\gamma)u_1(x)-p(\gamma)u'(\gamma)u_2(x) \tag{1.33}$$

The function $\tilde{u}(x)$ is a solution of (1.1a) with initial values

$$\tilde{u}(\gamma)=u(\gamma)-u(\gamma)u_1(\gamma)-p(\gamma)u'(\gamma)u_2(\gamma)=0 \tag{1.34a}$$

$$p(\gamma)\tilde{u}'(\gamma)=p(\gamma)[u'(\gamma)-u(\gamma)u_1'(\gamma)-p(\gamma)u'(\gamma)u_2'(\gamma)]=0 \tag{1.34b}$$

Hence the uniqueness implies that $\tilde{u}(x)=0$ for all $x\in[\gamma,\mu]$.

We can then say that the Sturm-Liouville equation (1.1a) is a linear differential equation. That is, if we define the differential operator

$$L = \frac{1}{r(x)}\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \right] \text{ on } \gamma < x < \mu \quad (1.35)$$

Sturm-Liouville theory

Any homogeneous, second-order, linear ordinary differential equation will be said to be in self-adjoin form if it is written as

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x) = -\lambda r(x)u(x) \quad (1.36a)$$

$$L[u] := -\lambda r(x)u, \quad x \in (\gamma, \mu) \quad (1.36b)$$

$p(x)$, $q(x)$, $r(x)$ are piecewise continuous functions in (γ, μ) and L is a self-adjoint operator differential (Hermitian or Sturm-Liouvillian) $r = r(x)$ is some known function, λ is a constant (the eigenvalue) and u are the unknowns to be determined .

Example 1: Let us consider the general form of linear eigenvalue problem

$$A_2(x)\frac{d^2u}{dx^2} + A_1(x)\frac{du}{dx} + \left[A_0(x) + \lambda\right]u = 0 \quad (1.37)$$

$A_2(x), A_1(x), A_0(x)$ are continuous functions in the interval $[\gamma, \mu]$.

where $A_2(x) > 0$ on this interval. This differential eigenvalue problem is not in self-adjoint form unless $A_1(x) = A_2'(x)$. However, we can transform equation (1.37) into self-adjoint form by multiplying throughout by the function $\phi(x) = \frac{p(x)}{A_2(x)}$, producing

$$p(x)\frac{d^2u}{dx^2} + \phi(x)A_1(x)\frac{du}{dx} + \phi(x)\left[A_0(x) + \lambda\right]u = 0 \quad (1.38)$$

Equation above is now self-adjoint form provided we choose

$$p'(x) = \phi(x)A_1(x) \quad (1.39)$$

By solving this first order differential equation for $p(x)$, we find

$$p(x) = \exp\left[\int \frac{A_1(x)}{A_2(x)} dx \right] \quad (1.40a)$$

Also by further comparison of equation (1.38) with the self-adjoint form of equation (1.36a), we identify the functions as

$$q(x)=\phi(x)A_0(x)=\frac{p(x)A_0(x)}{A_2(x)} \tag{1.41}$$

$$r(x)=\phi(x)=\frac{p(x)}{A_2(x)}$$

Here, $L=D\lfloor p(x)D\rfloor+q(x)$ is called a self-adjoin operator; where $D=\frac{d}{dx}$. The self-adjoint form enjoys certain operational advantages over other forms. A self-adjoint operator L is said to be symmetric on the interval $\lfloor\gamma,\mu\rfloor$ if and only if

$$\int_{\gamma}^{\mu}(\phi L[\psi]-\psi L[\phi])dx=0 \tag{1.42}$$

for any functions ϕ and ψ having continuous second order derivatives on the interval satisfying the boundary prescribed conditions associated with L.

We examine whether the self-adjoint operator $L=D^2$ is symmetric on $\lfloor 0,1\rfloor$ with respect to the following boundary conditions

(i) $\phi(0)=0$; $\phi(1)=0$ (1.43a)

(ii) $\phi(0)-\phi(1)=0$; $\phi'(1)=0$ (1.43b)

Substituting $L=\frac{d^2}{dx^2}\left(=D^2\right)$ in the above equation (1.36) and using integration by parts

$$\int_0^1\left[\phi(x)\psi''(x)-\psi(x)\phi''(x)\right]dx=\left[\phi(x)\psi'(x)-\phi'(x)\psi(x)\right]_0^1$$

$$=\left[\phi(1)\psi'(1)-\phi'(1)\psi(1)\right]-\left[\phi(0)\psi'(0)-\phi'(0)\psi(0)\right]$$

For the boundary conditions in (i), it follows that ϕ and ψ satisfy $\phi(0)=\psi(0)=0$ and $\phi(1)=\psi(1)=0$. Hence the right hand side of the above integral vanishes and we conclude that $L=D^2$ is symmetric in this case. In the case (ii) boundary conditions lead to $\phi(0)-\phi(1)=0$, $\psi(0)-\psi(1)=0$ and $\phi'(1)=\psi'(1)=0$. Based on these relations the above integral reduces to

$$\int_0^1 \left[\phi(x)\psi''(x)-\psi(x)\phi''(x)\right]dx=\psi(0)\phi'(0)-\phi(0)\psi'(0) \qquad (1.44)$$

the right-hand side $\neq 0$

Thus $L=D^2$ is not symmetric in this case.

Theorem 1.1: A self-adjoint operator $L=\frac{d}{dx}\left[\left(p(x)\frac{d}{dx}\right)\right]+q(x)$ is said to be symmetric on the interval

$$[\gamma,\mu] \text{ if } \left[p(x)W(\phi,\psi)(x)\right]_{\gamma}^{\mu}=0 \qquad (1.45)$$

For any function ϕ, ψ that satisfy the prescribed boundary conditions associated with L and have continuous second order derivatives on $[\gamma,\mu]$.

Proof: If ϕ and ψ satisfy the boundary conditions at $x=\gamma$ then

$$\text{(i)} \begin{bmatrix}\psi(\gamma) & \psi'(\gamma)\\ \phi(\gamma) & \phi'(\gamma)\end{bmatrix}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \qquad (1.46)$$

which follows that $\phi'(\gamma)\psi(\gamma)-\phi(\gamma)\psi'(\gamma)=0$

If ϕ and ψ satisfy the boundary conditions at $x=\mu$

$$\text{(ii) } \phi'(\mu)\psi(\mu)-\phi(\mu)\psi'(\mu)=0 \qquad (1.47)$$

$$\left[p(x)W(\phi,\psi)(x)\right]_{\gamma}^{\mu}=p(\mu)(\mu)\psi'(\mu)-p(\gamma)\phi'(\gamma)\psi'(\gamma)=0 \qquad (1.48)$$

The most important properties on associated with eigenvalues and eigenfunctions of such a self-adjoint symmetric operator is the orthogonality of the eigenfunctions.

Theorem 1.2: Let L be a symmetric operator on the interval $[\gamma,\mu]$ associated with the eigen-equation (1.36a)

If λ_m and λ_n are any two distinct eigenvalues of L with corresponding eigen functions ϕ_n and ϕ_m respectively, then ϕ_n and ϕ_m are orthogonal, i.e.,

$$\int_{\gamma}^{\mu} r(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n \tag{1.49}$$

Proof: The eigenfunctions ϕ_n and ϕ_m satisfy the relations

$$L\left[\phi_m(x)\right] = -\lambda_m r(x)\phi_m(x) \tag{1.50a}$$

$$L\left[\phi_n(x)\right] = -\lambda_n r(x)\phi_n(x) \tag{1.50b}$$

Multiply (1.50a) by $\phi_n(x)$ and (1.50b) by $\phi_m(x)$, subtract the resulting expressions and integrate over $\gamma < x < \mu$

$$\int_{\gamma}^{\mu} \left\{\phi_n L\left[\phi_m(x)\right] - \phi_m L\left[\phi_n(x)\right]\right\} dx = \left(\lambda_n - \lambda_m\right)\int_{\gamma}^{\mu} r(x)\phi_m(x)\phi_m(x)dx \tag{1.51}$$

Since L is symmetric so that $\phi L[\psi] = \psi L[\phi]$

$$\left(\lambda_n - \lambda_m\right)\int_{\gamma}^{\mu} r(x)\phi_m(x)\phi_m(x)dx = 0 \tag{1.52}$$

By hypothesis $\lambda_m \neq \lambda_n$; thus we deduce that the integral vanishes and the theorem is thus proved.

Theorem 1.3: The eigenvalues of a symmetric operator are all real.

Proof: Suppose there exists some complex eigenvalue λ_k with corresponding eigenfunction, $\phi_k(x)$ i.e., $L[\phi_k(x)] + \lambda_k r(x)\phi_k(x) = 0$. Since the operator L is composed of real functions, its complex conjugate $\overline{L}$ equals L. Therefore, by forming the complex conjugate of the above equation, we find

$$\overline{L[\phi_k(x)] + \lambda_k r(x)\phi_k(x)} = L[\overline{\phi_k(x)}] + \overline{\lambda_k}\, r(x)\overline{\phi_k(x)} = 0 \tag{1.53}$$

It follows that $\phi_k(x)$ and $\overline{\phi_k(x)}$ belong to distinct eigenvalues, λ_k and $\overline{\lambda_k}$ respectively, and hence are necessarily orthogonal due to the symmetry of L. This implies that

$$\int_{\gamma}^{\mu} r(x)\phi_k(x)\overline{\phi_k(x)}dx = \int_{\gamma}^{\mu} r(x)\left|\phi_k(x)\right|^2 dx = 0 \tag{1.54}$$

but since the integrand is positive, this integral $\neq 0$, which leads to a contradiction. Our assumption that a complex eigenvalue exists must be false, and the theorem is proved. Every pair of eigenvectors corresponding to different eigenvalues are orthogonal.

If λ_n and λ_m are any two distinct eigenvalues of L and with corresponding eigenfunctions ϕ_n and ψ_m, respectively, then ϕ and ψ are orthogonal;

$$\int_{\gamma}^{\mu} \phi_n(x)\psi_m(x)dx = 0, \quad n \neq m \tag{1.55}$$

Sturm-Liouville Differential Expressions [Baily *et al* (1991)]

Let I denote any interval of the real line R with endpoints γ and μ, where $-\infty \leq \gamma < \mu \leq \infty$. A compact, i.e., a bounded and closed interval, is denoted by $[\gamma, \mu] = \{x \in R, \ \ \gamma \leq x < \mu \}$

where,

$\ell(I)$ and $\ell^1(I)$ denote the space of complex valued measurable functions on I

for which $\int_I |u(x)| dx < \int_I |u|$ (1.56)

Likewise $\ell^2(I)$ denotes the space (of equivalence classes) of functions u such that

$$\int_I |u(x)|^2 dx < \infty \tag{1.57}$$

If $r(x)$ is a positive measurable function on I then $\ell^2(I; r)$ represents the weighted space of functions u satisfying $\int_I |u(x)|^2 r(x) dx < \infty$ (1.58)

notation $\ell oc(I)$ is used to denote the space of functions u satisfying $x \in \ell[\gamma, \mu]$ for all compact subintervals $[\gamma, \mu]$ of I.

Throughout our thesis work we assume that the coefficients p, q, r satisfy:

$$p, q, \ r: I \to R \tag{1.59a}$$

$$p^{-1}, q, r \in \ell oc(I) \tag{1.59b}$$

$p(x) > 0$ and $q(x) > 0$, almost everywhere on I. (1.59c)

Solution of (1.36a) requires a function u such that u and pu' are both absolutely continuous on all compact subintervals of I (so that the left-hand side of (1.36a) is defined i.e., on I and (1.36a) holds, the classical derivative u' may not be.

1.5 Classification of Sturm-Liouville systems

Regular Sturm-Liouville systems: Most of the eigenvalue problems studied thus fur have featured unmixed or separated boundary conditions. Problems of this type are characterized by $L[u]+\lambda r(x)u=0, \quad x\in(\gamma,\mu)$ (1.60a)

$$M_1[u]=\alpha_1 u(\gamma)+\beta_1 u'(\gamma)=0 \quad \left(\alpha_1^2+\beta_1^2\neq 0\right) \tag{1.60b}$$

$$M_2[u]=\alpha_2 u(\mu)+\beta_2 u'(\mu)=0 \quad \left(\alpha_2^2+\beta_2^2\neq 0\right) \tag{1.60c}$$

where, $L=D[p(x)D]+q(x)$ (1.60d)

Any eigenvalue problem belonging to this general class is called a Regular Sturm-Liouville system. Unmixed homogeneous boundary conditions which at either endpoint of the interval may assume one of the following forms:

First kind: $u=0$ (1.61a)

Second kind: $u'=0$ (1.61b)

Third kind: $hu+u'=0$ (h constant) (1.61c)

Regular end-point: An endpoint is called singular if it is not regular. Thus, an endpoint is singular if it is infinite, or the endpoint is finite but at least one of p, q, r is not integrable in any neighborhood of the endpoint.

Periodic Sturm-Liouville systems: We consider

$$L[u]+\lambda r(x)u=0, \tag{1.62a}$$

$$u(\gamma)=u(\mu), \quad u'(\gamma)=u'(\mu) \tag{1.62b}$$

where, $L=D[p(x)D]+q(x)$ and $p(\gamma)=p(\mu)$

Here, $p(x)$, $q(x)$, $r(x)$ are given sufficiently smooth functions of argument x, $\gamma\le x\le\mu$, which may be extended for all $|x|<\infty$ as periodic functions. Moreover, $p(x)$, $q(x)>0$ and the systems which contains such problems are called periodic Sturm-Liouville systems. We observe that the Sturm-Liouville boundary value problem (1.62a) and (1.62b) is self-conjugate and the eigenfunctions corresponding to distinct λ_n are orthogonal w.r.to. weight $r(x)$ on $[\gamma,\mu]$. In contrast to the BVP considered above, with each eigenvalue corresponding to a single eigenfunction (to within a constant co-efficient), in the case of periodic boundary conditions (1.62a), the same may correspond to two linearly independent eigenfunctions.

Singular Sturm-Liouville system:

A Sturm-Liouville is said to be singular if one or more of the following events occur on the interval $[\gamma,\mu]$:

(i) $p(\gamma)=0$ and/or $p(\mu)=0$

(ii) $p(x)$, $q(x)$ or $r(x)$ becomes infinite at $x=\gamma$ or $x=\mu$ or both.

(iii) Either γ or μ (or both) are infinite.

(iv) The singular endpoint $x=\mu$ is a limit-circle (***LC*** for short) if and only if for every solution $u(x) = \int_{\gamma}^{\gamma+\varepsilon} r(x)|u(x)|^2\, dx$ is finite. An endpoint which is not ***LC*** is called limit-point or ***LP*** for short.

To ensure that a singular Sturm-Liouville system has a symmetric operator, we require using theorem 1.1

$$(\phi L[\psi]-\psi L[\phi])dx=[p(x)W(\phi,\psi)(x)]_{\gamma}^{\mu}=0 \tag{1.63}$$

where ϕ and ψ are any continuous, twice differentiable functions satisfying the prescribed boundary conditions of the Sturm-Liouville system.

If there is a singularity at $x=\gamma$, we impose boundary conditions such that

$$\lim_{x\to\gamma^+} p(x)W(\phi,\psi)(x)=0^+ \tag{1.64}$$

$$p(\mu)W(\phi,\psi)(\mu)=0 \tag{1.65}$$

When the singularity arises specifically from $p(\gamma)=0$, then equation (1.64) is satisfied. For illustration, by prescribing the condition $u(x)$, $u'(x)$ finite as $x\to\gamma^+$ and equation (1.65) is satisfied by prescribing the condition of the form

$$\alpha_2 u(\mu)+\beta_2 u'(\mu)=0. \tag{1.66}$$

1.5.1 The basic approximation theorem [Baily *et al* (1991)]

Under the condition (1.59a), (1.59b) and (1.59c) and the assumption that each endpoint is either a regular or ***LC***, there exists an infinite number of eigenvalues. These are real countable, isolated and each eigenfunction is unique up to constant multiples. If each endpoint is in the ***NO*** case then, in addition, the eigenvalues are bounded below. Thus, they can be indexed such that

$$-\infty<\lambda_0<\lambda_2<\ldots\ldots\ldots\ldots\ldots\text{and } \lambda_n\to\infty. \tag{1.67a}$$

Furthermore, if ϕ_n denotes an eigenfunction corresponding to λ_n $n\in N_0=\{0,1,2,\ldots\ldots\}$, then ϕ_n has exactly n zeros in the open interval (γ,μ). [Atkinson (1964)]. If one or both end points is oscillatory, then the eigenvalues are not bounded below. With λ_n, $n\in Z=\{-2,-1,0,1,2,\ldots\}$ denoting the eigenvalues and ϕ_n be the corresponding eigenfunctions. We have, in this case,

$$\ldots\ldots\lambda_{-2}<\lambda_{-1}<\lambda_0<\lambda_1<\lambda_2\ldots\ldots\ldots, \tag{1.67b}$$

$\lambda_n\to-\infty$ as $n\to-\infty$ and each eigenfunctions ψ_n, $n\in Z=\{0,1,2,\ldots\ldots\}$ has infinitely many zero (γ,μ).

1.5.2 Some useful definitions

Full matrix: In numerical analysis and computer science, a sparse matrix or sparse array is a matrix in which most of the elements are zero. By contrast, if most of the elements are nonzero, then the matrix is considered Full or dense.

Non-normality ratio: The measures of non-normality of a complex square matrix signifies as non-normality ratio $H(A)$ and can be measured by the formula given as:

$$H(A):=\left(\rho\left(A^{*}A-AA^{*}\right)\right)^{\frac{1}{2}}\Big/\rho(A) \tag{1.68}$$

where A^{*} is the conjugate transpose of A and $\rho(A)$ implies for the Frobenius norm of A. An important evaluation for $H(A)$

$0 \le H(A) \le 2^{\frac{1}{4}}$, with $H(A)=0$ iff A is normal, i.e., $A^{*}A - AA^{*} = 0$ [Dragomirescu and Gheorghiu (2010)].

Condition number

The extent of this sensitivity is measured by the condition number. The condition number a measure of how close a matrix is to being singular: a matrix with large condition number is nearly singular, whereas a matrix with condition number close to 1 is far from being singular.

If the condition number $\kappa(A)=10^{p}$, then we may lose up to p digits of accuracy on top of what would be lost to the numerical method due to loss of precision from arithmetic methods. However, the condition number does not give the exact value of the maximum inaccuracy that may occur in the algorithm. It generally just bounds it with an estimate.

$$\kappa(A)=\begin{cases} \|A\|.\left\|A^{-1}\right\| & if\ A\ is\ invertible \\ \infty, & otherwise \end{cases} \tag{1.69}$$

$$\lambda_{\max}=\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots\ldots\ldots\ldots\ldots \ge \lambda_n = \lambda_{\min} > 0 \tag{1.70}$$

Then we have $\|A\|_2 = \lambda_{max}$ Since $\left\|A^{-1}\right\|$ has eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots\ldots\ldots, \frac{1}{\lambda_n}$

we find $\left\|A^{-1}\right\|_2 = \lambda_{\min}^{-1}$.

a) The condition number of Ain the two *norm is* $\kappa(A)=\frac{\lambda_{\max}}{\lambda_{\min}}$ (1.71a)

b) If A is normal matrix then, $\kappa(A)=\frac{|\lambda_{\max}|}{|\lambda_{\min}|}$ (1.71b)

Spectral radius= maximum of the eigenvalues, denoted by $\rho(A)=|\lambda_{\max}|$.

c) If A is non-symmetric matrix, $\kappa(A)=\frac{|\sigma_{max}|}{|\sigma_{min}|}=\frac{|\sqrt{\lambda_{max}}|}{|\sqrt{\lambda_{min}}|}$ (1.71c)

Non-normality condition: The non-normality ratio for a square complex matrix A is defined as $m_A(\lambda)$.

Multiplicity of eigenvalues

The algebraic multiplicity $m_A(\lambda)$ of an eigenvalues λ_i is its multiplicity as a root of the characteristic polynomial, that is, the largest integer k such that $\left(\lambda-\lambda_i\right)^k$ divides evenly the characteristic polynomial..

Geometric multiplicity of an eigenvalue is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the null space of $A-\lambda I$, where A is elements of matrix and I is identity matrix. $g_A(\lambda)=n-rank(A-\lambda I)$

The geometric multiplicity of an eigenvalue of a matrix cannot exceed its algebraic multiplicity.

$$1\le g_A(\lambda)\le m_A(\lambda)\le n$$

Dual vector spaces

Linear functional: A linear functional on V is a function $T:V\to F$ such that

$$T(\alpha_1 v_1+\alpha_2 v_2)=\alpha_1 T(v_1)+\alpha_2 T(v_2) \quad (1.72)$$

$\forall\ \alpha_2, \alpha_2\in F$ and $\forall\ v_1, v_2\in V$

Thus, a linear functional, is a linear transformation $V\to F$, where F is construed as a one-dimensional vector space over itself [Kreyszig (1978)].

Example: If $V\to F$ (column vectors) and y is a $1\times n$ row vector then the map $v\to yv$ is a linear functional on V.

Dual space: Given any vector space V over a field F, the dual space is defined as the set of all linear maps $T:V\to F$ (linear functional). It is itself a vector space with the following operations.

$$(\phi+\psi)(x)=\phi(x)+\psi(x) \tag{1.73a}$$

$$\phi(bx)=b(\phi(x)) \qquad \phi,\psi\in V^{*}\ , \quad x\in V,\ b\in F \tag{1.73b}$$

Thus, the collection $V*$ of all such linear functional is the dual space of V.

1.6 Piecewise polynomials or basis functions: A piecewise polynomials function is defined on $[\gamma,\mu]$ by

$p(x)=p_i(x),\quad x_i\le x\le x_{i+1},\quad i=0,1,2,3,\ldots\ldots\ldots,n-1$, where for each function $p_i(x)$ is a polynomial defined on $[x_i, x_{i+1}]$.The degree of $p(x)$ is the maximum degree of polynomial $p_i(x)$ for $i=0,1,2,3,\ldots\ldots\ldots,n-1$.

1.7 Bernstein polynomials

Bernstein polynomial basis, introduced 100 years ago [1912] as a means to constructively prove the ability of polynomials to approximate any continuous function, to any desired accuracy, over a fixed interval. Their slow convergence rate, and the lack of digital computers to efficiently construct them, caused the Bernstein polynomials to lie dormant in the theory rather than practice of approximation for the better part of a century. It became evident that the Bernstein coefficients of a polynomial provide valuable insight into its behavior over a given finite interval, yielding many useful properties and elegant algorithms that are now being increasingly adopted in other application domains. For a more flexible description of curves, we can use Bézier curves, which are easy to compute and store on CAD systems and have nice properties like being easily transformable. These results make up the foundation of research that is being used in computer graphics, computer animation and scientific visualization today. Many authors [Kreyszig E., (1979), Lorentz (1986), Farouqi and Rajan (1987), Bhatti and Bracken (2007), Weikang *et al* (2011)] have been studied and implemented Bernstein polynomial for solving differential equations. For each positive integer *n*, there is a sequence of Bernstein polynomials over the finite interval $[\gamma,\mu]$ is defined by Islam and Hossain (2015)

$$B_{i,n}(x)=\binom{n}{i}(x-\gamma)^{i}(\mu-x)^{n-i}\ ,\ \gamma\le x\le\mu \quad i=0,1,2,\cdots\cdots\cdots,n \tag{1.74}$$

where the binomial coefficients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}, \quad \gamma \le x \le \mu \quad i = 0,1,2,\cdots\cdots,n$$

Recursion's relation properties

$$B_{i,n}(x) = \frac{1}{\mu-\gamma}\left[(\mu-x)B_{i,n-1}(x) + (x-\gamma)B_{i-1,n-1}(x)\right] \tag{1.75}$$

Bernstein polynomials and their dual basis [Juttler (1998)]

Associated with Bernstein basis, there are the corresponding dual basis functions with respect to the usual inner product of the Hilbert space $L^2[0,1]$. Let P^n be the $(n+1)$ dimensional real linear space of all polynomials of maximal degree n in the variable x then

$$P^n = \left\{1, x, x^2, x^3, \ldots\ldots, x^n\right\}$$

$$\langle \phi(x)\psi(x) \rangle = \int_0^1 \phi(x)\psi(x)dx \quad \text{for} \quad \phi, \psi \in P^n. \tag{1.76}$$

The linear space P^n becomes the $n+1$ dimensional Hilbert space. Like any basis of space P^n, the Bernstein polynomials $\left\{B_n^0, B_n^1, B_n^2, ,\ldots\ldots, B_n^n\right\}$ have a unique basis consists of $n+1$ dual basis $\left\{D_n^0, D_n^1, D_n^2, ,\ldots\ldots, D_n^n\right\}$

$$D_n^i(x) = \sum_{j=0}^{n} a_{ij} B_{j,n} \tag{1.77a}$$

where,

$$a_{ij} = \frac{(-1)^{i+j}}{\binom{n}{i}} \sum_{l=0}^{\min(i,j)} (2l+1)\binom{n+l+1}{n-i}\binom{n-l}{n-i}\binom{n+l+1}{n-j}\binom{n-l}{n-j}, \tag{1.77b}$$

$i, j = 0,1,2,3,\ldots\ldots\ldots,n$

Juttler (1998) represented the dual basis function with respect to the Bernstein basis. The dual basis function must satisfy the relation of duality

$$\left(D_i^n(x)B_j^n(x)\right)=\delta_{ij}:=\begin{cases}1 & if\ i=j\\0 & if\ i\neq j\end{cases} \quad (1.78)$$

$a_{ij}\in R$ are the real unknown coefficients.

Lemma [Doha *et al* (2011)]: The inner product of two Bernstein polynomials estimate

$$\left\langle B_i^q(x), B_j^s(x)\right\rangle=\frac{\binom{q}{i}\binom{s}{j}}{(q+s+1)\binom{q+s}{i+j}} \quad (1.79)$$

Theorem 1.4 [Doha *et al* (2011)]**:** Any generalized Bernstein basis polynomials of degree n can be written as a linear combination of the generalized Bernstein basis polynomials of degree $n+1$:

$$B_{i,n}(x)=\frac{n-i+1}{n+1}B_{i,n+1}+\frac{i+1}{n+1}B_{i+1,\ n+1} \quad (1.80)$$

Some useful properties of Bernstein polynomials [Doha *et al* (2011), Islam and Hussain (2015)]

The above polynomials having degree n+1 satisfy the following properties

i) $B_{i,n}(x)=0$, if $i<0$ *or* $i>n$, (1.81a)

ii) It can be readily shown that each of the Bernstein polynomials is positive and also the sum of all the Bernstein polynomials is unity for all real x belonging to the interval [0, 1] that is, $B_{i,n}(\gamma)=0=B_{i,n}(\mu)$, $1\leq i\leq n-1$

$$\sum_{i=0}^{n}B_{i,n}(x)=1 \quad (1.81b)$$

iii) It can be easily shown that any given polynomial of degree n can be expanded in terms of a linear combination of the basis functions, that is,

$$\sum_{i=0}^{n}a_iB_{i,n}(x)\geq 1 \quad (1.81c)$$

iv) All Bernstein polynomials vanish at the points γ and μ, that is,

$$B_{i,n}(\gamma)=0 \text{ and } B_{i,n}(\mu)=0, \quad i=1,2,3..........,n-1$$

v) The product of two polynomials is defined as

$$B_{i,j}(x)\,B_{k,l}(x)=\frac{\binom{j}{i}\binom{l}{k}}{\binom{j+l}{i+k}}B_{i+k,j+l}(x) \tag{1.81d}$$

vi) By induction the r-th derivative can be written as

$$D^{r+1}B_{i,n}(x)=\frac{n!}{(n-r-1)!}\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\binom{r+1}{k}B_{i-k,n-r-1}(x) \tag{1.81e}$$

vii) The Bernstein polynomial of any degree $B_{i,n}(x)$ in terms of any higher degree basis $B_{i,n+r}(x)$ using the following lemma.

$$B_{i,n}(x)=\sum_{j=k}^{k+r}\frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}}B_{j,n+r}(x) \tag{1.81f}$$

viii) Bernstein polynomials of order n form a basis for the space of polynomials of degree less than or equal to n i.e., they span the space of polynomials and are linearly independent. If there exists constant $a_0, a_1, a_2, \ldots\ldots, a_n$ so that the identity

$$a_0B_{0,n}(x)+a_1B_{1,n}(x)+a_2B_{2,n}(x)+\ldots\ldots+a_nB_{n,n}(x)=0 \tag{1.81g}$$

holds for all x. Then all the $a_i's$ must be zero.

ix) We can define the n-th Bernstein polynomials for a function f on the interval [0,1]

x) $$B_{i,n}(f;x)=\sum_{i=0}^{n}f\left(\frac{i}{n}\right)\binom{n}{i}x^{i}(1-x)^{n-i} \tag{1.81h}$$

For simplicity we denote $B_{i,n}(x)$ as B_j throughout the study.

Derivatives of Bernstein basis polynomial [Doha *et al* (2011)]

The first derivative and second derivatives may be defined successively as follows:

$$B'_{i,n}(x)=n\left[B_{i-1,n-1}(x)-B_{i,n-1}(x)\right] \tag{1.82a}$$

$$B''_{i,n}(x)=n(n-1)\left[B_{i-2,n-2}(x)-2B_{i-1,n-2}(x)+B_{i,n-2}\right] \tag{1.82b}$$

For simplicity we take $B_{i,n}$ as B_i throughout this research work.

1.8 Weierstrass Approximation Theorem (1885) [Finlayson (1972)]

Let f be a continuous function defined on the closed interval [γ, μ]. For any $\varepsilon > 0$, there exists a polynomial function $p_n(x)$ such that for all x in [γ, μ], we have

$$|f(x)-p_n(x)| < \varepsilon . \tag{1.83}$$

To show this, we need to prove that this is true on the interval [γ, μ] = [0,1]. We define $g: C([\gamma,\mu]) \to C([0,1])$ by

$$(gf)(x) = g(\gamma+(\mu-\gamma)x) \tag{1.84}$$

Then g is linear and invertible with the inverse

$$\left(g^{-1}f\right)(x)=f\left(\frac{x-\gamma}{\mu-\gamma}\right) \tag{1.85}$$

Moreover, g is an isometry since, $\| gf \| = \| f \|$, and for any polynomial p, both gp and $g^{-1}p$ are polynomials. If p is dense in $C([0,1])$, then for any $f \in ([\gamma,\mu])$, we have $p_n \to gf \in C([0,1])$.

Hence $g^{-1}p$ converge to f in $C([\gamma,\mu])$. To show $p_n(x)$ is dense in $C([0,1])$, we will use Bernstein's proof which will not only suffice but will also give us an explicit sequence of polynomials that converge uniformly to $f \in C([0,1])$.

1.9 The Bernstein Approximation Theorem [Levasseur, 1978]: Every continuous function f defined on $[0,1]$ can be uniformly approximated as closely as desired by a polynomial function. For any $\varepsilon > 0$, there exists a positive integer N such that for all $x \in [0,1]$, an integer $n \geq N$ we have,

$$\left| f(x)-B_n(f;x)\right| < \varepsilon \tag{1.86}$$

where $B_n(f;x)$ is a polynomial on x similar to equation (1.81 h). Hence given any power-form polynomial of degree N, it can be uniquely converted into a Bernstein polynomial of degree n for $n \geq N$.

Bernstein polynomials approach to $f(x)$ i.e., $B_n(f;x) \to f(x)$ as $n \to \infty$, for each point x of continuity of the function $f(x)$ defined on the interval $[0,1]$.

To evaluate these Bernstein polynomials, $B_n(1;x)$, $B_{i,n}(x;x)$ and $B_{i,n}\left(1;x^2\right)$, we have taken the first and second derivative (with respect to x) of the binomial expansion of the polynomials [Estep and Donald (2002)].

$$\sum_{i=0}^{n}\binom{n}{i}x^i z^{n-i} = (x+z)^n \quad (1.87a)$$

$$\sum_{i=0}^{n}\left(\frac{i}{n}\right)\binom{n}{i}x^i z^{n-i} = x(x+z)^{n-1} \quad (1.87b)$$

$$\sum_{i=0}^{n}\left(\frac{i}{n}\right)^2\binom{n}{i}x^i z^{n-i} = \left(\frac{n-1}{n}\right)x^2(x+z)^{n-2} + \frac{x}{n}(x+z)^{n-1} \quad (1.87c)$$

$$\sum_{i=0}^{n}\left(\frac{i}{n}\right)^3\binom{n}{i}x^i z^{n-i} = \frac{(n-1)(n-2)}{n^2}x^3(x+z)^{n-3} + \frac{3(n-1)}{n^2}x^2(x+z)^{n-2} + \frac{1}{n^2}x(x+z)^{n-1} \quad (1.87d)$$

Evaluating these at $z = 1-x$, so as to satisfy the definition of Bernstein polynomial, we get

$$B_{i,n}\left(1;x^2\right) = 1 \quad (1.88a)$$

$$B_{i,n}(x;x) = x \quad (1.88b)$$

$$B_{i,n}\left(x;x^2\right) = \left(\frac{n-1}{n}\right)x^2 + \left(\frac{1}{n}\right)x \quad (1.88c)$$

Multiply each term by $\binom{n}{i} x^i (1-x)^{n-i}$ and sum from 0 to n

Using (1.87a), (1.87b), (1.87c), (1.87d) we have

$$\sum_{i=0}^{n} \left(\frac{i}{n} - x\right)^2 \binom{n}{i} x^i (1-x)^{n-i} = B_{i,n}\left(x; x^2\right) - 2xB_{i,n}(x;x) + x^2 B_{i,n}(1;x)$$

$$= \frac{1}{n} x(1-x) \tag{1.89}$$

for any fixed $x \in [0,1]$. We have estimated the sum of the polynomials $B_{i,n}(x)$ over all the values of i for which i/n is not close to x. We choose a number $\delta > 0$ and let G_δ denote the set of all values of i satisfying $\left|\frac{i}{n} - x\right| \geq \delta$. (1.90a)

Sum of the polynomials $B_{i,n}(x)$ over all $i \in G_\delta$

Evidently $\frac{1}{\delta^2}\left(\frac{i}{n} - x\right)^2 \geq 1$ (1.90b)

Using (1.90b)

$$\sum_{i \in G_\delta} \binom{n}{i} x^i (1-x)^{n-i} \leq \sum_{i \in G_\delta} \frac{1}{\delta^2}\left(\frac{i}{n} - x\right)^2 \binom{n}{i} x^i (1-x)^{n-i} \tag{1.91}$$

$$\sum_{i \in G_\delta} \frac{1}{\delta^2}\left(\frac{i}{n} - x\right)^2 \binom{n}{i} x^i (1-x)^{n-i} = \frac{x(1-x)}{n\delta^2} \tag{1.92}$$

Since $0 \leq x(1-x) \leq \frac{1}{4}$ on $[0,1]$ (1.93)

Thus we have $\sum_{i \in G_\delta} \binom{n}{i} x^i (1-x)^{n-i} \leq \frac{1}{4n\delta^2}$ (1.94)

We can write $\sum_{i \in G_\delta} = \sum_{i \in G_\delta} + \sum_{i \notin G_\delta}$

where,

$\left|\frac{i}{n}-x\right|<\delta$ depends upon the choice of δ.

$$f(x)-B_n(f;x)=\sum_{i=0}^{n}\left[f(x)-f\left(\frac{i}{n}\right)\right]\binom{n}{i}x^i(1-x)^{n-i} \tag{1.95}$$

which gives

$$f(x)-B_n(f;x)=\sum_{i\in G_\delta}\left[f(x)-f\left(\frac{i}{n}\right)\right]\binom{n}{i}x^i(1-x)^{n-i}$$

$$+\sum_{i\notin G_\delta}\left[f(x)-f\left(\frac{i}{n}\right)\right]\binom{n}{i}x^i(1-x) \tag{1.96}$$

$$f(x)-B_n(f;x)=\sum_{i\in G_\delta}\left[f(x)-f\left(\frac{i}{n}\right)\right]\binom{n}{i}x^i(1-x)^{n-i}$$

$$\sum_{i\notin G_\delta}\left[f(x)-f\left(\frac{i}{n}\right)\right]\binom{n}{i}x^i(1-x)^{n-i} \tag{1.97}$$

$$\left|f(x)-B_m(f;x)\right|\leq\sum_{i\in G_\delta}\left|f(x)-f\left(\frac{i}{n}\right)\right|\binom{n}{i}x^i(1-x)^{n-i}$$

$$+\sum_{i\notin G_\delta}\sum_{i\in G_\delta}\left|f(x)-f\left(\frac{i}{n}\right)\right|\left|\binom{n}{i}\right|x^i(1-x)^{n-i} \tag{1.98}$$

Since $f\in[0,1]$, it is bounded on $[0,1]$ and we have $\left|f(x)\right|\leq M$, for some $M>0$.

$$\left|f(x)-f\left(\frac{i}{n}\right)\right|\leq 2M \text{ for all } i, \tag{1.99}$$

and all $0\leq x\leq 1$ and so

$$\sum_{i\in G_\delta}\left|f(x)-f\left(\frac{i}{n}\right)\right|\left|\binom{n}{i}\right|x^i(1-x)^{n-i}\leq 2M\sum_{i\in G_\delta}\binom{n}{i}x^i(1-x)^{n-i} \tag{2.100}$$

On using (1.94)

$$\sum_{i\in G_\delta}\left|f(x)-f\left(\frac{i}{n}\right)\right|\left|\binom{n}{i}\right|x^i(1-x)^{n-i}\leq\frac{M}{2n\delta^2} \tag{2.101}$$

Since f continuous, it is also uniformly continuous on $[0,1]$. Thus, corresponding to any choice of $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|x - x_0| < \delta$ $\Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2}$ for all $x, x_0 \in [0, 1]$. Thus, for the sum $i \notin G_\delta$, we have

$$\sum_{i \notin G_\delta} \left| f(x) - f\left(\frac{i}{n}\right) \right| \binom{n}{i} x^i (1-x)^{n-i} \leq \frac{\varepsilon}{2} \sum_{i \notin < G_\delta} \binom{n}{i} x^i (1-x)^{n-i}$$

$$< \frac{\varepsilon}{2} \binom{n}{i} x^i (1-x)^{n-i} \tag{2.102}$$

and hence

$$\sum_{i \notin G_\delta} \left| f(x) - f\left(\frac{i}{n}\right) \right| \binom{n}{i} x^i (1-x)^{n-i} < \frac{\varepsilon}{2} \sum_{i \notin < G_\delta} \binom{n}{i} x^i (1-x)^{n-i}$$

$$< \frac{\varepsilon}{2} \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} \tag{2.103}$$

Using (1.81b)

$$\sum_{i \notin G_\delta} \left| f(x) - f\left(\frac{i}{n}\right) \right| \binom{n}{i} x^i (1-x)^{n-i} < \frac{\varepsilon}{2}$$

On combining the above two

$$\left| f(x) - B_n(f;x) \right| < \frac{M}{2n\delta^2} + \frac{\varepsilon}{2} \tag{2.104}$$

If we choose

$$N > \frac{M}{\left(\varepsilon \delta^2\right)}, \text{ then } \left| f(x) - B_n(f;x) \right| < \varepsilon \tag{2.105}$$

for all $n \geq N$ and this completes the proof.

1.10 Legendre polynomials [Atkinson and Kendall (1989)]

The Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre as the coefficients in the expansion of the Newtonian potential

$$\frac{1}{\left|X-X'\right|}=\frac{1}{\sqrt{r^{2}+r'^{2}-2rr'\cos\gamma}}=\sum_{k=0}^{\infty}\frac{r'^{k}}{r^{k+1}}L_{k}\cos\gamma \tag{2.106}$$

where r and r' are the lengths of the vectors X and X' respectively and γ is the angle between those vectors. The series converges when $r > r'$. The expression gives the gravitational potential associated to a point charge. The expansion using Legendre polynomials might be useful, for instance, where integrating this expression over a continuous mass or charge distribution.

Now we introduce Legendre polynomials through the generating function

$$h(t,x)=\frac{1}{\sqrt{1-2xt+t^{2}}}=\sum_{n=0}^{\infty}L_{n}(x)t^{n} \tag{2.107}$$

Legendre polynomials play a very important role in physics as they satisfy the Sturm-Liouville eigenvalue problem called Legendre's equation. Legendre polynomials $L_n(x)$, $(n=0,1,2.........,n-1)$ of degree n which is an eigen function of the Singular Sturm-Liouville eigenvalue problem given by,

$$\left(\left(1-x^{2}\right)L_{n}'(x)\right)'+n(n+1)L_{n}(x)=0 \tag{2.108}$$

$$\text{where } N=\begin{cases}\dfrac{n}{2}, & \text{when } n \text{ is even}\\[2mm] \dfrac{n-1}{2}, & \text{when } n \text{ is odd}\end{cases} \tag{2.108a}$$

on (-1,1).

provided L_n is bounded on [-1,1] i.e., $\left|L_n(x)\right| \le 1$.

The solution of the Legendre's equation (2.108) is called the Legendre polynomial of degree n and is denoted by $L_n(x)$. The general form of the Legendre polynomials over the interval [-1, 1] is defined by

$$L_{n}(x)=\sum_{r=0}^{N}(-1)^{r}\frac{(2n-2r)!}{2^{n}\,r!(n-r)!(n-2r)!}x^{n-2r}, \tag{2.109}$$

Properties of Legendre polynomials

(i) $L_n(\pm 1) = (\pm 1)^n$,

(ii) $L_n'(\pm 1) = \frac{1}{2}(\pm 1)^{n-1} n(n+1)$

(iii) $\int_{-1}^{1} L_n(x) L_r(x) dx = \frac{\delta_{nr}}{n+\frac{1}{2}} \; \forall \; r, n \geq 0$

(iv) Legendre polynomials are orthogonal with respect to the $L^2(-1,1)$ inner product. Also, these polynomials are complete in the sense that for any $v \in L^2(-1,1)$.

$$v(x) = \sum_{m=0}^{\infty} \tilde{v}_m L_m(x) \tag{2.110a}$$

$$\tilde{v} = (m+\frac{1}{2}) \int_{-1}^{1} v(x) L_m(x) dx \tag{2.110b}$$

where the sum converges to $L^2(-1,1)$ norm.

The Rodrigues' Formula of degree n is defined as:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \text{, where } n \geq 1 \tag{2.111}$$

Legendre polynomial which are orthogonal in the interval [-1,1] satisfy the following recurrence relation.

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x) \text{ , } n \geq 1 \tag{2.112}$$

1.11 Shifted Legendre polynomials:

$$L_n^*(x) = \tilde{L}_n\left(\frac{2x-\gamma-\mu}{\mu-\gamma}\right), \tag{2.113}$$

Here shifting the function $x \to \frac{2x-\gamma-\mu}{\mu-\gamma}$ (affine transformation) is chosen such that it objectively maps the interval $[\gamma, \mu]$ to the interval $[-1,1]$, where $L_n(x)$ are the classical Legendre polynomials. They may be generated by using the recurrence relation

$$(n+1)L^{*}_{n+1}(x)=(2n+1)\left(\frac{2x-\mu-\gamma}{\mu-\gamma}\right)L^{*}_{n}(x)-nL^{*}_{n-1}(x), \tag{2.114}$$

with $L^{*}_{0}(x)=1$, $L^{*}_{1}(x)=\dfrac{2x-\mu-\gamma}{\mu-\gamma}$. These polynomials are orthogonal of $[\gamma,\mu]$.

Evidently, using Rodrigues' Formula in equation (2.111) shifted Legendre polynomials over the interval $[0, 1]$ takes the following form:

$$L_{n}(x)=\frac{1}{n!}\frac{d^{n}}{dx^{n}}(x^{2}-1)^{n}, \tag{2.115}$$

An explicit expression for shifted Legendre polynomials is given by

$$L_{n}(x)=(-1)^{n}\sum_{p=0}^{n}\binom{n}{p}\binom{n+p}{p}(-x)^{p} \tag{2.116}$$

These polynomials are orthogonal on $|0, 1|$

$$\int_{0}^{1}L_{n}(x)L_{r}(x)dx=\frac{\delta_{nr}}{2n+1} \tag{2.117}$$

We modified the above shifted Legendre polynomials given in equation (2.115) as

$$\tilde{L}_{n}(x)=\left[\frac{1}{n!}\frac{d^{n}}{dx^{n}}(x^{2}-x)^{n}-(-1)^{n}\right](x-1) \tag{2.118}$$

so as to satisfy the homogeneous form of the Dirichlet boundary conditions $\tilde{L}_{n}(0)=\tilde{L}_{n}(1)$ to derive the matrix formulations of the fourth order Sturm-Liouville problems over the interval [0,1].

CHAPTER 2

Second Order Sturm-Liouville Problems Employing the Methods of Weighted Residual

2.1 Introduction

Sturm-Liouville problem is a second-order ordinary differential equation problem where two boundary conditions are specified, but no unique solution exists. These problems may be regular or singular at each endpoint of the underlying interval [Baily *et al* (1991)]. They are used to describe the vibration modes of various systems, such as the vibrations of a string. In 1836-1837, Sturm and Liouville published a series of papers on second order linear ordinary differential equations including boundary value problems. The influence of their work was such that this subject became known as Sturm-Liouville theory. Numerous papers, by mathematicians, physicists, engineers, and others, relating to this area have been written since then. Yet, remarkably, this subject is an intensely active field of research today.

Since many eigenvalue problems are of second order, for example Sturm-Liouville problems (SLEs), we also implemented a code for second order problems and paid special attention to the approximation of the boundary conditions in the singular case and periodic case separately. Also, for the weighted residual Galerkin, collocation and Spectral collocation method, numerical examples are given and empirical convergence orders have been presented. Finally, we have analyzed the pros and cons of the presented numerical methods using two polynomial basis functions for the solution of differential eigenvalue problems. We conclude that the combination of both techniques results in a very successful and efficient approach.

Special equations (minus boundary conditions) that fall into this classification of second order Sturm-Liouville problems are the followings:

i) $\dfrac{d^2u}{dx^2}+\lambda u=0\ ,\ \gamma<x<\mu$ (Helmholtz equation)

ii) $\left(1-x^{2}\right)\frac{d^{2}u}{dx^{2}}-2x\frac{du}{dx}+\lambda u=0, \quad -1<x<1$ (Legendre equation)

iii) $x^{2}\frac{d^{2}u}{dx^{2}}+x\frac{du}{dx}+\left(\lambda x^{2}-\nu^{2}\right)u=0, \quad 0<x<\mu$ (Bessel equation)

iv) $x\frac{d^{2}u}{dx^{2}}+(1-x)\frac{du}{dx}+\lambda u=0, \quad 0<x<\infty$ (Laguerre equation)

v) $\frac{d^{2}u}{dx^{2}}-2x\frac{du}{dx}+\lambda u=0, -\infty<x<\infty$ (Hermite equation)

Helmholtz equation appears in vibrating-string problems and in finding the temperature distribution in a rod. Legendre equation arises in certain problems displaying spherical geometry, Bessel's equation is closely associated with problems involving circular and cylindrical-shaped regions, and the equations of Laguerre and Hermite are conventional in certain quantum mechanics problems. Chawla and Shivakumar (1993) presented fourth-order finite-difference method for computing eigenvalues of second order two-point boundary value problems. The differential Transform method was applied to compute eigenvalues and eigenfunctions of second order regular SLEs by Chen and Ho (1996). The Weighted residual method using Chebyshev collocation points are investigated for approximate eigenvalues of second order SLEs by Ibrahim Celik (2005). Calculation of eigenvalues of Helmholtz equation using boundary method are presented by Reutskiy (2006). The polynomial-based Differential Quadrature (PDQ) and the Fourier expansion-based differential quadrature (FDQ) methods are found in the work of Ugur Yucel (2006) to compute eigenvalues of the second order Sturm–Liouville problems. Chanane (2005), Chanane and Boucherif (2014) used *Regularized Sampling Method* to compute the eigenvalues of regular Sturm-Liouville and the former author extended it to singular problems [Chanane (2007)]. Reutskiy (2010) proposed a new technique based on mathematically modelling the physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the eigenvalues. In recent years Taiser *et al* (2010) have presented a comparative study of Sinc Galerkin and Differential Transform method to solve second order SLEs.

Recently, Abbasbandy and Shirzadi (2011) applied the homotopy analysis method (HAM) to numerically approximate the eigenvalues of the second and fourth order SLEs. Recently Amodio and Settanni (2015) demonstrated variable step finite difference technique to solve some second order SLEs.

The second and fourth order Sturm-Liouville problems are more recurrently available in the literature utilizing various types of discretization method. The singular two-point boundary-value problems of second-order occur frequently in many practical models such as electro hydrodynamics and some thermal explosions.

Our aim is to develop Bernstein polynomial based collocation method using Chebyshev clustered grid points to generate a system of algebraic equations with unknown co-efficient in matrix form.

For regular second order Sturm-Liouville problems several studies have been carried out by many researchers to attain superior accuracy. Among them most of the literature have been devoted to the implementation of Chebychev and Legendre polynomials, using Chebychev clustered grids for different schemes. Chebychev-Fourier Spectral method utilizing trigonometric polynomials is found in the literature [Bojan and Andrej (2014)]. Recently, collocation and Spectral methods have shown great promise for solving second order singular Sturm-Liouville differential eigenvalue problems. Very recently Zhang *et al* (2017) implemented a new collocation method utilizing non-polynomial basis functions for solving second order boundary value problems. Celik (2005), Celik and Gokmen (2005) studied Chebychev collocation method to calculate the eigenvalues of regular and periodic Sturm-Liouville boundary value problems. Isik and Sezer (2013) utilized Bernstein polynomials to solve for a class of second order Lane-Emden type boundary value problems by applying collocation method. Collocation method along with Bernstein polynomials basis is implemented by Isık *et al* (2013) to give the approximate solution of a parabolic partial differential equations. Bernstein polynomials basis was employed to solve Abel's Integral equations [Alipour and Rostamy (2011)]. Double Exponential Sinc collocation method has been applied for solving second order singular SLEs

[Gaudreau *et al*, 2016] and energy eigenvalues of harmonic oscillator [Gaudreau *et al*, 2013].

A class of singular SLEs are studied by Baily *et al* (1991) applying the improved version of the algorithm of various proposed SLEIGN 2. Eigenvalues of singular Sturm-Liouville problems using collocation method [Auzinger (2006)] and modified Adomian Decomposition method [Singh and Kumar (2013)] are also studied.

We proposed Chebyshev-Legendre Spectral collocation method for solving second order linear and nonlinear eigenvalue problems exploiting Legendre derivative matrix. The Sturm-Liouville (SLE) problems have been formulated utilizing Chebyshev-Gauss-Lobatto (CGL) nodes instead of Legendre Gauss-Lobatto (LGL) nodes and Legendre polynomials are taken as basis function. We have discussed, in detail, the formulations of the present method for the Sturm-Liouville problems (SLE) with Dirichlet and mixed type boundary conditions. The accuracy of this method is demonstrated by computing eigenvalues of three regular and two singular SLEs. Nonlinear Bratu type problem is also tested in this study. The numerical results are in good agreement with the other available relevant studies.

Spectral methods namely Spectral Galerkin, Spectral collocation, Spectral Tao methods etc. are extensively used in the field of applied sciences and engineering due to the better performance and exponentially rapid convergent rate in preference to algebraic convergence rates for finite difference and finite element methods. Many researchers contributed to their works to the study of Spectral Chebyshev collocation method for computing eigenvalues of second order Sturm-Liouville problems. Not much works is found for the solution of Sturm Liouville eigenvalue problems applying Spectral collocation method using Legendre derivative matrix in the recent years. In this study, we have presented Spectral collocation method that offers accurate solutions which are put up with in terms of truncated series of smooth polynomial functions. The proposed scheme becomes simple, much efficient and preserves spectral accuracy which has many applicability's in physical and engineering models.

For the solutions of Sturm-Liouville eigenvalue problems several authors applied Spectral techniques to achieve the desired accuracy. Min and Gottlieb (2005) applied domain decomposition techniques for Spectral methods. To obtain the accuracy, the authors classified each sub domain by the finite degrees of Chebyshev and Legendre polynomials exploiting Legendre-Galerkin, Legendre-collocation, Legendre-collocation penalty, Chebyshev-collocation, and Chebyshev-collocation penalty methods and compared the results among these methods. The classical Liouville-Bratu-Gelfand [Bratu (1914), Gelfand (1963)] problem is concerned with positive solutions which was used to model a combustion problem in a numerical slab, the fuel ignition of the thermal combustion theory, and appeared in the Chandrasekhar model of the expansion of the universe. Bratu's equation is widely used to test nonlinear eigenvalue solvers. The non-linear Bratu problems are solved using various methods by different authors namely weighted residual [Aregbesola (2003)], Domain Decomposition [Min and Gottileb (2005)], B-Spline [Caglar *et al* (2010)], Laplace transformation Decomposition [Khuri (2004)], Decomposition [Liao and Tan (2007)], non-polynomial Spline [Jalilian (2010)], Parametric Spline [Zarebnial and Sarvari (2012)] and modified Adomian Decomposition [Singh and Kumar (2013)] methods etc.

Application of spectral methods are also available in detail for the solution of Boundary Value and eigenvalue problems [Shen and Tang (1996), Chen and Shizgal (2001), Lui (2011), Taher *et al* (2013), Shen (1996), Trefethen (2000), Weidman (1987)].

In this exertion, we prefer Chebyshev Gauss-Lobatto points to compensate for Legendre Gauss-Lobatto points. Since Legendre Gauss-Lobatto points are not explicitly defined and their estimation suffer round off errors for large n. Furthermore, discretization with Chebyshev grid points with fairly fewer nodes reduce CPU time with a minimum effort. Since Chebyshev polynomials are mutually orthogonal with respect to a singular weight function $w(x) = \left(1-x^2\right)^{-\frac{1}{2}}$, which leads to complexities in the study of the Chebyshev Spectral method. On the other hand, Legendre polynomials are mutually orthogonal in the standard L^2 inner product, with respect weight function $\theta(x)=1$, this criterion makes the Legendre spectral methods more attractive and much convenient for their analysis than that of the Chebyshev Spectral method.

We organize this work as follows. Chapter 2 is devoted to find the numerical approximations of eigenvalues for the second order linear Sturm-Liouville boundary value problems exploiting Galerkin WRM (Bernstein and Legendre polynomials as basis functions), Bernstein collocation method and Cheby-Legendre Spectral collocation method. We have derived rigorous matrix formulations in detail by Galerkin WRM for three different types of boundary conditions in section 2.2.1. We have also illustrated some completeness and convergence criteria along with some theorem for polynomial basis in short in section 2.2.2. Section 2.2.3 dealt with Non-Self-adjoint ordinary differential equations and their convergence conditions in brief. Numerical examples are considered in section 2.2.4 to verify reliability of the proposed formulation and the computed results are compared as well. Bernstein polynomials-based collocation technique along with their properties, some useful theorems and convergence criteria are presented in section 2.3.1 to 2.3.3. Numerical schemes for the solution of some problems are also described in section 2.3.4 in tabular form. In section 2.4, we have offered Chebyshev-Legendre Spectral collocation technique. Chebyshev polynomials and Legendre polynomials together with their properties are introduced in section 2.4.1. In section 2.4.2-2.4.3, we have discussed in brief about the Spectral collocation method the formulation of Spectral Legendre Operational Derivative matrix precisely. Formulation of Spectral collocation method and the techniques of imposing boundary conditions associated with SLEs have been demonstrated in section 2.4.4. Convergence and stability conditions are conferred in sections 2.4.5 and 2.4.6. Section 2.4.7 includes some numerical results which confirm the accuracy of the current method. This section comprises for the solution of nonlinear problem as well. Finally, conclusions for the proposed three different techniques are depicted in section 2.5.

2.2 The Galerkin Weighted Residual Method (WRM)

2.2.1 Formulation of the Galerkin WRM

We consider the linear second order regular Sturm-Liouville problem with different types boundary conditions

(i) SLE with Dirichlet boundary conditions

The general Sturm-Liouville problem is

$$a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0 u(x) = \lambda w(x)u(x), \tag{2.1}$$

where

$$a_1(x) = \frac{p'(x)}{p(x)}, \quad a_0(x) = \frac{q(x)}{p(x)}, \quad w(x) = \frac{r(x)}{p(x)} \tag{2.1a}$$

subject to the following two types of boundary conditions

Homogeneous boundary conditions

Type I: $u(\gamma)=0, \quad u(\mu)=0$ (2.1b)

Type II: $u'(\gamma)=u'(\mu)=0$ (2.1c)

Type III: $\alpha_1 u(\gamma)+\beta_1 u'(\gamma)=0, \quad \left(\alpha_1^2+\beta_1^2 \neq 0\right)$ (2.1d)

$$\alpha_2 u(\mu)+\beta_2 u'(\mu)=0, \; (\alpha_2^2+\beta_2^2 \neq 0)$$

Type IV: $u(\gamma)=u(\mu), \quad u'(\gamma)=u'(\mu)$ (2.1e)

Type V: $u(\gamma)=0$, u, u' is finite as $x \to \infty$. (2.1f)

where $\theta_0(x)$ is specified by the Dirichlet boundary conditions $B_i(\gamma)=0$ and $B_i(\mu)=0$ for each $i=1,2,3,\ldots\ldots,n-1$.

where $a_i = 0,1,2$ and w are all continuous and differentiable functions of x defined on the interval [0,1]. Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0,1], so the BVP (2.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(\mu-\gamma)x+\gamma$ and thus we have:

$$m_2(x)\frac{d^2u}{dx^2}+m_1(x)\frac{du}{dx}+m_0u(x)=\lambda\omega(x)u(x),\ \ 0<x<1 \tag{2.2a}$$

subject to the following two types of boundary conditions

$$u(\gamma)=0\,,\quad u(\mu)=0\ ;\ \frac{1}{\mu-\gamma}u'(0)=0\ ,\quad \frac{1}{\mu-\gamma}u'(1)=0 \tag{2.2b}$$

where,

$$m_1=\frac{1}{\mu-\gamma}a_1\left[(\mu-\gamma)x+\gamma\right],\quad m_0=a_0\left[(\mu-\gamma)x+\gamma\right],\ \ \omega=w\left[(\mu-\gamma)x+\gamma\right]$$

$$\begin{cases}\alpha_1u(0)+\dfrac{\alpha_2}{\mu-\gamma}u'(0)=0\ ,\ \left(\alpha_1^2+\beta_1^2\neq 0\right)\\[2ex] \beta_1u(1)+\dfrac{\beta_2}{\mu-\gamma}u'(1)=0\,,\ \ \left(\alpha_2^2+\beta_2^2\neq 0\right)\end{cases} \tag{2.2c}$$

Approximate solution of SLE (2.2a), in terms of Bernstein or Legendre polynomials basis be given as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}c_iB_i(x),\quad n\geq 1 \tag{2.3}$$

where $\theta_0(x)$ is specified by the essential boundary conditions which must satisfy the corresponding homogeneous boundary conditions such that $B_j(0)=B_j(1)=0$ for each $i=1,2,3,\cdots\cdots,n.$ Using (2.3) into equation (2.2a), the Galerkin weighted residual equations are:

$$\int_0^1\left[\frac{d^2\tilde{u}}{dx^2}+m_1\frac{d\tilde{u}}{dx}+m_0\tilde{u}-\lambda\omega\tilde{u}\right]B_j\ dx=0\ ,\ \ j=1,2,3,\cdots\cdots,n. \tag{2.4}$$

Formulation I

In this section we develop the matrix form with boundary conditions of type I

$$\int_0^1\frac{d^2\tilde{u}}{dx^2}B_j(x)dx\ =\left[B_j(x)\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d}{dx}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=-\int_0^1\frac{d}{dx}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{2.5}$$

$$\int_0^1 m_1(x)\frac{d\tilde{u}}{dx}B_j(x)dx=\Big[m_1(x)B_j(x)\tilde{u}(x)\Big]_0^1-\int_0^1\frac{d}{dx}\Big[m_1(x)B_j(x)\Big]\tilde{u}(x)dx \qquad (2.6)$$

Inserting, $B_j(\gamma)=B_j(\mu)=0$ in the above integrals, using equations (2.5), (2.6) in equation (2.4), we finally obtain the equation given by,

$$\int_0^1\Big[-\frac{d}{dx}\Big[B_j(x)\Big]\frac{d\tilde{u}}{dx} \quad -\frac{d}{dx}\Big[m_1(x)B_j(x)\Big]\tilde{u}+m_0(x)B_j\tilde{u}-\lambda\,\omega(x)B_j\tilde{u}\Big]dx \qquad (2.7)$$

$$\sum_{i=1}^{n-1}\Big[F_{i,j}-\lambda E_{i,j}\Big]c_i=0 \qquad (2.8)$$

where,

$$F_{i,j}=\int_0^1\Big[\Big\{-\frac{d}{dx}\Big[B_j(x)\Big]B_i'-\frac{d}{dx}\Big[m_1(x)B_j(x)\Big]B_i'\Big\}\Big]dx \qquad (2.8a)$$

$$E_{i,j}=\int_0^1\Big[\;\omega(x)B_iB_jdx\Big] \qquad (2.8b)$$

Finally, the eigenvalues are obtained in solving the system as below

$$F-\lambda E=0 \qquad (2.9)$$

Formulation II

In this section we develop the matrix form with boundary conditions of type II

$$\int_0^1\frac{d^2\tilde{u}}{dx^2}B_j(x)dx \;=\Big[B_j(x)\frac{d\tilde{u}}{dx}\Big]_0^1-\int_0^1\frac{d}{dx}\Big[B_j(x)\Big]\frac{d\tilde{u}}{dx}dx$$

$$=-\int_0^1\frac{d}{dx}\Big[B_j(x)\Big]\frac{d\tilde{u}}{dx}dx \qquad (2.10)$$

$$\int_0^1 m_1(x)\frac{d\tilde{u}}{dx}B_j(x)dx=\Big[m_1(x)B_j(x)\tilde{u}(x)\Big]_0^1-\int_0^1\frac{d}{dx}\Big[m_1(x)B_j(x)\Big]\tilde{u}(x)dx$$

$$=m_1(1)B_j(1)u(1)-m_1(0)B_j(0)u(0)-\int_0^1\frac{d}{dx}\Big[B_j(x)\Big]\frac{d\tilde{u}}{dx}dx \qquad (2.11)$$

Inserting $u'(\gamma)=u'(\mu)=0$ in the above integrals in, we finally obtain the equations utilizing equations (2.10), (2.11) in equation (2.4)

$$\int_0^1 \left[-\frac{d}{dx}\left[B_j(x) \right] \frac{d\tilde{u}}{dx} - \frac{d}{dx}\left[m_1(x) B_j(x) \right] \tilde{u} + m_0(x) B_j \tilde{u} - \lambda\, \omega(x) B_j \tilde{u} \right] dx$$

$$+ m_1(1) B_j(1) u(1) - m_1(0) B_j(0) u(0) = 0 \qquad (2.12)$$

Equivalently, $\sum_{i=1}^{n-1} \left[F_{i,j} - \lambda E_{i,j} \right] c_i = 0$ (2.13a)

where,

$$F_{i,j} = \int_0^1 \left[\left\{ -\frac{d}{dx}\left[B_j(x) \right] B_i' - \frac{d}{dx}\left[m_1(x) B_j(x) \right] B_i' \right\} \right] dx + m_1(1) B_j(1) u(1)$$

$$- m_1(0) B_j(0) u(0) \qquad (2.13b)$$

$$E_{i,j} = \int_0^1 \left[\omega(x) B_i B_j dx \right] \qquad (2.13c)$$

Finally, the eigenvalues are obtained in solving the system as below

$$F - \lambda E = 0 \qquad (2.14)$$

Formulation III

In this portion, we obtain the matrix formulation by applying the boundary conditions of type III.

Here we consider a linear fourth order differential equation given by

$$-\frac{d}{dx}\left(p(x) \frac{du}{dx} \right) + \left[\lambda r(x) - q(x) \right] u = 0, \quad \gamma < x < \mu \qquad (2.15)$$

where $p(x)$, $q(x)$ and $r(x)$ are specified continuous functions. We want to solve the boundary value problem (BVP) in equation (2.15) by the Galerkin method using Bernstein and Legendre polynomials as trial functions.

Since our aim is to use the Bernstein and Legendre polynomials as basis functions which are derived over the interval [0, 1], so the BVP (2.15) is to be converted to an equivalent problem on [0, 1] by replacing x by $(\mu - \gamma)x + \gamma$ and thus we have:

$$-\frac{d}{dx}\left(\tilde{p}(x) \frac{d\tilde{u}}{dx} \right) + \tilde{q}(x) \tilde{u} - \lambda \tilde{r}(x) \tilde{u} = 0, \quad 0 < x < 1 \qquad (2.16a)$$

where,

$$\begin{cases}\tilde{p}(x)=\dfrac{1}{(\mu-\gamma)^2}p[(\mu-\gamma)x+\gamma]\\ \tilde{q}(x)=q[(\mu-\gamma)x+\gamma]\\ \tilde{r}(x)=r[(\mu-\gamma)x+\gamma]\end{cases} \tag{2.16b}$$

We approximate the solution of SLE (2.16a), in terms of Bernstein or Legendre polynomials basis as: $\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}c_iB_i(x)$ (2.17)

where $\theta_0(x)$ is specified by the essential boundary conditions, $B_i(x)/L_i(x)$ are the Bernstein or Legendre polynomials corresponding to the homogeneous boundary conditions such that $B_i(0)=0$ and $B_i(1)=0$ for each $i=1,2,3,\ldots\ldots,n$.

Weighted residual equations corresponding to the equation (2.16a) is given by

$$\int_0^1\left[-\frac{d}{dx}\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right)+\tilde{q}(x)\tilde{u}-\lambda\tilde{r}(x)\tilde{u}\right]B_j(x)\,dx=0,\quad j=1,2,3,\ldots\ldots \tag{2.18}$$

$$\int_0^1\left[-\frac{d}{dx}\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right)\right]B_j(x)\,dx=-\left[\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right)B_j\right]_0^1$$

$$+\int_0^1\left[\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right)\right]\frac{dB_j(x)}{dx}dx \tag{2.19}$$

Integrating each term of equation (2.18) by parts and using equations (2.16b), we obtain the Galerkin weighted residual equation:

$$\sum_{i=0}^{n}\left[\int_0^1\left[\tilde{p}(x)\frac{dB_i}{dx}\frac{dB_j}{dx}+\tilde{q}(x)B_i(x)B_j(x)-\lambda\tilde{r}(x)B_i(x)B_j(x)\right]dx\right.$$

$$\left.-\frac{\beta_0(\mu-\gamma)\tilde{p}(1)B_i(1)B_j(1)}{\beta_1}+\frac{\alpha_0(\mu-\gamma)\tilde{p}(0)B_i(0)B_j(0)}{\alpha_1}\right]c_i=0 \tag{2.20}$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(F_{i,j}-\lambda E_{i,j}\right)c_i=0,\qquad i,j=1,2,3\ldots\ldots,n \tag{2.21}$$

where,

$$F_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dB_i}{dx} \frac{dB_j}{dx} + \tilde{q}(x) B_i(x) B_j(x) \right] dx - \frac{\beta_0 (\mu - \gamma) \tilde{p}(1) B_i(1) B_j(1)}{\beta_1}$$

$$+ \frac{\alpha_0 (\mu - \gamma) \tilde{p}(0) B_i(0) B_j(0)}{\alpha_1} \qquad (2.21a)$$

$$E_{i,j} = \int_0^1 \tilde{r}(x) B_i(x) B_j(x) dx \qquad (2.21b)$$

Finally, the eigenvalues are obtained in solving the system as below

$$F - \lambda E = 0 \qquad (2.21c)$$

2.2.2 Completeness of the set of eigenfunctions of Sturm-Liouville (SL) System

In previous sections, we have defined the Hilbert space H as a subspace of $\ell^2([\gamma, \mu], r(x), dx)$; with functions satisfying the boundary conditions of a Sturm-Liouville system defined on $[\gamma, \mu]$ .Claim is that the set of eigenfunctions of SL system forms a complete orthogonal basis of H:

Let $\left\{ \phi_n \middle| n = 1,2,3,\dots\dots \right\}$ be the set of normalized eigenfunctions of the Sturm-Liouville system. If f be a function in Hilbert space H, then

$$\lim_{n \to \infty} \left\| f - \sum_{k=1}^{n} c_k \phi_k \right\| = 0 \qquad (2.22a)$$

where $c_k = \int_\gamma^\mu \bar{\phi}_k f(x)\, r(x) dx$ (2.22b)

We consider the linear operator L with a field of definition F_L, that is, Lu is defined for $u \in F_L$. The inner product is

$(u,v)_L = (u, Lv) = \int_V u\, Lv\, dx$.

The operator is symmetric if for elements $u, v \in F_L$

$(u, Lv) = (v, Lu)$

The operator is positive definite if for any function in F_L, not identically zero, $(u, Lu) \geq 0$ and is positive bounded below if for any $u \in F_L$

$(u, Lu) \geq \beta(u, u)$, for $\beta > 0$

Uniform convergence: For any $\varepsilon > 0$, we can find n such that

$$\left| u(x) - \tilde{u}_n(x) \right| < \varepsilon \tag{2.23}$$

Convergence in energy requires $\left\| u - \tilde{u}_n \right\| < \varepsilon$ where energy is defined as

$|u| = (u, Lu)$.

The corresponding error bounds are point wise error in equation (2.23). A sequence u_n, converges weakly to an element u of a space if $\underset{n \to \infty}{Lt} (u_n, \phi) = (u, \phi)$ holds for all ϕ in the space. The Galerkin method sometimes yields sequences which converges weakly to a generalized solution.

A set of trial functions form a complete set of functions and they are complete in a space if any function in the space can be expanded in terms of the set of functions, for sufficiently large n [Finlayson (1972)].

$$\left\| u - \sum_{k=1}^{\infty} c_k \phi_k \right\| < \varepsilon \tag{2.24}$$

The following theorem [Mikhlin (1964)] holds:

Theorem. 2.2.1: If an orthonormal set of functions $\{\phi_k\}$ is complete in the sense of convergence in the mean, with respect to some class of functions, then the Fourier series of any function u of the given class

$$u = \sum_{k=1}^{\infty} (u, \phi_k)\phi \tag{2.25}$$

converges in the mean to this function.

A system of functions is said to be complete in energy if, for n sufficiently large

$$\left| u - \sum_{k=1}^{n} c_k \phi_k \right| < \varepsilon \tag{2.26}$$

The method to prove a set of functions is complete is to show that the only function orthogonal to each member of the set is the null function (Courant and Hilbert, 1953). We did not prove completeness here but summarize some known results. Mikhlin and Smolitskiy (1967) state the following theorem:

Theorem. 2.2.2: Let $\phi \in F_L$ where L is a positive definite operator and suppose that the sequence $\{L\phi_n\}$ is complete in the given Hilbert space *H.* The sequence $\{\phi_n\}$ is complete in the energy space H_L. Completeness in energy is required of trial functions, and this Theorem means that trial functions must be capable of representing ϕ and as well as derivatives $L\phi$.

If L and N are positive definite operators and the spaces H_L and H_N contain the same members, any system that is complete in H_L is complete in H_N and vice versa. This is useful for proving completeness. We consider the problem

$$Lu = -\left(p(x)\,u'\right)' + \lambda\, r(x)\, u = 0 \tag{2.27}$$

$$u(0) = u(1) = 0 \quad , \ p(x) \geq \beta > 0, \ r(x) > 0$$

We define

$$N(u) = -u'', \ u(0) = u(1) = 0 \tag{2.28}$$

then the spaces H_L and H_N consist of the same elements for which boundary conditions of the above types hold

$$\int_0^1 \left[u'(x)\right]^2 dx < \infty \tag{2.29}$$

Functions which are complete for equation (2.28) are then complete for equation (2.27). Examples are illustrated in the studies [Mikhlin and Smolitskiy (1967), Collatz (1966a)].

$$\phi_n = x^n(1-x) \text{ and } \phi_n = \sin n\pi x$$

Weierstrass's theorem 2.2.3 [Courant and Hilbert, 1953] says that any continuous function on $\gamma < x < \mu$ can be approximated uniformly by polynomials. The derivatives can be approximated as well. Collatz (1966a) showed that once we have an orthonormal system of functions it is possible to generate new systems by means of a weight function $p(x)$ which is positive and continuous on (γ, μ) and lies between positive bounds *m* and *M.*

We consider the problem

$$Lu=\sum_{i=0}^{2q}(-1)^{i}\frac{d^{i}}{dx^{i}}\left(p_{i}(x)\frac{d^{i}u}{dx^{i}}\right)=\lambda u \tag{2.30a}$$

$$u(0)=u'(0)=u''(0)=\ldots\ldots\ldots\ \ldots\ldots\ldots=u^{q-1}(0)=0 \tag{2.30b}$$

$$u(1)=u'(1)=u''(1)=\ldots\ldots\ldots\ldots\ldots\ldots=u^{q-1}(1)=0 \tag{2.30c}$$

The following system is complete in H_L (assuming $p_{2q}>0$ and p_i are such that Lu is positive definite)

$$\phi_n=x^{n+2q}(1-x)^{2q} \tag{2.31}$$

2.2.3 Non-Self-adjoint ordinary differential equations [Finlayson, 1972]

Non-self-adjoint ordinary differential equations must be solved with the Galerkin method (or another *WRM*) and convergence proofs are known [Mikhlin (1964)]. We consider the problem.

$$Lu=\sum_{i=0}^{2q}(-1)^{i}\frac{d^{i}}{dx^{i}}\left(p_{i}(x)\frac{d^{i}u}{dx^{i}}\right)-Ku=\lambda u \tag{2.32}$$

subject to the homogeneous boundary conditions of two types

$$u(0)=u'(0)=u''(0)=\ldots\ldots\ldots\ \ldots\ldots\ldots=u^{q-1}(0)=0 \tag{2.32a}$$

$$u(1)=u'(1)=u''(1)=\ldots\ldots\ldots\ \ldots\ldots\ldots=u^{q-1}(1)=0 \tag{2.32b}$$

Here, Ku is a linear differential operator of order $2q-1$. We take the coordinate function $L_0u=(-1)^{q}u^{2q}$, which are complete in a space H_0 with inner product (L_0u,v). The proof of convergence depends essentially on showing that the operator L_0 is positive bounded-below and using the consequences of that fact Mikhlin (1967) proved the following theorem.

Theorem 2.2.4: The Galerkin method applied to equation (2.32) gives a convergent sequence in H_0 provided:

(i) The problem has a unique solution:

(ii) The coordinate functions are in the field of definition of L_0 satisfy the boundary conditions (2.32).

For sufficiently large n the following inequalities hold

$$\left|\tilde{u}_n^{k}(x)-u^{k}(x)\right|<\varepsilon,\ \ k<q \tag{2.33a}$$

$$\int_0^1\left[\tilde{u}_n^{q}(x)-u^{q}(x)\right]^2 dx<\varepsilon \tag{2.33b}$$

Therefore, point wise convergence holds for the function and its first $q-1$ derivatives, whereas the q-th derivative converges in the mean. Trial functions can be taken as polynomials, such as

$$\tilde{u}_n(x)=x^{q}(1-x)^{q}\sum_{i=1}^{n}c_i x^{i-1} \tag{2.34}$$

2.2.4 Numerical Examples:

In this section we will present seven numerical examples of second order SLE problems, using the method outlined in the previous section. All the numerical calculations are carried out using MATLAB 13 by an intel(R) Core(TM) i5-4570 CPU with power 3.20 GHz CPU, equipped with 8 GB of Ram.

The convergence of our existing method is measured by the two errors

$$\text{Absolute error, } \delta_k=\left|\lambda^{exact}-\lambda^{(Gal.)}\right| \tag{2.35}$$

$$\text{Relative error, } \varepsilon_k=\left|\frac{\lambda^{exact}-\lambda^{(Gal.)}}{\lambda^{exact}}\right| \tag{2.36}$$

where $\lambda^{(Gal.)}$ denotes the approximate eigenvalues using *n-th* polynomials and $\delta\le 10^{-10}$ depends upon the problems.

Example 2.1: Let us consider one dimensional Helmholtz equation Reutskiy (2006).

$$\begin{cases}\dfrac{d^2u}{dx^2}=\lambda^2 u\\[2ex] u(0)=u(1)=0\end{cases} \tag{2.37a}$$

Table 2.1, displays the first ten eigenvalues for n=15, exploiting both Bernstein and Legendre polynomials, the smallest eigenvalue attains the accuracy up to 10^{-16} and error increases rapidly for higher eigenvalues than the lower values which is better than boundary method. As we increase the grid points or nodes from n=15 to n=30, the error decays fast for all the eigenvalues and consistent accuracy is obtained up to 10^{-13}. We observed that increasing of nodes reveal the stable behaviour of all the eigenvalues for n=30.

Table 2.1: Relative errors between the Galerkin and Boundary methods for example 2.1.

k	Exact eigenvalue	Rel. error Bernst.		Rel. error Legn.		Rel. error Reutskiy (2006)
		n=15	n=30	n=15	n=30	
1	π	9.895×10^{-16}	4.302×10^{-13}	9.891×10^{-16}	4.301×10^{-13}	1.7×10^{-12}
2	2π	2.286×10^{-13}	1.309×10^{-13}	2.261×10^{-15}	1.292×10^{-13}	1.6×10^{-12}
3	3π	6.325×10^{-13}	2.337×10^{-14}	2.261×10^{-15}	2.307×10^{-13}	1.5×10^{-12}
4	4π	8.309×10^{-11}	6.658×10^{-14}	1.018×10^{-14}	6.571×10^{-13}	9.7×10^{-13}
5	5π	7.547×10^{-8}	3.169×10^{-13}	3.135×10^{-10}	3.169×10^{-13}	9.0×10^{-13}
6	6π	7.271×10^{-7}	3.255×10^{-13}	5.368×10^{-9}	3.260×10^{-13}	5.8×10^{-13}
7	7π	2.082×10^{-4}	2.764×10^{-13}	4.546×10^{-6}	2.728×10^{-12}	9.2×10^{-13}
8	8π	6.253×10^{-4}	9.871×10^{-13}	2.074×10^{-5}	1.390×10^{-13}	1.8×-10^{-13}
9	9π	1.639×10^{-2}	1.390×10^{-13}	1.526×10^{-2}	9.742×10^{-12}	5.3×10^{-13}
10	10π	2.623×10^{-2}	4.876×10^{-12}	3.280×10^{-3}	4.875×10^{-12}	1.2×10^{-12}

From table 2.1, it is observed that our present approach accomplishes accurate results and is compatible to the existence new boundary approach for one dimensional Helmholtz equation.

Example 2.2: We compute the eigenvalues of the Sturm-Liouville problem worked out by Taiser *et al* (2010) given below.

$$\begin{cases} -\dfrac{d^2u}{dx^2} + cos^2 xu = \lambda u \\ u(0) = u(\pi) = 0 \end{cases} \tag{2.38}$$

It has been noticed in table 2.2 that the absolute errors achieved for n=10 for both the polynomials, our present method showing better accuracy than that of the Differential Transform method by Chen and Ho (1996), for n=10. Also, when we use n=32 grid points, the maximum absolute error obtained 10^{-6} which is more accurate than those of Sinc Galerkin method for n=32.

Table 2.2: Comparison of absolute errors of WRM with the Sinc Galerkin and differential Transform method for example 2.2.

Exact eigevalues	Absolute error Sinc Gal. n=32	Absolute error Legn.Gal. n=20	Absolute error Bernst. Gal. n=20	Absolute error Diff.Transf. n=10	Absolute error Legn.Gal. n=10	Absolute error Bernst. Gal. n=20
1.24242	1.42×10^{-5}	8.826×10^{-6}	8.826×10^{-6}	3.19×10^{-5}	8.826×10^{-6}	7.112×10^{-6}
4.49479	4.85×10^{-4}	3.079×10^{-6}	8.826×10^{-6}	1.70×10^{-4}	3.122×10^{-6}	1.477×10^{-6}
9.50366	9.91×10^{-3}	4.867×10^{-6}	4.867×10^{-6}	1.83×10^{-3}	2.365×10^{-5}	1.477×10^{-6}
16.50208	1.31×10^{-1}	1.901×10^{-6}	1.901×10^{-6}	3.81×10^{-1}	1.336×10^{-4}	4.634×10^{-4}

Example 2.3: We consider the following SLE with mixed boundary conditions studied by Chen and Ho (1996):

$$\begin{cases} \dfrac{d^2u}{dx^2} = \lambda u \\ u(0) - u'(0) = 0 \\ u(1) + u(1) = 0 \end{cases} \tag{2.39}$$

Here is k-th estimated eigenvalue corresponding to n and the differences between the k-th and $(k-1)$-th eigenvalues are given by $\left|\lambda_i^k - \lambda_i^{k-1}\right| < \varepsilon$, where, ε is very small and $\varepsilon \to 0$.

It is noticed from table 2.3, that the errors decreased with the increasing degree of n and differences between successive eigenvalues converge to zero as the node numbers increased and be given as follows:

Table 2.3: Absolute errors between the successive eigenvalues for example 2.3.

i	Exact eigenvalues Chen and Ho(1996)	Absolute error Bernst. $\lambda_i^6 - \lambda_i^5$	Absolute error present Legn. $\lambda_i^6 - \lambda_i^5$	Absolute error Bernst. $\left\|\lambda_i^{12} - \lambda_i^{11}\right\|$	Absolute Error Legn. $\left\|\lambda_i^{12} - \lambda_i^{11}\right\|$
1	1.71	2.718×10^{-9}	3.66×10^{-4}	0.0000	4.80×10^{-12}
2	13.49	0.0000	2.32×10^{-3}	0.0000	3.22×10^{-8}
3	43.36	4.346×10^{-3}	$8.70 10^{-1}$	1.00×10^{-9}	1.70×10^{-4}

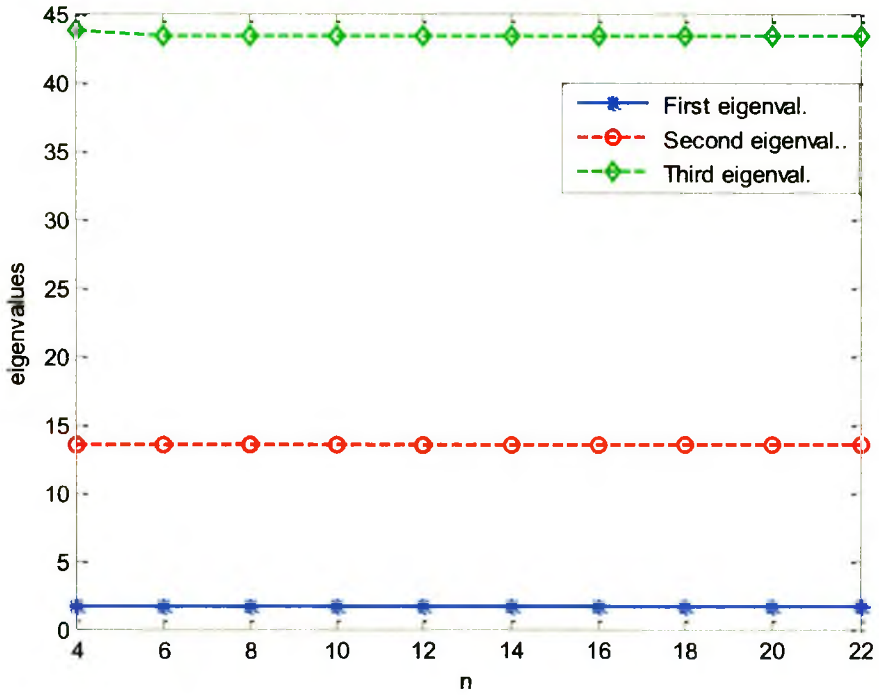

Figure 2.1: Convergence of eigenvalues $\lambda_1, \lambda_2, \lambda_3$ utilizing Bernstein polynomials.

Using Legendre polynomials, we obtain the errors for successive eigenvalues as follows:

$$\left|\lambda_1^{12} - \lambda_1^{11}\right| \leq 0.0000000000, \left|\lambda_2^{12} - \lambda_2^{11}\right| \leq 0.0000000001 \text{ and } \left|\lambda_3^{12} - \lambda_3^{11}\right| \leq 0.000001.$$

Using Bernstein polynomials, we obtain the errors for successive eigenvalues illustrated as follows:

$\left|\lambda_1^{12} - \lambda_1^{11}\right| \leq 0.000000000000$, $\left|\lambda_2^{12} - \lambda_2^{11}\right| \leq 0.000000000000$, $\left|\lambda_3^{12} - \lambda_3^{11}\right| \leq 0.000000001$.

Convergence of the first three eigenvalues are shown in figure 2.1.

Example 2.4: We consider the singular SLE studied by Reutskiy (2010)

$$\begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = (x+1)^{-2} u \\ \\ u(0) = u(1) = 0 \end{cases} \tag{2.40}$$

with the exact solution

$u_n = const \times \sqrt{1+x}\,\sin\left(n\pi \dfrac{ln(1+x)}{ln2}\right)$, $\lambda_n = k_n^2 = \dfrac{1}{4} + \left(\dfrac{n\pi}{\ln 2}\right)^2$ [Reutskiy (2010)].

The first five eigenvalues for different values of n utilizing Bernstein and Legendre polynomials are listed in table 2.4. Our computed results converge to the exact results up to ten significant digits. We conclude that our present method is much efficient and accurate and can compatible with the result of Reutskiy (2010).

Table 2.4: Comparison of eigenvalues for example 2.4

k	Exact eigenvalues	Absolute Error Legn. n=22	Absolute error Bernst. n=25	Reutskiy (2010) $\varepsilon = 10^{-8}$ N=500	Reutskiy (2010) $\varepsilon = 10^{-8}$ N=2000
1	20.79228846	20.79228846	20.79228846	20.79228846	20.79228845
2	82.41915382	82. 41915382	82.41915382	82.41915381	82.41915382
3	185.13059609	185.13059609	185.13059609	185.13059598	185.13059609
4	328.92661528	328.92661528	328.92661528	328.92661454	328.92661528
5	513.80721138	5 13.80721138	513.80721138	513.80720849	513.80721137

Example 2.5: We consider the following Sturm-Liouville problem studied by Chawla and Shivakumar (1993).

$$\begin{cases} \dfrac{d^2u}{dx^2} + \left[\lambda(1+x)^{-2} - (1+x)^2\right] u = 0 \\ \\ u(0) - u'(0) = 0 \;;\; u(1) + u'(1) = 0 \end{cases} \tag{2.41}$$

Relative error norms are listed in table 2.5. We computed approximate eigenvalues utilizing present method and compare the accuracy of our results with results attained by symmetric finite difference method for the smallest eigenvalue. The maximum absolute error achieved by the present method is about 4.54×10^{-9}, whereas error attained by Chawla and Shivakumar (1993) of order 10^{-7}. Besides using only eight Bernstein and Legendre polynomials, superior accuracy has been achieved whereas finite difference method attained less accuracy by using 64 grids. It is also observed that, relative errors remain constant and are not decreasing with the increased degree of polynomials.

Table 2.5: Comparison of relative errors of the smallest eigenvalue with known results of example 2.5 for different values of n.

Exact eigenval. Chawla and Shivakumar (1993)	n	No of grid points Chawla (1983) n	Rel. error Chawla (1983)	Degree of polyn. Legendre n	Degree of polyn. Bernstein n	Rel. error Legendre
	1	8	1.65×10^{-4}	8	1.1700×10^{-8}	4.548×10^{-9}
5.833767621	2	16	1.97×10^{-5}	12	4.097×10^{-9}	4.097×10^{-9}
	3	32	2.32×10^{-6}	16	4.099×10^{-9}	$4.098\times\times 10^{-9}$
	4	64	2.81×10^{-7}	20	4.099×10^{-9}	$4.098\times\times 10^{-9}$

Example 2.6: Let us consider the following periodic Sturm-Liouville problem [Aukulenko and Nesterov (2006)].

$$\begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = 0 \\ u(\pi) = u(-\pi) \\ u'(\pi) = u'(-\pi) \end{cases} \tag{2.42}$$

The equivalent eigenvalue of boundary value problem over [0,1] is,

$$\begin{cases} \dfrac{1}{4\pi^2}\dfrac{d^2u}{dx^2}+\lambda u=0 \\ u(1)=u(0) \\ u'(1)=u'(0) \end{cases} \tag{2.42a}$$

Employing the method for periodic boundary conditions illustrated in section 1.5, we approximate the solution in the form

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}c_iB_i(x) \tag{2.43}$$

The weighted residual equation becomes

$$\sum_{i=1}^{n}\left\{\int_0^1\left[\frac{dB_i}{dx}\frac{dB_j}{dx}+B_i(x)B_j(x)\right]\right\}c_i dx-B_i'(1)\phi_j(1)\left[p(1)-p(0)\right]$$

$$=\lambda\sum_{i=0}^{n}\int_0^1\left[B_i(x)B_j(x)\right]dx \tag{2.44}$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(F_{i,j}-\lambda E_{i,j}\right)c_i=0,\quad i,j=1,2,3\ldots\ldots,n \tag{2.45}$$

where,

$$F_{i,j}=\int_0^1\frac{1}{4\pi^2}\left[\frac{dB_i}{dx}\frac{dB_j}{dx}\right]c_i dx-\frac{1}{4\pi^2}\left\{B_i'(1)\phi_j(1)\left[p(1)-p(0)\right]\right\} \tag{2.45a}$$

$$E_{i,j}=\int_0^1\left[B_i(x)B_j(x)\right]dx \tag{2.45b}$$

Table 2.6 displays the first ten eigenvalues for n=30, exploiting both Bernstein and Legendre polynomials. The smallest eigenvalue attains the accuracy up to 10^{-11} and absolute error increases rapidly for higher eigenvalues than those of lower ones. As we increase the grid points or nodes from n=30 to n=35, the error reduces for the first three eigenvalues in case of Legendre polynomial. Although the error increases in case of Bernstein polynomials for the same.

Accuracy is obtained up to 10^{-13} .We observe that increasing of nodes using Legendre polynomials improve the results which is not obvious for the use of Bernstein polynomials.

Table 2.6: Maximum absolute errors for example 2.6.

k	Exact eigenvalues	Absolute error Legn. n=30	Absolute error Bernst. n=30	Absolute error Legn. n=35	Absolute error Bernst. n=35
1	1.000	0.000000	1.150×10^{-15}	0.000000	1.017×10^{-11}
2	4.000	3.089×10^{-15}	1.950×10^{-14}	0.000000	8.790×10^{-10}
3	9.000	4.026×10^{-11}	1.301×10^{-13}	1.776×10^{-15}	6.327×10^{-9}
4	16.000	5.067×10^{-7}	5.958×10^{-13}	6.446×10^{-11}	1.095×10^{-7}
5	25.000	2.068×10^{-4}	1.417×10^{-13}	3.091×10^{-7}	1.074×10^{-6}
6	36.000	7.138×10^{-3}	1.755×10^{-7}	1.035×10^{-3}	1.799×10^{-5}
7	49.000	1.202×10^{-2}	3.211×10^{-5}	6.073×10^{-3}	8.979×10^{-3}
8	64.000	2.699×10^{-1}	4.080×10^{-2}	7.244×10^{-2}	9.221×10^{-1}
9	81.000	…….	9.076×10^{-1}	8.678×10^{-2}	0.1975×10^{-1}
10	100.000	……..	0.970×10^{-1}	2.788×10^{-1}	……….

Example 2.7: Let us consider [Amidio and Settani, 2015].

$$\begin{cases} -\dfrac{d^2u}{dx^2}+xu=\lambda u \\ u(0)=0 \end{cases} \tag{2.46}$$

The eigenvalues are the zeros of the Airy function given by

$A_i(\lambda)=\left(J_{1/3}+J_{-1/3}\right)\left(\frac{2}{3}\lambda^{\frac{1}{3}}\right)$, where J_α is the Bessel function.

The above problem is given in the semi-infinite interval $0<x<\infty$, and u is finite as $x\to\infty$ .We use the interval from 0 to $\mu\to 2000, 3000,$ ……… and some lower spectrum is complex eigenvalues. This leads $u(0)=0$ and $u(\mu)\to 0$ when $\mu\to\infty$, $\gamma<x<\infty$.

We have used polynomials as trial functions which have been derived over the interval $\gamma<x<\mu$. We first converted the SLE in equation (2.46) over the domain $[0,\infty]$ to an equivalent problem on [1,0]. This exertion is performed by placing

$x=1-\frac{1}{e^{t}}$ so that $\frac{dx}{dt}=\frac{1}{1-t}$; $\frac{d\tilde{u}}{dx}=\frac{d\tilde{u}}{dt}\frac{dt}{dx}=(1-t)\frac{d\tilde{u}}{dt}$

$$\frac{d^{2}\tilde{u}}{dt^{2}}=\frac{d}{dx}\left(\frac{d\tilde{u}}{dx}\right)=\frac{d}{dt}\left(t\frac{d\tilde{u}}{dt}\right)\frac{dt}{dx}=(1-t)^{2}\frac{d^{2}\tilde{u}}{dt^{2}}+(1-t)\frac{d\tilde{u}}{dt}$$

Equation (2.46) transformed to

$$\begin{cases}(1-t)^{2}\frac{d^{2}\tilde{u}}{dt^{2}}-(1-t)\frac{d\tilde{u}}{dt}-=\lambda u\\ u(0)=0\end{cases} \tag{2.47}$$

The problem is regular in $\gamma=0$ although $\mu=1$ is LP. As the coefficients of the equation (2.47) are undefined in the endpoint of the interval $[\gamma,\mu]$, we have taken the interval $[a,b]$ with $\gamma<a<x<b<\mu$.

Furthermore, the problem is singular in 1, the interval is truncated to $b=1-\varepsilon$ with $\varepsilon=10^{-5}$. We have considered the interval the endpoint boundary conditions $u(\mu)\to 0$ as $\mu\to 1$ in $0<x<\mu$. Using the method for infinite domain illustrated in section 2.5, we approximate $u(t)$ as

$$\tilde{u}(t)=\theta_{0}(t)+\sum_{i=1}^{n}c_{i}B_{i}(t) \tag{2.48}$$

Here the boundary terms vanish because the boundary conditions imply $B_{j}(0)=0$ and consequently we set $B_{j}(b)=0$.

$$\sum_{i=1}^{n-1}\left[F_{i,j}+\lambda E_{i,j}\right]c_{i}=0 \tag{2.49}$$

where,

$$F_{i,j} = \int_0^b \left[\left\{ -\frac{d}{dx}\left[B_j(t) \right] B_i'(t) \right\} \right] dt + \left[B_j(t) B_i'(t) \right]_{t=0}^{t=b} \quad (2.49a)$$

$$E_{i,j} = \int_0^b \left[B_i(t) B_j(t) dt \right] \quad (2.49b)$$

The required eigenvalues are obtained solving the system

$$F + \lambda E = 0, \quad (2.50)$$

Sturm-Liouville problems in infinite range $\mu \to \infty$. We can further increase the range of the interval and equation (1.8a) gives

$$\int_\gamma^\mu \left[-\frac{d}{dx}\left(p(t)\frac{d\tilde{u}}{dx} \right) + q(t)\tilde{u} - \lambda r(t)\tilde{u} \right] \phi_j(t)dt = 0, \quad 1 < t < \infty \quad (2.51)$$

We list first four eigenvalues of Airy's equation. Amidio Settani (2015) reported two eigenvalues which are depicted in table 2.7.

We display the present numerical results with the Galerkin WRM, the relative error between the Galerkin WRM and the Chebychev path following method in table 2.7. The results with Galerkin WRM converge to at least eight significant figures. Galerkin WRM with Bernstein polynomial converges relatively slowly than Galerkin WRM but computational cost is less in comparison with the later.

Table 2.7: Absolute errors between the successive eigenvalues for example 2.7.

k	Bernstein/ Legendre n=12	k	Amidio and Settani (2015) λ_k	Bernstein $n=25$ λ_k	Legendre $n=20$ λ_k	$\left\lvert \frac{\lambda^{Cheby} - \lambda^{Gal.}}{\lambda^{Gal.}} \right\rvert$
1	2.3381	1	2.33810740	2.338107398	2.338107412	8.59×10^{-10}
2	4.0879	4	7.94413358	7.94413364	7.94413368	7.552×10^{-9}
3	5.5206					
4	6.7867					

2.3 The Bernstein Collocation Method

Collocation method

The collocation method forces the residual to vanish point wise at a set of preassigned points. More precisely, let $\{x_i\}_{i=0}^{n}$ be a set of Gauss-Lobatto points and let p_n real algebraic polynomials of degree $\leq n$ such that at the interior collocation points the residual

$$R_n(x_i) = Lu_n(x_i) - \lambda\, r(x_i) = 0 , \tag{2.52}$$

satisfies exactly the boundary conditions, i.e.,

$$u_n(x_0) = u_n(x_n) = 0 \qquad (i = 1,2,3,\ldots\ldots\ldots\ldots\ldots, n) \tag{2.53}$$

$$u_n(x) = \sum_{i=0}^{n-1} u_n(x_i)\, p_i(x_i) = 0 \tag{2.54}$$

The collocation points x_i in $[\gamma,\mu]$ is defined as

$$x_i = \frac{\mu-\gamma}{2}\left(\frac{\gamma+\mu}{\mu-\gamma} + \cos\left(\frac{i\pi}{n}\right)\right), \quad i = 1,2,3,\ldots\ldots\ldots\ldots\ldots, n \tag{2.55}$$

2.3.1 Recurrence relations

Theorem 2.3.1: Any generalized Bernstein basis polynomials of degree n can be written as a linear combination of the generalized Bernstein basis polynomials of degree $n+1$.

$$(n+1)B_{i,n}(x) = (n-i+1)B_{i,n+1}(x) + (i+1)B_{i+1,n+1}(x) \tag{2.56}$$

For details of the above we refer [Akyuz, Dascıoglu and Isler (2013)].

Theorem 2.3.2: The first derivatives of n-th degree generalized Bernstein basis polynomials can be written as a linear combination of the generalized Bernstein basis polynomials of degree n:

$$B'_{i,n}(x) = \frac{1}{\mu-\gamma}\Big[(n-i+1)B_{i-1,n}(x) + (2i-n)B_{i,n}(x) - (i+1)B_{i+1,n}(x)\Big] \tag{2.57}$$

Proof: By utilizing Theorem 2.1 the following functions can be written as

$$n\,B_{i,n-1}(x) = (n-i)\,B_{i,n}(x) + (i+1)\,B_{i+1,n}(x)\ , \tag{2.58a}$$

which follows that

$$n\,B_{i-1,n-1}(x) = (n-i+1)\,B_{i-1,n}(x) + i\,B_{i,n}(x)\ , \tag{2.58b}$$

Substituting these relations into the right-hand side of the property (1.86) in chapter 1, the desired relation is obtained.

Theorem 2.3.3: There is a relation between generalized Bernstein basis polynomials matrix and their derivatives in the form

$$B^{(k)}(x) = B(x)Q^{k}\ ;\ k = 1,2,3,\ldots\ldots\ldots\ldots, n. \tag{2.59}$$

Here the elements of $(n+1)\times(n+1)$ matrix, $i,j = 0,1,2,3,\ldots\ldots\ldots\ldots,n$ can be worked out as

$$q_{i,j} = \frac{1}{\mu-\gamma}\begin{cases} n-i\ , & if \quad j=i+1 \\ 2i-n\ , & if \quad j=i \\ -i\ , & if \quad j=i-1 \\ 0\ , & otherwise \end{cases}$$

From theorem 2.3.2 and the property of Bernstein polynomials revealed in section 1.8

Proof [Akyuz *et al* (2013)]:

We have $B'_{0,n}(x) = \dfrac{1}{\mu-\gamma}\left[-n\,B_{0,n}(x) - B_{1,n}(x)\right]$ (2.60a)

$$B'_{1,n}(x) = \frac{1}{\mu-\gamma}\left[n\,B_{0,n}(x) + (2-n)B_{1,n} - 2B_{2,n}(x)\right] \tag{2.60b}$$

$$B'_{2,n}(x) = \frac{1}{\mu-\gamma}\left[(n-1)\,B_{1,n}(x) + (4-n)B_{2,n} - 3B_{3,n}(x)\right] \tag{2.60c}$$

$$B'_{n-1,n}(x) = \frac{1}{\mu-\gamma}\left[2\,B_{n-2,n}(x) + (n-2)B_{n-1,n} - nB_{n,n}(x)\right] \tag{2.60d}$$

$$B'_{n,n}(x) = \frac{1}{\mu-\gamma}\left[\,B_{n-1,n}(x) + n\,B_{n,n}(x)\right] \tag{2.60e}$$

Hence the matrix relation is attained as

$$B'(x) = B(x)Q\ , \tag{2.61}$$

where

$$B(x)=\left[B_{0,n}(x) \quad B_{1,n}(x) \quad B_{2,n}(x)........B_{n,n}(x)\right], \tag{2.61a}$$

$$Q(x)=\begin{bmatrix} -n & n & 0 & .... & 0 & 0 & 0 \\ -1 & 2-n & n-1 & .... & 0 & 0 & 0 \\ 0 & -2 & 4-n & .... & 0 & 0 & 0 \\ 0 & 0 & -3 & .... & 0 & 0 & 0 \\ \vdots & & & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & & n-4 & 2 & 0 \\ 0 & 0 & 0 & & 1-n & n-2 & 1 \\ 0 & 0 & 0 & & 0 & -n & n \end{bmatrix} \tag{2.61b}$$

Similarly, the second derivatives

$$B''(x)=B'(x)Q=B(x)Q^{2} \tag{2.62}$$

Hence, we get derivatives of the unknown function in the form

$$B^{k}(x)=B^{(k-1)}(x)Q=B(x)Q^{(k)} \tag{2.63}$$

Thus, we completed the proof of the above theorem in detailed.

2.3.2 Bernstein polynomial approximations [Pirabaharan and Chandrakumar (2016)]

Let $V_n(x)=\left\{B_{0,n}(x) \quad B_{1,n}(x) \quad B_{2,n}(x)........B_{n,n}(x)\right\}^{T}$, (2.64)

and $V_n \subset H$ be the set of all Bernstein polynomial of degree n. Let g be an arbitrary element in H. Since $V_n(x)$ is a finite dimensional and closed subspace, therefore, V_n is a complete subset of H. Therefore, g has the best unique approximation out of V_n, thus, there exists unique coefficient c_i, $i=0, 1, 2, 3,..............,n$ such that

$$g(x)=\sum_{i=0}^{n} c_i B_{i,n}(x)=C^{T} B_n(x) \text{ [Kreyszig, (1978)]} \tag{2.65}$$

where, $C^{T}=\left[c_0, c_1, c_2, ...c_n\right]$ is thus computed out of

$$\left\langle g, B_n \right\rangle=\int_0^1 g(x)\, B_n dx \tag{2.66}$$

We define

$Q = \left\langle B_n , B_n \right\rangle$ is a $(n+1)\times(n+1)$ dual matrix of B_n.

$$Q_{i+1,j+1} = \int_0^1 B_{i,n}(x)B_{j,n}(x)\,dx = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{j+i}}$$

$$B_{i,n}(x) = \binom{n}{i} x^i \left\{ \sum_{k=0}^{n-i} \binom{n}{i} \binom{n-i}{k} \right\} x^k$$

$$B_{i,n}(x) = \binom{n}{i} \left\{ \sum_{i=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} x^{i+k} \right\} \tag{2.67}$$

$$= \left\{ \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{i} x^{i+k} \right\}, \tag{2.68}$$

$i, j = 0,1,2,3,\ldots\ldots\ldots\ldots,n,$

Using (2.18) we have

$$V_n B_n(x) = M p_n(x) \tag{2.69}$$

where,

$$M_{i+1,j+1} = \begin{cases} 0 \;, & i > j \\ (-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i} \;, & i \le j \end{cases} \tag{2.70a}$$

Matrix M is a $(n+1)\times(n+1)$ upper triangular matrix and $|M| \neq 0$. Therefore M is invertible.

For the function approximation the following lemma illustrated in the articles [Alipour and Rostamy (2011), Pirabaharan and Chandrakumar (2016)].

Lemma: Suppose that the function $g:[0,1] \to R$ is $n+1$ times continuously differentiable i.e., $g \in C^{n+1}([0,1])$, also

$V_n = \{B_{0,n}(x) \;\; B_{1,n}(x) \; B_{2,n}(x) \ldots\ldots B_{n,n}(x)\}^T$.

If $C^T B$ be the best approximation g out of V, then

$$\left\|g-C^T B\right\|_{L^2[0,1]} \le \frac{l}{(n+1)!\sqrt{2n+3}}; \tag{2.71}$$

where, $l = \max_{x\in[0,1]}\left|g^{n+1}(x)\right|$, $C = \left[c_{.0}, c_{1.}, c_{2.}, \dots\dots\dots\dots, c_n\right]^T$

Proof [Yousefi *et al* (2011)]: Since, the set $\{x_1, x_2, x_3, \dots\dots\dots, x_n\}$ is a basis polynomials space of degree *n*.

Therefore, we define the function

$$u_1(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) + \dots\dots\dots\dots\dots\dots + \frac{x^n}{n!}u^n(0)\ . \tag{2.72}$$

From the Taylor expansion, we have

$$\left|u(x)-u_1(x)\right| = \int_0^1 \left|u^{n+1}(\alpha_x)\frac{x^{n+1}}{(n+1)!}\right| \tag{2.73}$$

where, $\alpha_x \in (0,1)$. Since $C^T B$ be the finest estimation of u out of V, $u_1 \in V_n$.

$$\left\|u-C^T\right\|^2_{L^2(0,1)} \le \left\|u-u_1\right\|^2_{L^2(0,1)} = \int_0^1 \left|u(x)-u_1(x)\right|^2 dx$$

$$= \int_0^1 \left|u^{n+1}(\alpha_x)\right|^2 \left(\frac{x^{n+1}}{(n+1)!}\right)^2 dx$$

$$\le \frac{x^{n+1}}{(n+1)!^2}\int_0^1 x^{2n+2}\,dx = \frac{m}{(n+1)!^2\,(2n+3)}$$

Taking the square root on both sides,

$$\left\|u-C^T\right\|^2_{L^2(0,1)} \le \frac{m}{(n+1)!^2\,(2n+3)} \tag{2.74}$$

2.3.3 Second order Sturm-Liouville problems

Matrix Formulation of Collocation method

Consider the following general second order nonsingular Sturm-Liouville problem $\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+q(x)u=\lambda u, \qquad 0< x<1$ (2.75a)

Eqivalently,

$$p(x)\frac{d^2u}{dx^2}+p'(x)\frac{du}{dx}+q(x)u=\lambda u \tag{2.75b}$$

Here $[0, 1]$ is finite interval; $p(x)$, $q(x)$ are all piecewise continuous functions and $p(x)>0$, subject to the homogeneous boundary conditions of $u^{m}(0)=0$,

$$u^{m}(1)=0, \quad \text{for} \quad m=0, 1 \tag{2.75c}$$

the method can be developed for the problem defined in the domain [0, 1] to obtain the solution in terms of shifted Bernstein polynomials

$$R(x)\approx p(x)C^{\eta^T}B^{\eta}(x)Q^2+p'(x)C^{\eta^T}B^{\eta}(x)Q'$$
$$+C^{\eta^T}q(x)B^{\eta}(x)-\lambda C^{\eta^T}B^{\eta}(x) \tag{2.76}$$

where

$$B^{\eta}(x)=\left[B_{0.n}, B_{1.n}, B_{2.n}, \ldots\ldots\ldots\ldots\ldots\ldots B_{n,n}\right] \tag{2.76a}$$

$$C^{\eta}(x)=\left[c_{.0}, c_{1.}, c_{2.}, \ldots\ldots\ldots\ \ldots\ldots\ldots, c_{n}\right]^T \tag{2.76b}$$

Moreover, the matrix forms of the conditions become

$$C^{\eta}B^{\eta}(0)=0 \text{ and } C^{\eta}B^{\eta}(1)=0 \tag{2.77}$$

From the boundary condition properties of Bernstein polynomials, we get $C^{\eta}=0$.

Collocation points are defined by, $x_k=\frac{1}{2}\left(1-\cos\left(\frac{i\pi}{n}\right)\right), \qquad 0<x<1$ (2.78)

Now we can write approximate eigensolution as $u(x)=\sum_{j=2}^{n-1} B_j(x)c_j$ (2.79)

Substituting (2.79) into (2.76) and evaluating at $x=x_i$; $\forall \quad i=0,1,2,.............,n$

Then we obtain the fundamental matrix equation for equation (2.77)

$$P\,B^{\eta}Q^{2}C^{\eta}+TB^{\eta}Q'C^{\eta}+SB^{\eta}C^{\eta}=0 \tag{2.80}$$

$$P=\begin{bmatrix} p(x_0) & 0 & \cdots & 0 \\ 0 & p(x_1) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(x_n) \end{bmatrix}, \tag{2.80a}$$

$$T=\begin{bmatrix} p'(x_0) & 0 & \cdots & 0 \\ 0 & p'(x_1) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p'(x_n) \end{bmatrix}. \tag{2.80b}$$

$$S=\begin{bmatrix} q(x_0)+\lambda & 0 & \cdots & 0 \\ 0 & q(x_1)+\lambda & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q(x_n)+\lambda \end{bmatrix}. \tag{2.80c}$$

The above expression (2.77) can be written is matrix form as

$$A\ +\lambda DC^{\eta}=0 \tag{2.81}$$

This equation corresponds to a system of $n+1$ linear algebraic equations with unknown Bernstein coefficients $c_0,c_1,c_2,........,c_n$.

where,

$$A=\begin{bmatrix} B''_{1,n}(x_1) & B''_{2,n}(x_1) & B''_{3,n}(x_1) & .... & B''_{n-2,n}(x_1) & B''_{n-1,n}(x_1) \\ B''_{1,n}(x_2) & B''_{2,n}(x_2) & B''_{3,n}(x_2) & .... & B''_{n-2,n}(x_2) & B''_{n-1,n}(x_2) \\ .... & .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... & .... \\ B''_{1,n}(x_{n-1}) & B''_{2,n}(x_{n-1}) & B''_{3,n}(x_{n-1}) & .... & B''_{n-2,n}(x_{n-1}) & B''_{n-1,n}(x_{n-1}) \end{bmatrix}. \tag{2.81a}$$

$$D=\begin{bmatrix} B''_{1,n}(x_1) & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & B''_{2,n}(x_2) & 0 & \cdots\cdots & B''_{n-2,n}(x_2) & B''_{n-1,n}(x_2) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 & 0 & \cdots\cdots & 0 & B''_{n-1,n}(x_{n-1}) \end{bmatrix} \tag{2.81b}$$

2.3.4 Numerical examples

In this section we present three numerical examples of second order SLE problems, using the Bernstein collocation method depicted in this section. The computed eigenvalues exploiting Bernstein collocation method are compared with the WRM Galerkin method for the first two examples for brevity.

Example 2.8: Let us consider one dimensional Helmholtz equation [Reutskiy, 2006]

$$\begin{cases} \dfrac{d^2 y}{dx^2} = \lambda^2 u \\ u(0) = u(1) = 0 \end{cases} \tag{2.82}$$

Table 2.8, lists first ten eigenvalues of the problem for n=30. It also illustrates the results using Galerkin method and Boundary method [Reutskiy (2009)]. We noticed that the smallest eigenvalue attains the accuracy up to 10^{-12} and error increases rapidly for higher eigenvalues than the lower values. We accomplished that Bernstein Galerkin, Legendre Galerkin attain superior accuracy than those of Bernstein collocation and Boundary method [Reutskiy (2009)] for one dimensional Helmholtz equation.

Example 2.9: We consider the Sturm-Liouville problem studied by Celik (2005) given by

$$\begin{cases} \dfrac{d^2 y}{dx^2} + \lambda u = (x+0.1)^{-2} u \\ u(0) = u(\pi) = 0 \end{cases} \tag{2.83}$$

Table 2.9 lists the first 20 eigenvalues for different values of n. We observed that Galerkin method exploiting Bernstein polynomials achieves reasonable accuracy for first nine eigenvalues whereas Bernstein collocation method show better performance than that of Galerkin method. However, the last four eigenvalues using Chebychev corrected collocation method achieves better results than that of Bernstein collocation method.

Table 2.8: Comparison of eigenvalues of example 2.8 for different methods.

n	Exact eigenval.	Bernstein coll n=30	Rel. error Bernstein Galerkin n=30	Rel. error Legn Galerkin n=30	Rel. error Reutskiy (2009)
1	π	8.439×10^{-12}	4.30×10^{-13}	4.301×10^{-13}	1.7×10^{-12}
2	2π	8.439×10^{-12}	1.309×10^{-13}	1.292×10^{-13}	1.6×10^{-12}
3	3π	1.549×10^{-10}	2.337×10^{-14}	2.307×10^{-13}	$1.5\text{-}\times10^{-12}$
4	4π	2.278×10^{-8}	6.658×10^{-14}	6.571×10^{-13}	9.7×10^{-13}
5	5π	1.477×10^{-9}	3.169×10^{-13}	3.169×10^{-13}	9.0×10^{-13}
6	6π	5.144×10^{-8}	3.255×10^{-13}	3.260×10^{-13}	5.8×10^{-13}
7	7π	9.940×10^{-9}	2.764×10^{-13}	2.728×10^{-12}	9.2×10^{-13}
8	8π	1.371×10^{-7}	9.871×10^{-13}	1.390×10^{-13}	1.8×10^{-13}
9	9π	4.879×10^{-6}	1.390×10^{-13}	9.742×10^{-12}	5.3×10^{-13}
10	10π	7.371×10^{-5}	4.876×10^{-12}	4.875×10^{-12}	1.2×10^{-12}

Example 2.10: The Boyd equation considered by Baily *et al* (1993), Auzinger (2006), Singh and Kumar (2013).

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \lambda u(x) + \dfrac{1}{x}u(x) \\ u(0) = u(1) = 0 \end{cases} \tag{2.84}$$

The endpoint 1 is regular, the endpoint 0 is singular and is the LC case. In table 2.10, the first five eigenvalues are displayed. We listed the present numerical results with the polynomial collocation [Auzinger *et al* (2006)], the relative error between the Bernstein collocation and the polynomial collocation as well. Auzinger *et al* (2006) computed approximate eigenvalues denoted by $\lambda_k{}^*$ and λ^{ref}. using different tolerances. Here N_k^* is the number of grid points with default tolerance abs. Tol=10^{-6}, rel. Tol=10^{-3} and N_k^{ref} is the respective number of grid points with stricter tolerance abs. Tol= rel. Tol =10^{-8}. The numerical results employing current method are converged to at least 9 significant figures and relative errors obtained by Bernstein collocation method are much smaller. From table 2.10, we observe that eigenvalues work out by our present approach agrees well with the other methods.

Table 2.9: Comparison of eigenvalues of example 2.9 for different methods.

k	Exact eigenvalues	Eigenval. Bern coll. n=35	Absolute error Bernstein coll.	Absolute error Bernstein Galerkin (present)	Eigenval. Corrected coll. Celik (2005)	Absolute error Cheby Celik (2005)
1	1.51986582	1.5198659	0.000000	0.0000000	1.5198659	0.0000000
2	4.9433098	4.9433098	0.000000	0.0000000	4.9433098	0.000000
3	10.284663	10.284662	0.00000	0.000000	10.284662	0.000000
4	17.559958	17.559957	0.000000	0.000000	17.559957	0.000000
5	26.782863	26.782863	0.000000	0.000000	26.782863	0.000000
6	37.964426	37.964584	0.000000	0.000000	37.964584	0.000000
7	51.113358	51.113358	0.000000	0.000000	51.113358	0.000000
8	66.236448	66.236448	0.000000	0.000000	66.236448	0.000000
9	83.338962	83.338962	0.000000	0.000000	83.338962	0.000000
10	102.42499	102.42499	0.000000	0.0000	102.42499	0.00000
11	123.49771	123.49771	0.0000000	0.0012	123.49771	0.00000
12	146.55961	146.55961	0.000000	0.11801	146.55961	0.00000
13	171.61264	171.61265	0.000000	……..	171.61265	0.00000
14	198.65837	198.65837	0.00000	.. ……	198.65838	0.00000
15	227.69803	227.69803	0.00000	………	227.69803	0.00000
16	258.73262	258.73258	0.0000	………	258.73262	0.00000
17	291.76293	291.76206	0.000759	……..	291.76282	0.00000
18	326.78963	326.79782	0.007192	………	326.78962	0.00000
19	363.81325	363.74147	0.07177	………	363.81338	0.00013
20	402.83424	403.30631	0.4727	……..	402.83237	0.00187

Table 2.10: Comparison of eigenvalues obtained by present method for example 2.10 with other methods

k	Present method λ_k	Eigenval. Auzinger *et al* (2006) λ^{ref}	λ_k^*	Rel. error $\frac{\lambda_k - \lambda^{ref}}{\lambda^{ref}}$	Rel error $\frac{\lambda_k^* - \lambda^{ref}}{\lambda^{ref}}$	N_k^*	N_k^{ref}	n_k^*
1	7.37398502	7.37398502	7.37398502	4.827×10^{-10}	4.827×10^{-10}	32	153	20
2	36.3360196	36.3360196	36.3360196	1.265×10^{-10}	1.451×10^{-9}	32	267	20
3	85.2925821	85.2925821	85.2925811	6.906×10^{-11}	1.111×10^{-8}	32	425	20
4	154.0986237	154.098624	154.098619	1.6872×10^{-9}	2.786×10^{-8}	32	583	20
5	242.7055592	242.705559	242.7055545	9.580×10^{-10}	5.851×10^{-8}	32	741	20

2.4 Chebyshev-Legendre Spectral Collocation Method

2.4.1 Legendre and Chebyshev polynomials

In this part before moving onto our main study, we discuss some basic properties to support our scheme. In this study we are concerned with approximating solutions of second order Sturm-Liouville problems using Legendre polynomials as trial functions and Chebychev Gauss-Lobatto nodes for collocation.

The Legendre polynomials of degree n defined on [-1, 1] is given as [Atkinson and Kendall (1989)]

$$L_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}, \tag{2.85a}$$

where,

$$N = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even} \\ \frac{n-1}{2}, & \text{when } n \text{ is odd} \end{cases} \tag{2.85b}$$

The Rodrigues' formula of degree n is defined as:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1), \quad \text{where } n \geq 1 \tag{2.86}$$

The n-th order Legendre differential equation is given by,

$$-\left((1-x^2)L_n'(x)\right)' = n(n+1)L_n(x) \tag{2.87}$$

Provided L_n is bounded on [-1,1] i.e., $\left|L_n(x)\right| \le 1$.

Properties of Legendre polynomials

(i) $L_n(\pm 1) = (\pm 1)^n$,

(ii) $L_n'(\pm 1) = \frac{1}{2}(\pm 1)^{n-1} n(n+1)$

(iii) $\int_{-1}^{1} L_n(x)L_r(x)dx = \frac{\delta_{nr}}{n+\frac{1}{2}}$,

The Legendre polynomials are orthogonal with respect to the L^2 (-1,1) inner product. Also, these polynomials are complete in the sense that for any

$$u(x) = \sum_{n=0}^{\infty} \tilde{u}_n L_n(x) \tag{2.88}$$

where, (2.89a)

$$\tilde{u}_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} u(x)L_n(x)dx \tag{2.89b}$$

where, the sum converges to $L^2(-1,1)$ norm. Legendre polynomials which are orthogonal in the interval [-1, 1] satisfy the following recurrence relation.

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), \tag{2.90}$$

Chebyshev polynomials [Lui (2011)] of degree n over an interval $[\gamma, \mu]$ are defined by

$$T_n(x) = cos\left(ncos^{-1}\left(\frac{2}{\mu-\gamma}\left(x-\frac{\gamma+\mu}{2}\right)\right)\right), \quad n = 0,1,2......., \quad \gamma \le x \le \mu \tag{2.91}$$

The collocation points x_k in $[\gamma, \mu]$ are defined as

$$x_k = \frac{\gamma-\mu}{2}\left(\frac{\gamma+\mu}{\mu-\gamma} + \cos\left(\frac{k\pi}{n}\right)\right), \tag{2.92}$$

$T_n(x)$ is bounded on [-1,1]. Chebyshev Gauss-Lobatto nodes are the zeroes of the orthogonal polynomial $(1-x^2)\,T_n'(x)$. These nodes are placed symmetrically around $x=0$ and denser near the end points $x=\pm 1$. The spacing between the collocation points near the boundaries is of order $O(N^{-2})$, in contrast with $O(N^{-1})$ for finite differences or finite elements.

Root finding using Newton Raphson's iterative method

To compute the zeros of orthogonal polynomials, from three-term recurrence relation sometimes difficult as becomes very large which leads the method suffer from round-off errors. An alternative approach is to compute the zeros of $L_n^{(r)}(x)$ numerically, where $r<n$ is the order of derivative. If x_k^0 is an initial approximation to the k-th zero of p_n,

$$x_k^r = x_k^r + D\left(x_k^r\right)$$

where $D = -p_n(x)/p_n'(x)$.

Newton's method under some standard assumptions can achieve rapid convergence quantified as q-quadratic, namely, the number of significant figures in the approximates doubles with each iteration.

2.4.2 Legendre-Chebyshev Spectral Collocation method

In Spectral collocation methods, there are basically two steps to obtaining a numerical approximation to a solution of differential equation. Firstly, an appropriate finite or discrete representation of the solution must be chosen. However, it is well known that the Lagrange interpolation polynomial based on equally spaced points does not give a satisfactory approximation to general smooth functions. superior results are obtained by relating the collocation points to the structure of classical orthogonal polynomials. Secondly, we necessitate to obtain a system of algebraic equations from discretization of the original equation. In our present study we implement the spectral collocation method is in physical space and approximates derivative values by direct differentiation of the Lagrange interpolating polynomial at a set of Gauss-Lobatto points.

This proposed scheme works well for differential equations with any type of boundary conditions. In practice the Gauss-Lobatto points are taken in order to be able to prescribe function values at the boundary. The Gauss points are all located in the internal of the domain. As the weight function for Legendre polynomials is unity, for combination with weak formulations of differential equations, Legendre polynomials are more suitable then Chebyshev polynomials. Polynomial interpolations based on Chebychev nodes are often used to approximate smooth function. We define second order Sturm-Liouville eigenvalue problem as previously by:

$$Lu_n := -\frac{d}{dx}\left(p(x)\frac{du(x)}{dx} \right) + q\,(x)u(x) = \lambda r(x)u\,(x), \tag{2.93a}$$

$p(x)$, $q(x)$, $\sigma(x) > 0$ are piecewise continuous functions and Lu_n is a self-adjoint operator for the left-hand side of equation (2.93a). Hence eigenvalues of a self-adjoint equation are all real.

Here we consider again the following homogeneous Sturm-Liouville boundary value problem (2.93a) specified as

$$\frac{d}{dx}\left(x^{\alpha}\frac{du}{dx} \right) = \lambda r(x)u(x),\; \gamma < x < \mu \tag{2.94b}$$

$$\left.\begin{aligned} u(\gamma) &= 0 \\ u(\mu) &= 0 \end{aligned}\right\} \tag{2.94c}$$

For every $\alpha > 0$, the problem (2.94b) is called singular.

Let x_k be the set of Gauss-Lobatto nodes with two end points x_0 and x_n, where $k = 0,1,2,\ldots\ldots\ldots,n$ and let p_n be the set of all real algebraic polynomials of degree $\le n$. The spectral collocation method for equation (2.94a) is to find the $u_n \in p_n$ such that the residual $R_n(x) = Lu_n(x) - \lambda r(x)$ equal to zero at the interior collocation points. The Spectral methods are particularly attractive due to the following approximation properties. The "distance" between the solution $u(x)$ of the above problem and its spectral approximation $\tilde{u}_n$ is of order $\frac{1}{n^s}$, i.e.,

$|u-\tilde{u}_n| \le \frac{c}{n^s}$, where s is the regularity index. Moreover, if $u(x)$ is infinitely derivable, the above distance vanishes faster than any power of $\frac{1}{n}$, which is termed as spectral accuracy. In other words, while spectral methods use trial (shape) and test functions, defined globally and very smooth, in finite elements methods these functions are defined only locally and are less smooth.

2.4.3 Legendre Pseudo Spectral Differentiation matrices:

Spectral collocation methods, also known as Pseudo Spectral methods, are obtained when the test functions in the variational formulation are Dirac functions based on a pre-determined set of collocation points. The present method is known as *nodal* method based on interpolation formulas that utilize Lagrange polynomials. Here unknowns are the actual sampled values of the function and so no transformation is needed.

$$S_L(x) = (1-x^2)L_n'(x) \tag{2.95}$$

$$S_L'(x) = -2x L_n'(x) + (1-x^2)L_n''(x) \tag{2.96}$$

Using equation (2.87), we have

$$S_L'(x) = -n(n+1)L_n(x) \tag{2.97}$$

Lagrange polynomial for the nodes $\{x_0, x_1, x_2, ..., x_n\}$ be defined as

$$l_i(x) = \frac{(1-x^2)L_n'(x)}{S_L'(x_i)\,(x-x_i)} \tag{2.98}$$

Since $\{x_k\}$ are the roots of $\left(1-x^2\right)L_n'(x)$.

$$l_i(x_i) = \frac{1}{S_L'(x_i)} \lim_{x\to x_i} \frac{\left(1-x^2\right)L_n'(x)}{x-x_i} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{S_L'(x_i)} \lim_{x\to x_i} \frac{(1-x^2)L_n''(x) - 2x L_n'(x)}{1} \quad \text{[using L' Hospital rule]}$$

$$=\frac{1}{S'_L(x_i)}\lim_{x\to x_i} S'_L(x)=\frac{1}{S'_L(x_i)}S'_L(x_i)=1 \tag{2.99}$$

We can define

$$\delta_{ik}=\begin{cases}0, & i\neq k\\ 1, & i=k\end{cases} \tag{2.100}$$

Again, differentiating equation (2.98)

$$l'_i(x)=\frac{1}{S'_L(x_i)}\frac{\left(x-x_i\right)\left\{L''_n(x)-2xL'_n(x)\right\}-(1-x^2)L'_n(x)}{\left(x-x_i\right)^2}$$

$$=\frac{1}{S'_L(x_i)}\frac{(x-x_i)S'_L(x)-S_L(x)}{(x-x_i)^2}$$

$$l'_i(x)=\frac{1}{S'_L(x_i)}\left[\frac{S'_L(x)}{x-x_i}-\frac{S_L(x)}{\left(x-x_i\right)^2}\right] \tag{2.101}$$

For any continuous function u, we define Legendre interpolate of u by $I^L_n u$, can be expressed as the unique polynomial in p_n such that,

$$\left(I^L_n u\right)(x)=\sum_{i=0}^{n}u_i l_i(x),\quad 0\le k\le n \quad \text{and} \left(I^L_n u\right)'(x)=\sum_{i=0}^{n}u_i l'_i(x) \tag{2.102}$$

From equation (2.102)

$$l'_i\left(x_k\right)=\frac{1}{S'_L(x_i)}\frac{S'_L\left(x_k\right)}{\left(x_k-x_i\right)}=\frac{1}{-n(n+1)L_n(x_i)}\frac{-n(n+1)L_n\left(x_k\right)}{L_n\left(x_k-x_i\right)},$$

Since $S_L\left(x_k\right)=0$

$$=\frac{L_n\left(x_k\right)}{L_n\left(x_i\right)\left(x_k-x_i\right)},\quad i\neq k \tag{2.103}$$

Also, $D_{i,i}=l'_i\left(x_i\right)=\frac{1}{S'_L\left(x_i\right)}\lim_{x\to x_i}\frac{S'_L(x)\left(x-x_i\right)-S_L(x)}{\left(x-x_i\right)^2}=\frac{S''_L\left(x_i\right)}{2S'_L\left(x_i\right)}$

$$=\frac{L'_n\left(x_k\right)}{2L_n\left(x_k\right)} \tag{2.104}$$

Equation (2.104) is obtained using equation (2.97) and L' Hospital Rule.

$D_{k,k} = 0$ for $1 \le k \le n-1$

$$D_{0,0} = -D_{n,n} = \frac{n(n+1)}{4}. \tag{2.105}$$

Equation (2.105) is obtained using properties of Legendre polynomials illustrated in section 1.11.

It can be shown that the $(n+1)\times(n+1)$ Legendre Pseudo spectral derivative matrix D, which computes the derivative exactly at the Legendre Gauss-Lobatto nodes, gives the derivative of the interpolate of u. Using equations (2.103), (2.104) and (2.105), the Legendre Pseudo spectral derivative can be written together as,

$$D_{k,i} = \begin{cases} \dfrac{1}{x_k - x_i}\dfrac{L_n(x_k)}{L_n(x_i)}, & i \ne k \\ 0, & 1 \le i = k \le n-1 \\ \dfrac{n(n+1)}{4}, & i = k = 0 \\ -\dfrac{n(n+1)}{4}, & i = k = n; \end{cases} \tag{2.106}$$

2.4.4 Formulation of second order SLEs:

In this section, we use Legendre orthogonal polynomials to approximate the solution as a weighted sum of polynomials of second order Sturm-Liouville problems. We collocate at Chebychev clustered grid points to generate a system of equations to approximate the weights for the polynomials. For it the equation (2.93a) are to be put in the form as follows:

$$u''(x_k) + p(x_k)u'(x_k) + q(x_k)u - \lambda r(x_k)u = 0, \tag{2.107}$$

Boundary conditions of mixed type

General boundary conditions (mixed type) are written as:

$$\alpha_1 u_n + \beta_1 \sum_{k=0}^{n} (D^1)_{nk} u_k = 0 \tag{2.108a}$$

$$\alpha_2 u_0 + \beta_2 \sum_{k=0}^{n} (D^1)_{0k} u_k = 0 \tag{2.108b}$$

Equation (2.108a) leads to

$$\beta_1 (D^1)_{n0} u_0 + \left[\alpha_1 + \beta_1 (D^1)_{nn}\right] u_n = -\beta_1 \sum_{k=1}^{n-1} (D^1)_{nk} u_k \tag{2.109a}$$

Similarly, from (2.108b), we have

$$\left[\alpha_2 + \beta_2 (D^1)_{00}\right] u_0 + \beta_2 (D^1)_{0n} u_n = -\beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k , \tag{2.109b}$$

$$\gamma \le x \le \mu$$

$$u_0 = \frac{-\beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k - \beta_2 (D^1)_{0n} u_n}{\alpha_2 + \beta_2 (D^1)_{00}} \tag{2.110}$$

Substituting equation (2.110) into equation (2.109a),

$$\beta_1 (D^1)_{n0} \times \left[\frac{-\beta_2 (D^1)_{0n} u_n}{(\alpha_2 + \beta_2 (D^1)_{00}} - \frac{\beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k}{(\alpha_2 + \beta_2 (D^1)_{00}} \right] + \left[\alpha_1 + \beta_1 (D^1)_{nn}\right] u_n$$

$$= -\beta_1 \sum_{k=1}^{n-1} (D^1)_{nk} u_k \tag{2.111}$$

Now let,

$$a_1 = \beta_1 (D^1)_{n0} ,\ b_1 = \alpha_1 + \beta_1 (D^1)_{nn} ,\ c_1 = \alpha_2 + \beta_2 (D^1)_{00} ,\ d_1 = \beta_2 (D^1)_{0n} \tag{2.112}$$

Using equation (2.112), equation (2.111) becomes

$$a_1 \times \left[\frac{-d_1 u_n}{c_1} - \frac{\beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k}{c_1} \right] + b_1 u_n = -\beta_1 \sum_{k=1}^{n-1} (D^1)_{nk} u_k$$

$$\frac{(b_1 c_1 - a_1 d_1)}{c_1} u_n = \frac{-\beta_1 c_1 \sum_{k=1}^{n-1} (D^1)_{nk} u_k}{c_1} + \frac{a_1 \beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k}{c_1}$$

$$u_n = \frac{\beta_1 c_1 \sum_{k=1}^{n-1} (D^1)_{nk} u_k}{a_1 d_1 - b_1 c_1} - \frac{a_1 \beta_2 \sum_{k=1}^{n-1} (D^1)_{0k} u_k}{a_1 d_1 - b_1 c_1} \tag{2.113a}$$

$$\text{where, } u_n = \sum_{k=1}^{n-1} \theta_{nk} u_k \tag{2.113b}$$

$$= \frac{c_1\beta_1(D^1)_{nk} - a_1\beta_2(D^1)_{0k}}{a_1d_1 - b_1c_1} \tag{2.113c}$$

Also, from equation (2.110)

$$u_0 = -\frac{d_1}{c_1}u_n - \frac{\beta_2 \sum_{k=1}^{n-1}(D^1)_{0k} u_k}{c_1}$$

$$= -\frac{d_1}{c_1}\left\{\frac{\beta_1 c_1 \sum_{k=1}^{n-1}(D^1)_{nk} u_k - a_1\beta_2 \sum_{k=2}^{n-1}(D^1)_{0k} u_k}{a_1d_1 - b_1c_1}\right\} - \frac{\beta_2 \sum_{k=1}^{n-1}(D^1)_{0k} u_k}{c_1}$$

On simplification

$$u_0 = -\frac{\left(\beta_1 d_1 \sum_{k=1}^{n-1}(D^1)_{nk} - b_1\beta_2 \sum_{k=1}^{n-1}(D^1)_{0k}\right)u_k}{(a_1d_1 - b_1c_1)} \tag{2.114a}$$

$$u_0 = = \sum_{k=1}^{n-1} \theta_{0k} u_k \tag{2.114b}$$

$$\text{where } \theta_{0k} = \frac{b_1\beta_2(D^1)_{0k} - \beta_1 d_1 (D^1)_{nk}}{(a_1d_1 - b_1c_1)} \tag{2.114c}$$

Let the constants β_1 and β_2 be nonzero.

Using equations (2.113b) and (2.114b), the equation (2.107) reduces to

$$\sum_{k=1}^{n-1}\left[(D^2)_{i,k} + p(x_i)(D^1)_{i,k} + \left\{q(x_i) - \lambda r(x_i)u(x_i)\right\}\delta_{i,k}\right]u_k$$

$$+\left[(D^2)_{i,0} + p(x_i)(D^1)_{i,0} + q(x_i) - \lambda r(x_i)\right]\left[\sum_{k=1}^{n-1}\theta_{0k}u_k\right]$$

$$+\left[(D^2)_{i,n} + p(x_i)(D^1)_{i,n} + q(x_i) - \lambda r(x_i)\right]\left[\sum_{k=1}^{n-1}\theta_{nk}u_k\right] = 0, \qquad 1 \le i \le n-1$$

$$= \sum_{k=1}^{n-1} \left[(D^2)_{i,k} + p(x_i)(D^1)_{i,k} + \{q(x_i) - \lambda r(x_i) u(x_i)\}\delta_{i,k} \right] u_k$$

$$- \left[(D^2)_{i0} + p(x_i)(D^1)_{i0} + q(x_i) \right] \theta_{0k} u_k$$

$$- \left[(D^2)_{in} + p(x_i)(D^1)_{in} + q(x_i) \right] \theta_{nk} u_k + \lambda r(x_i)(\theta_{0k} + \theta_{nk}) u_k = 0 \quad (2.115)$$

The Spectral collocation solution for the eigenvalues for the Sturm-Liouville problem in equation (2.107) with the general boundary conditions (2.108a) and (2.108b) takes the matrix equation form as given by

$$A\tilde{u} = \lambda B\tilde{u} \quad (2.116)$$

where,

$$a_{i,k} = (D^2)_{i,k} + p(x_i)(D^1)_{i,k} + \{q(x_i) - \lambda r(x_i) u(x_i)\}\delta_{i,k}$$

$$- \left[(D^2)_{i0} + p(x_i)(D^1)_{i0} + q(x_i) \right] \theta_{0k}$$

$$- \left[(D^2)_{in} + p(x_i)(D^1)_{in} + q(x_i) \right] \theta_{nk} \quad (2.117a)$$

$$b_{i,k} = b_{i,k} = r(x_i)(\theta_{0k} + \theta_{nk}) \quad (2.117b)$$

Here, $\tilde{u} = \left[u_1, u_2, \dots\dots\dots, u_{n-1} \right]^T$, $A = (a_{i,k}), B = (b_{i,k})$ are $(n-1)\times(n-1)$ matrices, and $\delta_{i,k}$ is the Kronecker delta. Solving equation (2.116) the required eigenvalues are obtained.

Sturm-Liouville problems(SLEs) with Dirichlet boundary conditions:

Boundary conditions are: $u(\gamma) = 0;\ u(\mu) = 0$ (2.118)

The spectral differentiation matrix for the SLE (2.107) incorporating the boundary conditions (2.118)

$$\sum_{k=1}^{n-1} \left[(D^2)_{i,k} + p(x_i)(D^1)_{i,k} + \{q(x_i) - \lambda \mu(x_i)\}\delta_{i,k} \right] u_k = 0 \quad (2.119)$$

Dirichlet boundary conditions (2.118) satisfies the following linear system

$$A\tilde{u} = \lambda B\tilde{u} \quad (2.120)$$

$$\widetilde{u} = \left[u_1, u_2, \ldots\ldots\ldots u_{n-1} \right]^T \tag{2.121a}$$

$$b_{i,k} = \mu(x_i)\delta_{i,k} \tag{2.121b}$$

Solving equation (2.120) required eigenvalues are obtained.

For nonlinear Bratu problem, we have first computed the initial values on neglecting the non-linear terms and using the above method Then using the Newton's iterative method we found the numerical approximations for desired nonlinear BVP.

2.4.5 Convergence analysis

The Legendre polynomials, $L_k(x)$, $k = 0,1,2\ldots\ldots\ldots\ldots$ form eigenfunctions of singular Sturm-Liouville problems given as

$$\left((1-x^2)L_k'(x) \right)' + k(k+1)L_k(x) = 0 \tag{2.122}$$

If $\{L_k(x)\}$ be a set of orthogonal polynomials with respect to weight function $w(x)$, then

$$\int_{-1}^{1} L_k(x)L_l(x)w(x)dx = 0, \quad for\ k \neq l \tag{2.123}$$

We consider $\phi(x)$ be the functions in the Hilbert space $L_w^2(-1,1)$ such that

$$\int_{-1}^{1} |\phi(x)|^2 w(x)dx < \infty. \tag{2.124}$$

For any continuous functions $\phi(x)$ and $\psi(x)$ in $L_w^2(-1,1)$, we have

$$\int_{-1}^{1} |\phi(x)|^2 w(x)dx < \infty. \tag{2.125}$$

Suppose, $u(x)$ be the eigenfunction of the Sturm-Liouville problem in the Hilbert space then the series expansion in the case of Legendre polynomials is

$$u(x) = \sum_{k=0}^{\infty} \widetilde{u}_k L_k(x) \tag{2.126}$$

Approximate solution in terms of truncated Legendre series is

$$u(x) = \sum_{k=0}^{n} \tilde{u}_k L_k(x) \tag{2.127}$$

where the coefficients, $\tilde{u}_k = \dfrac{\int_{-1}^{1} u(x) L_k(x) w(x) dx}{\|L_k(x)\|^2}$, (2.128)

since $w(x) = 1$, for Legendre polynomials

we have,

$$\|L_k\|^2_{L^2_w(-1,1)} = \int_{-1}^{1} |L_k|^2 dx \tag{2.129}$$

If P_n be the orthogonal projection operator onto the Legendre polynomial space Π_n, then

$$(P_n \tilde{u}, v)_w = (u, v)_w, \qquad \forall\, v \in \Pi_n \qquad \forall\, v \in \Pi_n \tag{2.130}$$

The completeness of $\{L_k(x)\}$ implies that,

$$\|u - P_n u\| \to 0 \text{ as } n \to \infty.\ \forall\, u \in L^2_w(-1,1). \tag{2.131}$$

Hence, following the above convergence result, if $\forall\, u \in C^\infty$, the produced error approaches to zero as $n \to \infty$ and with exponential rate $O(\lambda^m e^{-\gamma n})$, $\gamma > 0$, for the m-the eigenvalue [Lui (2011), Taher *et al*(2013)].

2.4.6 Condition number of Legendre Collocation

Let n be a positive integer and $u \in p_n$ be a non-trivial solution of

$$-u'' = \lambda u \tag{2.132}$$

Then $c_1 \le \lambda \le c_2 n^4$, where c_i are constants independent of n. From the eigenvalue relation, we obtain

$$-u''(x_k) = \lambda u(x_k) \text{on } (-1, 1), \quad k = 1,2,3,\ldots\ldots\ldots\ldots, n-1 \tag{2.133}$$

where, $\{x_k\}$ are the Legendre Gauss-Lobatto points. Then

$$-\sum_{k=0}^{n} u''(x_k) u(x_k) w_k = \lambda \sum_{k=0}^{n} u(x_k)^2 w_k \tag{2.134}$$

w_k are the weights corresponding to Gauss-Lobatto points. Since all polynomials of degree by p_{2n-1} can be integrated exactly using Gaussian Quadrature

$$-\int_{-1}^{1} u'' u = \int_{-1}^{1} (u')^2 = \lambda [u,u]_n \tag{2.135}$$

From the equivalence of norm $[\cdot,\cdot]_n^{1/2}$, we obtain

$$-c_3 \frac{\int_{-1}^{1} u'^2}{\int_{-1}^{1} u^2} = c_4 \frac{\int_{-1}^{1} u'^2}{\int_{-1}^{1} u^2} \tag{2.136}$$

2.4.7 Numerical Experiments :

In this section we have presented six numerical examples of second order linear Sturm-Liouville problems in brief, using the method outlined in the previous section. One nonlinear Bratu type BVP is also illustrated concisely. The convergence of our existing method is measured by the absolute and relative error

$$\text{Absolute error, } \delta_k = \left| \lambda^{exact} - \lambda^{(coll.)} \right| \tag{2.137a}$$

$$\text{Relative error, } \varepsilon_k = \left| \frac{\lambda^{exact} - \lambda^{(coll.)}}{\lambda^{exact}} \right| \tag{2.137b}$$

Nonlinear BVP is calculated by the absolute error of two consecutive iterations such that of two consecutive iterations such that

$$\left| \tilde{u}_n^{N+1} - \tilde{u}_n^{N} \right| < \delta$$

where δ is less than 10^{-10} and δ is the Newton's iteration number.

Example 2.11: Let us consider one dimensional Helmholtz equation [Reutskiy (2006)].

$$\begin{cases} \dfrac{d^2 u}{dx^2} = \lambda^2 u \\ u(0) = 0 \\ u(1) = 0 \end{cases} \tag{2.138a}$$

We transfer the equation (2.138a) by changing the variables $x = \frac{1}{2}t + \frac{1}{2}$, the Sturm Liouville problem transforms to

$$\begin{cases} 4\dfrac{d^2 u}{dt^2} = \lambda u \quad , t \in (-1,1) \\ u(-1) = 0 \\ u(1) = 0 \end{cases} \tag{2.138b}$$

The Differential eigenvalue problems in matrix form can be written together with boundary conditions as

$$\sum_{k=1}^{n-1} \left[(D^2)_{i,k} - \lambda \delta_{i,k} \right] u_k = 0 \tag{2.139}$$

Table 2.11, lists first ten eigenvalues for *n*=20. Smallest eigenvalue attains the accuracy up to 10^{-14} and error increases rapidly for higher eigenvalues than the lower values which is better than boundary method. As we increase the grid points or nodes from *n*=20 to *n*=30, the error decays very fast for all the eigenvalues and accuracy is obtained up to 10^{-15} .We observed that increasing of nodes reveal the stable behaviour of all the eigenvalues for *n*=30.

From table 2.11, it is observed that our present approach attains more accurate results than the new boundary approach for one dimensional Helmholtz equation.

Example 2.12: Here, we consider the SLE with Neumann boundary conditions [Tao Tang (2006)]

$$\begin{cases} \dfrac{d^2 u}{dx^2} = \lambda u \\ u'(0) = 0 \\ u'(1) = 0 \end{cases} \tag{2.140}$$

The exact smallest eigenvalues is 2.4137.

We define spectral radius as

$$\rho(A) = \max\{|\lambda| : |A - \lambda I| = 0\} \tag{2.141a}$$

$$\kappa(A) = \max\{|\lambda| : |A - \lambda I| = 0\} / \min\{|\lambda| : |A - \lambda I| = 0\} \tag{2.141b}$$

Table 2.11: Comparison of absolute errors between the new boundary method (ε -procedure) [Reutskiy (2006)] and present method for example 2.11

k	Exact eigenvalues	rel. error present method n=20	rel. error present method n=30	rel. error Reutskiy (2006) $\varepsilon = 10^{-6}$
1	π	3.251×10^{-14}	3.251×10^{-14}	1.7×10^{-12}
2	2π	3.209×10^{-13}	3.223×10^{-13}	1.6×10^{-12}
3	3π	2.344×10^{-13}	2.344×10^{-13}	1.5×10^{-12}
4	4π	3.906×10^{-14}	8.371×10^{-15}	9.7×10^{-13}
5	5π	2.790×10^{-13}	7.924×10^{-14}	9.0×10^{-13}
6	6π	2.353×10^{-10}	4.278×10^{-14}	5.8×10^{-13}
7	7π	5.236×10^{-8}	1.594×10^{-14}	9.2×10^{-13}
8	8π	3.706×10^{-6}	8.371×10^{-15}	1.8×10^{-13}
9	9π	1.450×10^{-4}	1.240×10^{-14}	5.3×10^{-13}
10	10π	2.168×10^{-3}	1.671×10^{-12}	1.2×10^{-12}

The authors (Tao Tang, 2006) showed that there exists two constants c_1 and c_2 in problem (2.140) independent of n such that $0 < c_1 \le -\lambda \le c_2 N^4$.

We have proved it using our proposed algorithm.

$$\rho(A) \approx 0.01984\, N^4$$

$$\kappa(A) \approx 0.00515\, N^4$$

We noticed that condition numbers attained with the current method are smaller than those of the Chebychev Spectral collocation method stated in and the Spectral radius as well which are displayed in Figure 2.1.

Example 2.13: We consider the SLE studied by Celik (2005) as below

$$\begin{cases} \dfrac{d^2u}{dx^2} + (\lambda - e^x)u = 0 \\ u(0) = 0 \\ u(\pi) = 0 \end{cases} \tag{2.142a}$$

changing the variables $x = \frac{\pi}{2}t + \frac{1}{2}$, the Sturm-Lioville problem (2.412a) transforms to the following form:

$$\begin{cases} \dfrac{4}{\pi^2}\dfrac{d^2u}{dx^2} - u\,e^{\frac{\pi}{2}(t+1))} = \lambda u, \quad t \in (-1,1) \\ u(-1) = u(1) = 0 \end{cases} \tag{2.142b}$$

The Differential eigenvalue problems in matrix form can be written together with boundary conditions as

$$\sum_{k=1}^{n-1} \left[(D^2)_{i,k} - e^{\frac{\pi}{2}\left(t_k + 1\right)} \right] u_k = \lambda \delta_{i,k} u_k \tag{2.143}$$

Absolute errors obtained by using PDQ, FDQ, Chebychev collocation and our present method are illustrated table 2.12, for *n*=40. Yucel (2006), showed that FDQ approach gives better convergence than that of PDQ. We achieve almost the same accuracy in our existing method is. Accordingly, our present approach is in good agreement with the other three other methods and much accurate.

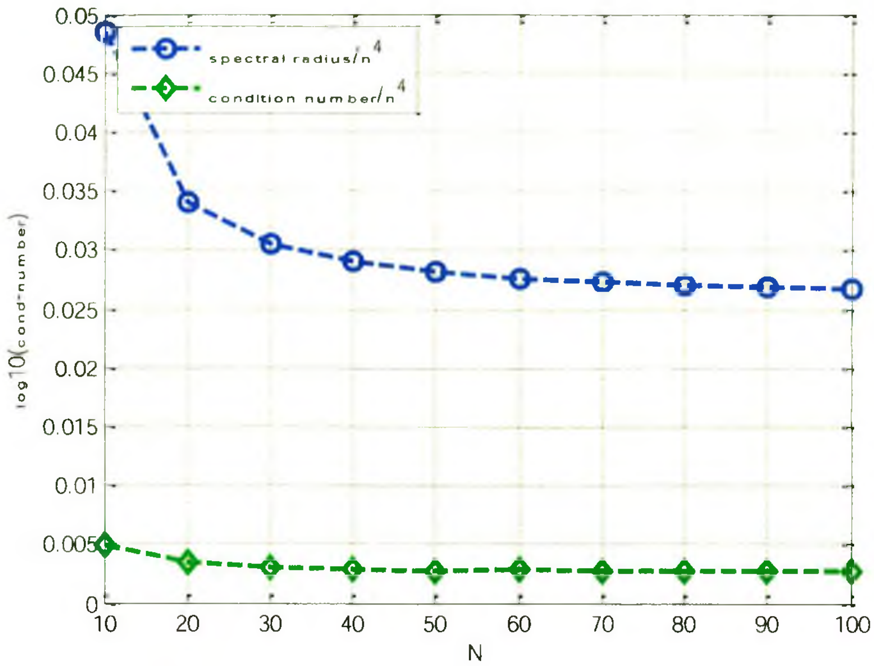

Figure 2.1: The spectral radius and condition number associated with the Legendre Spectral methods.

Example 2.14: This SLE is taken from the article worked out by Chen and Ho (1996)

$$\begin{cases} \dfrac{d^2u}{dx^2} = \lambda u \\ u(0) - u'(0) = 0 \\ u(1) + u'(1) = 0 \end{cases} \tag{2.144a}$$

Change the boundary points from 0 to 1 into -1 to 1, leads the SLE as follows

$$\begin{cases} 4\dfrac{d^2u}{dx^2} = \lambda u \\ u(-1) - u'(-1) = 0 \\ u(1) + u'(1) = 0 \end{cases} \tag{2.144b}$$

Here λ_k^n is k-th estimated eigenvalue corresponding to n and the differences between the k-th and (k-1)-th eigenvalues are given by $\left|\lambda_i^k - \lambda_i^{k-1}\right| < \varepsilon$, where ε is very small and $\varepsilon \to 0$.

It is evident from table 2.13, that the differences between successive eigenvalues converge to zero as the node number inceased and is given as follows:

$\left|\lambda_1^{12} - \lambda_1^{11}\right| \le 0.000001$, $\left|\lambda_2^{12} - \lambda_2^{11}\right| \le 0.000001$ and $\left|\lambda_3^{12} - \lambda_3^{11}\right| \le 0.000001$

Chen and Ho (1996) computed absolute differences between the successive eigenvalues and found that these differences tend to zero as he increased the order of derivatives. We also calculate the absolute differences and relative errors of the first three eigenvalues for n=5,6 and n=11, 12. It is clear that absolute differences diminish by zero as the node numbers are increased. Convergence of the first three eigenvalues are depicted in Figure 2.2.

Example 2.15: Consider the singular Sturm-Liouville boundary value problem illustrated in the article of Singh and Kumar (2013).

$$\begin{cases} -\dfrac{d^2 u}{dx^2} = \lambda^2 \dfrac{1}{x} u(x) \quad , \quad 0 < x \le 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases} \tag{2.145}$$

The exact eigenvalues are computed by solving the equation of the Bessel function $J_1(2\lambda) = 0$, for λ and eigenfunction is $u(x) = \sqrt{x}\, J_1\left(2\lambda\sqrt{x}\right)$.

We compared nine approximate eigenvalues for n=20 nodes with those tabulated using Adomian Decomposition method [Singh and Kumar (2013)]. From table 2.14, it has been noticed that the first eight numerically attained eigenvalues by our present method are correct up to figures eight significant which is yields reasonable accuracy.

Table 2.12: Comparison of absolute errors between the Chebychev's collocation and present method for example 2.13.

k	Exact eigenvalues	Chebychev coll. Celik (2005) N=40	Spectral coll. Present n=40	Absolute Error Cheby. Coll. N=40	Absolute error Yucel (2006) PDQ N=40	Absolute error Yucel (2006) FDQ N=40	Absolute Error Spect. Coll.(present)
1	4.8966694	4.8966694	4.896694	0.00000	0.0000	0.0000	0.000000
2	10.045190	10.045190	10.045190	0.00000	0.0000	0.0000	0.000000
3	16.019267	16.019267	16.019267	0.00000	0.0000	0.0000	0.000000
4	23.266271	23.266271	23.266271	0.00000	0.0000	0.0000	0.000000
5	32.263707	32.263707	32.263707	0.00000	0.0000	0.0000	0.000000
6	43.220020	43.220020	88.132119	0.00000	0.0000	0.0000	0.000000
7	56.181594	56.181594	56.181594	0.00000	0.0000	0.0000	0.000000
8	71.152998	71.152998	71.152998	0.00000	0.0000	0.0000	0.000000
9	88.132119	88.132119	88.132119	0.00000	0.0000	0.0000	0.000000
10	107.11668	107.11668	107.11668	0.00000	0.0000	0.0000	0.000000
11	128.10502	128.10502	128.10502	0.00000	0.0000	0.0000	0.000000
12	151.09604	151.09604	151.09604	0.00000	0.0000	0.0000	0.000000
13	176.08900	176.08900	176.08900	0.00000	0.0000	0.0000	0.000000
14	203.08337	203.08337	203.08337	0.00000	0.0000	0.0000	0.000000
15	232.07881	232.07881	232.07881	0.00000	0.000	0.0000	0.000000
16	263.07507	263.07507	263.07507	0.00000	0.010	0.0000	0.00000
17	296.07196	296.07198	296.07196	0.00002	0.020	0.0000	0.00000
18	331.06934	331.06940	331.06935	0.00005	0.0500	0.0000	0.00001
19	368.06713	368.06769	368.06702	0.00052	0.5600	0.0000	0.000011
20	407.06524	407.04923	407.06672	0.01502	16.01	0.030	0.00148

Table 2.13: Absolute errors between the successive eigenvalues for example 2.14.

Eigenvalue index i	Exact eigevalues Chen and Ho (1996)	Absolute error present $\lvert\lambda_i^6 - \lambda_i^5\rvert$	Absolute error present $\lvert\lambda_i^{12} - \lambda_i^{11}\rvert$
1	1.71	2.96×10^{-4}	1.50×10^{-12}
2	13.49	5.19×10^{-2}	5.97×10^{-8}
3	43.36	5.29×10^{-1}	7.16×10^{-4}

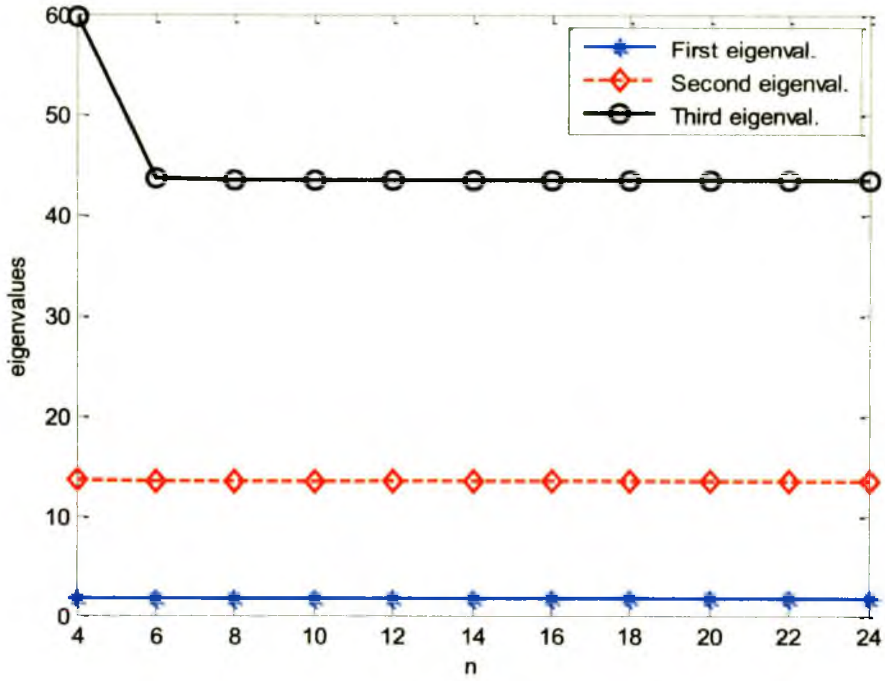

Figure 2.2: Convergence of the eigenvalues.

Table 2.14: Comparison of eigenvalues obtained by present method with ADM [Singh and Kumar (2013)] for example 2.15.

k	Exact eigenvalues Auzinger (2006) λ_k	Present (Spectral coll.)	ADM [Singh and Kumar (2013)]
1	1.9158529	1.9158529851	1.9158529
2	3.5077933	3.5077933349	3.5077933
3	5.0867340	5.0867340674	5.0867340
4	6.6618459	6.6618459681	6.6618459
5	8.2353150	8.2353150254	8.2353150
6	....	9.8079292552	9.8079292
7	....	11.3800421902	11.380042
8	....	12.9518360438	12.9518360
9	....	14.5235414260	14.5235964

Example 2.16: The Boyd equation considered by Baily *et al* (1991), Singh and Kumar (2013).

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \lambda u(x) + \dfrac{1}{x}u(x) \\ u(0) = 0 \\ u(1) = 0 \end{cases} \tag{2.146a}$$

We have calculated the first five eigenvalues and compare our approximate results with Adomian Decomposition method studied by the said authors. From table 2.15, we observed that eigenvalues work out by our present approach agrees well with the Adomian Decomposition method, SLEIGN 2.

Example 2.17: We consider the Bratu's boundary value problem in one dimensional planar coordinates studied by some authors [Chen and Ho (1996), Aregbesola (2003), Khuri (2004), Liao and Tan (2007), Caglar *et al*, Jalilian (2010), Zarebnial and Sarvari (2012), Trefethen (2000)] in the form:

$$\begin{cases} \dfrac{d^2u}{dx^2} = \lambda e^{u(x)} \\ u(0) = 0 \\ u(1) = 0 \end{cases} \tag{2.147a}$$

Table 2.15: Comparison of solutions obtained by the present Spectral method with other various methods for example 2.16.

k	Current method n=20	Adomian Decomposition	Baily *et al* (1991)	
			SLEIGN 2	Transcendental equation
1	7.37398502	7.3739850	7.37399	7.3740
2	36.3360196	36.3360196	36.33602	36.3360
3	85.2925821	85.2925820	85.29258	85.2925
4	154.0986237	154.0986237	154.09862	154.099
5	242.7055594	242.7055594	242.70555	242.705

For $\lambda > 0$, the exact solutions of equation (2.147a) is found as

$$u(x) = -2\ln\left[\frac{\cosh\left\{\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right\}}{\cosh\left(\frac{\theta}{4}\right)}\right] \tag{2.148}$$

Solving the above equation $\theta = \sqrt{2\lambda}\,\mathrm{coh}(\theta/4)$, the values θ are computed.

The maximum absolute errors in solutions of Bratu nonlinear problem are compared with methods investigated by some authors [Khuri (2004), Liao and Tan (2007), Caglar *et al* (2010), Jalilian (2010), Zarebnial and Sarvari (2012)]. The computed solutions are tabulated in tables 2.16 and 2.17 for n = 10. Table 2.16 reveals that the absolute errors of the solutions for λ=1, are quite accurate and are in good agreement with the other numerical methods.

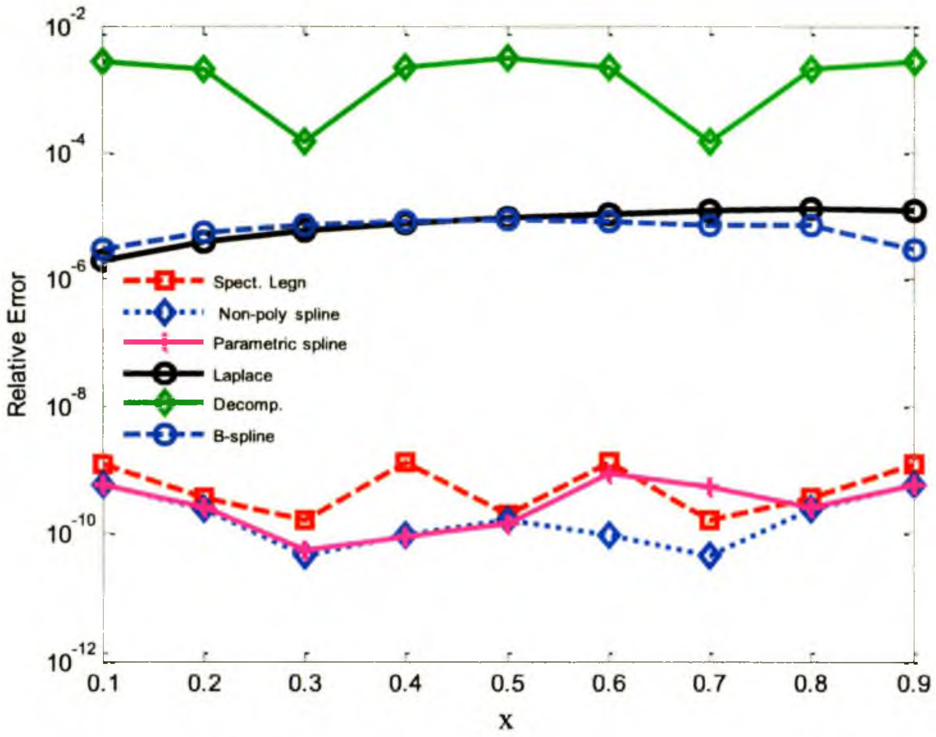

Figure 2.3: Comparison of our result obtained using spectral collocation method with the non-poly-spline, parametric spline and Laplace transformation decomposition methods for n=10 and λ=1.

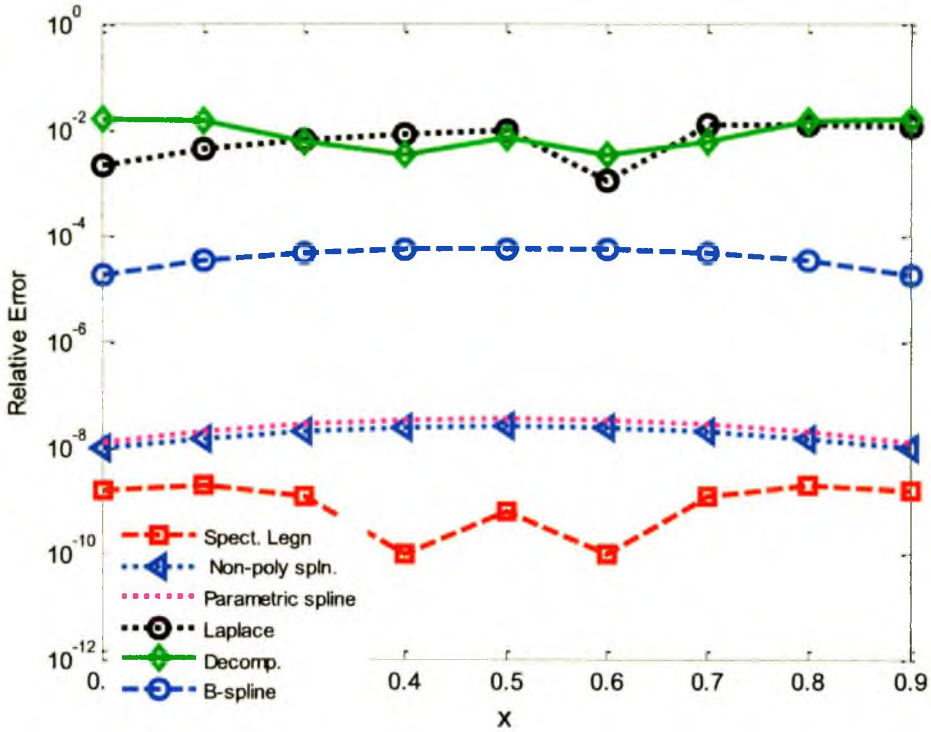

Figure 2.4: Comparison of our result obtained using spectral collocation method with the non-poly spline, parametric spline and B-spline methods for n=10 and λ=2.

Table 2.16: Comparison of solutions obtained by present method for Bratu equation with other methods for λ=1, of example 2.17.

x	Exact eigenvalues	Present method n=10	Parametric spline N=10	Non polyn. Spline N=10	Laplace	Decomp.	B-spline
0.1	0.0498467900	1.24×10^{-9}	5.87×10^{-10}	5.77×10^{-10}	1.98×10^{-6}	2.68×10^{-3}	2.98×10^{-6}
0.2	0.0891899350	3.64×10^{-10}	2.58×10^{-10}	2.47×10^{-10}	3.94×10^{-6}	2.02×10^{-3}	5.46×10^{-6}
0.3	0.1176090956	3.99×10^{-11}	5.59×10^{-11}	4.56×10^{-11}	5.85×10^{-6}	1.52×10^{-4}	7.33×10^{-6}
0.4	0.1347902526	1.29×10^{-9}	8.77×10^{-11}	9.64×10^{-11}	7.70×10^{-6}	2.20×10^{-3}	8.50×10^{-6}
0.5	0.1405392142	2.04×10^{-10}	1.38×10^{-10}	1.66×10^{-10}	9.47×10^{-6}	3.01×10^{-3}	8.89×10^{-6}
0.6	0.1347902526	1.29×10^{-9}	8.77×10^{-10}	9.64×10^{-11}	1.11×10^{-5}	2.20×10^{-3}	8.50×10^{-6}
0.7	0.1176090956	1.60×10^{-10}	5.59×10^{-10}	4.56×10^{-11}	1.26×10^{-5}	1.52×10^{-4}	7.33×10^{-6}
0.8	0.0891899350	3.64×10^{-10}	2.58×10^{-10}	2.47×10^{-10}	1.35×10^{-5}	2.02×10^{-3}	5.46×10^{-6}
0.9	0.0498467900	1.24×10^{-9}	5.87×10^{-10}	5.77×10^{-10}	1.20×10^{-5}	2.68×10^{-3}	2.98×10^{-6}

Table 2.17: Comparison of solutions obtained by present method for Bratu equation with other methods for λ=2, of example 2.17.

x	Exact eigenvalues	Current method n=10	Parametric Spline	B-spline	Non polyn. Spline n=10	Laplace	Decomposition
0.1	0.1144107440	1.44×10^{-9}	1.25×10^{-8}	1.72×10^{-5}	9.71×10^{-9}	1.98×10^{-6}	1.52×10^{-2}
0.2	0.2064191156	1.98×10^{-9}	1.95×10^{-8}	3.26×10^{-5}	1.41×10^{-8}	3.98×10^{-6}	1.47×10^{-2}
0.3	0.2738793116	1.18×10^{-9}	2.73×10^{-8}	4.49×10^{-5}	1.98×10^{-8}	5.85×10^{-6}	5.89×10^{-3}
0.4	0.3150893646	9.84×10^{-11}	3.31×10^{-8}	5.28×10^{-5}	8.50×10^{-6}	7.70×10^{-6}	3.25×10^{-3}
0.5	0.3289524214	6.03×10^{-10}	3.53×10^{-8}	5.56×10^{-5}	8.89×10^{-6}	9.47×10^{-6}	6.98×10^{-3}
0.6	0.3150893646	9.84×10^{-11}	3.31×10^{-8}	5.28×10^{-5}	8.50×10^{-6}	1.11×10^{-5}	3.25×10^{-3}
0.7	0.2738793116	1.18×10^{-9}	2.73×10^{-8}	4.49×10^{-5}	7.33×10^{-6}	1.26×10^{-5}	5.89×10^{-3}
0.8	0.2064191156	1.98×10^{-9}	1.95×10^{-8}	3.26×10^{-5}	5.46×10^{-6}	1.35×10^{-5}	1.47×10^{-2}
0.9	0.1144107440	1.44×10^{-9}	1.25×10^{-8}	1.72×10^{-5}	9.71×10^{-9}	1.20×10^{-5}	1.52×10^{-2}

It is also noticed that for the case of λ=2 in table 2.17, the absolute errors in the present method are reduced and are more convergent than all other methods. Therefore, as the value of λ increases the solutions are more accurate and reliable and our method is more efficient. The absolute errors achieved by our current scheme method are depicted in figures 2.3 and 2.4 and compared to the other methods.

2.5 Conclusions

In this study, a novel formulation of the Weighted Residual method using both Bernstein and Legendre polynomials is proposed. The main reason why the Galerkin method is chosen is its flexibility and simple implementation. Excellent agreement and better performance is achieved even with small number of basis polynomials which sometimes minimize the cost of computational time for some second order BVPS. The disadvantage of the current method is that, in case of huge number of eigenvalues computation, higher eigen modes are less convergent than the lower modes and with increasing of the degree of polynomials, the computational time highly increases. In spite of this disadvantage, we can conclude that for a relatively small n, i.e., $n = 10$, fairly accurate numerical results are obtained using the proposed method.

In table 2.1, relative errors applying the technique of Galerkin exploiting Bernstein and Legendre polynomials proves the reliability and efficiency than that of other existing numerical method. Furthermore, the smallest eigenvalue which characterizes potentially the most visual structures of the dynamical systems arises in vibration of a deformable bodies can be computed very accurately applying Galerkin WRM. In table 2.2, absolute errors computed for the smallest eigenvalue are much smaller than those of Sinc Galerkin and Differential Transform method. For Sturm-Liouville problem with mixed boundary conditions, Galerkin method using Bernstein polynomials eigenvalues depicted in table 2.3, achieves better accuracy than those of Legendre polynomials. Also from table 2.5, SLE with same type of boundary conditions all the eigenvalues converge the error reaches up to 10^{-9} which are much superior than those of finite difference method. For singular SLE, the eigenvalues calculated by the Galerkin method accomplishes high accuracy i.e., converge to ten significant digits illustrated in tables 2.4 and 2.7. In case of periodic SLE demonstrated in table 2.6, lower eigenvalues converge rapidly to the exact results and for higher eigenvalues absolute error is less than 5%. The results shown in table 2.1 to table 2.7 indicate that Galerkin method using Bernstein and Legendre polynomials produces very accurate results compared to the other available numerical methods.

We observed that relative errors and absolute errors comprise of are much smaller and much competent with as well as WRM Galerkin and available numerical studies. Dirichlet boundary conditions illustrated in tables 2.8-2.9 Bernstein collocation techniques computes higher eigenvalues more efficiently than those obtained by Galerkin method.

Our proposed method is much superior in the sense of accuracy and applicability especially for second order problems. From these comparisons we see that eigenvalues obtained by the present method competes very well with other methods. In this study, the Spectral collocation method is applied for solving linear, nonlinear second order eigenvalue problems respectively. Eigenvalues obtained using Spectral collocation method attains much more accuracy than those of

Bernstein Galerkin and Bernstein collocation method for most of the problems. Besides, the present Spectral method is computationally efficient and much competent with the other earlier published works. Furthermore, this method with the aid of Matlab 13 code is well suited for both regular as well as singular Sturm-Liouville problems. Finally, the computational stable convergence for some eigenvalue problems is achieved.

CHAPTER 3

Eigenvalues of Fourth Order Sturm-Liouville Problems Exploiting the Methods of Weighted Residual

3.1 Introduction

The mathematical models or differential equations that govern a number of problems ranging from structural stability to vibration and control are classified as eigenvalue problems which play a crucial role in many fields of engineering as well as in pure and applied mathematics. The fourth order eigenvalue problems occur in the study of buckling of beam-columns and plates deflection theory and the theory of shear flows of viscous Newtonian incompressible fluids. A profoundly understanding of the class of fourth order eigenvalue problems is a precondition for vibration and buckling analyses of structures. Design optimization of structures to prevent failure due to instability (buckling) and vibration introduces the problem of determining optimal physical parameters such that load carrying capacity or the fundamental natural frequency is maximized. The instability of such viscoelastic and inelastic flow has been and continues to be one of the most constantly pursued topics in fluid mechanics. Differential equations that govern the boundary value problems associated with vibration and buckling may be represented as Sturm-Liouville differential equations.

Numerical approximations for higher order eigenvalue problems are challenging because of the higher order derivatives and boundary conditions involving higher order derivatives of the unknown function. Our aim is to develop Bernstein polynomial based collocation and Legendre-Lagrange polynomial based spectral collocation method to calculate the eigenvalues of fourth order Sturm-Liouville problems. We use Chebyshev clustered grid points to generate a system of algebraic equation with unknown co-efficient in matrix form. For implementing Spectral collocation technique, we have computed some Legendre differentiation matrices to attain all higher order derivatives.

In this study all the unknown coefficients are also expressed in terms of known co-efficient of the boundary conditions and thus handling boundary conditions is much easier.

Existence and uniqueness conditions of higher order boundary value problems and characterization of the associated eigenvalue problems are revealed in [Agarwal (1986), Wong and Agarwal (1996)]. Chawla (1983) presented fourth-order finite-difference method, computing eigenvalues of fourth order Sturm-Liouville problems. Twizell and Matar (1992) developed finite difference method for approximating the eigenvalues of fourth-order boundary value problems.

Although no numerical method is worked out, some authors Abbasbandy and Shirazdi (2011), Shi and Cao (2012), Ycel and Boubaker (2012), Gamel and Sameeh (2012), Taher *et al* (2013), Huang *et al* (2013) paid their attention to develop various techniques for finding eigenvalues of fourth order Sturm Liouville BVPs. They applied different algorithms to minimize the convergence rates.

Chanane (1998, 2002, 2010) introduced a novel series representation for the boundary/characteristic function associated with fourth-order Sturm-Liouville problems using the concepts of Fliess series, iterated integrals and also Extended Sampling method. Jia *et al* (2005) approximated the eigenvalues of fourth order BVP for a class of crosswise vibration equation of beam using Galerkin method and obtained the estimation of errors using the trigonometric polynomials that satisfies all the boundary conditions directly. The Adomian decomposition method (ADM) to solve eigenvalues of fourth-order Sturm-Liouville problems was used by Attili and Lesnic (2006). Syam and Siyyam (2009) developed a Variational Iteration method (VIM) for finding the eigenvalues of fourth-order non-singular Sturm-Liouville problems.

Recently Taher *et al* (2013) applied an efficient technique using Chebychev spectral collocation method where Chebychev differentiation matrix is defined and computed the eigenvalues of fourth order Sturm-Liouville problems. Since

This chapter has been devoted to find the numerical solutions of the fourth order exploiting piecewise continuous and differentiable polynomials such as Bernstein and Legendre polynomials with various types of boundary conditions. The matrix formulation of the general linear fourth order Sturm-Liouville problems by utilizing the technique of Galerkin WRM incorporated with the boundary conditions have been discussed in section 3.2. In section 3.3, we have considered numerical examples to verify the efficiency of the proposed method. Section 3.4 has been offered for introduction of the WRM of Collocation using Bernstein polynomials as basis function. Brief description and Matrix formulation of the current scheme has been derived precisely in section 3.4.1 and 3.4.2. Efficiency of the current method is established by considering a few numerical examples in section 3.4.3. Cheby-Legendre Spectral collocation method and description of the scheme is presented in section 3.5 and 3.5.1. Accuracy of this proposed technique is illustrated through various examples in section 3.5.2.

The approximate solutions converge to the exact solutions monotonically even with desired large significant digits. Finally, we have given the conclusions of this chapter.

3.2 Matrix Formulation

Consider the following general fourth order nonsingular Sturm-Liouville problem (SLE)

$$\frac{d^2}{dx^2}\left[p(x)\frac{d^2u}{dx^2}\right]-\frac{d}{dx}\left[q(x)\frac{du}{dx}\right]+r(x)u=\lambda w(x)u, \quad \gamma\le x\le\mu \tag{3.1}$$

where, $p(x)$, $q(x)$, $r(x)$ and $w(x)$ are all piecewise continuous functions and $p(x)$, $w(x)>0$ subject to some specified conditions and at these conditions mean that equation (3.1) is regular, i.e., nonsingular.

We consider the equation governing the equilibrium of a beam subjected to an axial force P, according to Euler-Bernoulli beam theory, is

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)+P\frac{d^2u}{dx^2}=0\,,\ \gamma\le x\le\mu \tag{3.2}$$

Here E is the Young's Modulus characterizing the material from which the beam is formed, and $I(x)$ is the second moment of area of cross the section, which is an eigenvalue equation with $\lambda=P$ as the eigenvalue, which represents buckling load. Several different types of boundary conditions are commonly prescribed in the study of beam deformations:

a) Clamped-clamped boundary conditions:

$$u(0)=0,\ \frac{du}{dx}(0)=0,\ u(l)=0,\ \frac{du}{dx}(l)=0,\qquad 0<x<l$$

b) Clamped-free boundary conditions:

$$u(0)=0\,,\quad \frac{du}{dx}(0)=0,\quad \frac{d^2u}{dx^2}(l)=0,\quad \frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)(l)=0,\quad 0<x<l$$

c) Simple support or hinged end:

$$u(0)=0\,,\quad u(l)=0,\quad \frac{d^2u}{dx^2}(0)=0,\quad EI\frac{d^2u}{dx^2}(l)=0,\quad 0<x<l$$

d) Free-free end:

$$EI\frac{d^2u}{dx^2}(0)=0,\quad EI\frac{d^2u}{dx^2}(l)=0\,,\quad \frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)(0)=0,\quad \frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)(l)=0$$

$$0<x<l$$

e) Clamped-hinged boundary conditions:

$$u(0)=0\,,\quad u(1)=0\,,\quad \frac{d^2u}{dx^2}(0)=0,\quad \frac{du}{dx}(1)=0,\quad 0<x<l$$

Using Leibnitz rule of differentiation, we can rewrite the equation (3.1) in the form as a general fourth order eigenvalue problem over the finite interval $[\gamma,\mu]$.

The Sturm-Liouville problem (3.1) has an infinite number of sequence of eigenvalues $\{\lambda_i\}_{i\ge1}$ which are bounded from below and the eigenvalues can be ordered as an increasing sequence, i.e

$\lambda_0 < \lambda_1 < \lambda_2$ ....

$\lim_{i\to\infty} \lambda_i = \infty$ and each eigenvalue has multiplicity at most 2 [Taher *et al* (2013)].

$$\frac{d^4 u}{dx^4} + a_3 \frac{d^3 u}{dx^3} + a_2 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = \lambda \sigma u, \quad \gamma < x < \mu \tag{3.3}$$

subject to the following two types of boundary conditions

Type I: $u(0)=0,\ u(1)=0;\ u'(0)=0\ ,\ u'(1)=0;$ (3.3a)

Type II : $u(0)=0, \quad u(1)=0\ ; \quad u''(0)=u''(1)=0$. (3.3b)

where,

$$a_3(x) = \frac{2p'(x)}{p(x)},\ a_2(x) = \frac{p''(x)-q(x)}{p(x)},\ a_1(x) = -\frac{q'(x)}{p(x)},\ a_0(x) = -\frac{r(x)}{p(x)},$$

$$\sigma(x) = \frac{w(x)}{p(x)}$$

where, $a_0, a_1, a_2, a_3, \sigma$ are all continuous functions of x defined on the interval $[\gamma, \mu]$. Let us consider the fourth order SLE (3.3) which can be transformed as the following equation:

$$\frac{d^4 u}{dx^4} + m_3 \frac{d^3 u}{dx^3} + m_2 \frac{d^2 u}{dx^2} + m_1 \frac{du}{dx} + m_0 u = \lambda \omega u, \qquad 0 < x < 1 \tag{3.4}$$

where, $m_0, m_1, m_2, m_3, \omega$ are all continuous functions of x defined on the interval $[0,1]$ by replacing x by $\frac{x-\gamma}{\mu-\gamma}$ in the approximate solution $\tilde{u}(x)$.

Type I:

$$u(0)=0, \quad u(1)=0\ ; \quad \frac{1}{\mu-\gamma} u'(0) = 0\ , \quad \frac{1}{\mu-\gamma} u'(1) = 0.$$

Type II

$$u(0)=0\ , \quad u(1)=0\ ; \quad \frac{1}{(\mu-\gamma)^2} u''(0) = 0\ , \quad \frac{1}{(\mu-\gamma)^2} u''(1) = 0.$$

where,

$$m_3 = \frac{1}{(\mu-\gamma)^3} a_3[(\mu-\gamma)x+\gamma] , \quad m_2 = \frac{1}{(\mu-\gamma)^2} a_2[(\mu-\gamma)x+\gamma]$$

$$m_1 = \frac{1}{\mu-\gamma} a_1[(\mu-\gamma)x+\gamma], \quad m_0 = a_0[(\mu-\gamma)x+\gamma], \quad \omega = \sigma[(\mu-\gamma)x+\gamma]$$

To approximate the solution of SLE (3.3a), we express in terms of Bernstein or Legendre polynomial basis as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_{i,n}(x), \; n \ge 1 \tag{3.5}$$

where,

$B_{i,n}(x) = 0$ denotes the Bernstein polynomial and it satisfies all the essential boundary conditions in $[0,1]$ and $\theta_0(x) = 0$, is specified by the Dirichlet boundary conditions,

$B_{i,n}(0) = B_{i,n}(1) = 0$, for each $i = 1, 2, 3, \ldots, n-1$.

Using (3.5) into equation (3.4) and imposing boundary conditions of Type I, the Galerkin weighted residual equations are:

$$\int_0^1 \left[\tilde{u}^{(4)}(x) + m_3(x)\tilde{u}^{(3)} + m_2(x)\tilde{u}'' + m_1(x)\tilde{u}' + m_0(x)\tilde{u} - \lambda\omega(x)\tilde{u} \right] B_j \, dx = 0 \tag{3.6}$$

$$j = 1, 2, 3, \cdots\cdots, n.$$

Now integrating each term of (3.6) by parts, we have

$$\int_0^1 \frac{d^4\tilde{u}}{dx^4} B_j(x)dx = \left[B_j(x)\frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= \left[B_j(x)\frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{du}{dx} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.7}$$

$$\int_0^1 m_3(x)\frac{d^3\tilde{y}}{dx^3}B_j(x)dx=\left[m_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d}{dx}\left[m_3(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=\left[m_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d}{dx}\left[m_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$+\int_0^1\frac{d^2}{dx^2}\left[m_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=\int_0^1\frac{d^2}{dx^2}\left[m_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx+\left[m_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1 \tag{3.8}$$

$$\int_0^1 m_2(x)\frac{d^2\tilde{u}}{dx^2}B_j(x)dx=\left[m_2(x)B_j(x)\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d}{dx}\left[m_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=-\int_0^1\frac{d}{dx}\left[m_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.9}$$

Since, $\left[m_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1=0$, $\left[m_2(x)B_j(x)\frac{d\tilde{u}}{dx}\right]_0^1=0$

by the Dirichlet boundary conditions.

Similarly,

Inserting $B_j(0)=B_j(1)=0$ in the above integrals, we finally obtain the equations (3.7), (3.8) and (3.9). Substituting these equations into equation (3.6) and after rearranging the terms we have

$$\int_0^1\left\{-\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}+\frac{d^2}{dx^2}\left[m_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}-\frac{d}{dx}\left[m_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right.$$

$$-\left[m_1(x)B_j(x)\right]\frac{d\tilde{u}}{dx}+m_0(x)B_j\tilde{u}-\lambda\,\omega(x)B_j\tilde{u}\Big\}dx-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1=0 \tag{3.10}$$

Also from equation (3.5), we have

$$\tilde{u}(0)=\sum_{i=1}^{n-1}c_i\,B_i(0),\qquad \tilde{u}(1)=\sum_{i=1}^{n-1}c_i\,B_i(1) \tag{3.11a}$$

$$\frac{d^2\tilde{u}}{dx^2}=\sum_{i=1}^{n-1}c_i\frac{d^2B_i}{dx^2},\quad \frac{d^3\tilde{u}}{dx^3}=\sum_{i=1}^{n-1}c_i\frac{d^3B_i}{dx^3}\quad\text{and}\quad \frac{d^4\tilde{u}}{dx^4}=\sum_{i=1}^{n-1}c_i\frac{d^4B_i}{dx^4} \tag{3.11b}$$

Using equations (3.11a) and (3.11b) into equation (3.10) we obtain

$$\sum_{i=1}^{n-1}\Bigg[\int_0^1\Bigg[-\frac{d^3B_j}{dx^3}\frac{dB_i}{dx}+\frac{d^2}{dx^2}\left[m_3(x)B_j(x)\right]\frac{dB_i}{dx}-\frac{d}{dx}\left[m_2(x)B_j(x)\right]\frac{dB_i}{dx}$$

$$-\frac{d}{dx}\left[m_1(x)B_j(x)\right]B_i(x)+m_0(x)B_i(x)B_j(x)\Bigg]dx\Bigg]c_i$$

$$+\sum_{i=1}^{n-1}\left\{-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=1}+\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=0}\right\}c_i$$

$$=\lambda\sum_{i=1}^{n-1}\int_0^1\left[\omega(x)B_iB_j\,dx\right]c_i \tag{3.12}$$

Finally, the eigenvalues are obtained in matrix form as below

$$\sum_{i=1}^{n}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0 \tag{3.13}$$

$$F_{i,j}=\int_0^1\Bigg\{\Bigg[-\frac{d^3}{dx^3}\left[B_j(x)\right]B_i'(x)+\frac{d^2}{dx^2}\left[m_3(x)B_j(x)\right]B_i'(x)$$

$$-\frac{d}{dx}\left[m_2(x)B_j(x)\right]B_i'(x)-\left[m_1(x)B_j(x)\right]B_i'(x)\Bigg]+m_0B_i(x)B_j(x)\Bigg\}dx$$

$$-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2 B_i}{dx^2}\right]_{x=1}+\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2 B_i}{dx^2}\right]_{x=0} \tag{3.13a}$$

$$E_{i,j}=\int_0^1 \omega(x)B_i(x)B_j(x)\,dx \tag{3.13b}$$

Equivalently, eigenvalues can be obtained by solving the system

$$F-\lambda E=0, \tag{3.14}$$

$$\lambda I = F E^{-1} \tag{3.14a}$$

We define

$$A = F E^{-1} \tag{3.15}$$

where the matrices F and E are defined by (3.13a) and (3.13b) respectively.

Type II: $u(0)=0$, $u(1)=0$, $u''(0)=0$, $u''(1)=0$

$$\int_0^1 \frac{d^4\tilde{u}}{dx^4}B_j(x)\,dx=\left[B_j(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\left[B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=\left[B_j(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{du}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.16}$$

$$\int_0^1 m_3(x)\frac{d^3\tilde{u}}{dx^3}B_j(x)\,dx=\left[m_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d}{dx}\left[m_3(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= \left[m_3(x) B_j(x) \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d}{dx}\left[m_3(x) B_j(x) \right] \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[m_3(x) B_j(x) \right] \frac{d\tilde{u}}{dx} dx$$

$$= -\left[\frac{d}{dx}\left[m_3(x) B_j(x) \right] \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[m_3(x) B_j(x) \right] \frac{d\tilde{u}}{dx} dx \qquad (3.17)$$

$$\int_0^1 m_2(x) \frac{d^2 \tilde{u}}{dx^2} B_j(x) dx = \left[m_2(x) B_j(x) \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[m_2(x) B_j(x) \right] \frac{d\tilde{u}}{dx} dx$$

$$= -\int_0^1 \frac{d}{dx}\left[m_2(x) B_j(x) \right] \frac{d\tilde{u}}{dx} dx \qquad (3.18)$$

by the Dirichlet boundary conditions.

Similarly, inserting $B_j(0) = B_j(1) = 0$ in the above integrals, we finally have obtained the equations (3.16) to (3.18).

Substituting (3.16), (3.17), (3.18) into equation (3.6) and after rearranging the terms the eigenvalues are obtained in matrix form as:

$$\sum_{i=1}^{n-1} \left[\int_0^1 \left\{ \left[-\frac{d^3}{dx^3}\left[B_j(x) \right] \frac{dB_i}{dx} + \frac{d^2}{dx^2}\left[m_3(x) B_j(x) \right] \frac{dB_i}{dx} - \frac{d}{dx}\left[m_2(x) B_j(x) \right] \frac{dB_i}{dx} \right. \right. \right.$$

$$\left. \left. + m_0(x) B_j(x) B_j(x) \right] - \lambda \left[\omega(x) B_i B_j \right] \right\} c_i \, dx$$

$$- \sum_{i=1}^{n-1} \left\{ \left[\frac{d}{dx}\left[m_3(x) B_j(x) \right] \frac{dB_i}{dx} \right]_0^1 - \left[\frac{d^2}{dx^2}\left[B_j(x) \right] \frac{dB_i}{dx} \right]_0^1 \right\}_0^1 c_i = 0 \qquad (3.19)$$

$$\sum_{i=1}^{n-1} \left[F_{i,j} - \lambda E_{i,j} \right] c_i = 0 \qquad (3.20)$$

where

$$F_{i,j} = \int_0^1 \left\{ \left[-\frac{d^3}{dx^3}\left[B_j(x) \right] \frac{dB_i}{dx} + \frac{d^2}{dx^2}\left[m_3(x) B_j(x) \right] \frac{dB_i}{dx} - \frac{d}{dx}\left[m_2(x) B_j(x) \right] \frac{dB_j}{dx} \right. \right.$$

$$\left. + m_0(x) B_j(x) \tilde{u} \right\} dx + \left[\frac{d^2}{dx^2}\left[B_j(x) \right] \frac{dB_i}{dx} \right]_0^1 - \left[\frac{d}{dx}\left[m_3(x) B_j(x) \right] \frac{dB_i}{dx} \right]_0^1 \qquad (3.20a)$$

$$E_{i,j} = \int_0^1 \omega(x) B_i(x) B_j(x) dx \tag{3.20b}$$

Equivalently, eigenvalues can be obtained by solving the system

$$F - \lambda E = 0 \tag{3.21}$$

Here the matrices F and E are defined by (3.20a) and (3.20b).

3.3 Test Examples:

In this section we have presented several numerical examples of fourth order Sturm-Liouville problems, using the method outlined in the previous section with different boundary conditions. The convergence and effectiveness of the method are confirmed by comparing numerical results with the exact and other existing numerical results. The convergence of our existing method is measured by the relative error given below.

$$\varepsilon_k = \left| \frac{\lambda_n^{Exact} - \lambda_n^{(Gal)}}{\lambda_n^{exact}} \right| < 10^{-12}.$$

where, λ_n^{Exact} denotes the approximate eigenvalues using the n-th polynomials and ε_k depends upon the problems.

Example 3.1(a): We first consider the Sturm-Liouville BVP examined by Yucel and Boubaker (2012) Gamel and Sameeh (2012) and Chebychev spectral collocation method Taher *et al* (2013).

$$\frac{d^4u}{dx^4} - \lambda u(x) = 0 \quad , \qquad 0 < x < 1 \tag{3.22a}$$

$$\begin{aligned} u(0) &= u'(0) = 0 \\ u(1) &= u''(1) = 0 \end{aligned} \tag{3.22b}$$

which corresponds to the case $a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0$, $\gamma = 0$ and $\mu = 1$ in equation (3.3).

The exact solution of (3.22a) is obtained by solving

$\tanh\left(\sqrt{\lambda}\right) - \tan\left(\sqrt{\lambda}\right) = 0.$

Using the method illustrated in section 3.2, we approximate $\tilde{u}(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1} c_i B_i(x). \tag{3.23}$$

Here $\theta_0(x)=0$ as specified by the Dirichlet boundary conditions of equation (3.22b). Also $B_{i,n}(0)=0$ and $B_{i,n}(1)=0$

$i=1,2,3,\dots\dots\dots,n$. The weighted residual equation (3.6) becomes

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0\ ,\ j=1,2,3,\dots\dots\dots,n-1 \tag{3.24}$$

where,

$$F_{i,j}=\int_0^1 -\frac{d^3B_j}{dx^3}\frac{dB_i}{dx}dx+\left[\frac{d^2B_i(0)}{dx^2}\frac{dB_j(0)}{dx}-\frac{d^2B_j(1)}{dx^2}\frac{dB_i(1)}{dx}\right] \tag{3.24a}$$

$$E_{i,j}=\int_0^1 B_i B_j dx \tag{3.24b}$$

We define eigenvalue matrix as

$$\lambda I=F E^{-1} \tag{3.25}$$

Comparison of eigenvalues obtained by the existing method with the other numerical methods have been displayed in table 3.1. Exact eigenvalues and relative errors are for the first ten eigenvalues are tabulated in table 3.2 using different degrees of polynomials with the relative error for the differential quadrature method [Ycel and Boubaker (2012)], Chebychev method [Gamel and Sameeh (2012)] and Chebychev Spectral collocation method [Taher *et al* (2013)]. We have increased Bernstein polynomials from n=20 to n=30 and have exploited 30 Legendre polynomials, the maximum error achieved for both the polynomials is about 10^{-14}, which shows the better performance of the current technique. Using the same degree of polynomials in the case of Bernstein and Legendre polynomials, the smallest eigenvalue attains the accuracy 10^{-11} and 10^{-9} respectively. If we further increase the degree of both the polynomials the accuracy of this method does not improve as expected. The observed CPU time for Bernstein polynomials is 3.78 seconds Legendre polynomials is 4.707 seconds for degree of polynomials, n=20. Again, using n=30, CPU time for Bernstein polynomials requires 7.532 seconds.

Furthermore, in table 3.1, the first seven eigenvalues using Legendre polynomials are very close to the exact result and the computed values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues. At the same time, it has also been observed in table 3.2 that all 10 eigenvalues obtained using Bernstein polynomials converge more rapidly than those of Legendre polynomials. In fact, relative error decays as the of degree of polynomials increased in the case of Bernstein basis. But on the other hand, estimated eigenvalues using Legendre polynomials show less convergent especially for the higher eigenvalues It is clearly observed that eigenvalues obtained by Galerkin-Bernstein method are most accurate and Galerkin-Legendre results are much more accurate than the other results.

Example 3.1(b): We consider the Sturm-Liouville BVP worked out by Syam & Siyyam (2009), Gamel and Sameeh (2012):

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0 \tag{3.26a}$$

$$\begin{cases} u''(0) = u''(1) = 0 \\ u'''(0) = u'''(1) = 0 \end{cases} \tag{3.26b}$$

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_i(x). \tag{3.27}$$

Here $\tilde{u}(x)$ in equation (3.27) for the case of Bernstein polynomials is not satisfied by the Dirichlet boundary conditions i.e., $B_j(0) \neq 0$ and $B_j(1) \neq 0$.

Legendre polynomial basis function is

$$L_n(x) = \left[\frac{1}{n!} \frac{d^n}{dx^n} (x^2 - x)^n \right] \tag{3.28}$$

Here $\tilde{u}(x)$ in equation (3.28) does not satisfy the Dirichlet boundary conditions for Legendre basis i.e., $L_j(0) \neq 0$ and $L_j(1) \neq 0$.

The weighted residual equation (3.6) becomes

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0,\; j=1,2,3,\ldots\ldots\ldots,n-1 \tag{3.29}$$

$$F_{i,j}=-\int_0^1 \frac{d^3 B_j}{dx^3}\frac{dB_i}{dx}dx+\left[\frac{d^2 B_i(1)}{dx^2}\frac{dB_j(1)}{dx}-\frac{d^2 B_j(0)}{dx^2}\frac{dB_i(0)}{dx}\right], \tag{3.29a}$$

$$E_{i,j}=\int_0^1 B_i B_j\,dx \tag{3.29b}$$

Table 3.3 shows the comparison of our result obtained using n=20, for Bernstein and Legendre polynomials, with the first five eigenvalues of the problem with Gamel and Sameeh (2012), Syam and Siyyam (2009).

Example 3.1(c): We consider the Sturm-Liouville BVP which is taken from Attili and Lesnic (2006)

$$\frac{d^4 u}{dx^4}-\lambda u(x)=0, \tag{3.30a}$$

$$\begin{aligned} &u''(0)=u'''(0)=0\\ &u(1)=u'(1)=0 \end{aligned} \tag{3.30b}$$

Here,

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}c_i B_{i,n}(x) \tag{3.31}$$

For Legendre polynomial we modified the above basis as

$$L_n(x)=(1-x)+\left[\frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n-(-1)^n\right](x-1) \tag{3.32}$$

Table 3.1: Comparison of eigenvalues with various numerical methods for example 3.1(a).

Results of Gamel and Sameeh (2012) $\lambda_k^{(Cheby)}$	Results of Attili and Lesnic (2006)	Results of Abbasbandy and Shirazdi (2011)	Results of Syam and Siyyam(2011)	Eigenvalue (Bernstein) (present) $\lambda_k^{(Gal.)}$
237.72106753	237.72106753	237.72106753	237.72106754	237.72106753
2496.48743786	2496.48743785	2496.48743785	2496.48743843	2496.48743786
10867.58221704	10867.59367146	10867.58221697	10867.58221699	10867.58221698
31780.09645409	31475.48355038	31780.09645277	31780.09650785	31780.09645408
....	....	74000.84934655	74000.85036550	74000. 84934930
....	....	148634.47747229	148634.47728684	148634. 47773948

Table 3.2: Observed relative errors of eigenvalues for example 3.1(a).

k	Exact eigenvalues	Relative error WRM Legendre n=30	Relative error WRM Bernstein n=20	Relative error Bernstein n=30	Relative errors (Spect. Cheby. coll.) n=30	Relative error (Cheby Coll.)	Relative errors (Diff. Quad.) n=20	Relative errors (Diff. Quad.) n=30
1	237.72106753	4.69×10^{-12}	4.69×10^{-12}	4.68×10^{-12}	2.03×10^{-9}	4.70×10^{-12}	7.59×10^{-9}	7.59×10^{-9}
2	2496.48743786	1.27×10^{-12}	1.27×10^{-12}	1.28×10^{-12}	7.93×10^{-10}	3.05×10^{-12}	4.44×10^{-8}	4.45×10^{-8}
3	10867.58221698	1.02×10^{-13}	1.02×10^{-13}	1.10×10^{-13}	2.33×10^{-10}	5.10×10^{-12}	1.94×10^{-9}	1.71×10^{-8}
4	31780.09645408	3.39×10^{-14}	3.88×10^{-14}	3.41×10^{-14}	8.60×10^{-9}	8.60×10^{-9}	4.50×10^{-8}	2.36×10^{-8}
5	74000.84934915	7.43×10^{-14}	1.99×10^{-12}	7.21×10^{-14}	7.51×10^{-11}		3.97×10^{-5}	2.99×10^{-8}
6	148634.47728577	1.96×10^{-15}	3.05×10^{-9}	1.04×10^{-14}	2.24×10^{-10}		1.43×10^{-4}	4.77×10^{-8}
7	269123.43482664	6.49×10^{-15}	2.26×10^{-13}	7.33×10^{-14}			4.08×10^{-3}	9.61×10^{-10}
8	451247.99471928	5.19×10^{-14}	1.64×10^{-5}	7.59×10^{-14}			1.11×10^{-2}	1.74×10^{-8}
9	713126.24789600	1.32×10^{-11}	1.36×10^{-4}	4.41×10^{-12}			9.02×10^{-2}	3.16×10^{-6}
10	1075214.1034736	1.61×10^{-9}	4.10×10^{-3}	1.61×10^{-11}			2.06×10^{-2}	9.31×10^{-6}

Here $L_n(0) \neq 0$ which implies $u(0) \neq 0$ and $u(1)=0$ and $\theta_0(1)=0$ as specified by the essential boundary condition.

The weighted residual, equation (3.6) becomes

$$\sum_{i=1}^{n-1}\left[F_{i,j} - \lambda E_{i,j}\right]c_i = 0, j = 1,2,3,........,n-1 \quad (3.33a)$$

$$F_{i,j} = -\int_0^1 \frac{d^3 B_j}{dx^3}\frac{dB_i}{dx}dx - \left[\frac{dB_i(1)}{dx}\frac{d^2 B_j(1)}{dx^2}\right] - \left[\frac{d^2 B_j(0)}{dx^2}\frac{dB_i(0)}{dx}\right] \quad (3.33a)$$

$$E_{i,j} = \int_0^1 B_i B_j dx \quad (3.33b)$$

Table 3.4 demonstrates the comparison of our result obtained by using $n=20$ for Bernstein polynomials, Legendre polynomials for $n=25$ with the first nine eigenvalues of the problem with the results of Attili and Lesnic (2006). We observed that our current method is in good agreement with the reference studies.

Table 3.3: Comparison of eigenvalues for example 3.1(b).

	$\lambda_k^{(Galerkin)}$ Bernstein n=20	$\lambda_k^{(Galerkin)}$ Legendre n=20	Gamel and Sameeh Cheby-coll. (2012)	Results of Syam and Siyyam (2009)
1	500.563901740	500.563901740	500.563901740	500.563901756
2	3803.53708050	3803.53708050	3803.53708058	3803.53708049
3	14617.6301311	14617.6301311	14617.6301777	14617.6301311
4	39943. 7990057	39943.7990057	....	39943.7990057
5	89135.4076574	89135.4076574	....	89135.4076571

Example 3.2(a): We consider the Sturm-Liouville BVP taken from the articles of Attili and Lesnic (2006) and Taher *et al* (2013), respectively.

$$\frac{d^4 y}{dx^4} = 0.02x^2\frac{d^2 y}{dx^2} + 0.04\,x\frac{dy}{dx} - (0.0001x^4 - 0.02)u(x) + \lambda u(x) \quad (3.34a)$$

$$\begin{cases} u(0) = u''(0) = 0 \\ u(5) = u''(5) = 0 \end{cases} \quad (3.34b)$$

Table 3.4: Comparison of eigenvalues for example 3.1(c).

k	Computed eigenvalue Bernstein polynomial $\lambda_k^{(Gal.)}$ n=20	Computed eigenvalue Legendre polynomial $\lambda_k^{(Gal.)}$ n=25	Results of Attili and Lesnic (2006)
1	12.3623633683259	12.3623633683262	12.3623633683262
2	485.518818513372	485.518818513371	485.518818513372
3	3806.54626639151	3806.54626639145	3806.54626639145
4	14617.2733051187	14617.2733051188	14617.2733051100
5	39943.8317785095	39943.8317785095	39943.8317790386
6	89135.4050714239	89135.4050714232	89135.4050444342
7	173881.315656105	173881.315656106	173881.315656105
8	308208.452093651	308208.452093656	308208.438655408
9	508481.543266068	508481.543299331	508481.270992137

The equivalent Sturm-Liouville BVP over $[0,1]$ is,

$$\frac{1}{5^4}\frac{d^4u}{dx^4} = 0.02\times 25x^2\left(\frac{1}{5^2}\frac{d^2u}{dx^2}\right) + 0.04\times 5x\left(\frac{1}{5}\frac{du}{dx}\right)$$

$$-(0.0001\times 5^4x^4 - 0.02)u(x) + \lambda u(x) \quad (3.35a)$$

$$\begin{cases} u(0) = u''(0) = 0 \\ u(1) = u''(1) = 0 \end{cases} \quad (3.35b)$$

Here $\theta_0(x) = 0$ as specified by the Dirichlet boundary conditions of equation (3.35b). Also $B_{i,n}(0) = 0$ and $B_{i,n}(1) = 0$

The weighted residual equation becomes

$$\sum_{i=1}^{n-1}\left[F_{i,j} - \lambda E_{i,j}\right]c_i = 0 \quad , j = 1,2,3,\ldots\ldots\ldots,n-1\ , \ i = 1,2,3,\ldots\ldots\ldots,n. \quad (3.36)$$

where,

$$F_{i,j}=\int_0^1\left[-\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{dB_i}{dx}+25\left[xB_j(x)\right]\frac{dB_i}{dx}-\left(\frac{25}{2}\right)\frac{d}{dx}\left[x^2B_j(x)\right]\frac{dB_i}{dx}\right.$$

$$\left.-(0.0625x^4-0.02)B_iB_j\right]dx+\left(\frac{25}{2}\right)\left[x^2B_j(x)\frac{dB_i}{dx}\right]_0^1$$

$$+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_0^1 \tag{3.36a}$$

$$E_{i,j}=\int_0^1 B_i\,B_j\,dx \tag{3.36b}$$

If we replace x by $\frac{5}{2}(x+1)$ in $\tilde{u}(x)$, then we get desired approximate solution of the SLE (3.34a). Comparison of our result obtained using $n=20$ for both Bernstein and Legendre polynomial, in table 3.5. Among the first six eigenvalues of the problem with the of Attili and Lesnic (2006), Syam and Siyyam (2009), Yucel and Boubaker (2012), Gamel and Sameeh (2012) and Taher *et al* (2013). The observed CPU time is 5.33 seconds for Bernstein polynomials, and 6.9334 seconds for Legendre polynomials.

Example 3.2(b): Consider the Sturm-Liouville BVP worked out by Attili and Lesnic (2006), Chanane (2010), Yucel and Boubaker (2012), Taher *et al* (2013).

$$\frac{d^4u}{dx^4}=0.02x^2\frac{d^2u}{dx^2}+0.04\,x\,\frac{du}{dx}-\left(0.0001\,x^4-0.02\right)u(x)+\lambda\,u(x), \tag{3.37a}$$

$$\begin{cases} u(0)=u'(0)=0 \\ u(5)=u'(5)=0 \end{cases} \tag{3.37b}$$

Here,

$$F_{i,j}=\int_0^1\left[-\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{dB_i}{dx}+25\left[xB_j(x)\right]\frac{dB_i}{dx}-\left(\frac{25}{2}\right)\frac{d}{dx}\left[x^2B_j(x)\right]\frac{dB_i}{dx}\right.$$

$$\left.-(0.0625x^4-0.02)B_iB_j\right]dx+\left(\frac{25}{2}\right)\left[x^2B_j(x)\frac{dB_i}{dx}\right]_0^1 \tag{3.38a}$$

$$E_{i,j} = \int_0^1 B_i B_j dx \tag{3.38b}$$

Table 3.6 depicts the comparison of our result obtained using the degree of polynomial n=20, for the first six eigenvalues of the problem using Bernstein and Legendre polynomials with the results of (Attili and Lesnic, Chanane, Yucel and Boubaker, Taher *et al* [2006, 2010, 2012, 2013] respectively).

Table 3.5: Comparison of eigenvalues of example 3.2(a) with several methods

k	Current method $n=22$ Bernst.	Results of Taher *et al* (2013)	Results of Gamel and Sameeh (2012)	Result of Attili and Lesnic (2006)	Results of Yucel and Boubaker (2012)	Results of Syam and Siyyam (2009)	Present method Legn n=20
1	0.21505086437	0.21505086432	0.21505086437	0.2150508643697	0.21505086437	0.21505086437	0.2150508644
2	2.75480993468	2.75480993362	2.7548099346829	2.7548099346829	2.75480993468	2.75480993468	2.7548099347
3	13.21535154056	13.21535154059	13.215351546416	13.215351540558	13.2153515406	13.2153515406	13.215351542
4	40.95081975916	40.95081975814	40.950820029821	40.950819759137	40.9508197591	40.9508197591	40.950819759
5	99.05347806349	99.05347803835	....	99.053478138138	99.0534780633	99.0534781381	99.053478067
6	204.35573226893	204.35573547934	....	204.35449348957	204.355732256	204.3544934895	204.35573425
CPU Time	**5.33 secs.**						**6.933 secs.**

Table 3.6: Comparison of eigenvalues of example 3.2(b) with various methods

k	Bernstein n=20 $\lambda_k^{(galerkin)}$	Legendre n=20 $\lambda_k^{(galerkin)}$	Taher *et al* (2013)	Yucel and Boubaker (2012)	Chanane (2010)	Attili and Lesnic (2006)
1	0.86690250239970	0.86690250239971	0.86690250239196	0.86690250224260	0.86690250239947	0.8669025023997106
2	6.35768644814590	6.35768644814590	6.35768644814386	6.35768644843984	6.35768644817446	6.357686448145815
3	23.99274685030238	23.99274685030234	23.99274685032633	23.9927468509660	23.99274695066747	23.992746850281375
4	64.97866759050172	64.978667595017305	64.97866759484157	64.97866761311830	64.97863591597007	64.97866759571622
5	144.2806269274497	144.28062692790312	144.28062688384347	144.2806269273480	....	144.28062803844648
6	280.6009633049182	280.6009633067139	280.60096699712966	280.60096374439620	....	280.58602048195377
CPU Time	**6.0408** sec	**7.459** sec				

3.4 The Bernstein Collocation Method

3.4.1 Description of the scheme for fourth order SLEs

We consider the following fourth order Sturm-Liouville eigenvalue problem

$$\frac{d^4u}{dx^4} + p(x)\frac{d^2u}{dx^2} - q(x)\frac{du}{dx} + r(x)u = \lambda u(x), \quad x_1 < x < x_n \tag{3.39a}$$

Starting with the hinged boundary conditions, our task is to construct a polynomial of degree n satisfies n-1 boundary conditions

We have four boundary conditions

$$u(0) = u(1) = 0 \quad ; \quad u''(0) = u''(1) = 0 \tag{3.39b}$$

Let us seek the solution of (3.39a) in terms of Bernstein polynomials as

$$u(x) = \sum_{j=0}^{n} B_j(x)c_j \quad , \quad 0 < x < 1 \tag{3.40}$$

Residual for shifted Bernstein polynomial

$$R(x) \approx C^{\eta^T} B^{\eta}(x)Q^{(4)} + p(x)C^{\eta^T} B^{\eta}(x)Q^{(2)}$$
$$+C^{\eta^T} r(x)B^{\eta}(x) - C^{\eta^T} q(x)B^{\eta}(x)Q'(x) - \lambda C^{\eta^T} B^{\eta}(x) \tag{3.41}$$

where $\left\{B_j\right\}$ is again the Bernstein basis set corresponding to the full set of nodes $\{x_k\}$ .The Bernstein polynomial satisfies $u(x_1)=u(x_n)=0$.

3.4.2 Matrix Formulation

As before we use the Bernstein polynomial in Chapter 1, equation (1.74), and require the differential equation to be satisfied at the interior grid points, yielding

$$\sum_{j=2}^{n-1} B_j^4(x_k)c_j = \lambda c_k, \quad k = 3,4,5,\ldots\ldots\ldots n-2 \tag{3.42}$$

The hinged boundary conditions imply

$$\sum_{j=2}^{n-1} B_j''\left(x_n\right)c_j = 0 \text{ and } \sum_{j=2}^{n-1} B_j''\left(x_n\right)c_j = 0 \tag{3.42a}$$

Equations (3.42) - (3.42a) form a linear system of n-2 equations. To put the discrete equation (3.42a) in the form of an algebraic eigenvalue problem we eliminate c_2 and c_{n-1} [Weidman and Reedy (2000)]. We define

$$B_1''(x_1)c_1(x_1)+\sum_{j=2}^{n-1} B_j''(x_1)c_j = 0 \tag{3.42b}$$

$$B_1''(x_n)c_1(x_n)+\sum_{j=2}^{n-1} B_j''(x_n)c_j = 0 \tag{3.42c}$$

Equations (3.42b) and (3.42c) can be written in matrix form as

$$\begin{bmatrix} B_3''(x_1) & B_4''(x_1) & B_5''(x_1)\ldots\ldots & B_{n-2}''(x_1) \\ B_3''(x_n) & B_4''(x_n) & B_5''(x_n)\ldots\ldots & B_{n-2}''(x_n) \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_{n-2} \end{bmatrix}+$$

$$\begin{bmatrix} B_2''(x_1) & B_{n-1}''(x_1) \\ B_2''(x_n) & B_{n-1}''(x_n) \end{bmatrix}\begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix}=0 \tag{3.43}$$

Equation (3.43) can be written as $M_1C^{\eta}+M_2C^{\omega}=0$ (3.44)

$C^{\omega}=-M_2^{-1}M_1C^{\eta}$

where,

$$M_1=\begin{bmatrix} B_3''(x_1) & B_4''(x_1) & B_5''(x_1)\ldots\ldots & B_{n-2}''(x_1) \\ B_3''(x_n) & B_4''(x_n) & B_5''(x_n)\ldots\ldots & B_{n-2}''(x_n) \end{bmatrix} \tag{3.44a}$$

$$M_2=\begin{bmatrix} B_2''(x_1) & B_{n-1}''(x_1) \\ B_2''(x_n) & B_{n-1}''(x_n) \end{bmatrix}, \quad C^{\eta}=\begin{bmatrix} c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_{n-2} \end{bmatrix}, \quad C^{\omega}=\begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix} \tag{3.44b}$$

$$\sum_{j=2}^{n-1} B_j^4(x_k)c_j = \sum_{j=3}^{n-2} B_j^4(x_k)c_j + \begin{bmatrix} B_2^{iv}(x_3) & B_{n-1}^{iv}(x_3) \\ B_2^{iv}(x_4) & B_{n-1}^{iv}(x_4) \\ B_2^{iv}(x_5) & B_{n-1}^{iv}(x_5) \\ \vdots & \vdots \\ B_2^{iv}(x_{n-2}) & B_{n-1}^{iv}(x_{n-2}) \end{bmatrix} \begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix} \tag{3.45}$$

$k=3,4,\ldots\ldots\ldots\ldots,n-2$

$$\widetilde{B}^4 C^\eta = \lambda C^\eta \text{ , where } \quad C^\eta = \left[c_3, c_4, \ldots\ldots\ldots, c_{n-2}\right]^T \tag{3.45a}$$

Also, equation (3.45) implies that

$$\widetilde{B}^{(4)} C^\eta = \overline{B}^{(4)} C^\eta + M_3 C^\omega \tag{3.46}$$

where,

$$M_3 = \begin{bmatrix} B_2^{iv}(x_3) & B_{n-1}^{iv}(x_3) \\ B_2^{iv}(x_4) & B_{n-1}^{iv}(x_4) \\ B_2^{iv}(x_5) & B_{n-1}^{iv}(x_5) \\ \vdots & \\ B_2^{iv}(x_{n-2}) & B_{n-1}^{iv}(x_{n-2}) \end{bmatrix} \tag{3.46a}$$

On using equation (3.44), equation (3.46) becomes

$$\widetilde{B}^{(4)} C^\eta = \overline{B}^4 C^\eta - \left(M_3 M_2^{-1} M_1\right) C^\eta \tag{3.47}$$

The differential eigenvalue problem (3.39a) now becomes the algebraic eigenvalue problem and be written as

$$\lambda I = \left(\widetilde{B}^{(4)} B^{-1}\right) \tag{3.48}$$

Here $\overline{B}^4$ is the interior (n-4) × (n-4) submatrix, corresponding to rows and columns 3 to n-2 of the standard fourth-derivative Bernstein polynomial is taken as n=30 for these two problems.

3.4.3 Test examples

In this section, the proposed WRM of collocation is used to solve two examples in order to prove its efficiency and accuracy. It is to be noted that the maximum degree of polynomials in Bernstein basis is 30.

Example 3.3: We consider the following fourth order eigenvalue problem.

$$\frac{d^4 y}{dx^4} = \lambda u(x) \tag{3.49a}$$

$$\begin{cases} u(0) = u(1) = 0 \\ u'(0) = u'(1) = 0 \end{cases} \tag{3.49b}$$

Table 3.7: Comparison of eigenvalues applying present techniques with exact ones for example 3.3.

k	Exact eigenval. $\lambda^{Exact.}$	$\lambda_k^{Coll.}$ Bernstein n=10	$\lambda_k^{Coll.}$ Bernstein n=20	$\lambda_k^{Coll.}$ Bernstein n=30	$\lambda_k^{(Gal.)}$ Bernstein n=20	Results of Huang *et al* (2013)
1	500.563902	500.563902	500.563902	500.563902	500.563902	500.563902
2	3803.53708	3731.96160	3803.53709	3803.53708	3803.53708	3803.53708
3	14617.6301	12768.6876	14617.6359	14617.6301	14617.6301	14617.6301
4	39943.7990	....	39948.3633	39943.7990	39943.7990	39943.7994
5	89135.4077	....	88946.9687	89135.4077	89135.4077	89135.4223
CPU time			**2.177** sec	**3.9771** sec	**3.3214** sec	

The numerical results of eigenvalues for clamped-clamped boundary conditions are displayed in table 3.7. In order to verify the convergence of the proposed method, we have calculated the first five characteristic values of the Sturm–Liouville equation (3.49) by taking different n values in the approximate solution of equation (3.40). We compared our numerical results with the exact ones, method worked out by Huang *et al* (2013) and the Galerkin WRM as well. We have noticed that the numerical results employing the current collocation technique have a rapid convergence, with n increasing from 10 to 30, the errors between the numerical and exact results drastically decrease and the results when taking n = 30 are identical to the exact ones, which indicates that the present approach competes well with other methods. Furthermore, table 3.7 reveals that last two eigenvalues attained by the present method are more convergent to the exact results than those of the results calculated by Huang *et al* (2013). Computational time and performance to achieve the desired accuracy computed by Gal WRM and Collocation method are almost the same. Table 3.7 demonstrates that the present numerical results are in good agreement with the exact results.

Example 3.4: We consider the Sturm-Liouville eigenvalue problem worked out by [Attili and Lesnic (2006), Yucel and Boubaker (2012), Taher *et al* (2012)].

$$\frac{d^4 u}{dx^4} = 0.02x^2 \frac{d^2 u}{dx^2} + 0.04\, x \frac{du}{dx} - (0.0001\, x^4 - 0.02) u(x) + \lambda u(x) \qquad (3.50a)$$

$$\begin{cases} u(0) = u(5) = 0 \\ u'(0) = u'(5) = 0 \end{cases} \qquad (3.50b)$$

Table 3.8 lists first six eigenvalues for n=20. All the computed eigenvalues are very close to the results of [Taher *et al* (2013), Yucel and Boubaker (2012), Attili and Lesnic (2006)].

The condition number assesses the stability of the numerical method with respect to matrix inversion. Although discretization methods achieve high accuracy, eventually lead to ill-conditioned system as well. Figure 3.1 illustrates that condition number for the fourth order Chebychev differentiation matrix rises to something of order 10^4 to 10^8; whereas this number varies between 10^2 to 10^8 for Bernstein collocation method. Conversely in case of Bernstein Galerkin WRM, the condition numbers increase slight rapidly than those of the said collocation methods with the increase of polynomials. From Figure 3.1, it has been observed that collocation matrix is of better conditioning than those of Galerkin and Chebychev differentiation matrix.

Table 3.8: Comparison of eigenvalues of example 3.4 for various methods.

	Computed Eigenvalues				
k	Bernstein Coll. present	WRM Galerkin present	Taher *et al* (2013)	Yucel and Boubaker (2012)	Attili and Lesnic (2006)
1	0.866902502393833	0.86690250239970	0.86690250239196	0.86690250224260	0.866902502399711
2	6.357686448139751	6.35768644814590	6.35768644814386	6.35768644843984	6.357686448145815
3	23.99274685029107	23.99274685030238	23.99274685032633	23.9927468509660	23.992746850281375
4	64.97866759500305	64.97866759050172	64.97866759484157	64.97866761311830	64.97866759571622
5	144.2806269274242	144.2806269274497	144.28062688384347	144.2806269273480	144.28062803844648
6	280.6009632838048	280.6009633049182	280.60096699712966	280.6009637443962	280.58602048195377

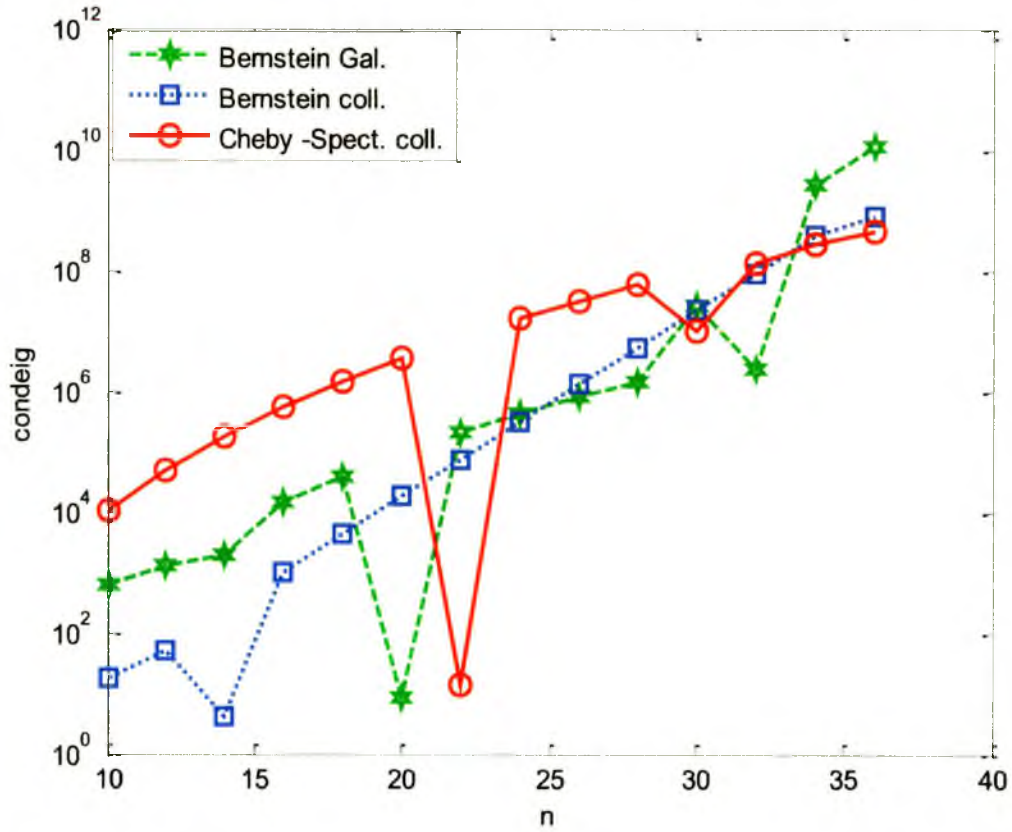

Figure 3.1: Logarithm of the condition number of the Bernstein Galerkin matrix (A), Bernstein collocation matrix $\left(\widetilde{B}^{4}\right)$ and Chebyshev Differentiation matrix [Taher *et al* (2012)].

3.5 The Chebychev-Legendre Spectral Collocation Method

In this section, the numerical solution $u(x)$ for the fourth order SLEs at Chebyshev Gauss Lobatto points in equation (2.92) of section 2, are presented and it is based on Legendre approximations of equation (2.95). We now collocate the equation (3.1) at the grid points as before and be given by

$$p(x_k)u^{iv}(x_k)+p''(x_k)u''(x_k)-q(x_k)u''(x_k)-q'(x_k)u(x_k)u'$$
$$r(x_k)u(x_k)-\lambda\, u(x_k)=0 \qquad x\in(\gamma,\mu) \tag{3.51}$$

Here we consider a fourth order Sturm Liouville eigenvalue problem with constant co-efficient for brevity given by the following equation:

$$\frac{d^4u}{dx^4}=\lambda u \tag{3.52a}$$

$$\begin{cases} u(-1)=u(1)=0 \\ u'(-1)=u'(1)=0 \end{cases} \tag{3.52b}$$

3.5.1 Methodology

Suppose $\{y_k\}$ be the set of $(n-1)$ vector of values of u sampled at $x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots, x_n$. Let us assume that p be the unique polynomial of degree $\leq n+2$ corresponding to the full set of Chebyshev nodes $\{x_k\}$ as defined by (2.92) with $p(\pm1)=p'(\pm1)=0$ and $p(x_k)=y_k$.

For $k=1,2,3,\ldots\ldots\ldots\ldots, n-1$.

$$\text{Set } v_k = p^4(x_k) \tag{3.53}$$

We obtain v as a byproduct of our usual Legendre differentiation matrix D_n if

we set $p(x)=\left(1-x^2\right)h(x)$

Using Leibnitz rule of differentiation, we get

$$p(x) = \left(1-x^2\right) h^{iv}(x) - 8xh'''(x) - 12h''(x) \tag{3.54}$$

A polynomial h of degree $\leq n$ with $h(\pm 1) = 0$ corresponds to a polynomial p of degree $\leq n+2$ with $p(\pm 1) = 0$ and $p'(\pm 1) = 0$. Thus, we can carry out the spectral differentiation as given below

i) Let h be the unique polynomial of degree $\leq n$ with $h(\pm 1) = 0$ and

$$h\left(x_k\right) = \frac{y_k}{1-x_k^2}, \quad k = 1, 2, 3 \ldots\ldots, n-1 \tag{3.55}$$

ii) Set $v_k = \left(1-x_k^2\right) h^{iv}(x_k) - 8x_k h'''(x_k) - 12h''(x_k)$ (3.56)

At the matrix level let $\tilde{D}_n^2, \tilde{D}_n^3, \tilde{D}_n^4$ be the matrices obtained by taking the indicated powers of $\tilde{D}_n$ and stripping away the first and last rows and columns. Thus, our spectral bi-harmonic operator is

$$L = \left[diag\left(1-x_k^2\right) \tilde{D}_n^4 - 8 diag\left(x_k\right) \tilde{D}_n^3 - 12 \tilde{D}_n^2 \right] \times diag\left(\frac{1}{\left(1-x_k^2\right)} \right), \tag{3.57}$$

$$k = 1, 2, 3 \ldots\ldots, n-1$$

Therefore, we get a system of linear equation for u ,

$$Lu = \lambda I u \tag{3.58}$$

We now handle with the hinged boundary conditions in Chebychev differentiation matrix is difficult part in this method. Here we aim to develop a scheme where we handling boundary conditions in an easier way. Let us consider the following hinged boundary conditions:

$$\begin{cases} u(-1) = u(1) = 0 \\ u''(-1) = u''(1) = 0 \end{cases} \tag{3.59}$$

For this reason, we shall solve (3.52a) subject to (3.59) with the method of explicit

enforcement of boundary conditions. The Interpolating polynomial is taken to be

$$p_{n-1} = \sum_{i=2}^{n-1} u_i \phi_i(x) \tag{3.60}$$

Where $\{\phi_i(x)\}$ is again the Lagrangian basis set of equation (2.98) corresponding the Chebychev nodes in equation (2.92). This interpolant satisfies $p(\pm 1) = 0$. We require the equation (3.52) to be satisfied at the interior $n-4$ grid points:

$$\sum_{i=2}^{n-1} u_i \phi_i^4(x_k) = \lambda u_k, \quad k = 3,4,5,\ldots\ldots\ldots\ldots n-2 \tag{3.61}$$

The hinged boundary conditions imply

$$\phi_1''(-1)u_1(-1) + \sum_{i=2}^{n-1} \phi_i''(x_1)u_i = 0 \tag{3.61a}$$

$$\phi_1''(1)u_1(1) + \sum_{i=2}^{n-1} \phi_i''(x_n)u_i = 0 \quad , \tag{3.61b}$$

$k=3,4,5,\ldots\ldots\ldots\ldots\ldots n-2$ are interior nodes

From (3.61a)

$$\phi_2''(x_1)u_2 + \phi_{n-1}''(x_1)u_{n-1} + \sum_{i=3}^{n-2} \phi_i''(x_1)u_i = 0$$

from (3.61b) $\phi_2''(x_n)u_2 + \phi_{n-1}''(x_n)u_{n-1} + \sum_{i=3}^{n-2} \phi_i''(x_1)u_i = 0$ (3.62)

where,

$$u = [u_2 \ldots\ldots\ldots u_{n-1}]^T$$

$$\tilde{D}_{1,i} = \phi_i''(x_1), \quad i = 2,3\ldots\ldots\ldots\ldots, n-1,$$

$$\tilde{D}_{k-1,i} = \phi_i^{iv}(x_k) \quad i=2,3\ldots\ldots\ldots\ldots,n-1 \quad , \quad k=3,4,5,\ldots\ldots\ldots\ldots\ldots n-2$$

$$\tilde{D}_{n-2,i} = \phi_i''(x_n) \quad , \quad i=2,3\ldots\ldots\ldots\ldots,n-1$$

From (3.61a) and (3.61b) $\begin{bmatrix} \phi_3''(x_1) & \phi_4''(x_1) & \phi_5''(x_1) \ldots\ldots\ldots\ldots & \phi_{n-2}''(x_1) \\ \phi_3''(x_n) & \phi_4''(x_n) & \phi_5''(x_n) \ldots\ldots\ldots\ldots & \phi_{n-2}''(x_n) \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ \vdots \\ u_{n-2} \end{bmatrix}$

$$+\begin{bmatrix} \phi_2''(x_1) & \phi_{n-1}''(x_1) \\ \phi_2''(x_n) & \phi_{n-1}''(x_n) \end{bmatrix} \begin{bmatrix} u_2 \\ u_{n-1} \end{bmatrix} = 0 \tag{3.63}$$

Equation (3. 63) can be written as $M_1 y^* + M_2 y^{**} = 0$ (3.64)

$$y^{**} = -M_2^{-1} M_1 y^* \tag{3.65a}$$

where $M_1 = \begin{bmatrix} \phi_4''(x_1) & \phi_5''(x_1) & \phi_6''(x_1) \ldots\ldots\ldots\ldots & \phi_{n-3}''(x_1) \\ \phi_4''(x_n) & \phi_5''(x_n) & \phi_6''(x_n) \ldots\ldots\ldots\ldots & \phi_{n-3}''(x_n) \end{bmatrix}$ (3.65b)

$$M_2 = \begin{bmatrix} \phi_2''(x_1) & \phi_{n-1}''(x_1) \\ \phi_2''(x_n) & \phi_4''(x_n) \end{bmatrix}, \quad u^* = \left[u_3, u_4, \ldots\ldots\ldots\ldots, u_{n-2} \right]^{\mathrm{T}},$$

$$u^{**} = \left[u_2 \quad u_{n-1} \right]^T \tag{3.65c}$$

$$\sum_{i=2}^{n-1} \phi_i^{iv}(x_k) u_i = \sum_{i=3}^{n-2} \phi_i^{iv}(x_k) u_i + \begin{bmatrix} \phi_2^{iv}(x_4) & \phi_{n-1}^{iv}(x_4) \\ \phi_2^{vi}(x_5) & \phi_{n-1}^{iv}(x_5) \\ \phi_2^{vi}(x_6) & \phi_{n-1}^{iv}(x_6) \\ \vdots & \\ \phi_2^{iv}(x_{n-3}) & \phi_{n-1}^{iv}(x_{n-3}) \end{bmatrix} \tag{3.66}$$

$$\tilde{D}^4 u^* = D^4 u^* + M_3 u^{**}, \qquad k = 3, 4, \ldots\ldots\ldots\ldots, n-2 \tag{3.67}$$

The differential eigenvalue problem now becomes the algebraic eigenvalue problem

$$\tilde{D}^4 u^* = \lambda u^* \tag{3.68}$$

where, $u^* = \left[u_3, u_4, \ldots\ldots\ldots\ldots, u_{n-2} \right]^T$ (3.68a)

$$\tilde{D}^4 = \bar{D}^{(4)} + M_3 = \bar{D}^{(4)} + \left(M_3 M_2^{-1} \right) M_1 \tag{3.69}$$

Here $\overline{D}^{(4)}$ is the interior sub matrix, corresponding to rows and columns 3 to $n-2$, of the standard fourth-derivative Legendre $\overline{D}^{(4)}$ which is computed by Legendre differentiation matrix.

3.5.2 Numerical Applications

To exhibit the effectiveness and power of the Legendre Spectral method, we have tested four Sturm–Liouville equations eigenvalue problems. These problems have been preferred because they are commonly discussed in literature reported by other workers.

Example 3.5: We consider the Sturm-Liouville BVP studied by Yucel and Boubaker (2012), Gamel and Sameeh (2012) and Taher *et al* (2013).

$$\frac{d^4u}{dx^4} - \lambda u(x) = 0 \quad , \; 0 < x < 1 \tag{3.70a}$$

$$\begin{cases} u(0) = u(\pi) = 0 \\ u''(0) = u''(\pi) = 0 \end{cases} \tag{3.70b}$$

Changing the variables $x = \frac{\pi}{2}t + \frac{1}{2}$, the Sturm-Liouville problem (3.70a) transforms into the interval [-1, 1].

$$\begin{cases} \frac{d^4u}{dt^4} - \lambda u(t) = 0 \\ u(-1) = u(1) = 0, \qquad -1 < t < 1 \\ u''(-1) = u''(1) = 0 \end{cases} \tag{3.70c}$$

Since, the interpolating polynomials have to satisfy the differential equation at each interior node. We obtain the following collocation equation

$$\frac{16}{\pi^4} p^{(4)}(t_k) - \lambda\, p(t_k) = 0$$

$$\left(\frac{16}{\pi^4}\tilde{D}^{(4)} - \lambda I\right)u = 0 \ , \tag{3.71}$$

The first forty-one eigenvalues and related absolute error are given in table 3.9. The exact eigenvalues in reference of Taher *et al* (2013) of equation (3.71a) are $\lambda_k = k^4$, $k = 1,2,3............n$. For brevity we use 40, 60, 80 and 97 nodes. Numerical Results attained applying of our present scheme in comparison with Cheby-Legendre collocation method revealed almost the same convergence. Furthermore, present scheme produced the same result for n=97 whereas this number is n=100 in article [Taher *et al* (2013)]. Therefore, our current method is much competent with the Chbychev scheme.

Example 3.6: The following SLE problem was examined by Yucel and Boubaker (2012), Gamel and Sameeh (2012) and Taher *et al* (2013).

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0 \ , \qquad 0 < x < 1 \tag{3.72a}$$

$$\begin{cases} u(0) = u'(0) = 0 \\ u(1) = u''(1) = 0 \end{cases} \tag{3.72b}$$

Relative errors achieved by our present method illustrated in table 3.10 which are smaller than Cheby-Spectral collocation [Taher *et al* (2012)], PDQ, FDQ [Yucel and Boubaker (2012)] and Cheby-Collocation method [Gamel and Sameeh (2012)]. Taher *et al* (2012) reported the first eigenvalues and the last one reaches the accuracy up to 10^{-10}, whereas our present Spectral collocation technique attains the accuracy up to 10^{-14} for the same. Implementation of Legendre-Spectral collocation technique demonstrates that it is much superior than other numerical discretization techniques namely Cheby-Spectral collocation, PDQ and FDQ.

Table 3.9: Comparison of absolute errors of Legendre spectral collocation with the results of Chebychev's Spectral collocation method for example 3.5.

k	Exact Eigenval. λ^{exact}	Cheby Spect. coll. n=40	Legn Spect. coll. n=40.	Cheby spect. coll. n=60	Legn spect. coll. n=60	Cheby Spect. coll. n=80	Legn. Spect. coll. n=80	Cheby Spect. coll. n=100	Legn. Spect. coll n=97	Absolute error Cheby.	Absolute error Legn.
1	1	1	1	1	1	1	1	1	1	0.00000	0.00000
2	16	16	16	16	16	16	16	16	16	0.00000	0.00000
3	81	81	81	81	81	81	81	81	81	0.000000	0.000000
4	256	256	256	256	256	256	256	256	256	0.00000	0.00000
5	625	625	625	625	625	625	625	625	625	0.000000	0.000000
6	1296	1296	1296	1296	1296	1296	1296	1296	1296	0.00000	0.00000
7	2401	2401	2401	2401	2401	2401	2401	2401	2401	0.000000	0.000000
8	4096	4096	4096	4096	4096	4096	4096	4096	4096	0.000000	0.000000
9	6561	6561	6561	6561	6561	6561	6561	6561	6561	0.00000	0.00000
10	10000	10000	10000	10000	10000	10000	10000	10000	10000	0.00000	0.00000
11	14641	14641	14641	14641	14641	14641	14641	14641	14641	0.000000	0.000000
12	20736			20736	20736	20736	20736	20736	20736	0.00000	0.00000
13	28561			28561	28561	28561	28561	28561	28561	0.000000	0.000000
14	38416			38416	38416	38416	38416	38416	38416	0.00000	0.00000
15	50625			50625	50625	50625	50625	50625	50625	0.000000	0.000000
16	65536			65536	65536	65536	65536	65536	65536	0.000000	0.000000
17	83521			83521	83521	83521	83521	83521	83521	0.00000	0.00000
18	104976			104976	104976	104976	104976	104976	104976	0.00000	0.00000
19	130321			130321	130321	130321	130321	130321	130321	0.000000	0.000000
20	160000			160000	160000	160000	160000	160000	160000	0.00000	0.00000
21	194481			194481		194481	194481	194481	194481	0.000000	0.000000
22	234256					234256	234256	234256	234256	0.00000	0.00000
23	279841					279841	279841	279841	279841	0.000000	0.000000
24	331776					331776	331776	331776	331776	0.000000	0.000000
25	390625					390625	390625	390625	390625	0.00000	0.00000
26	456976					456976	456976	456976	456976	0.00000	0.00000
27	531441					531441	531441	531441	531441	0.000000	0.000000
28	614656					614656	614656	614656	614656	0.00000	0.00000
29	707281					707281	707281	707281	707281	0.000000	0.000000
30	810000					810000	810000	810000	-810000	0.00000	0.00000
31	923521					923521	923521	923521	923521	0.000000	0.000000
32	1048576							1048576	1048576	0.000000	0.000000
33	1185921							1185921	1185921	0.00000	0.00000
34	1336336							1336336	1336336	0.00000	0.00000
35	1500625							1500625	1500625	0.000000	0.000000
36	1679616							1679616	1679616	0.00000	0.00000
37	1874161							1874161	1874161	0.000000	0.000000
38	2085136							2085136	2085136	0.00000	0.00000
39	2313441							2313441	2313441	0.000000	0.000000
40	2560000							2560000	2560000	0.000000	0.000000
41	2825761							2825761	2825761	0.00000	0.00000

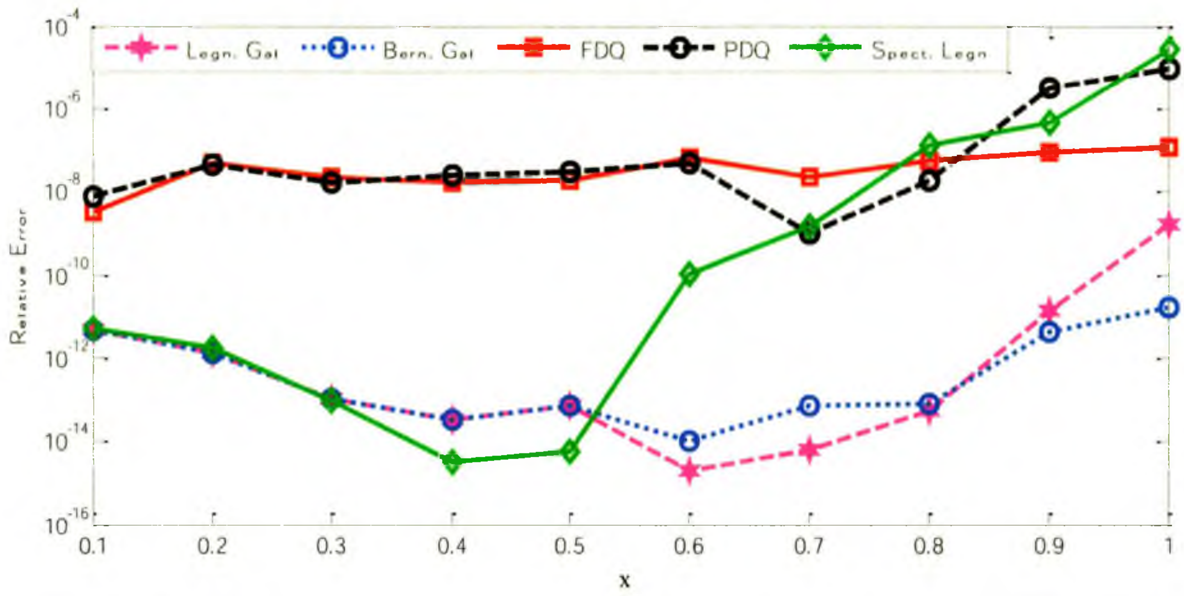

Figure 3.2: Comparison relative errors obtained using present WRM to various numerical methods.

The relative errors of our present methods are compared to the relative errors achieved by other methods are depicted in figure 3.2 for n=30.

Example 3.7: We consider the following SLE worked out by Gamel and Sameeh (2012), Syaam and Siyyam (2011).

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0 \tag{3.73a}$$

$$\left.\begin{aligned} u''(0) = u''(1) = 0 \\ u'''(0) = u'''(1) = 0 \end{aligned}\right\} \tag{3.73b}$$

Table 3.11 displays the first six eigenvalues using collocation nodes (n=20) and compared to the results of VIM [Syaam and Siyyam (2011)], Cheby Spectral [Taher *et al* (2012)], Bernstein Galerkin WRM, Legendre Galerkin WRM. All six eigenvalues are much closer to the other methods. Our present work is much compatible with the available numerical techniques.

Table 3.10: Comparison of eigenvalues and relative errors of example 3.6 for different methods.

k	Exact eigenvalues λ_k	Rel. error Bernst. n=30	Legn. Spect. n=30	Legn. Spect. n=35	Rel. error Cheby. Spect. Coll n=30	Rel. errors Cheby. Coll.	Rel. error PDQ n=30
1	237.7210653	4.68×10^{-12}	5.28×10^{-12}	7.89×10^{-11}	2.03×10^{-9}	4.7×10^{-12}	7.59×10^{-9}
2	2496.48743786	1.28×10^{-12}	1.78×10^{-12}	1.50×10^{-11}	7.93×10^{-10}	3.04×10^{-12}	4.45×10^{-8}
3	10867.58221698	1.10×10^{-13}	9.19×10^{-14}	2.39×10^{-12}	2.33×10^{-10}	5.10×10^{-12}	1.71×10^{-8}
4	31780.09645408	3.40×10^{-14}	3.09×10^{-15}	6.73×10^{-13}	8.61×10^{-9}	8.61×10^{-9}	2.36×10^{-8}
5	74000.84934915	7.26×10^{-14}	5.89×10^{-14}	1.47×10^{-12}	7.51×10^{-11}		2.99×10^{-8}
6	148634.4772857	1.04×10^{-14}	1.01×10^{-10}	4.21×10^{-14}	2.24×10^{-10}		4.77×10^{-8}
7	269123.43482664	7.33×10^{-14}	1.48×10^{-9}	1.49×10^{-13}			9.6×10^{-10}
8	451247.99471928	7.59×10^{-14}	1.34×10^{-7}	2.09×10^{-13}			1.74×10^{-8}
9	713126.24789600	4.41×10^{-12}	4.509×10^{-7}	2.21×10^{-13}			3.16×10^{-6}
10	1075214.1047396	1.61×10^{-11}	2.702×10^{-5}	1.141×10^{-9}			9.31×10^{-6}

Example 3.8: We consider the Sturm-Liouville boundary value problem studied by Yucel and Boubaker (2012), Taher *et al* (2013), Attili and Lesnic (2006), Chanane (2011).

$$\frac{d^4u}{dx^4}=0.02x^2\frac{d^2u}{dx^2}+0.04\,x\frac{du}{dx}-(0.0001\,x^4-0.02)u(x)+\lambda u(x) \quad (3.74a)$$

$$\begin{cases} u(0)=u(5)=0 \\ u'(1)=u'(5)=0 \end{cases} \quad (3.74b)$$

by changing the variables $x=\frac{5}{2}t+\frac{5}{2}$, the Sturm-Lioville problem (3.74a) transforms into the interval $[-1,1]$, the transformed equation becomes

$$\frac{16}{625}\frac{d^4u}{dt^4}=0.0032\times\left(\frac{5}{2}t+\frac{5}{2}\right)^2\frac{d^2u}{dt^2}+0.016\left(\frac{5}{2}t+\frac{5}{2}\right)\frac{du}{dt}$$

$$-\left\{0.0001\times\left(\frac{5}{2}t+\frac{5}{2}\right)^4-0.02\right\}u(x)+\lambda u(t) \qquad (3.75)$$

From table 3.12, we observe that computed first six eigenvalues for clamped boundary conditions, utilizing n=20, are in well agreement with the results attained by other various numerical methods existing in the literature. Computational cost is only about 0.355 seconds for this computations.

Table 3.11: Comparison of eigenvalues for example 3.7 with several methods.

	Present Legn. Spectral Coll.	$\lambda_k^{(galerkin)}$ Legn.	$\lambda_k^{(galerkin)}$ Bernst.	Result of (Cheby.) Coll.	Result of Syaam and Siyyam (2011)
1	500.563901740	500.563901740	500. 563901740	500.563901740	500.563901756
2	3803.53708049	3803.53708049	3803. 53708049	3803.53708058	3803.53708049
3	14617.6301311	14617.6301311	14617. 6301311	14617.6301777	14617.6301311
4	39943.7990057	39943.7990057	39943. 7990057		39943.7990057
5	89135.4077270	89135.4077270	89135.4076573		89135.4076570
6	173881.317531	173881.317531	173881.317531		173881.315471

3.6 Conclusions

We have discussed in details the formulations of Sturm-Liouville problem by the Galerkin weighted residual method using Bernstein and Legendre polynomials as basis functions. It is evident that eigenvalues obtained using Bernstein polynomials give much accurate results than those of Legendre polynomials. It is also observed that eigenvalues obtained by Galerkin-Bernstein method are most accurate and Galerkin-Legendre results are much compatible with the other results achieved by various methods. We have noticed that Bernstein polynomials converge slowly and computational cost is more for Bernstein Galerkin method than that of the Bernstein collocation method. Regardless of disadvantage, we can conclude that for a relatively small n, i.e., n = 20, moderately precise numerical results are obtained using the proposed method. Therefore, we may conclude that Galerkin-Bernstein polynomial and Galerkin-Legendre polynomial scheme perform well with degree of polynomials not greater than 30 than all other previously published works available in the literature. Furthermore, Bernstein collocation method is of well-conditioned than those of our other two proposed schemes i.e., Galerkin and Spectral collocation. In tables 3.9-3.12, the cost of computational time is much smaller in the case of Legendre spectral method than that of Galerkin WRM and Bernstein collocation method. Despite of some shortcomings, our proposed methods are much superior in the sense of accuracy and applicability especially for higher order problems. Furthermore, in table 3.2, the first seven eigenvalues using Legendre polynomials are very close to the exact result and the computed values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues. At the same time, it has also been observed in table 3.2 that all 10 eigenvalues obtained using Bernstein polynomials converge more rapidly than those of Legendre polynomials. In fact, relative error decays as the of degree of polynomials increased in the case of Bernstein basis. But on the other hand, estimated eigenvalues using Legendre polynomials show less convergent especially for the higher eigenvalues. It can be seen clearly from the table 3.2 that when degree of the Legendre polynomial is 20, the Galerkin results reach the best precision and as the degree of the Legendre polynomial further increased the accuracy of this method is not improved. It can be obviously observed that eigenvalues obtained by Galerkin-Bernstein method are most accurate and Galerkin-Legendre results are much more accurate than the other results.

Table 3.12: Comparison of eigenvalues for example 3.8 for various method

Present method $\lambda_k^{(coll)}$	Present method $\lambda_k^{(galerkin)}$	Taher*et al*(2013)	Yucel and Boubaker(2012)	Chanane(2010)	Attili and Lesnic(2006)
0.86690250239970	0.86690250239970	0.86690250239196	0.86690250224260	0.86690250239947	0.8669025023997106
6.35768644814590	6.35768644814590	6.35768644814386	6.35768644843984	6.35768644817446	6.357686448145815
23.99274685030238	23.99274685030238	23.99274685032633	23.9927468509660	23.99274695066747	23.992746850281375
64.97866759050172	64.97866759050172	64.97866759484157	64.97867761311830	64.97863591597007	64.97866759571622
144.2806269274497	144.2806269274497	144.28062688384347	144.2806269273480	....	144.28062803844648
280.6009633049182	280.6009633049182	280.60096699712966	280.60096374439620		280.58602048195377

CHAPTER 4

Eigenvalues of Sixth Order Boundary Value Problems Using the Technique of WRM

4.1 Introduction

The literature on the numerical solution of sixth-order BVP is scarce. A few literatures are found on computation of eigenvalues of higher order BVPs. Existence and uniqueness solutions of such higher order BVPs are listed in Agarwal (1986) but no numerical approach are illustrated therein.

In fluid dynamics, hydrodynamic stability is the field which analyses the stability and the onset of instability of fluid flows. The study of hydrodynamic stability aims to find out if a given flow is stable or unstable, and if so, how these instabilities will cause the development of turbulence. Stability and instability against small perturbations of such patterns of flow can be realized only for certain ranges of parameters characterizing them. When an infinitely small variation of the present state changes only by an infinitely small quantity of the state at some future time, the condition of the system, whether at rest or in motion, is said to be stable. A system is stable if no mode of disturbance exists and unstable even for the existence of only one mode of disturbance. When the value of the chosen parameter takes certain value and all others have their preassigned values, instability sets in at this value. The foundations of hydrodynamic stability, both theoretical and experimental, were laid most notably by Helmholtz, Kelvin, Rayleigh and Reynolds during the nineteenth century. These foundations have given many useful tools to study hydrodynamic stability. These include Reynolds number, the Euler equation and the Navier–Stokes equations. When studying flow stability, it is useful to understand more simplistic systems, e.g. incompressible and in viscid fluids which can then be developed further onto more complex flows [Philip Gerald Drazin (1934–2002)]. Stability characteristics are found in detail for a model of flow in a slowly-varying channel by use of the WKBJ approximation in the ref. [Drazin, 1974]. Since the 1980s, more computational methods are being used to model and analyze the more complex flows. Lord Rayleigh studied this problem and obtained a straightforward criterion. Rayleigh–Bénard convection is a type of natural convection, occurring in a plane horizontal layer of fluid heated from below, in which the fluid develops a regular pattern of convection cells known as Bénard cells. Rayleigh–Bénard convection is one of the most commonly studied convection phenomena because of its analytical and experimental accessibility.

Hydrodynamic and hydro-magnetic stability, instability and over stability have been studied rigorously by Chandrasekhar (1981) and we may observe that sixth order eigenvalue problems arise when instability of layers of fluid heated from below. From the literature review, we find that many researchers have attempted to solve the sixth order eigenvalue problems by several techniques. For this, Baldwin (1987) has studied with asymptotic expansions and global phase integral method while the finite difference method has been used by Twizell (1988), Twizell and Boutayeb (1990) to find the solutions of sixth order boundary and eigenvalues along with the Bénard-Layer problems. Besides this, Wang *et al* (2003), Lesnic and Attili (2006), Siyyam and Syam (2011) have paid their attentions to find the solutions of sixth order eigenvalue problems by the methods of local adaptive differential quadrature with Lagrange polynomials, Adomian decomposition, and Variational iteration method respectively. Furthermore, Rayleigh number has been studied rigorously by Gheorghiu and Dragomirescu (2009) using shifted Chebyshev, shifted Legendre-Galerkin spectral and Chebyshev collocation methods, and they have achieved a good agreement with exact solutions. Mdalla and Syam (2014) applied Chebyshev collocation-path following method for solving sixth order Sturm-Liouville problems. Very recently Amodio and Settanni (2015) discussed the solution of regular and singular Sturm-Liouville problems by means of high-order finite difference schemes. They described a method to define a discrete problem and its numerical solution by means of linear algebra techniques.

We note that Bernstein polynomials [Doha *et al* (2011)] vanish at the two end points of the interval and this property makes it attractive for implementation in the Galerkin WRM. On the other hand, imposing the boundary conditions over the higher order eigenvalue problems are quite complicated. This difficulty can easily be overcome for any higher order problems without reducing the order of the equations by applying Galerkin weighted residual method and all kind of derivative boundary conditions can be imposed directly in the weak form of the integrand. These two criteria partially motivate our interest to compute the eigenvalues of the BVPs with Galerkin WRM using Bernstein and Legendre polynomials. Legendre-Galerkin method for the solution of sixth order BVPs are worked out by Bhrawy (2009).

However, we summarize the Chapter as follows. Section 4.1 and 4.2 are devoted to hydrodynamic stability problems and brief description of the parameters that arise Galerkin weighted residual method and formulation of the general sixth order Sturm-Liouville problem have been conferred in section 4.4, respectively. Stability and error analysis of the present work are discussed in section 4.5. In section 4.6, we consider some numerical examples to verify the efficiency of the proposed method. Section 4.7 is devoted for the formulation of Bernstein collocation method to calculate eigenvalues of sixth order BVPs. We depict the numerical results of four Sturm-Liouville eigenvalue problems in section 4.8. A description of the Spectral collocation technique for discretization of the sixth-order differential equation is presented in Section 4.9. A few numerical examples are also illustrated in section 4.10. Conclusions of the Galerkin WRM, Bernstein collocation and Legendre Spectral collocation methods are given in section 4.11.

When Benard Layer has non-uniform destabilizing steady state temperature profile, convection sets in at a level where the local gradient sufficiently exceeds the adiabatic gradient for the inhibiting effects of viscosity and thermal conduction to be overcome. If this level is not at a boundary the linearized equations governing the motion leads to Boussinesq approximation to the boundary value problem.

$$\left(D^2 - a^2\right)^3 u + Ra^2 (1 - x^2) u = 0, \text{ where } u \to 0 \quad x \to \infty.$$

Here x is a dimensionless boundary layer co-ordinate, u and a are a dimensionless vertical velocity number and horizontal wave number respectively, Ra stands for critical value or Rayleigh number and $D = \frac{d}{dx}$ as well.

4.2 Brief Summary of Hydrodynamic Stability Problems:

Hydrodynamic problems face complexities as the order of the differential equations raises rapidly. These problems are usually solved applying discretization methods. Although these methods accomplish high accuracy, eventually lead to ill-conditioned system as well. The condition number [section 1.5.2 in chapter 1]
for the fourth order Chebychev differentiation matrix raises to something of order and for the eighth order differentiation matrix attains something of order $O\left(10^{30}\right)$ illustrated in [Gheorghiu and Dragomirescu (2009)]. The authors reduced the order of differentiation of the sixth order problems and transformed the equation into a second order system to implement the spectral method. In spite of this transformation, the condition number of the second order differentiation matrix raises almost of $O\left(10^{18}\right)$ with superior cost of computation of algebraic eigenvalues.

Our main objective of this study is ascertaining a new method with various types of boundary conditions and to minimize the condition number as well as the cost of computations. In the context of solving Bénard types of problems, brief descriptions of hydrodynamic parameters can be described as well. Rayleigh number R arises in the problem of thermal instability of a horizontal layer of fluid heated from below. When R exceeds a threshold value R_c then it is called critical Rayleigh number i.e., when $R \geq R_c$, instability occurs. The minimum value of R is obtained for the corresponding value of the wave number, that gives the length scale of modes for $R > R_c$ is excited. The smallest eigenvalue, i.e. the critical value or the Rayleigh number varies with variable gravity. Details of the hydrodynamic stability equations are found in the monograph of Straughan (2003). Gheorghiu and Dragomirescu

(2009) considered the sixth order differential equation with Dirichlet and Hinged boundary conditions given by

$$\left(D^2 - A^2\right)^3 W = -R\,A\left[1 + \varepsilon\, g(z)\right]A^2 W \tag{4.1a}$$

$$W(0) = DW(0) = D^2W(0) = 0 \tag{4.1b}$$

$$W(1) = DW(1) = D^2W(1) = 0 \tag{4.1c}$$

Throughout this paper we use the notation $D(.) := d(.)/dz$. We refer non-dimensional eigen parameter of the problem $RA := R^2$ is termed as Rayleigh number of the problem (4.1), ε is the scale parameter and $\varepsilon g(z)$ represents the gravity variation. Here, A refers to wave number.

We consider the general form of sixth order nonsingular eigenvalue problem with variable coefficients is of the form

$$\left(p(x)\,u'''(x)\right)''' - \left(q(x)u''(x)\right)'' + \left(r(x)u'(x)\right)' - \left(s(x) - \lambda\sigma(x)\right)u(x) = 0 \tag{4.2a}$$

subject to the homogeneous boundary conditions of two types

Type I: $u^{m}(\gamma)=0, \quad u^{m}(\mu)=0, \quad \text{for} \quad m = 0, 2, 4$ (4.2b)

Type II: $u^{m}(\gamma)=0, \quad u^{m}(\mu)=0, \quad \text{for} \quad m = 0, 1, 3$ (4.2c)

where, u is continuous function of x defined in the interval $[\gamma, \mu]$.

Equation (4.2a) has infinite sequence of eigenvalues $\left(\lambda_p\right)_{p\geq 1}$ which are bounded from below by a constant λ_0 i.e., $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\ldots\ldots\ldots\lambda_p \leq \ldots$ with $\underset{p\to\infty}{Lt}\,\lambda_p = \infty$ and each eigenvalue has multiplicity at most three [Gheorghiu and Dragomirescu (2009)].

The interval $[\gamma, \mu]$ can be transformed into the unit interval [0, 1] by any linear transformation.

4.3 Outline of Sixth Order Eigenvalue Problems

We have considered the following general sixth order nonsingular Sturm-Liouville problem

$$\frac{d^3}{dx^3}\left[p(x)\frac{d^3u}{dx^3}\right]-\frac{d^2}{dx^2}\left[q(x)\frac{d^2u}{dx^2}\right]+\frac{d}{dx}\left[r(x)\frac{du}{dx}\right]-\left(s(x)-\lambda\sigma(x)\right)u(x)=0 \qquad (4.3)$$

where, $p(x)$, $q(x)$, $r(x)$, $s(x)$ and $\sigma(x)$ are all piecewise continuous functions and $p(x)$, $\sigma(x)>0$ subject to some specified conditions and at these conditions mean that equation (4.3) is regular, i.e., nonsingular.

Using Leibnitz rule of differentiation, we can rewrite the equation (4.3) in the following form as a general sixth order SLE over the finite interval $[\gamma,\mu]$.

$$\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0y=\lambda wu \ , \qquad (4.4)$$

where $a_5(x)=\dfrac{3p'(x)}{p(x)}$, $a_4(x)=\dfrac{3p''(x)-q(x)}{p(x)}$, $a_3(x)=\dfrac{p'''(x)-2q'(x)}{p(x)}$

$a_2(x)=\dfrac{r(x)-q''(x)}{p(x)}$, $a_1(x)=\dfrac{r'(x)}{p(x)}$, $a_0(x)=-\dfrac{s(x)}{p(x)}$, $w(x)=-\dfrac{\sigma(x)}{p(x)}$

where, $a_0, a_1, a_2, a_3, a_4, a_5, w$ are all continuous functions of x defined on the interval $[\gamma,\mu]$.

4.4 Formulation of the Galerkin WRM

Let us consider the sixth order Sturm-Liouville (4.4) subject to the two types of boundary conditions

Type I: $u(\gamma)=0,\ u(\mu)=0,\ u''(\gamma)=0,\ u''(\mu)=0\ ,\ u^{iv}(\gamma)=0\ ,\ u^{iv}(\mu)=0$ (4.5a)

Type II: $u(\gamma)=0,\ u(\mu)=0,\ u'(\gamma)=0,\ u'(\mu)=0,\ u'''(\gamma)=0,\ u'''(\mu)=0$ (4.5b)

To approximate the solution of equation (4.4), we expressed in terms of polynomials as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}c_i B_{i,n}(x) \qquad (4.6)$$

where, $B_{i,n}(x)$ denotes the Bernstein or Legendre polynomials and it satisfies all the Dirichlet boundary conditions in $[\gamma,\ \mu]$ and $\theta_0(x)=0$, is specified by the Dirichlet boundary conditions, $B_{i,n}(\gamma)=B_{i,n}(\mu)=0$, for each $i=1,2,3,\cdots\cdots,n-1$. Thus the estimated solution takes the following form:

$$\tilde{u}(x)=\sum_{i=1}^{n-1}c_i B_{i,n}(x) \tag{4.6a}$$

Using (4.6a) into equation (4.4), the Galerkin weighted residual equations are:

$$\int_0^1\left[\frac{d^6\tilde{u}}{dx^6}+a_5(x)\frac{d^5\tilde{u}}{dx^5}+a_4(x)\frac{d^4\tilde{y}}{dx^4}+a_3(x)\frac{d^3\tilde{u}}{dx^3}+a_2(x)\frac{d^2\tilde{u}}{dx^2}+a_1(x)\frac{d\tilde{u}}{dx}+a_0(x)\tilde{u}-\lambda w(x)\tilde{u}\right]B_j\,dx=0 \tag{4.7}$$

$$j=1,2,3,\ldots\ldots\ldots,n$$

4.4.1 Formulation I

In this section, we formulate the matrix form with boundary conditions of type I. Integrating term by parts the terms up to second derivative on the left-hand side of (4.7), we have

$$\int_0^1\frac{d^6\tilde{u}}{dx^6}B_{j,n}(x)dx=\left[B_j(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\int_0^1\frac{d}{dx}\left[B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{4.8}$$

Similarly,

$$\int_0^1 a_5(x)\frac{d^5\tilde{u}}{dx^5}B_j(x)dx = \left[a_5(x)B_j(x)\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_5(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_5(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$-\left[\frac{d^3}{dx^3}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^4}{dx^4}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (4.9)$$

$$\int_0^1 a_4(x)\frac{d^4\tilde{u}}{dx^4}B_j(x)dx = \left[a_4(x)B_j(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^3}{dx^3}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (4.10)$$

$$\int_0^1 a_3(x)\frac{d^3\tilde{u}}{dx^3}B_j(x)dx = \left[a_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_3(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (4.11)$$

$$\int_0^1 a_2(x)\frac{d^2\tilde{u}}{dx^2}B_j(x)dx = \left[a_2(x)B_j(x)\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (4.12)$$

$$\int_0^1 a_1(x)\frac{d\tilde{u}}{dx}B_j(x)dx=\Big[a_1(x)B_j(x)\tilde{u}\Big]_0^1-\int_0^1\frac{d}{dx}\Big[a_1(x)B_j(x)\Big]\tilde{u}(x)dx \tag{4.13}$$

$$=-\int_0^1\frac{d}{dx}\Big[a_1(x)B_j(x)\Big]\tilde{u}(x)dx \tag{4.14}$$

Equations (4.8) to (4.14) are obtained using the boundary conditions in (4.5a) together with boundary conditions $B_{i,n}(0)=B_{i,n}(1)=0$

Also, from equation (4.6), we have

$$\tilde{u}(0)=\sum_{i=1}^{n-1}c_i B_i(0),\quad \tilde{u}(1)=\sum_{i=1}^{n-1}c_i B_i(1) \tag{4.15a}$$

$$\frac{d^2\tilde{u}}{dx^2}=\sum_{i=1}^{n-1}c_i\frac{d^2B_i}{dx^2}\qquad \frac{d^4\tilde{u}}{dx^4}=\sum_{i=1}^{n-1}c_i\frac{d^4B_i}{dx^4} \tag{4.15b}$$

Substituting equations (4.15*a*) and (4.15b) into equation (4.7), using approximation for $\tilde{u}(x)$ given in equation (4.6) and after rearranging the terms for the resulting equations we get a system of equations in the matrix form as

$$\sum_{i=1}^{n-1}\Big[F_{i,j}-\lambda E_{i,j}\Big]c_i=0,\quad j=1,2,3,.........,n-1 \tag{4.16}$$

$$F_{i,j}=\int_0^1\Bigg\{\Bigg[-\frac{d^5}{dx^5}\Big[B_j(x)\Big]B_i'(x)+\frac{d^4}{dx^4}\Big[a_5(x)B_j(x)\Big]B_i'(x)-\frac{d^3}{dx^3}\Big[a_4(x)B_j(x)\Big]B_i'(x)$$

$$-\frac{d}{dx}\Big[a_2(x)B_j(x)\Big]B_i'(x)+\frac{d}{dx}\Big[a_1(x)B_j(x)\Big]B_i'(x)\Bigg]$$

$$+a_0B_i(x)B_j(x)-\lambda w(x)B_iB_j\Bigg\}dx+\left[\frac{d^2}{dx^2}\Big[B_j(x)\Big]\frac{d^3B_i}{dx^3}\right]_{x=0}^1$$

$$+\left[\frac{d^4}{dx^4}\Big[B_j(x)\Big]\frac{dB_i}{dx}\right]_{x=0}^1-\left[\frac{d}{dx}\Big[a_5(x)B_j(x)\Big]\frac{d^3B_i}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\Big[a_4(x)B_j(x)\Big]\frac{dB_i}{dx}\right]_{x=0}^1$$

$$-\left[\frac{d}{dx}\left[a_3(x)B_j(x)\right]\frac{dB_i}{dx}\right]_0^1 \tag{4.16a}$$

$$E_{i,j}=\int_0^1 w(x)\,B_i(x)\,B_j(x)\,dx \tag{4.16b}$$

Equivalently, eigenvalues can be obtained by solving the system

$$F-\lambda E=0, \tag{4.17}$$

where the matrices F and E are defined by (4.16a) and (4.16b).

4.4.2 Formulation II

In this portion we obtain the matrix formulation by applying the boundary conditions of type II.

In the same way of portion (4.2.1), integrating by parts the term consisting sixth, fifth, fourth, third and second derivatives on the left-hand side of (4.7), and applying the boundary conditions prescribed in type II equation (4.5b), we get a system of equations in matrix form as

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0 \quad j=1,2,3,\dots\dots\dots,n-1 \tag{4.18}$$

where

$$F_{i,j}=\int_0^1\left\{\left[-\frac{d^5}{dx^5}\left[B_j(x)\right]B_i'(x)+\frac{d^4}{dx^4}\left[a_5(x)B_j(x)\right]B_i'(x)-\frac{d^3}{dx^3}\left[a_4(x)B_j(x)\right]B_i'(x)\right.\right.$$
$$\left.+\frac{d^2}{dx^2}\left[a_3(x)B_j(x)\right]B_i'(x)-\frac{d}{dx}\left[a_2(x)B_j(x)\right]B_i'(x)+\frac{d}{dx}\left[a_1(x)B_j(x)\right]B_i'(x)\right]$$
$$\left.+a_0B_i(x)B_j(x)-\lambda w(x)B_iB_j\right\}dx \tag{4.18a}$$

$$-\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^4B_i}{dx^4}\right]_{x=1}+\left[\frac{d}{dx}\left[B_j(x)\right]\frac{d^4B_i}{dx^4}\right]_{x=0}-\left[\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[a_5(x)B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[a_5(x)B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=1}+\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2B_i}{dx^2}\right]^1_{x=0}$$

$$E_{i,j}=\int_0^1 w(x)B_i(x)B_j(x)dx \quad (4.18b)$$

Equivalently, eigenvalues can be obtained by solving the system

$$F-\lambda E=0 \quad (4.19)$$

4.5 Stability and Convergence Criteria for Galerkin WRM

Stability and convergence issues for the polynomial approximations have been well studied by several authors [Chandrasekhar (1981), Straughan (2003), Gheorghiu and Dragomirescu (2009)]. Here we have a short review on the stability and convergence conditions. Let $\tilde{u}_n(x)$ and $u(x)$ the approximate and the exact solution and respectively. Let

$$\tilde{u}_n(x,c)=\phi_0(x)+\sum_{i=1}^{n-1}c_i\phi_i(x), \quad (4.20)$$

provided that the functions $\phi_i(x)$ are linearly independent. Now the Galerkin approximation converges to the solution of the eigenvalue problem as follows:

i) ***Completeness conditions***: It illustrates that the sequence of approximate eigenfunctions converge to the exact solution as the number of degrees of freedom increases indefinitely.

Mathematically,

$$\left|u(x)-\tilde{u}_n(x)\right|\to 0 \text{ as } n\to\infty,\ \gamma<x<\mu, \quad (4.21)$$

ii) ***Continuity conditions***

The Bernstein Approximation Theorem (Qian *et al*, 2011): Every continuous function

f defined on $[0,1]$ can be uniformly approximated as closely as desired by a polynomial function. For any $\varepsilon > 0$, there exists a positive integer M, such that for all $x \in [0,1]$, an integer $m \geq M$ we have,

$$\left| f(x) - B_m(f;x) \right| < \varepsilon \tag{4.22}$$

where $B_m(f;x)$ is a polynomial on x similar to equation (3b). Hence, given any power-form polynomial of degree n, it can be uniquely converted into a Bernstein polynomial of degree m for $m \geq n$.

Bernstein polynomials approach to $f(x)$ i.e., $B_m(f;x) \to f(x)$ as $m \to \infty$, for each point x of continuity of the function $f(x)$ defined on the interval $[0,1]$.

We noted that uniform convergence requires the maximum value of $\left| u(x) - \tilde{u}_n(x) \right|$ in the domain vanish as $n \to \infty$. The residual error is to be minimized with the eigensolution given as

$$E(x) = u - \tilde{u}. \tag{4.23}$$

Also, the convergence of the eigenvalues by Galerkin WRM is measured by the relative error

$$\varepsilon_k = \left| \frac{\lambda^{Exact} - \lambda^{(Approx.)}}{\lambda^{Exact}} \right| < \delta \tag{4.24}$$

where $\lambda^{(Approx.)}$ denotes the approximate solution using n-th polynomials and $\delta \leq 10^{-10}$ depends upon the problems.

The condition number is essential for estimating the errors such as round-off error arises from many sources in the numerical solution. The condition number measures the stability or sensitivity of a matrix in the field of numerical analysis. If the condition number of a matrix is large, it is called ill conditioned, whereas, for the small condition number, the matrix is well conditioned. The condition number of positive definite matrix A given by Shen and Tang (2006) is

$cond(A) = \kappa(n) = \dfrac{\max(abs(A))}{\min(abs(A))}$ is an increasing function of n which assesses the stability of the numerical method with respect to matrix inversion. The advantage of utilizing Bernstein polynomials is that with only a few basis functions sometimes attain superior accuracy for some eigenvalue problems, guarantees that the present method is stable.

4.6 Numerical examples

To test the applicability of the proposed method, we consider two linear SLEs with boundary conditions of the type (4.2b) and two SLEs with boundary conditions of the type (4.2c). For all the examples, the eigenvalues obtained by the proposed method and are compared with the numerically accepted eigenvalues computed previously by various numerical methods available in the literature. Rayleigh numbers are estimated for the particular case in which the gravity variation is $\varepsilon h(x)=-\varepsilon\,(2x-1)$ are given.

All the numerical calculations are carried out using MATLAB 13 by Intel(R) Core(TM) i5-4570 CPU with power 3.20 GHz CPU, equipped with 8 GB of Ram.

Example 4.1: Consider the Sturm-Liouville problem worked out by Siyyam and Syam (2011), Allame *et al* (2015):

$$\frac{d^6 \tilde{u}}{dx^6} = \lambda u(x) \tag{4.25a}$$

subject to the boundary conditions

$$\left.\begin{aligned} u(0) = u''(0) = u^{iv}(0) = 0 \\ u(1) = u''(1) = u^{iv}(1) = 0 \end{aligned}\right\} \tag{4.25b}$$

The exact eigenvalues of (4.25a) are $\lambda_m = -(m\pi)^6, \quad m = 1,2,3,........$ (4.26)

Here $\tilde{u}(x)$ in equation (4.20) satisfies the Dirichlet boundary conditions i.e., $B_j(\gamma) = 0$ and $B_j(\mu) = 0$. Also $\theta_0(x) = 0$ is specified by the Dirichlet boundary conditions. We have the matrix form of equation (4.25a) as

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0\ ;\quad j=1,2,3,\ldots\ldots\ldots,n-1 \tag{4.27}$$

where,

$$F_{i,j}=\frac{d^2B_j(1)}{dx^2}\frac{d^3B_i(1)}{dx^3}-\frac{d^2B_j(0)}{dx^2}\frac{d^3B_i(0)}{dx^3}+\frac{d^4B_j(1)}{dx^4}\frac{dB_i(1)}{dx}$$
$$-\frac{d^4B_j(0)}{dx^4}\frac{dB_i(0)}{dx}-\int_0^1\frac{d^5}{dx^5}\left[B_j(x)\right]\frac{dB_i}{dx}dx \tag{4.27a}$$

$$E_{i,j}=\int_0^1 B_iB_j\,dx \tag{4.27b}$$

We define

$$A=FE^{-1} \tag{4.28}$$

The proposed method stated in section 4.3 is tested on this problem. Here the domain [0, 1] is subdivided into 15, 20 and 25 equally subintervals, respectively and the numerical results are illustrated in table 4.1. The maximum absolute error achieved by the present method is about 2.365×10^{-16}, whereas error attained by VIM (2011) is of order 10^{-13} and which shows the better performance of the current technique. It is also noticed that relative errors for the last four eigenvalues for n=25 are less than those obtained for n=15 and n=20. Besides these, the relative errors are much smaller and consistent for all six eigenvalues for n=25. This indicates the fact that the increasing of the degree of polynomials leads more accurate and efficient results. Besides, the ratio of the errors attained by the proposed method is compared with VIM (2011), which is displayed in table 4.1.

Furthermore, the accuracy of the current method is lost (errors starts increasing rapidly) for smaller eigenvalues for the degree of polynomials $n>35$ and some of the eigenvalues are complex. The behaviour of the first eigenvalue and the behaviour of the condition numbers of the eigenvalues with increasing degree of polynomials (=n or N) of the matrix A have been displayed in Figure 4.1 and Figure 4.2 respectively. From Figure 4.1, it is observed that the converges to the same rate with increasing n.

Example 4.2: In this example, a circular ring structure with constraints [Gutierrez and Laura (1995), Wu and Liu (2000), Wang *et al* (2003)], which has rectangular cross-sections of constant width and parabolic variable thickness is studied. Considering half of the ring structure, this eigenvalue problem is formulated by the following sixth-order differential equation:

$$\beta_1 w^{(6)} + \beta_2 w^{(5)} + \beta_3 w^{(4)} + \beta_4 w^{(3)} + \beta_5 w^{(2)} + \beta_6 w^{(1)}$$

$$= \Omega^2 (f \frac{d^2}{dx^2} + f^{(1)} \frac{d}{dx} - \pi^2 f) w \tag{4.29}$$

where,

$w^{(r)} = \frac{d^r w}{dx^r}$, Ω is the dimension frequency, w is the tangential displacement, and

$$\beta_1 = \frac{\phi}{\pi^4},\ \beta_2 = \frac{3\phi^{(1,)}}{\pi^4}$$

$$\beta_3 = \left(2\phi/\pi^2\right) + \left(3\phi^{(2)}/\pi^4\right),$$

$$\beta_4 = \left(4\phi^{(1)}/\pi^2\right) + \left(\phi^{(3)}/\pi^4\right),$$

$$\beta_5 = \phi + 3\phi^{(2)}/\pi^2,$$

$$\beta_6 = \phi^{(1)} + 3\phi^{(3)}/\pi^2,$$

$$\phi = [f(x)]^3\ ;\ f = f(x) = -4(r-1)x^2 + 4(r-1)x + 1;\ x \in [0,1],$$

$\phi^{(i)} = \frac{d^i w}{dx^i}$, $f^{(i)} = \frac{d^i \phi}{dx^i}$, and r is the variable related to the thickness of the cross-section of the ring.

Equation (4.29a) can be written as

$$\left(\beta_1(x)\frac{d^6}{dx^6} + \beta_2(x)\frac{d^5}{dx^5} + \beta_3(x)\frac{d^4}{dx^4} + \beta_4(x)\frac{d^3}{dx^3} + \beta_5(x)\frac{d^2}{dx^2} + \beta_6 \frac{d}{dx}\right)w$$

$$= \lambda^2 \left(f \frac{d^2}{dx^2} + f^{(1)} \frac{d}{dx} - \pi^2 \right) w \tag{4.30a}$$

Boundary conditions are

$$w(0) = w^{(1)}(0) = w^{(3)}(0)\ ;\ \ w(1) = w^{(1)}(1) = w^{(3)}(1) = 0 \tag{4.30b}$$

$$\sum_{i=1}^{n-1} \left[F_{i,j} - \lambda E_{i,j} \right] c_i = 0 \qquad j = 1,2,3,\ldots\ldots\ldots, n-1 \tag{4.31}$$

where,

$$F_{i,j} = \int_0^1 \left\{ \left[-\frac{4}{\pi^2} \frac{d^5}{dx^5} \left[B_j(x) \right] B_i'(x) + \frac{d^4}{dx^4} \left[\beta_2(x) B_j(x) \right] B_i'(x) - \frac{d^3}{dx^3} \left[\beta_3(x) B_j(x) \right] B_i'(x) \right.\right.$$

$$\left.\left. + \frac{d^2}{dx^2} \left[\beta_4(x) B_j(x) \right] B_i'(x) - \frac{d}{dx} \left[\beta_5(x) B_j(x) \right] B_i'(x) + \frac{d}{dx} \left[\beta_6(x) B_j(x) \right] \right] \right\} dx$$

$$+ \frac{4}{\pi^2} \left[\frac{d^2 B_j(1)}{dx^2} \frac{d^3 B_i(1)}{dx^3} - \frac{d^2 B_j(0)}{dx^2} \frac{d^3 B_i(0)}{dx^3} + \frac{d^4 B_j(1)}{dx^4} \frac{d B_i(1)}{dx} - \frac{d^4 B_j(0)}{dx^4} \frac{dB_i(0)}{dx} \right]$$

$$- \left[\frac{d}{dx} \left[\beta_2(x) B_j(x) \right] \frac{d^3 B_i}{dx^3} \right]_{x=1} + \left[\frac{d}{dx} \left[\beta_2(x) B_j(x) \right] \frac{d^3 B_i}{dx^3} \right]_{x=0}$$

$$- \left[\frac{d^3}{dx^3} \left[\beta_3(x) B_j(x) \right] \frac{dB_i}{dx} \right]_{x=1} + \left[\frac{d^3}{dx^3} \left[\beta_3(x) B_j(x) \right] \frac{dB_i}{dx} \right]_{x=0}$$

$$- \left[\frac{d}{dx} \left[\beta_4(x) B_j(x) \right] \frac{dB_i}{dx} \right]_{x=1} + \left[\frac{d}{dx} \left[\beta_4(x) B_j(x) \right] \frac{dB_i}{dx} \right]_{x=0} \tag{4.31a}$$

$$E_{i,j} = \int_0^1 \left[-\frac{d}{dx} \left[f(x) B_j(x) \right] \frac{dB_i}{dx} - \left(f^{(1)} B_j \right) \frac{dB_i}{dx} - \pi^2 B_i B_j \right] dx \tag{4.31b}$$

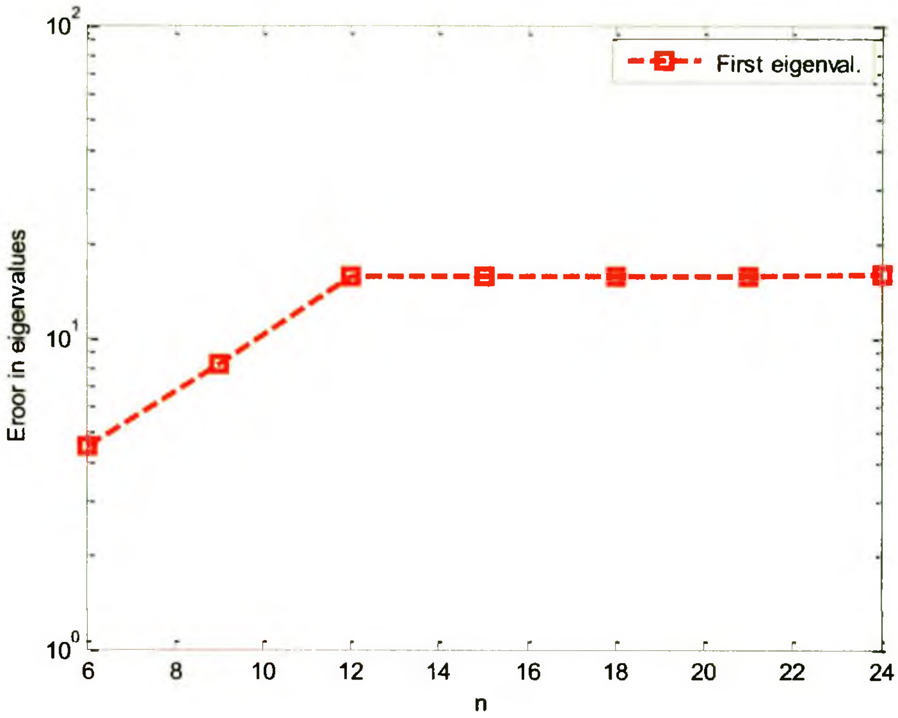

Figure 4.1: The convergence rate of the smallest eigenvalues.

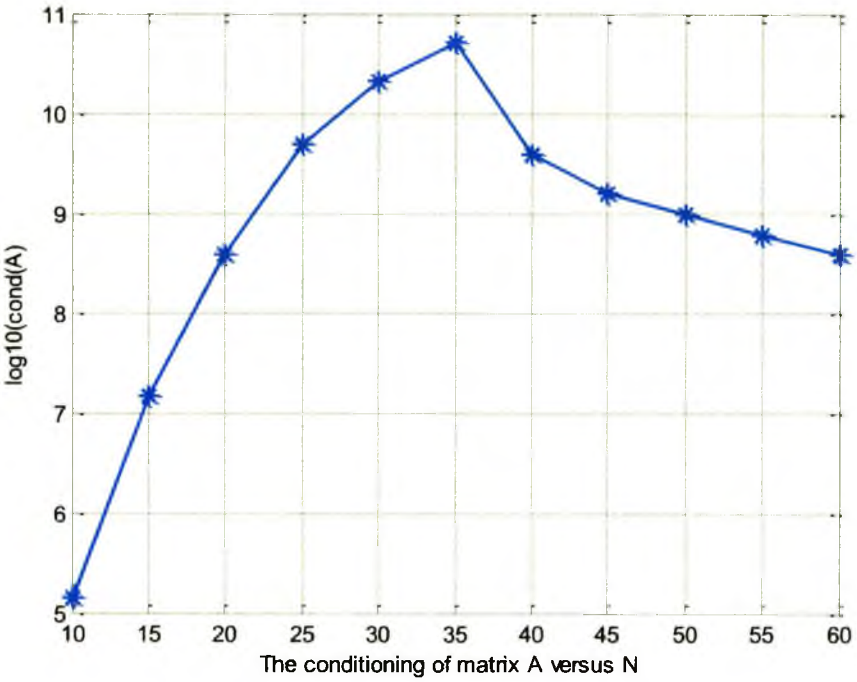

Figure 4.2: Logarithm of the condition number of the eigenvalues with increasing *N*.

Table 4.1: Observed relative errors of eigenvalues using Bernstein polynomials for example 4.1.

k	Exact eigenvalues $\eta_k = -\frac{\sqrt[6]{\lambda_k}}{\pi}$	Computed eigenvalues $n=15$	Computed eigenvalues $n=20$	Computed eigenvalues $n=25$	Relative error present Bernstein $n=15$	Relative error present Bernstein $n=20$	Relative error present Bernstein $n=25$	Ratio error present N=20	Ratio VIM Syam and Siyaam (2011)
1	961.3891935753043	961.3891935753044	961.3891935753044	961.3891935753051	2.365×10^{-16}	2.365×10^{-16}	9.460×10^{-16}	1.00000	1.00000
2	61528.90838881947	61528.90838881949	61528.90838881948	61528.90838881948	2.365×10^{-16}	2.365×10^{-16}	1.825×10^{-16}	2.00000	1.99999
3	700852.7221163968	700852.7222783775	700852.7221163969	700852.7221116397	2.311×10^{-10}	1.661×10^{-16}	3.322×10^{-16}	3.0000	2.99999
4	3937850.136884446	3937850.183683134	3937850.136884551	3937850.136884447	3.116×10^{-5}	2.057×10^{-12}	3.548×10^{-16}	4.0000	4.00000
5	15021706.14961413	15022174.20219283	15021706.14964502	15021706.14961413	1.188×10^{-8}	3.388×10^{-14}	1.240×10^{-16}	5.0000	4.99999
6	44854574.21544939	44862417.66944139	44854574.89196720	44854574.21544942	1.749×10^{-4}	1.5082×10^{-8}	4.983×10^{-16}	6.0000	5.99999

Table 4.2: Observed relative errors and eigenvalues for example 4.1.

k	Exact eigenvalues $\eta_k = -\frac{\sqrt[6]{\mu_k}}{\pi}$	Computed eigenvalues	Computed eigenvalues	Relative error present (Legn) n=15	Relative errork present (Legn) n=20	Relative error present (Legn) n=25	Ratio error (present) n=20	Ratio VIM Syam and Siyaam (2011)
1	961.3891935753043	961.3891935753044	961.3891935753044	2.3651×10^{-16}	2.365×10^{-16}	2.365×10^{-16}	1.000000000000000	1.000000000000000
2	61528.90838881947	61528.90838881949	61528.90838881948	3.5476×10^{-16}	2.3651×10^{-16}	2.365×10^{-16}	2.000000000000000	1.999999999999
3	700852.7221163968	700852.7222783775	700852.7221163969	1.7258×10^{-13}	1.661×10^{-16}	1.661×10^{-16}	3.000000000000000	2.9999999999999
4	3937850.136884446	15022174.20219283	15021706.149645018	2.199×10^{-7}	2.056×10^{-12}	1.240×10^{-16}	4.000000000010800	4.000000000000
5	15021706.14961413	3937850.183683134	3937850.136884551	1.188×10^{-8}	3.5478×10^{-16}	3.548×10^{-16}	5.000000000102800	4.9999999999987
6	44854574.21544939	44862417.66944139	44854574.89196729	1.749×10^{-4}	6.406×10^{-11}	4.983×10^{-16}	6.000000904948380	5.9999999999214

Eigenvalues obtained by various methods and present Galerkin method utilizing Bernstein and Legendre polynomials are listed in tables 4.3 and 4.4 respectively for different values of r. Since this example does not have exact solution, we verified accuracy of our results with the method of GDQR [Wu and Liu (2000)]. Moreover, we observed that the present method converges to the same frequency parameter i.e., r =1.0, 1.1, 1.2 and 1.3 as compared with GDQR at n=10, 11.

Table 4.3: Comparison of fundamental frequencies for example 4.2 using different methods.

r	DQM	Rayleigh Ritz	*GDQR* [Wu and Liu (2000)]					[Gutierrez and Laura (1995)]				
			7[a]	8[a]	9[a]	10[a]	11[a]	N=7[a]	N=8[a]	N=9[a]	N=10[a]	N=11[a]
1.0	2.2686(12[a])	2.274	2.2631	2.2669	2.2667	2.2667	2.2667	2.2624	2.2647	2.2669	2.2668	2.2667
1.1	2.417(12[a])	2.416	2.4133	2.4137	2.4137	2.4137	2.4137	2.4185	2.4136	2.4137	2.4137	2.4137
1.2	2.561(12[a])	2.557	2.5597	2.5565	2.5567	2.5568	2.5568	2.5583	2.5576	2.5570	2.5569	2.5568
1.3	2.701(12[a])	2.697	2.7139	2.6944	2.6962	2.6966	2.6966	2.7019	2.6995	2.6976	2.6972	2.6968
1.4	2.839(12[a])	2.834	2.8946	2.8242	2.8318	2.8336	2.8335	2.8452	2.8400	2.8364	2.8353	2.8341
1.5	2.976(12[a])	2.970	3.1297	2.9407	2.9623	2.9681	2.9678	2.9878	2.9791	2.9738	2.9715	2.9694

[a] *Number of polynomials used*, N [Wu and Liu (2000), Wang *et al* (2003)]

From table 4.4, it is noticed that the present method converges to five significant figures for the values of r=1.0, 1.1, 1.2 and 1.3. Accuracy of the present method slightly deviates for higher values of r =1.4 and r =1.5, still accuracy is much closer to GDQR [Wu and Liu (2000)] other than DQM [Gutierrez and Laura (1995)], Ritz and La-DQM [Gutierrez and Laura (1995)].

Example 4.3: We considered one dimensional sixth order Benard layer eigenvalue problem which has analytical eigenvalues are calculated by Baldwin (1987).

$$\left(\frac{d^2}{dx^2}-a^2\right)^3 u(x)+Ra^2(1-x^2)u(x)=0,\quad 0<x<10 \tag{4.32}$$

Table 4.4: Listed fundamental frequencies for example 4.2.

	Degree of Polynomial Present (Bernstein)					Degree of Polynomial Present (Legendre)				
r	n= 30	n=40	n=9	n=10	n=11	n=7	n=8	n=9	n=10	n=11
1.0	2.2677	2.2670	2.2669	2.2667	2.2667	2.2669	2.2669	2.2667	2.2667	2.2667
1.1	2.4135	2.4139	2.4137	2.4137	2.4137	2.4137	2.4137	2.4137	2.4137	2.4137
1.2	2.5573	2.5571	2.5568	2.5568	2.5568	2.5568	2.5568	2.5568	2.5568	2.5568
1.3	2.6964	2.6968	2.6966	2.6966	2.6966	2.6966	2.6966	2.6966	2.6966	2.6966
1.4	2.8330	2.8337	2.8327	2.8334	2.8334	2.8327	2.8327	2.8334	2.8334	2.8334
1.5	2.9676	2.9679	2.9644	2.9675	2.9675	2.9644	2.9644	2.9675	2.9675	2.9677

Here we have introduced two sets of boundary conditions for even and odd modes for the same Bénard layer eigenvalue problem in equation (4.32).

Set 1:

$$\left.\begin{array}{l} u(0) = u''(0) = u^{4}(0) = 0 \\ u(10) = u''(10) = u^{4}(10) = 0 \end{array}\right\} \qquad (4.32a)$$

Set 2:

$$\left.\begin{array}{l} u(0) = u^{3}(0) = u^{5}(0) = 0 \\ u(10) = u^{3}(10) = u^{5}(10) = 0 \end{array}\right\} \qquad (4.32b)$$

We have the matrix form of equation (4.32) exploiting the weighted residual technique illustrated in section 4.4 and utilizing the boundary conditions of set 1.

$$\sum_{i=1}^{n-1} \left[F_{i,j} - \lambda E_{i,j} \right] c_i = 0 \qquad j = 1,2,3,\ldots\ldots\ldots, n-1 \qquad (4.33)$$

where

$$F_{i,j} = \int_0^{10} \left\{ \left[-\frac{d^5}{dx^5}\left[B_j(x) \right] B_i'(x) + 3a^2 \frac{d^3}{dx^3}\left[B_j(x) \right] B_i'(x) - 3a^4 \frac{d}{dx}\left[B_j(x) \right] B_i'(x) \right.\right.$$

$$\left.\left. - a^6 \frac{d}{dx}\left[B_j(x) \right] B_i'(x) \right] \right\} dx + \left[\frac{d^2 B_j(10)}{dx^2} \frac{d^3 B_i(10)}{dx^3} - \frac{d^2 B_j(0)}{dx^2} \frac{d^3 B_i(0)}{dx^3} \right] \qquad (4.33a)$$

$$+\left[\frac{d^4 B_j(10)}{dx^4}\frac{dB_i(10)}{dx}-\frac{d^4 B_j(0)}{dx^4}\frac{dB_i(0)}{dx}\right]-3a^2\left[\frac{d^2 B_j(10)}{dx^2}\frac{d\tilde{B}_i(10)}{dx}-\frac{dB_j(0)}{dx}\frac{dB_i(0)}{dx}\right]$$

$$E_{i,j}=\int_0^{10}\left(1-x^2\right)B_iB_j\,dx \qquad (4.33b)$$

Again we have the matrix form of equation (4.32) as for boundary conditions of set 2.

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0,\quad j=1,2,3,.........,n-1 \qquad (4.34)$$

where,

$$F_{i,j}=\int_0^{10}\left\{\left[-\frac{d^5}{dx^5}\left[B_j(x)\right]B_i'(x)+3a^2\frac{d^3}{dx^3}\left[B_j(x)\right]B_i'(x)-3a^4\frac{d}{dx}\left[B_j(x)\right]B_i'(x)\right.\right.$$

$$\left.\left.-a^6\frac{d}{dx}\left[B_j(x)\right]B_i'(x)\right]\right\}dx-\left[\frac{dB_j(10)}{dx}\frac{d^4B_i(10)}{dx^4}-\frac{dB_j(0)}{dx}\frac{d^4B_i(0)}{dx^4}\right]$$

$$-\left[\frac{d^3B_j(10)}{dx^3}\frac{d^2B_i(10)}{dx^2}-\frac{d^3B_j(0)}{dx^3}\frac{d^2B_i(0)}{dx^2}\right]+3a^2\left[\frac{dB_j(10)}{dx}\frac{d\tilde{B}_i^{2}(10)}{dx^2}-\frac{dB_j(0)}{dx}\frac{dB_i^{2}(0)}{dx}\right]$$

$$E_{i,j}=\int_0^{10}\left(1-x^2\right)B_iB_j\,dx$$

We have listed the first six even and odd mode critical values of R, for the corresponding given values of A using $n=25$ in table 4.5. Observe that the critical values of R using the present method with Bernstein and Legendre polynomials show the better performance and eigenvalues attained are smaller and fairly close to the results of Baldwin (1987) and specially for the first three critical values.

Also the percentage of relative errors for R in our method for the smallest mode is about 0.000032% which is much smaller than the computed results by Twizell and Boutayeb (1990). Illustrated results also revealed that Galerkin weighted residual method works well and evaluates smaller eigenvalues with good accuracy other than the larger ones. Considerable accuracy is obtained if the degree of polynomials is much higher i.e., $n \geq 24$ for the Benard layer eigenvalue problem. Although slow convergent rate of Bernstein polynomials for some particular problems with complicated boundary conditions makes it unattractive still this drawback is to be compensated for achieving better accuracy.

Table 4.5: First six critical values for example 4.3 for even modes (n=2,4,6) and odd modes (n=1,3,5) listed with relative error and percentage of relative error for example 4.3.

n	Baldwin (1987) R	Baldwin A	Twizell Boutayeb (1990) R	Twizell Boutayeb (1990) A	Percent. rel. error	Present method R	Present method A	Rel. error R	Percentage rel. error present
1	9.78136567	0.72605	9.77836945	0.72603	0.0306	9.78136254	0.72605	3.19×10^{-7}	0.00003
2	411.720155	1.6791	411.515421	1.6790	0.0497	411.702291	1.6793	4.33×10^{-5}	0.004
3	3006.709534	2.7379	3003.053226	2.7374	0.121	3006.25406	2.7379	1.56×10^{-4}	0.0152
4	11382.695328	3.8130	11356.55701	3.8112	0.2296	11360.3771	3.8109	2.0×10^{-3}	0.200
5	30916.2534	4.8916	30800.6998	4.8882	0.373	30765.0081	4.8911	4.89×10^{-3}	0.489
6	68778.1170	5.971	68397.491	5.965	0.5334	68410.1362	5.9651	4.20×10^{-3}	0.420

Figure 4.3 displays the convergence of eigenvalues achieved from analytical [Baldwin (1987)], Finite Difference method [Twizell (1990)] and the current Galerkin weighted residual method.

Example 4.4: We considered the two-point boundary value problem studied by Gheorghiu and Dragomirescu (2009) governing the linear stability provided with the Dirichlet and hinged boundary conditions given as

$$\left(D^2 - a^2\right)^3 u = -Ra\left(1 + \varepsilon h(x)\right) a^2 u \tag{4.35}$$

$$\left.\begin{array}{l} u(0) = Du(0) = D^2 u(0) = 0 \\ u(1) = Du(1) = D^2 u(1) = 0 \end{array}\right\} \tag{4.35a}$$

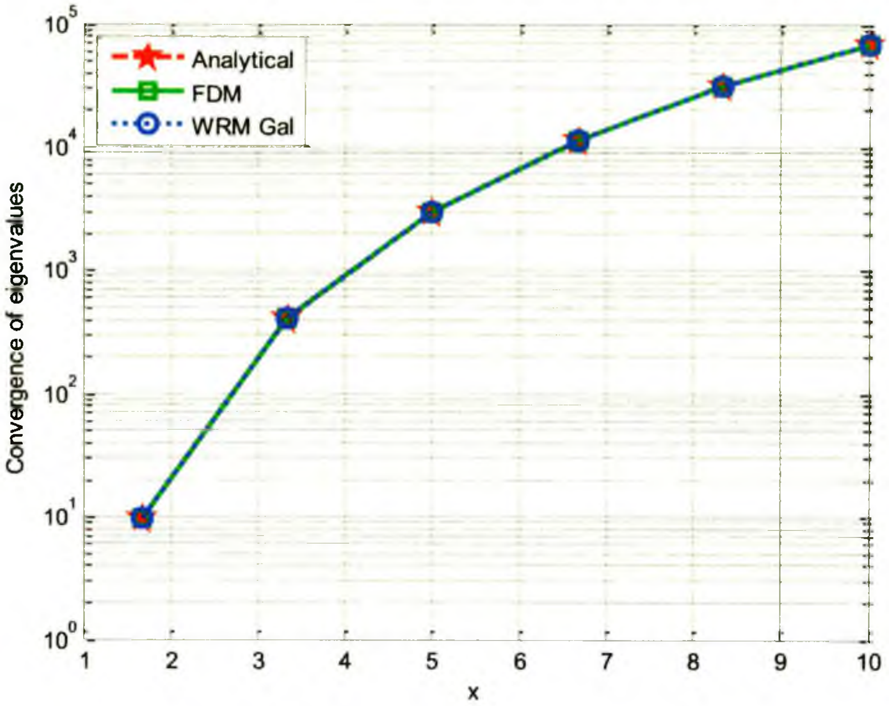

Figure 4.3: Convergence of eigenvalues using analytical method [Baldwin (1987), Finite difference method [Twizell and Boutayeb (1990)] and present method.

Numerical evaluations of the Rayleigh number for the particular case in which the gravity variation is $\varepsilon h(x)=(2x-1)$ are given.

We have the matrix form of equation (4.35) as for boundary conditions of set 1.

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0 \qquad j=1,2,3,\dots\dots\dots,n-1 \tag{4.36}$$

where,

$$F_{i,j}=\int_0^1\left\{\left[-\frac{d^5}{dx^5}\left[B_j(x)\right]B_i'(x)+3a^2\frac{d^3}{dx^3}\left[B_j(x)\right]B_i'(x)-3a^4\frac{d}{dx}\left[B_j(x)\right]B_i'(x)\right.\right.$$
$$\left.\left.-a^6\frac{d}{dx}\left[B_j(x)\right]B_i'(x)\right]\right\}dx+\left[\frac{d^2B_j(1)}{dx^2}\frac{d^3B_i(1)}{dx^3}-\frac{d^2B_j(0)}{dx^2}\frac{d^3B_i(0)}{dx^3}\right]$$

$$+\left[\frac{d^4B_j(1)}{dx^4}\frac{dB_i(1)}{dx}-\frac{d^4B_j(0)}{dx^4}\frac{dB_i(0)}{dx}\right]-3a^2\left[\frac{d^2B_j(1)}{dx^2}\frac{dB_i(1)}{dx}-\frac{dB_j(0)}{dx}\frac{dB_i(0)}{dx}\right] \quad (4.36a)$$

$$E_{i,j}=\int_0^1(1-\varepsilon x)B_iB_j\,dx \quad (4.36b)$$

Table 4.6: Computational time for example 4.4 for current method versus Spectral method [Gheorghiu and Dragomirescu (2009)].

n	*Ra* Cbeby. Spectral (2009)	CPU time sec. Cbeby. Spectral	Degree of poly. present	*Ra:* Present WRG (Bernstein)	CPU time (sec.) WRG (Bernstein)	*Ra:* Present WRG (Legn.)	CPU time sec. WRG (Legn.)
4	658.692	16.66	6	658.1283	1.2668	658.1283	1.175
5	657.6668	41.80	7	658.1283	1.3030	657.5084	1.732
7	657.5219	386.12	9	657.5134	2.0691	657.5133	2.292
9	657.5129	2679.83	11	657.5134	2.0823	657.5133	2.277
11	657.5127	17834.19	13	657.5133	2.6905	657.5133	2.941

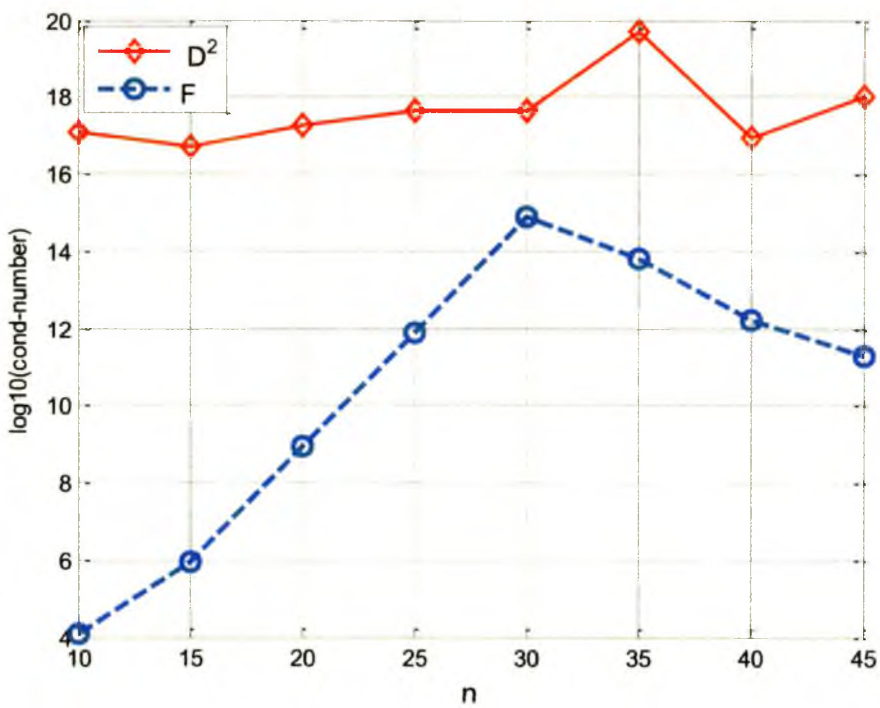

Figure 4.4(a): Logarithm of the condition number of the Chebyshev differentiation matrix $\left(D^2\right)$ and Galerkin WRM matrix F (present).

The figure 4.4(a) indicates that the present scheme offers relatively smaller condition numbers compared to different spectral methods [Gheorghiu and Dragomirescu (2009)].

Table 4.7: Numerical estimates for the Rayleigh number for various values of the parameters ε and a^2 attained by different Spectral methods Gheorghiu and Dragomirescu (2009) and weighted residual method for example 4.4.

ε	a^2	Present *Ra*: WRG (Bernst.)	Present *Ra*: WRG (Legn.)	*Ra*: SCP	*Ra*: SLP	*Ra*: TS	*Ra*: WRG (SB1)	*Ra*: WRG (SB2)	*Ra*: *CC*	*Ra*: WRP1	*Ra*: WRP2
0.00	4.92	657.513	657.5133	657.512	675.05	657.51	658.54	714.55	657.5133	658.59	662.80
0.01	4.92	660.817	660.817	660.747	678.45	660.81	661.88	718.62	660.8173	662.03	666.05
0.03	4.92	667.525	667.525	667.653	685.33	667.52	668.53	725.49	667.5254	668.84	673.19
0.33	4.92	787.288	787.288	787.363	808.30	787.28	792.54	859.84	787.2880	792.70	797.38
0.20	5.0	730.565	730.565	730.459	749.95	730.56	732.95	794.85	730.5647	733.06	737.45
0.20	9.0	829.392	829.392	829.440	846.70	829.39	832.09	886.91	829.3918	831.80	836.02
0.50	7.5	930.924	930.924	930.982	952.07	930.92	946.05	1013.30	930.9239	946.05	950.55
0.50	9.0	994.472	994.4721	994.393	1015.27	994.47	1010.50	1077.01	994.4721	1010.60	1015.20
0.75	10	1251.092	1251.092	1251.178	1276.05	1251.09	1313.80	1392.61	1251.0924	1314	1319.70

Figure 4.4(b), illustrates the two curves $\log_{10}\left(cond(A)\right)$(versus ε, for $0 \leq \varepsilon < 1$ and for convenience we increase the degree of polynomials from *n*= 10 to *n* = 20. It is noticed that they are quite parallel and horizontal for $0.0 \leq \varepsilon \leq 0.33$. For $0.33 < \varepsilon \leq 0.5$ they moderately decrease than the first and between the range $0.5 < \varepsilon \leq 7.5$, the curves decrease more than the first two. This indicates that the increasing of *n* for different values of ε, does not degenerate the condition number of the eigenvalues and thus reveal the stability of the present method. It is also observed in figure 4.4(b) that condition number in the present method varies between the range 10^5 to $10^{8.5}$ whereas this number increases to more than 10^{18} in the article studied by Gheorghiu and Dragomirescu (2009).

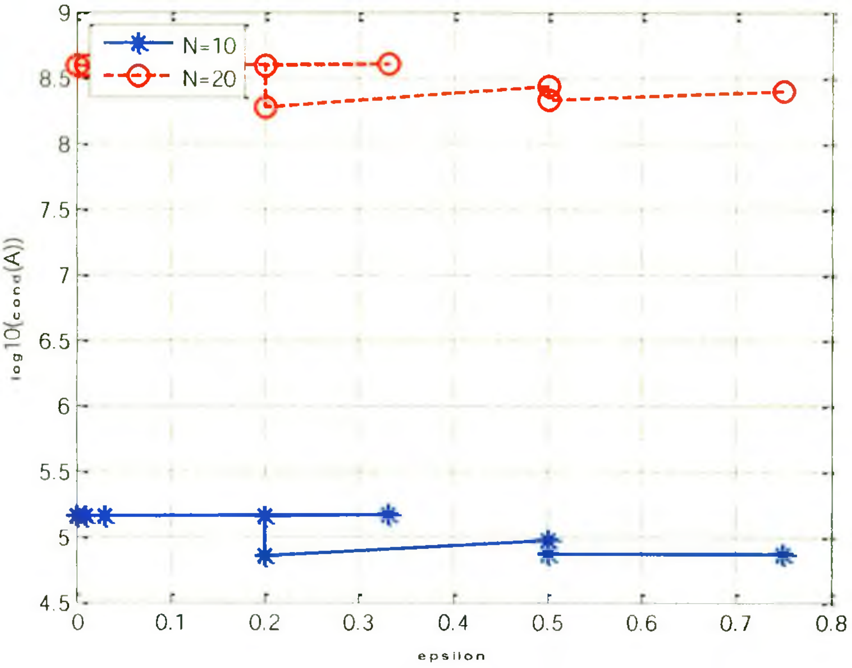

Figure 4.4(b): The conditioning of matrix *A* with respect to *n* and epsilon.

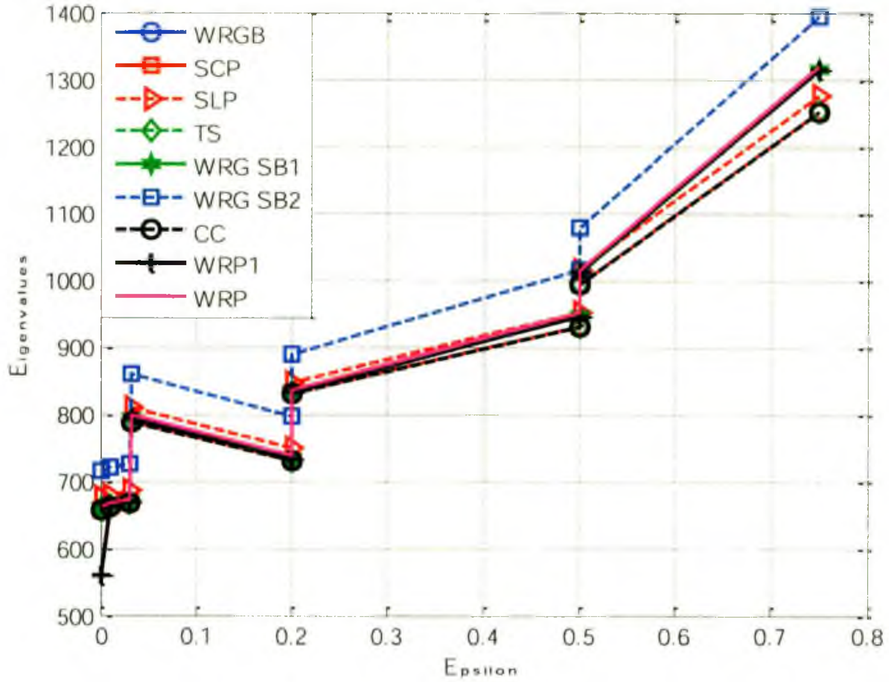

Figure 4.5: Comparison of eigenvalues for various numerical techniques (SCP, SLP, TS, CC and SB1, SB2, GP1, WGP2, WRGB).

The numerical appraisals are obtained for different significant values of the scale parameter ε and the wave number a are compared to other spectral methods based on shifted Chebyshev polynomials (SCP), shifted Legendre polynomials (SLP) and trigonometric series (TS), weighted residual Petrov-Galerkin method (WRPG1 and WRPG2) with the Heinrich's basis are exhibited in table 4.7. The estimated eigenvalues are obtained by Galerkin WRM using both the Bernstein and Legendre polynomials exhibited in table 4.6 for degree $n=10$ and are compared with other spectral methods illustrated in the article (Gheorghiu and Dragomirescu, 2009).

It is observed in table 4.7 that Rayleigh numbers (*Ra*), obtained by applying our proposed Galerkin WRM are very close to the results computed by spectral methods specially SCP and CC [Gheorghiu and Dragomirescu (2009)]. We observed that our present scheme is much accurate for $n=10$. On the other hand, our scheme produces stable results and performs better when degree of polynomials increased. Performance of eigenvalues for various numerical techniques (SCP, SLP, TS, CC and SB1, SB2, WGP1, WGP2, WRGB) is depicted in Figure 4.5.

Furthermore, CPU time for the current work is compared with the results of Gheorghiu and Dragomirescu (2009) are also displayed in table 4.6. It has been observed that CPU time applying WRM using Shen basis increases more than exponentially and much more expensive whereas, CPU time for the present method is much smaller with increasing *n*. Since the co-efficient matrix in Galerkin WRM is sparse and have symmetric banded matrix, which minimize the computational effort. It is worth noting here that the current work attains less computational cost than the work in Gheorghiu and Dragomirescu (2009) given in table 4.6.

The shortcoming of the Galerkin WRM is that, for the huge number of eigenvalues computation higher eigenvalues are less convergent than the lower ones and for increasing the degree of polynomials the computational time highly increases, without leading to a significant improvement of the numerical values.

4.7 The Bernstein Collocaltion Method

4.7.1 Problem Formulation

We consider the following sixth order Strum-Liouville boundary value problem

$$\frac{d^3}{dx^3}\left[p(x)\frac{d^3u}{dx^3}\right]-\frac{d^2}{dx^2}\left[q(x)\frac{d^2u}{dx^2}\right]+\frac{d}{dx}\left[r(x)\frac{du}{dx}\right]-(s(x)-\lambda\sigma(x))u(x)=0 \qquad (4.37a)$$

$$\left.\begin{aligned} u(x_1)=u''(x_1)=u^{iv}(x_1)=0\\ u(x_n)=u''(x_n)=u^{iv}(x_n)=0\end{aligned}\right\} \qquad x_1<x<x_n \qquad (4.37b)$$

Equation (4.37b) can be written as

$$\begin{bmatrix} B''_4(x_1) & B''_5(x_1) & B''_6(x_1)\ldots\ldots\ldots\ldots & B''_{n-3}(x_1)\\ B''_4(x_n) & B''_5(x_n) & B''_6(x_n)\ldots\ldots\ldots & B''_{n-3}(x_n)\end{bmatrix}\begin{bmatrix} c_4\\ c_5\\ c_6\\ \vdots\\ c_{n-3}\end{bmatrix}$$

$$+\begin{bmatrix} B''_2(x_1) & B''_{n-1}(x_1)\\ B''_2(x_n) & B''_{n-1}(x_n)\end{bmatrix}\begin{bmatrix} c_2\\ c_{n-1}\end{bmatrix}+\begin{bmatrix} B''_3(x_1) & B''_{n-2}(x_1)\\ B''_3(x_n) & B''_{n-2}(x_n)\end{bmatrix}\begin{bmatrix} c_3\\ c_{n-2}\end{bmatrix}=0 \qquad (4.38)$$

Equation (4. 38) can be written as

$$M_1C^{\eta}+M_2C^{\omega}+M_3C^{\tau}=0 \qquad (4.39)$$

where,

$$M_1=\begin{bmatrix} B''_4(x_1) & B''_5(x_1) & B''_6(x_1)\ldots\ldots\ldots\ldots & B''_{n-3}(x_1)\\ B''_4(x_n) & B''_5(x_n) & B''_6(x_n)\ldots\ldots\ldots & B''_{n-3}(x_n)\end{bmatrix}, \qquad (4.39a)$$

$$M_2=\begin{bmatrix} B''_2(x_1) & B''_{n-1}(x_1)\\ B''_2(x_n) & B''_{n-1}(x_n)\end{bmatrix}, \quad M_3=\begin{bmatrix} B''_3(x_1) & B''_{n-2}(x_1)\\ B''_3(x_n) & B''_{n-2}(x_n)\end{bmatrix} \qquad (4.39b)$$

Similarly, for the boundary conditions

$$u^{iv}(x_1)=u^{iv}(x_n)=0$$

$$\begin{bmatrix} B_4^{iv}(x_1) & B_5^{iv}(x_1) & B_6^{iv}(x_1) \cdots\cdots & B_{n-3}^{iv}(x_1) \\ B_4^{iv}(x_n) & B_5^{iv}(x_n) & B_6^{iv}(x_n) \cdots\cdots & B_{n-3}^{iv}(x_n) \end{bmatrix} \begin{bmatrix} c_4 \\ c_5 \\ c_6 \\ \vdots \\ c_{n-3} \end{bmatrix}$$

$$+\begin{bmatrix} B_2^{iv}(x_1) & B_{n-1}^{iv}(x_1) \\ B_2^{iv}(x_n) & B_{n-1}^{iv}(x_n) \end{bmatrix} \begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix} + \begin{bmatrix} B_3^{iv}(x_1) & B_{n-2}^{iv}(x_1) \\ B_3^{iv}(x_n) & B_{n-2}^{iv}(x_n) \end{bmatrix} \begin{bmatrix} c_3 \\ c_{n-2} \end{bmatrix} = 0$$

$$M_5 C^{\eta} + M_6 C^{\omega} + M_4 C^{\tau} = 0 \qquad (4.40)$$

where,

$$M_5 = \begin{bmatrix} B_4^{iv}(x_1) & B_5^{iv}(x_1) & B_6^{iv}(x_1) \cdots\cdots & B_{n-3}^{iv}(x_1) \\ B_4^{iv}(x_n) & B_5^{iv}(x_n) & B_6^{iv}(x_n) \cdots\cdots & B_{n-3}^{iv}(x_n) \end{bmatrix}, \qquad (4.40a)$$

$$M_6 = \begin{bmatrix} B_2^{iv}(x_1) & B_{n-1}^{iv}(x_1) \\ B_2^{iv}(x_n) & B_{n-1}^{iv}(x_n) \end{bmatrix}, \quad M_4 = \begin{bmatrix} B_3^{iv}(x_1) & B_{n-2}^{iv}(x_1) \\ B_3^{iv}(x_n) & B_{n-2}^{iv}(x_n) \end{bmatrix}; \qquad (4.40b)$$

Equation (4.39) becomes

$$C^{\tau} = -M_3^{-1}\left(M_1 C^{\eta} + M_2 C^{\omega}\right)$$

$$C^{\omega} = \frac{\left(M_5 - M_4 M_1 M_3^{-1}\right)}{\left(M_4 M_2 M_3^{-1} - M_6\right)} C^{\eta} = (L \backslash K) C^{\eta}$$

$$C^{\tau} = \left[M_3^{-1} M_1 - M_3^{-1} M_2 \left(L \backslash K\right)\right] C^{\eta}$$

where

$K = M_5 - M_4 M_1 M_3^{-1}$ and $L = M_4 M_2 M_3^{-1} - M_6$

$$p_{n-1}=\sum_{j=2}^{n-1} B_j^{vi}(x_k)c_j=\sum_{j=4}^{n-3} B_j^{vi}(x_k)c_j+\begin{bmatrix} B_2^{vi}(x_4) & B_{n-1}^{vi}(x_4) \\ B_2^{vi}(x_5) & B_{n-1}^{vi}(x_5) \\ B_2^{vi}(x_6) & B_{n-1}^{vi}(x_6) \\ \vdots & \\ B_2^{vi}(x_{n-3}) & B_{n-1}^{vi}(x_{n-3}) \end{bmatrix}\begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix}$$

$$+\begin{bmatrix} B_3^{vi}(x_4) & B_{n-2}^{vi}(x_4) \\ B_3^{vi}(x_5) & B_{n-2}^{vi}(x_5) \\ B_3^{vi}(x_6) & B_{n-2}^{vi}(x_6) \\ \vdots & \\ B_3^{vi}(x_{n-3}) & B_{n-2}^{vi}(x_{n-3}) \end{bmatrix}\begin{bmatrix} c_3 \\ c_{n-2} \end{bmatrix}$$

$$\tilde{B}^6 C^\eta = B^6 C^\eta + M_7 C^\omega + M_8 C^\tau \text{ , } i=4,5,3,\ldots\ldots\ldots\ldots\ldots\ldots,n-3 \qquad (4.41)$$

$$= B^6 C^\eta + M_7 (L\setminus K) + M_8\left[-M_3^{-1} M_1 C^\eta - M_3^{-1} M_2 C^\omega\right]$$

$$= B^6 C^\eta + M_7 (L\setminus K) + M_8\left[-M_3^{-1} M_1 C^\eta - M_3^{-1} M_2 (L\setminus K) C^\eta\right]$$

$$\left[B^6 + M_7 (L\setminus K) - M_8 M_3^{-1} M_1 - M_8 M_3^{-1} M_2\right] C^\eta$$

For fourth order derivative

$$\sum_{j=2}^{n-1} B_j^{iv}(x_i)c_j=\sum_{j=4}^{n-3} B_j^{iv}(x_i)c_j+\begin{bmatrix} B_2^{iv}(x_4) & B_{n-1}^{iv}(x_4) \\ B_2^{iv}(x_5) & B_{n-1}^{iv}(x_5) \\ B_2^{iv}(x_6) & B_{n-1}^{iv}(x_6) \\ \vdots & \\ B_2^{iv}(x_{n-3}) & B_{n-1}^{iv}(x_{n-3}) \end{bmatrix}\begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix}$$

$$+\begin{bmatrix} B_3^{iv}(x_4) & B_{n-2}^{iv}(x_4) \\ B_3^{iv}(x_5) & B_{n-2}^{iv}(x_5) \\ B_3^{iv}(x_6) & B_{n-2}^{iv}(x_6) \\ \vdots & \\ B_3^{iv}(x_{n-3}) & B_{n-2}^{iv}(x_{n-3}) \end{bmatrix} \begin{bmatrix} c_3 \\ \\ c_{n-2} \end{bmatrix}$$

$$\widetilde{B}^4 C^\eta = B^4 C^\eta + M_{34} C^\tau + M_{33} C^\omega = B^4 C^\eta + M_{34} C^\tau + M_{33} C^\omega$$

where

$$\widetilde{B}^4 C^\eta = B^4 C^\eta + M_{34} C^\tau + M_{33} C^\omega B^4$$

$$= B^4 C^\eta - M_{34}\left[M_3^{-1} M_1 + M_3^{-1} M_2 (L \setminus K)\right] C^\eta + M_{33} (L \setminus K) C^\eta$$

$$\widetilde{B}^4 C^\eta = \left\{ B^4 - M_{34}\left[M_3^{-1} M_1 + M_3^{-1} M_2 (L \setminus K)\right] + M_{33} (L \setminus K) \right\} C^\eta \tag{4.42}$$

For second derivative

$$\sum_{j=2}^{n-1} B_j''(x_k) c_j = \sum_{j=4}^{n-3} B_j''(x_k) c_j + \begin{bmatrix} B_2''(x_4) & B_{n-1}''(x_4) \\ B_2''(x_5) & B_{n-1}''(x_5) \\ B_2''(x_6) & B_{n-1}''(x_6) \\ \vdots & \vdots \\ B_2''(x_{n-3}) & B_{n-1}''(x_{n-3}) \end{bmatrix} \begin{bmatrix} c_2 \\ \\ c_{n-1} \end{bmatrix}$$

$$+\begin{bmatrix} B_3''(x_4) & B_{n-2}''(x_4) \\ B_3''(x_5) & B_{n-2}''(x_5) \\ B_3''(x_6) & B_{n-2}''(x_6) \\ \vdots & \vdots \\ B_3''(x_{n-3}) & B_{n-2}''(x_{n-3}) \end{bmatrix} \begin{bmatrix} c_3 \\ \\ c_{n-2} \end{bmatrix} \tag{4.43}$$

$$B'' C^\eta + M_{55} y^\omega + M_{66} y^\tau = B'' C^\eta + M_{55} \times (L \setminus K) C^\eta$$

$$+ M_{66} \times \left\{ -M_3^{-1}\left(M_1 C^\eta + M_2 \times (L \setminus K) C^\eta \right) \right\}$$

$$= B^{(2)} C^\eta + \left[M_{55} \times (L \setminus K) + M_{66} \times \left\{ -M_3^{-1}\left(M_1 C^\eta + M_2 \times (L \setminus K) \right) \right\} \right] C^\eta \tag{4.44}$$

For first derivative matrix

$$\sum_{j=2}^{n-1} B'_j(x_k)c_j = \sum_{j=4}^{n-3} B'_j(x_k)c_j + \begin{bmatrix} B'_2(x_4) & B'_{n-1}(x_4) \\ B'_2(x_5) & B'_{n-1}(x_5) \\ B'_2(x_6) & B'_{n-1}(x_6) \\ \vdots & \vdots \\ B'_2(x_{n-3}) & B'_{n-1}(x_{n-3}) \end{bmatrix} \begin{bmatrix} c_2 \\ c_{n-1} \end{bmatrix}$$

$$+ \begin{bmatrix} B'_3(x_4) & B'_{n-2}(x_4) \\ B'_3(x_5) & B'_{n-2}(x_5) \\ B'_3(x_6) & B'_{n-2}(x_6) \\ \vdots & \vdots \\ B'_3(x_{n-3}) & B'_{n-2}(x_{n-3}) \end{bmatrix} \begin{bmatrix} c_3 \\ c_{n-2} \end{bmatrix}$$

$$= M_{77} C^{\omega} + M_{88} C^{\tau} + \sum_{j=4}^{n-3} B'_j(x_k)c_j$$

$$= \left[M_{77} \times (L \setminus K) + M_{88} \times \left\{ -M_3^{-1} \left(M_1 y^{\eta} + M_2 \times (L \setminus K) \right) \right\} \right] C^{\eta} + B'C^{\eta} \qquad (4.45)$$

4.7.2 Test examples:

In this section we presented two numerical examples to show the efficiency of the presented method with the other available studies in literature.

Example 4.5: We consider the two-point boundary value problem studied by Gheorghiu and Dragomirescu (2009) governing the linear stability provided with the Dirichlet and hinged boundary conditions given as

$$\left(D^2 - a^2\right)^3 u = -Ra(1 + \varepsilon h(x))\, a^2 u \qquad (4.46a)$$

$$\left.\begin{aligned} u(0) = Du(0) = D^2 u(0) = 0 \\ u(1) = Du(1) = D^2 u(1) = 0 \end{aligned}\right\} \qquad (4.46b)$$

Numerical evaluations of the Rayleigh number for the particular case in which the gravity variation is $\varepsilon h(x) = (2x-1)$ are given. We compared our numerically calculated eigenvalues employing Bernstein collocation method with the present

Galerkin method and the other numerous techniques worked out by Gheorghiu and Dragomirescu (2009) as well.

Table 4.8.: Numerical estimates for the Rayleigh number for various values of the parameters ε and a^2 attained by different spectral methods and collocation method for example 4.5.

k	ε	a^2	Present Bern. coll.	*Ra*: WRG (Bernst.)	Present Ra: WRG (Legn.)	*Ra*: SCP	*Ra*: SLP	*Ra*: TS	*Ra*: CC	*Ra*: WRP1	*Ra*: WRP2
1	0.00	4.92	657.8576	657.513	657.5133	657.512	675.05	657.51	657.5133	658.59	662.80
2	0.01	4.92	661.3436	660.817	660.817	660.747	678.45	660.81	660.8173	662.03	666.05
3	0.03	4.92	668.1581	667.525	667.525	667.653	685.33	667.52	667.5254	668.84	673.19
4	0.33	4.92	790.0108	787.288	787.288	787.363	808.30	787.28	787.2880	792.70	797.38
5	0.20	5.0	730.8647	730.565	730.565	730.459	749.95	730.56	730.5647	733.06	737.45
6	0.20	9.0	826.6681	829.392	829.392	829.440	846.70	829.39	829.3918	831.80	836.02
7	0.50	7.5	948.1362	930.924	930.924	930.982	952.07	930.92	930.9239	946.05	950.55
8	0.50	9.0	995.0445	994.472	994.4721	994.393	1015.27	994.47	994.4721	1010.60	1015.20
9	0.75	10	1252.0596	1251.092	1251.092	1251.178	1276.05	1251.09	1251.0924	1314	1319.70
CPU time			**3.98 seconds**	**2.67 seconds**	**2.87 seconds**	**2679.83 seconds**					

Table 4.8 displays nine critical values for number of grids, n=35. From table 4.9, comparing to the other methods it is fairly clear that Bernstein collocation method provide closest results and it is clearly seen that Bernstein collocation method takes slightly greater CPU time than that of the Galerkin method. Besides these Bernstein collocation scheme converges rapidly with the increasing number of grid points.

Gheorghiu and Dragomirescu (2009) applied D^2 instead of $D^{(6)}$ strategy to minimize the rapid worsening of the condition number.

Table 4.9: Computational time for the smallest *Ra* applying current Galerkin and collocation method versus spectral method of example 4.5.

No. of poly. n	Cheby. Spect. Ra	CPU time (sec) Cheby. Spect	Degree of Poly.	WRG Ra:	CPU (sec) WRG. Bernst.	No. of grids	Ra: Bern coll.	CPU (sec) Bernst. coll.
4	658.692	16.66	6	658.1283	1.267	30	664.042	3.357
5	657.668	41.80	7	658.1283	1.303	31	663.652	3.359
7	657.5219	386.12	9	657.5134	2.069	32	663.218	3.460
9	657.5129	2679.83	11	657.5134	2.082	33	662.712	3.638
11	657.5127	17834.19	13	657.5133	2.691	34	658.537	3.782

From figure 4.6, we observed that condition numbers in case of Bernstein collocation differentiation matrix $B^{(6)}$ that we have been exploited are smaller than Chebychev collocation differentiation matrix D^{2} for n increases up to 60. Again condition numbers begin to increase for Bernstein collocation matrix $B^{(6)}$ and Chebychev Spectral matrix $D^{(6)}$ for $n > 60$.

Example 4.6: We consider one dimensional sixth order Benard layer eigenvalue problem which has analytical eigenvalues are calculated by Baldwin (1987), Twizell and Boutayeb (1990).

$$\left(\frac{d^{2}}{dx^{2}} - a^{2}\right)^{3} u(x) + Ra^{2}(1 - x^{2})u(x) = 0, \quad 0 < x < 10 \tag{4.47}$$

$$\begin{cases} u(0) = u^{2}(0) = u^{4}(0) = 0 \\ u(10) = u^{2}(10) = u^{4}(10) \end{cases} \tag{4.47a}$$

$$\begin{cases} u(0) = u^{3}(0) = u^{5}(0) = 0 \\ u(10) = u^{3}(10) = u^{5}(10) = 0 \end{cases} \tag{4.47b}$$

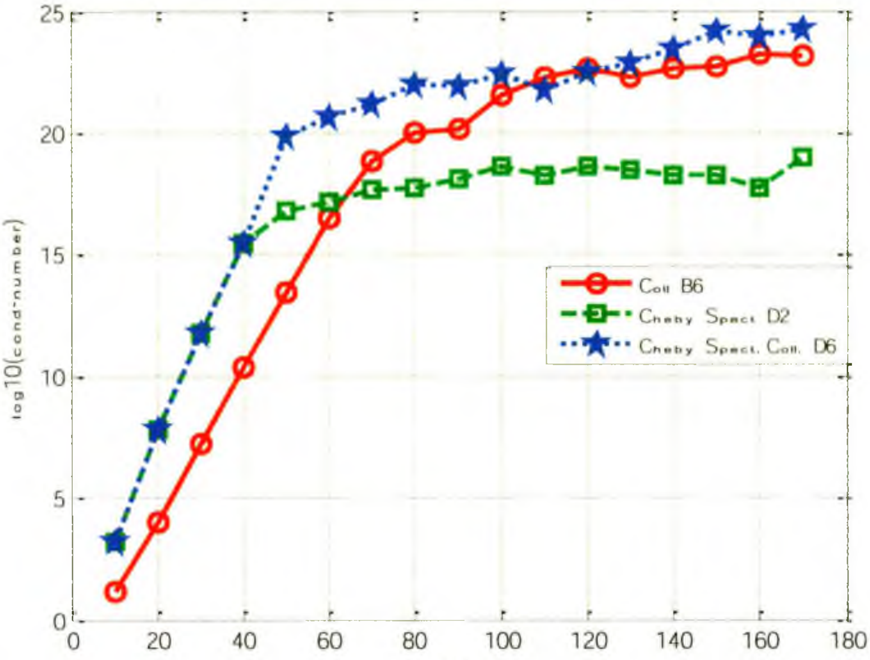

Figure 4.6: Logarithm of the condition number of the differentiation matrices for Chebychev and Bernstein polynomials.

We have demonstrated the first six even and odd modes critical values of R, for the corresponding given values of A exploiting 100 collocation points which have been displayed in table 4.10. We noticed that percentage of relative errors using the Galerkin for the smallest critical values are comparatively better than Bernstein collocation method and the worked out by Twizell and Boutayeb (1990). In spite of this Bernstein collocation method competes well with other methods for the higher critical values. Furthermore, relative error of the first six Rayleigh numbers (eigenvalues) attained by FDM [Twizell and Boutayeb (1990)] and present methods (WRM Gal and Bern Coll.) with the analytical results [Baldwin (1987)] have been displayed in figure 4.7.

Relative errors have been calculated by using the formula given as follows:

$$\text{Relative error}=\left|\frac{\lambda_k^{coll.}-\lambda^{Analy.}}{\lambda^{Analy.}}\right| \quad \text{and} \quad \text{Relative error}=\left|\frac{\lambda_k^{FDM.}-\lambda^{Analy.}}{\lambda^{Analy.}}\right|.$$

Table 4.10: Comparison of first six critical values with percentage of relative error for even modes and odd modes of example 4.6 for different methods.

Eigen index	Baldwin (1987)		Twizel & Boutayeb (1990)			Present methods			
	Analytical	Baldwin	Rayleigh Number	Wave number	Percent. rel. error	Rayleigh Number Bern coll.	Present method	Percent. rel. error Bern coll.	Percent. rel. error WRM Gal.
k	R	A	R	A	%	R	A	%	%
1	9.78136567	0.72605	9.7786945	0.72603	2.73×10^{-2}	9.7849560266	0.72605	2.13×10^{-3}	3.0×10^{-5}
2	411.720155	1.6791	411.515421	1.6790	5.0×10^{-2}	411.58375824	1.6791	3.33×10^{-2}	4.0×10^{-3}
3	3006.70954	2.7379	3003.053226	2.7374	1.2×10^{-1}	3005.17244941	2.7378	5.11×10^{-2}	1.5×10^{-2}
4	11382.69538	3.8130	11356.55701	3.8112	2.3×10^{-1}	11376.683195	3.8130	5.28×10^{-2}	2.1×10^{-1}
5	30916.2534	4.8916	30800.6998	4.8882	3.7×10^{-1}	30852.474289	4.8916	2.06×10^{-1}	4.9×10^{-1}
6	68778.117	5.971	68397.491	5.965	5.3×10^{-1}	68705.3443353	5.97013	1.06×10^{-1}	4.2×10^{-1}

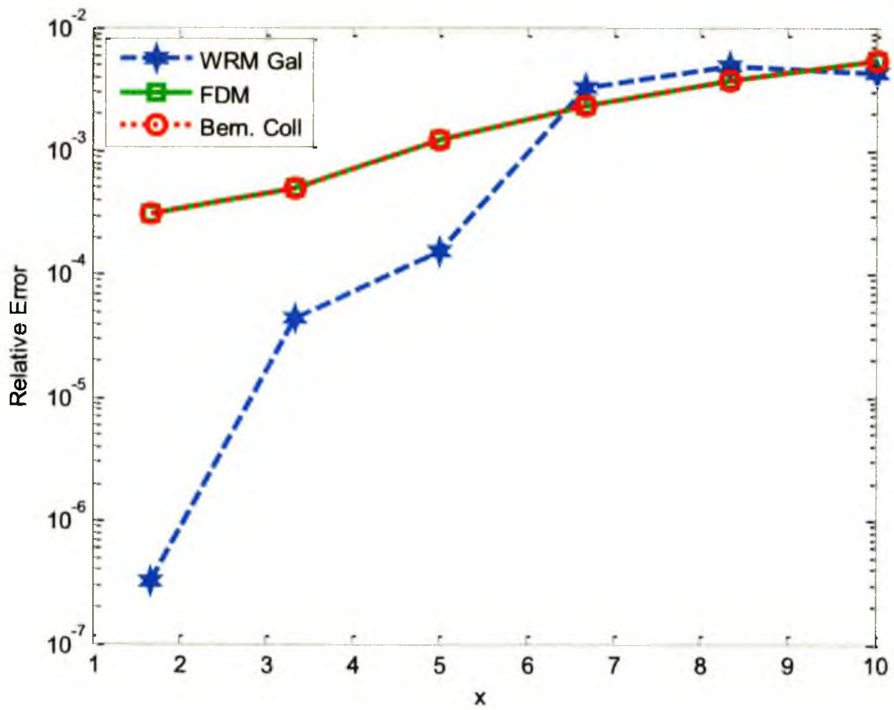

Figure 4.7: Relative errors for eigenvalues applying FDM, WRM Gal, Bern coll. and Analytical method.

4.8 The Chebychev-Legendre Spectral Collocation Method

4.8.1 Formulation of Sixth order SLEs

$$\frac{d^3}{dx^3}\left[p(x)\frac{d^3u}{dx^3}\right]-\frac{d^2}{dx^2}\left[q(x)\frac{d^2u}{dx^2}\right]+\frac{d}{dx}\left[r(x)\frac{du}{dx}\right]$$

$$-(s(x)-\lambda\sigma(x))u(x)=0 \qquad (4.48)$$

We have six boundary conditions

$$\begin{cases} u(x_1)=u''(x_1)=0 \\ u(x_n)=u''(x_n)=0 \end{cases} \qquad (4.48a)$$

$$\begin{cases} u(x_1)=u^{iv}(x_1)=0 \\ u(x_n)=u^{iv}(x_n)=0 \end{cases} \qquad (4.48b)$$

Equation (4.48a) can be put in the form:

$$\phi_1''(x_1)u_1(x_1)+\sum_{i=2}^{n-1}\phi_i''(x_1)u_i=0 \qquad (4.49a)$$

$$\phi_1''(x_1)u_1(x_1)+\sum_{i=2}^{n-1}\phi_i''(x_k)u_i=0 \qquad (4.49b)$$

$k=4,5,\ldots\ldots\ldots,n-3$ are interior nodes

From (4.49a) and (4.49b)

$$\phi_2''(x_1)u_2+\phi_{n-1}''(x_1)u_{n-1}+\phi_3''(x_1)u_3+\phi_{n-2}''(x_1)u_{n-2}+\sum_{i=4}^{n-3}\phi_i''(x_1)u_i=0 \qquad (4.50a)$$

$$\phi_2''(x_n)u_2+\phi_{n-1}''(x_n)u_{n-1}+\phi_3''(x_n)u_3+\phi_{n-2}''(x_n)u_{n-2}+\sum_{i=4}^{n-3}\phi_i''(x_n)u_i=0 \qquad (4.50b)$$

From (4.50a) and (4.50b)

$$\begin{bmatrix} \phi_4''(x_1) & \phi_5''(x_1) & \phi_6''(x_1)\ldots\ldots\ldots & \phi_{n-3}''(x_1) \\ \phi_4''(x_n) & \phi_5''(x_n) & \phi_6''(x_n)\ldots\ldots\ldots & \phi_{n-3}''(x_n) \end{bmatrix}\begin{bmatrix} u_4 \\ u_5 \\ u_6 \\ \vdots \\ u_{n-3} \end{bmatrix}$$

$$+\begin{bmatrix}\phi_2''(x_1) & \phi_{n-1}''(x_1)\\ \phi_2''(x_n) & \phi_4''(x_n)\end{bmatrix}\begin{bmatrix}u_2\\ u_{n-1}\end{bmatrix}+\begin{bmatrix}\phi_3''(x_1) & \phi_{n-2}''(x_1)\\ \phi_3''(x_n) & \phi_{n-2}''(x_n)\end{bmatrix}\begin{bmatrix}u_3\\ u_{n-2}\end{bmatrix}=0 \qquad (4.51a)$$

Similarly, using another set of boundary condition for x_1 and x_n nodes, we obtain the matrix form of equation (4.48b) as

$$\begin{bmatrix}\phi_4^{iv}(x_1) & \phi_5^{iv}(x_1) & \phi_6^{iv}(x_1)\ldots\ldots\ldots\ldots & \phi_{n-3}^{iv}(x_1)\\ \phi_4^{iv}(x_n) & \phi_5^{iv}(x_n) & \phi_6^{iv}(x_n)\ldots\ldots\ldots & \phi_{n-3}^{iv}(x_n)\end{bmatrix}\begin{bmatrix}u_4\\ u_5\\ u_6\\ \vdots\\ u_{n-3}\end{bmatrix}$$

$$+\begin{bmatrix}\phi_2^{iv}(x_1) & \phi_{n-1}^{iv}(x_1)\\ \phi_2^{iv}(x_n) & \phi_{n-1}^{iv}(x_n)\end{bmatrix}\begin{bmatrix}u_2\\ u_{n-1}\end{bmatrix}+\begin{bmatrix}\phi_3^{iv}(x_1) & \phi_{n-2}^{iv}(x_1)\\ \phi_3^{iv}(x_n) & \phi_{n-2}^{iv}(x_n)\end{bmatrix}\begin{bmatrix}u_3\\ u_{n-2}\end{bmatrix}=0 \qquad (4.51b)$$

Equation (4.51b) can be written as $M_1u^* + M_2u^{**} + M_3u^{***} = 0$ (4.52)

where

$$M_1=\begin{bmatrix}\phi_4''(x_1) & \phi_5''(x_1) & \phi_6''(x_1)\ldots\ldots\ldots\ldots & \phi_{n-3}''(x_1)\\ \phi_4''(x_n) & \phi_5''(x_n) & \phi_6''(x_n)\ldots\ldots\ldots & \phi_{n-3}''(x_n)\end{bmatrix} \qquad (4.52a)$$

$$M_2=\begin{bmatrix}\phi_2''(x_1) & \phi_{n-1}''(x_1)\\ \phi_2''(x_n) & \phi_{n-1}''(x_n)\end{bmatrix} \qquad (4.52b)$$

$$M_3=\begin{bmatrix}\phi_3''(x_1) & \phi_{n-2}''(x_1)\\ \phi_3''(x_n) & \phi_{n-2}''(x_n)\end{bmatrix} \qquad (4.52c)$$

$$u^* = [u_4, u_5, \ldots\ldots\ldots, u_{n-3}]^T,\quad u^* = \begin{bmatrix}u_4\\ u_5\\ u_6\\ \vdots\\ u_{n-3}\end{bmatrix},\quad u^{**} = \begin{bmatrix}u_2\\ u_{n-1}\end{bmatrix},\quad u^{***} = \begin{bmatrix}u_3\\ u_{n-2}\end{bmatrix}$$

Similarly, from equation (4.52b)

$$M_5u^* + M_6u^{**} + M_4u^{***} = 0 \qquad (4.53)$$

where

$$M_5 = \begin{bmatrix} \phi_4^{iv}(x_1) & \phi_5^{iv}(x_1) & \phi_6^{iv}(x_1) \ldots\ldots\ldots & \phi_{n-3}^{iv}(x_1) \\ \phi_4^{iv}(x_n) & \phi_5^{iv}(x_n) & \phi_6^{iv}(x_n) \ldots\ldots\ldots & \phi_{n-3}^{iv}(x_n) \end{bmatrix}$$

$$M_6 = \begin{bmatrix} \phi_2^{iv}(x_1) & \phi_{n-1}^{iv}(x_1) \\ \phi_2^{iv}(x_n) & \phi_{n-1}^{iv}(x_n) \end{bmatrix}, \qquad M_4 = \begin{bmatrix} \phi_3^{iv}(x_1) & \phi_{n-2}^{iv}(x_1) \\ \phi_3^{iv}(x_n) & \phi_{n-2}^{iv}(x_n) \end{bmatrix}$$

Solving equations (4.52) and (4.53), we have

$$u^{***} = -M_3^{-1}\left(M_1 u^{*} + M_2 u^{**}\right) \tag{4.54}$$

$$u^{**} = \frac{\left(M_5 - M_4 M_1 M_3^{-1}\right)}{\left(M_4 M_2 M_3^{-1} - M_6\right)} u^{*} = (L \setminus K) u^{*} \tag{4.54a}$$

$$u^{***} = \left[M_3^{-1} M_1 - M_3^{-1} M_2 (L \setminus K)\right] u^{*} \tag{4.54b}$$

where $K = M_5 - M_4 M_1 M_3^{-1}$ and $L = M_4 M_2 M_3^{-1} - M_6$

$$p_{n-1} = \sum_{i=2}^{n-1} \phi_i^{vi}(x_k) u_i = \sum_{i=4}^{n-3} \phi_i^{vi}(x_k) u_i + \begin{bmatrix} \phi_2^{vi}(x_4) & \phi_{n-1}^{vi}(x_4) \\ \phi_2^{vi}(x_5) & \phi_{n-1}^{vi}(x_5) \\ \phi_2^{vi}(x_6) & \phi_{n-1}^{vi}(x_6) \\ \vdots & \vdots \\ \phi_2^{vi}(x_{n-3}) & \phi_{n-1}^{vi}(x_{n-3}) \end{bmatrix} \begin{bmatrix} u_2 \\ u_{n-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \phi_3^{vi}(x_4) & \phi_{n-2}^{vi}(x_4) \\ \phi_3^{vi}(x_5) & \phi_{n-2}^{vi}(x_5) \\ \phi_3^{vi}(x_6) & \phi_{n-2}^{vi}(x_6) \\ \vdots & \vdots \\ \phi_3^{vi}(x_{n-3}) & \phi_{n-2}^{vi}(x_{n-3}) \end{bmatrix} \begin{bmatrix} u_3 \\ u_{n-2} \end{bmatrix} \tag{4.55}$$

$$\tilde{D}^6 u^{*} = \bar{D}^6 u^{*} + M_7 u^{**} + M_8 u^{***} \tag{4.56}$$

$$=\left[\overline{D}^{6}+M_7(L\setminus K)-M_8M_3^{-1}M_1u^{*}-M_3^{-1}M_2(L\setminus K)\right]u^{*}$$

$$\overline{D}^{6}u^{*}+M_7\,(L\setminus K)-M_8\left[-M_3^{-1}M_1y^{*}-M_3^{-1}M_2\,(L\setminus K)\right]u^{*} \tag{4.57}$$

The differential eigenvalue problem now becomes the algebraic eigenvalues problem

$$\tilde{D}^{6}u^{*}=\lambda u^{*} \tag{4.58}$$

where, $u^{*}=\left[u_4,u_5,\ldots\ldots\ldots\ldots,u_{n-3}\right]^{T}$ (4.58a)

$$\tilde{D}^{6}=\bar{D}^{(6)}+M_7\,(L\setminus K)-M_8\left[-M_3^{-1}M_1u^{*}-M_3^{-1}M_2\,(L\setminus K)\right] \tag{4.58b}$$

where

$$\tilde{D}^{4}u^{*}=D^{4}u^{*}+M_{34}u^{***}+M_{33}u^{**}$$

$$=D^{4}u^{*}-M_{34}\left[M_3^{-1}M_1+M_3^{-1}M_2(L\setminus K)\right]u^{*}+M_{33}(L\setminus K)u^{*} \tag{4.59}$$

$$\tilde{D}^{2}u^{*}=D^{2}u^{*}+M_{55}u^{**}+M_{66}u^{***}$$

$$=D^{2}u+-M_{55}\times(L\setminus K)u+M_{66}\left[M_3^{-1}\left(M_1+M_3^{-1}M_2(L\setminus K)\right)\right]u^{*} \tag{4.60}$$

$$\tilde{D}u^{*}=Du^{*}+\left[M_{77}\times(L\setminus K)+M_{88}M_3^{-1}\left(M_1+M_3^{-1}M_2(L\setminus K)\right)\right]u^{*} \tag{4.61}$$

4.8.2 Numerical Applications

We illustrated and discussed eigenvalues of three boundary value problems. The approach has been validated by convergence studies and comparison studies with existing results.

Example 4.7: We studied a sixth order Sturm-Liouville boundary value problem investigated by Siyyam and Syam (2011):

$$\frac{d^{6}u}{dx^{6}}=\lambda u(x) \tag{4.62a}$$

subject to the boundary conditions

$$\left.\begin{array}{l} u(0)=u''(0)=u^{iv}(0)=0 \\ u(1)=u''(1)=u^{iv}(1)=0 \end{array}\right\} \quad (4.62b)$$

Analytical eigenvalues for this problem are obtained by $\eta_k = \frac{\sqrt[6]{-\lambda_k}}{\pi}$, $k = 1,2,\ldots\ldots\ldots, 6$, the computed approximate eigenvalues are compared with those obtained by analytical results of equation (4.62a) illustrated in table 4.11. Variational method by [Syam and Siyyam, 2011], Chebychev Collocation-Path following method by [Mdallal and Syam (2014)]. The results indicate Chebychev Collocation-path following method and WRM Galerkin perform slightly better than the present algorithm. Condition numbers of eigenvalues versus grid points are depicted in Figure 4.7.

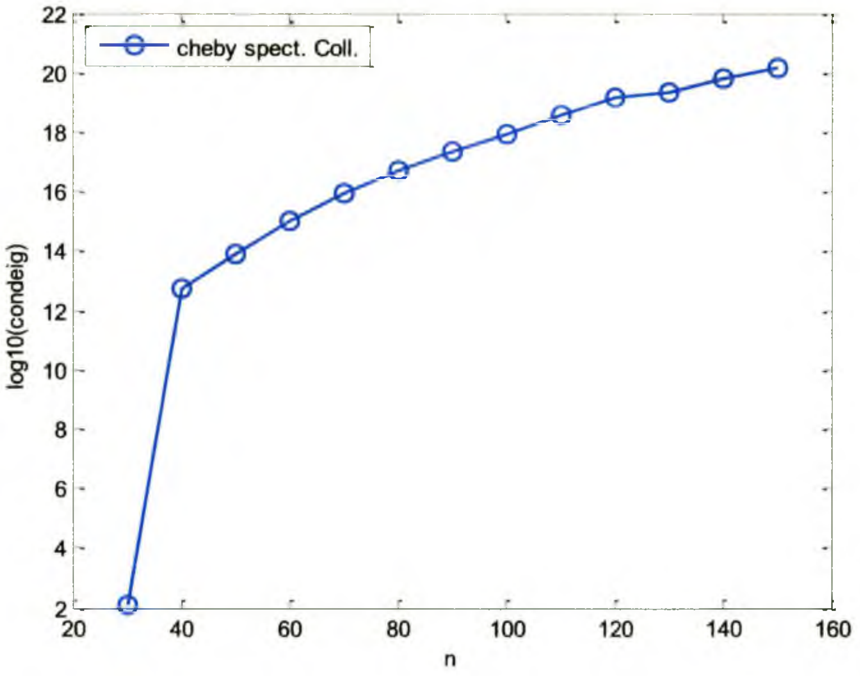

Figure 4.7: Logarithm of the condition number of the eigenvalues with increasing *n*.

Example 4.8: In this example, a circular ring structure with constraints [Gutierrez and Laura (1995), Wu and Liu (2000), Wang *et al* (2003)], which has rectangular cross-sections of constant width and parabolic variable thickness has been studied.

Considering half of the ring structure, this eigenvalue problem is formulated by the following sixth-order differential equation:

$$\beta_1 w^{(6)} + \beta_2 w^{(5)} + \beta_3 w^{(4)} + \beta_4 w^{(3)} + \beta_5 w^{(2)} + \beta_6 w^{(1)}$$

$$= \Omega^2 (f \frac{d^2}{dx^2} + f^{(1)} \frac{d}{dx} - \pi^2 f) w \qquad (4.63)$$

$w^{(r)} = \frac{d^r w}{dx^r}$, Ω is the dimension frequency, w is the tangential displacement

$\beta_1 = \frac{\phi}{\pi^4}$,

$\beta_2 = \frac{3\phi^{(1.)}}{\pi^4}$

$\beta_3 = (2\phi/\pi^2) + (3\phi^{(2)}/\pi^4)$,

$\beta_4 = \left(4\phi^{(1)}/\pi^2\right) + \left(\phi^{(3)}/\pi^4\right)$,

$\beta_5 = \phi + 3\phi^{(2)}/\pi^2$,

$\beta_6 = \phi^{(1)} + 3\phi^{(3)}/\pi^2$,

where,

$\phi = [f(x)]^3$; $f = f(x) = -4(r-1)x^2 + 4(r-1)x + 1$; $x \in [0,1]$, $f^{(i)} = \frac{d^i \phi}{dx^i}$, and r is the variable related to the thickness of the cross-section of the ring. Equation (4.63) can be written as

$$\left(\beta_1(x)\frac{d^6}{dx^6} + \beta_2(x)\frac{d^5}{dx^5} + \beta_3(x)\frac{d^4}{dx^4} + \beta_4(x)\frac{d^3}{dx^3} + \beta_5(x)\frac{d^2}{dx^2} + \beta_6 \frac{d}{dx} \right) w$$

$$= \Omega^2 \left(f \frac{d^2}{dx^2} + f^{(1)} \frac{d}{dx} - \pi^2 \right) w \qquad (4.64a)$$

Boundary conditions are

$$w(0) = w^{(1)}(0) = w^{(3)}(0); \quad w(1) = w^{(1)}(1) = w^{(3)}(1) = 0 \qquad (4.64b)$$

From table 4.12, we verify accuracy of our results with the method of *GDQR* [Wu and Liu (2000)] and La-DQM [Wang *et al* (2003)] increasing the nodes from 35 to 65. Furthermore, we observe that the present method converges to the same frequency parameter i.e., r =1.0, 1.1, 1.2 1.3 and 1.4 and are compared with *GDQR*, *La-DQM* at n=11. From table 4.12, it is noticed that the present method converges to five significant figures for the values of r=1.0, 1.1, 1.2 and 1.3 to GDQR. Accuracy of the present method slightly deviates for higher values of r =1.5 still accuracy is much closer to *GDQR* [Wu and Liu (2000)] other than *La-DQM* [Wang *et al* (2003)].

Table 4.11: The computed approximate eigenvalues η_k $(k = 1,2,........,6)$ of example 4.7

k	Cheby coll. Mdallal and Syam (2014)	Galerkin present Islam *et al* (2017)	Analytical	Present Legn. Spect.	Numerical (VIM) Syam and Siyyam(2011)
1	1.000000000000000	1.000000000000000	1.000000000000000	1.0000000007864	1.0000000000000
2	2.000000000000000	2.000000000000000	2.000000000000000	1.9999999999796	1.9999999999999
3	3.000000000000000	3.000000000000000	3.000000000000000	2. 9999999999365	2.9999999999999
4	4.000000000000000	4.000000000000000	4.000000000000000	4.0000000000327	4.0000000000005
5	5.000000000000000	5.000000000000000	5.000000000000000	5.0000000000478	4.9999999999987
6	6.000000000000000	6.000000000000000	6.000000000000000	6.0000000000393	5.9999999999214

Example 4.9: We consider one dimensional sixth order Benard layer eigenvalue problem which has analytical eigenvalues are calculated by Baldwin (1987).

$$\left(D^2 - a^2\right)^3 u(x) + Ra^2 (1 - x^2) u(x) = 0, \quad 0 < x < 10 \qquad (4.65a)$$

Here we have introduced two sets of boundary conditions for even and odd modes for the same Benard layer eigenvalue problem

Set 1:

$$\begin{cases} u(0) = u^2(0) = u^4(0) = 0 \\ u(10) = u^2(10) = u^4(10) \end{cases} \qquad (4.65b)$$

Set 2:

$$\begin{cases} u(0) = u^{3}(0) = u^{5}(0) = 0 \\ u(10) = u^{3}(10) = u^{5}(10) = 0 \end{cases} \tag{4.65c}$$

Here we have computed the critical Rayleigh numbers applying Cheby-Legendre collocation technique for even modes only for conciseness. The approximate relative error and percentage of relative error have been compared with the analytical method Baldwin (1987), Finite difference method Twizell & Boutayeb (1990), WRM Galerkin. The first 3 critical numbers for even modes are depicted in table 4.13.

Table 4.12: Fundamental frequencies of example 4.8 employing numerous techniques.

	La-DQM	*GDQR*	*GAL. WRM*	*Legn. Spectral* (present)						
r	$N=11^{a}$	$N=11^{a}$	*n*=11	*n*=35	*n*=40	*n*=45	*n*=50	*n*=55	*n*=60	*n*=65
1.0	2.2681	2.2677	2.2670	2.2670	2.2669	2.2668	2.2668	2.2668	2.2668	2.2667
1.1	2.4150	2.4135	2.4140	2.4139	2.4132	2.4137	2.4137	2.4137	2.4137	2.4137
1.2	2.5582	2.5577	2.5571	2.5571	2.5569	2.5569	2.5568	2.5568	2.5568	2.5568
1.3	2.6966	2.6964	2.6969	2.6968	2.9667	2.6967	2.6966	2.6966	2.6966	2.6966
1.4	2.8328	2.8330	2.8333	2.8337	2.8335	2.8335	2.8335	2.8335	2.8335	2.8335
1.5	2.9643	2.9676	2.9676	2.9679	2.9678	2.9678	2.9677	2.9677	2.9677	2.9677

Table 4.13: First six critical values listed for example 4.9 for even modes (*n*=2,4,6) with relative error and percentage of relative error.

n	Baldwin (1987) *R*	Baldwin *A*	Twizell & Boutayeb (1990) *R*	Twizell & Bouta-yeb *A*	Percent. of rel. error (1990)	Present method *R*	Current Spect. coll. *A*	Rel. error present *n*=60	Percent. of rel. error Spect. coll. %	Percent. rel. error WRM Gal. %
2	411.720155	1.6791	411.515421	1.6790	0.0497	411.720155	1.6793	1.14×10^{-9}	1.4×10^{-7}	4.0×10^{-3}
4	11382.6954	3.8130	11356.5570	3.8112	0.2296	11382.69535	3.8109	2.43×10^{-9}	2.4×10^{-7}	2.1×10^{-1}
6	68778.117	5.971	68397.491	5.965	0.5334	68778.089	5.965	4.04×10^{-7}	1.0×10^{-5}	4.2×10^{-1}
CPU time						**8.963 sec.**				

4.9 Conclusions

This study demonstrates direct formulations and application of Galerkin WRM, Bernstein collocation and Spectral collocation methods to solve eigenvalue problems governed by sixth order differential equations with different types of boundary conditions. Having compared our results with the exact eigenvalues in the first example of SLE, the computed eigenvalues are depicted to be much more accurate and compatible. When WRM is applied for the said problem, computed eigenvalues agree very well with those of GDQR [Wu and Liu (2000)]. Besides, the present work compensates for the complexities that arise in the implementation of GDQR [Wu and Liu (2000)] and difficulties of including additional nodes. For the Bernard layer eigenvalue problem, the estimated critical Rayleigh numbers in this paper show moderated performance, corresponding to the fixed wave numbers. It is worth mentioning that the percentage of relative error for all even and odd eigen modes is less than 1%. The critical numbers are smaller than the asymptotic expansion [Baldwin (1987)] which prove the accuracy and stability of the proposed method. In the last example, linear stability of the other numerical methods applied to the various numerical methods, we noticed that our proposed method is much superior in the sense of accuracy and applicability. The leading advantage of utilizing Galerkin WRM method is that, a great number of trial functions can be used in the approximation.

Galerkin methods often exhibit the lowest condition number dependence on matrix size and yields fairly accurate results for most of the problems. Besides the lower part of the spectrum, which is highly important, is accurately approximated. It is fairly clear that the Chebyshev-Legendre collocation method and Galerkin and collocation method based on Bernstein polynomials provide the closest results for some class of the boundary value problems.

Since in collocation method computed results using equally spaced nodes may yield less accuracy than the results obtained by Gauss-Lobatto nodes. Meanwhile the collocation method accomplishes relatively well-conditioned matrix which reveals that the method is stable. Although CPU time for the Bernstein collocation method is much smaller with increasing n which minimizes the computational cost but its numerical large number discretization brings in ill-conditioned and any answer swamped with numerical error in finite precision for some complicated problems.

In this Chapter we have examined three numerical examples applying Spectral collocation technique and compared our results with Galerkin method. We observed that computed approximate eigenvalues exploiting Spectral collocation technique in table 4.11 are very close to the WRM Galerkin and other existing numerical methods. It has been noticed that in the case of vibration of ring structure problems estimated by the other numerical techniques with the increasing nodes illustrated in table 4.12. Finally, for Benard Layer problem percentage of relative error for even modes attained by Spectral collocation method is smaller than WRM Galerkin and is much smaller than those of finite difference method studied by some authors.

CHAPTER 5

Eigenvalue Computations of Eighth, Tenth and Twelfth Order Boundary Value Problems Using Galerkin WRM

5.1 Introduction

Numerical computation of eigenvalues of eighth, tenth and twelfth order boundary value problems take place when the horizontal layer of fluid heated from below subject to the action of rotation. The fluid at the bottom will be lighter than that at the top and, in this situation, the layer will be potentially unstable. The role played by viscosity is to inhibit a tendency on the part of the fluid to redistribute itself. This role is affected by any additional effect of rotation and the rotation will introduce new factors into the ensuing thermal instability. In liquid, instability sets in mostly as overstability when rotation is present, but it sets in as stationary convection under the influence of a magnetic field [Chandrashekhar (1981)]. When instability sets in as over stability, the differential equations occur is of eighth order. Tenth and twelfth order equations arise when instability sets in as ordinary convection and as over stability respectively due to acts of a uniform magnetic field across the fluid in the same direction as gravity. The necessary and sufficient condition for stability is known as Rayleigh criterion. The stability of fluid flow is determined by numerical value of the non-dimensional parameters, referred as Taylor numbers which gives a measure of extent to which Rayleigh's criteria is violated.

There are a few literatures on the approximate solution of higher order boundary value problems specially the associated eigenvalues problems. Wang *et al* (2003) developed a numerical method namely local adaptive differential quadrature method implementing Lagrange polynomials for solving an eighth order boundary value problem and sixth order eigenvalue problems with multi boundary conditions. Twizell *et al* (1994) extended finite difference method employing direct numerical technique and second order finite difference technique to compute eigenvalues of eighth, tenth and twelfth order boundary value problems. Siddiqi and Akram (2007) applied non polynomial spline for the numerical solution of tenth order boundary value problems. Also the twelfth degree splines were utilized for the solutions of twelfth order BVPs investigated by Siddiqi and Twizell (1997). Application of Adomian decomposition method for twelfth order BVPs is exemplified in the work done by Ahmed and Saleh (2011). Very recently, Viswanadham and Ballem (2015) used Galerkin based Septic B-Splines method to solve tenth order BVPs. Dragomirescu and Gheorghiu (2010) studied linear Electro-hydrodynamic stability problem of an eighth order differential equation by both analytical and numerical methods. The later one is implemented through Spectral Galerkin and collocation method taking Chebychev and Legendre polynomials as basis functions. Islam and Hossain (2015) and Islam *et al* (2015) worked out eighth, tenth and twelfth order linear and nonlinear BVPs employing Galerkin WRM. The authors used Legendre polynomials in the former article and Bernstein polynomials as basis function in the later case. The aim of this study is to investigate the higher order eigenvalue problems utilizing Galerkin weighted residual method and the effect of solution due to direct implementation of polynomial basis. In the offered method basis functions are satisfied by the Dirichlet type boundary conditions while all the essential type boundary conditions are directly incorporated in the weak form of the Galerkin residual equation. Bernstein and Legendre polynomials are used as basis function for numerical computations of eigenvalues which are referred to as Rayleigh numbers for the corresponding values of the wave numbers. Our computational results reveal that the current method is much competent with the other numerical/analytical methods available in literature.

In the present chapter, first we derive the matrix formulation for solving linear eighth order eigenvalue problem by the Galerkin weighted residual method with Bernstein and Legendre polynomials basis. For brevity we only demonstrated the formulation for eighth order eigenvalue only with one type of boundary condition in section 5.3. To verify the reliability and efficiency of the proposed method, some numerical examples of eighth, tenth and twelfth order BVPs, available in the literature, have been presented in section 5.4. In section 5.5, we mentioned the conclusions of this chapter.

5.2 Problem Description

We consider the eighth order linear eigenvalue problems of the following form:

$$\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=\lambda r(x)u \quad (5.1)$$

$$\gamma < x < \mu$$

Subject to the homogeneous boundary conditions

Type I : $u_i^m(\gamma)=0,\quad u_i^m(\mu)=0,\quad$ for $\quad m=0,\ 2\ ,4,6$ (5.1a)

Type II : $u_i^m(\gamma)=0\ ,\ u_i^m(\mu)=0,\quad$ for $\quad m=1\ ,3,\ 5,\ 7$ (5.1b)

Here, $a_i(x)$, $i=0,1,2\cdots\cdots,7$ and $r(x)$ are all continuous functions defined on $[\gamma,\mu]$ and $i=0,\ 1,\ 2,3,4,5,6$. The Galerkin WRM forces the residual to vanish by requiring residual of equation (5.1)

$$R_n(x)=L\tilde{u}_n(x)-\lambda\tilde{u}_n(x)\neq 0 \quad (5.2)$$

Here u is continuous function of x defined on the interval $[\gamma,\mu]$. For deriving the matrix formulation of eighth order BVP on [0,1], Bernstein and Legendre polynomials are to be utilized as a set of polynomial basis to satisfy the essential boundary conditions at the two end points of the interval.

5.3 Matrix Derivation using Galerkin method

The eigenvalues of the boundary value problem (5.1) is solved with both cases of the boundary conditions of Type I and Type II.

Since we intend to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0,1], so the eigenvalue equation (5.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(\mu-\gamma)x+\gamma$ and thus we have:

$$\frac{d^8u}{dx^8}+m_7\frac{d^7u}{dx^7}+m_6\frac{d^6u}{dx^6}+m_5\frac{d^5u}{dx^5}+m_4\frac{d^4u}{dx^4}+m_3\frac{d^3u}{dx^3}+m_2\frac{d^2u}{dx^2}+m_1\frac{du}{dx}+m_0u=\lambda wu, \quad (5.3)$$

$$0 < x < 1$$

$$u(0)=0\,,\quad u(1)=0\,,\quad \frac{1}{(\mu-\gamma)^2}u''(0)=0\,,\quad \frac{1}{(\mu-\gamma)^2}u''(1)=0\,,\quad \frac{1}{(\mu-\gamma)^4}u^{iv}(0)=0$$

$$,\quad \frac{1}{(\mu-\gamma)^4}u^{iv}(1)=0\,,\quad \frac{1}{(\mu-\gamma)^6}u^{vi}(0)=0\,,\quad \frac{1}{(\mu-\gamma)^6}u^{vi}(1)=0 \tag{5.3a}$$

$$u(0)=0\,,\quad u(1)=0\,,\quad \frac{1}{\mu-\gamma}u'(0)=0,\quad \frac{1}{\mu-\gamma}u'(1)=0,\quad \frac{1}{(\mu-\gamma)^3}u'''(0)=0,$$

$$\frac{1}{(\mu-\gamma)^3}u'''(1)=0,\quad \frac{1}{(\mu-\gamma)^5}u^{5}(0)=0,\quad \frac{1}{(\mu-\gamma)^5}u^{5}(1)=0 \tag{5.3b}$$

where $m_7=\frac{1}{(\mu-\gamma)^7}a_7[(\mu-\gamma)x+\gamma]$, $m_6=\frac{1}{(\mu-\gamma)^6}a_6[(\mu-\gamma)x+\gamma]$

$$m_5=\frac{1}{(\mu-\gamma)^5}a_5[(\mu-\gamma)x+\gamma]\,,\quad m_4=\frac{1}{(\mu-\gamma)^4}a_4[(\mu-\gamma)x+\gamma]$$

$$m_3=\frac{1}{(\mu-\gamma)^3}a_3[(\mu-\gamma)x+\gamma]\,,\quad m_2=\frac{1}{(\mu-\gamma)^2}a_2[(\mu-\gamma)x+\gamma]$$

$$m_1=\frac{1}{\mu-\gamma}a_1[(\mu-\gamma)x+\gamma],\quad m_0=a_0[(\mu-\gamma)x+\gamma]\,,\quad w=r[(\mu-\gamma)x+\gamma]$$

In this section we first develop the matrix formulation for eighth order linear eigenvalue problem incorporating the boundary conditions of Type I. To approximate the solution of eigenvalue problem (5.3), we express in terms of Bernstein or Legendre polynomial basis as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}c_iB_i(x) \tag{5.4}$$

where $\theta_0(x)=0$ is specified by the homogeneous boundary conditions $B_i(0)=0$ and $B_i(1)=0$ for each $i=1,2,3,\ldots\ldots,n-1$.

Now the approximate solution of equation (5.3) be

$$\tilde{u}(x)\approx\tilde{u}_n(x)=\sum_{i=1}^{n-1}c_iB_i(x) \tag{5.5}$$

where $\left\{B_j\right\}$ are test functions. In Galerkin method test functions are same as trial functions. Using (5.4) into equation (5.3), the Galerkin weighted residual equations are:

$$\int_0^1 \left[\frac{d^8\tilde{u}}{dx^8} + m_7 \frac{d^7\tilde{u}}{dx^7} + m_6 \frac{d^6\tilde{u}}{dx^6} + m_5 \frac{d^5\tilde{u}}{dx^5} + m_4 \frac{d^4\tilde{u}}{dx^4} + m_3 \frac{d^3\tilde{u}}{dx^3} + m_2 \frac{d^2\tilde{u}}{dx^2} \right.$$

$$\left. + m_1 \frac{d\tilde{u}}{dx} + m_0 \tilde{u} - \lambda w \tilde{u} \right] B_j \, dx = 0 \qquad (5.6)$$

Now integrating each term of equation (5.6) by parts, we have

$$\int_0^1 \frac{d^8\tilde{u}}{dx^8} B_j(x)\,dx = \left[B_j(x) \frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \int_0^1 B_j'(x) \frac{d^7\tilde{u}}{dx^7} dx = -\left[B_j'(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \int_0^1 B_j^{(2)}(x) \frac{d^5\tilde{u}}{dx^5} dx$$

$$= -\left[B_j'(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[B_j^{(2)}(x) \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \int_0^1 B_j^{(3)}(x) \frac{d^5\tilde{u}}{dx^5} dx$$

$$= -\left[B_j'(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[B_j^{(2)}(x) \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[B_j^{(3)}(x) \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \int_0^1 B_j^4(x) \frac{d^4\tilde{u}}{dx^4} dx$$

$$= -\left[B_j'(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[B_j^{(2)}(x) \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[B_j^{(3)}(x) \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \left[B_j^{(4)}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$- \int_0^1 B_j^{(5)}(x) \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[B_j'(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[B_j^{(2)} \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[B_j^{(3)}(x) \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \left[B_j^{(4)}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$- \left[B_j^{(5)}(x) \frac{d^2\tilde{u}}{dx^2} \right] + \int_0^1 B_j^{(6)}(x) \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[B_j'(x)\frac{d^6\tilde{u}}{dx^6}\right]_{x=1} + \left[B_j'(x)\frac{d^6\tilde{u}}{dx^6}\right]_{x=0} + \left[B_j^2(x)\frac{d^5\tilde{u}}{dx^5}\right]_{x=1} - \left[B_j^2(x)\frac{d^5\tilde{u}}{dx^5}\right]_{x=0}$$

$$- \left[B_j^{(3)}(x)\frac{d^4\tilde{u}}{dx^4}\right]_{x=1} + \left[B_j^{(3)}(x)\frac{d^4\tilde{u}}{dx^4}\right]_{x=0} + \left[B_j^{(4)}(x)\frac{d^3\tilde{u}}{dx^3}\right]_{x=1} - \left[B_j^{(4)}(x)\frac{d^3\tilde{u}}{dx^3}\right]_{x=0}$$

$$- \left[B_j^{(5)}(x)\frac{d^2\tilde{u}}{dx^2}\right]_{x=1} + \left[B_j^{(5)}(x)\frac{d^2\tilde{u}}{dx^2}\right]_{x=0} + \left[B_j^{(6)}(x)\frac{d\tilde{u}}{dx}\right]_{x=1}$$

$$- \left[B_j^{(6)}(x)\frac{d\tilde{u}}{dx}\right]_{x=0} - \int_0^1 B_j^{(7)}(x)\frac{d\tilde{u}}{dx}dx \qquad (5.7)$$

$$\int_0^1 a_7(x)\frac{d^7\tilde{u}}{dx^7}B_j(x)dx = \left[a_7(x)B_j(x)\frac{d^6\tilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \int_0^1 \frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$

$$- \int_0^1 \frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$

$$- \left[\frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^3}{dx^3}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^3}{dx^3}\left[a_7(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^4}{dx^4}\left[a_7(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^3}{dx^3}\left[a_7(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^4}{dx^4}\left[a_7(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$

$$-\left[\frac{d^2}{dx^2}\left[a_7(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^3}{dx^3}\left[a_7(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$-\left[\frac{d^4}{dx^4}\left[a_7(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^5}{dx^5}\left[a_7(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.8)$$

Similarly,

$$\int_0^1 a_6(x)\frac{d^6\tilde{u}}{dx^6}B_j(x)dx = \left[a_6(x)B_j(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_6(x)B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$-\left[\frac{d}{dx}\left[a_6(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= \left[\frac{d}{dx}\left[a_6(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[a_6(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_6(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]$$

$$-\left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right] + \int_0^1 \frac{d^4}{dx^4}\left[a_6(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_6(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]$$

$$-\left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right] + \left[\frac{d^4}{dx^4}\left[a_6(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^5}{dx^5}\left[a_6(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.9)$$

$$\int_0^1 a_5(x)\frac{d^5\tilde{u}}{dx^5}B_j(x)dx = \left[a_5(x)B_j(x)\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^2}{dx^2}\left[a_5(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.10)$$

$$\int_0^1 a_4(x)\frac{d^4\tilde{u}}{dx^4}B_j(x)dx = \left[a_4(x)B_j(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_4(x)B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_4(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^3}{dx^3}\left[a_5(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{5.11}$$

$$\int_0^1 a_3(x)\frac{d^3\tilde{u}}{dx^3}B_j(x)dx = \left[a_3(x)B_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_3(x)B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[a_3(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{5.12}$$

$$\int_0^1 a_2(x)\frac{d^2\tilde{u}}{dx^2}B_j(x)dx = \left[a_2(x)B_j(x)\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[a_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[a_2(x)B_j(x)\right]\frac{d\tilde{u}}{dx}dx \tag{5.13}$$

Inserting $B_j(0) = B_j(1) = 0$ in the above integrals of equation (5.6), we finally obtain by substituting the equations (5.7)–(5.13) into equation (5.6) and using approximation for $\tilde{u}(x)$ given in equation (5.5) and after applying the boundary conditions given in equation (5.3a) and rearranging the terms for the resulting equations, we get a system of equations in matrix form as

$$\sum_{i=1}^{n-1}\left[F_{i,j} - \lambda E_{i,j}\right]c_i = 0 \tag{5.14}$$

$$F_{i,j} = \int_0^1 \Bigg[-\frac{d^7 B_j}{dx^7}\frac{dB_i}{dx} + \frac{d^6}{dx^6}\left[a_7(x)B_j(x)\right]\frac{dB_i}{dx} - \frac{d^5}{dx^5}\left[a_6(x)B_j(x)\right]\frac{dB_i}{dx}$$

$$+\frac{d^4}{dx^4}\left[a_5(x)B_j(x)\right]\frac{dB_i}{dx} - \frac{d^3}{dx^3}\left[a_4(x)B_j(x)\right]\frac{dB_i}{dx} + \frac{d^2}{dx^2}\left[a_3(x)B_j(x)\right]\frac{dB_i}{dx}$$

$$-\frac{d}{dx}\left[a_2(x)B_j(x)\right]\frac{dB_i}{dx} + \left[a_1(x)B_j(x)\right]\frac{dB_i}{dx} + a_0(x)B_j(x)\tilde{u} - \lambda w(x)B_j\tilde{u}\Bigg]dx$$

$$-\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=0}^{1}+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0}^{1}$$

$$+\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^{1}-\left[\frac{d}{dx}\left[a_7(x)B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=0}^{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[a_7(x)B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0}^{x=1}-\left[\frac{d^5}{dx^5}\left[a_7(x)B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[a_6(x)B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0}^{x=1}+\left[\frac{d^4}{dx^4}\left[a_6(x)B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^{x=1}$$

$$-\left[\frac{d}{dx}\left[a_5(x)B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0}^{x=1}+\left[\frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^{x=1}$$

$$-\left[\frac{d}{dx}\left(a_3(x)B_j(x)\right)\frac{dB_i}{dx}\right]_{x=0}^{x=1} \qquad (5.14a)$$

where,

$$E_{i,j}=\int_0^1\left[w(x)B_iB_j\,dx\right] \qquad (5.14b)$$

Finally, the eigenvalues are obtained in solving the system (5.14) as below

$$F-\lambda E=0 \qquad (5.15)$$

Hence

$$\lambda I=F\,E^{-1} \qquad (5.16)$$

Similarly, for the boundary conditions of the Type II, the formulation can be obtained easily.

5.4 Numerical examples

To test the competency of the proposed method, four linear eigenvalue problems with boundary conditions of the type I and type II have been worked out. For all the examples, the eigenvalues accomplished by the proposed technique are compared with the numerically accepted eigenvalues computed a few numerical methods.

Example 5.1: We consider the Sturm-Liouville problem worked out by Taher *et al* (2014).

$$\frac{d^8 u}{dx^8} = \lambda u(x) \tag{5.17a}$$

subject to the boundary conditions

$$\begin{cases} u(0)=u''(0)=u^{iv}(0)=u^{vi}(0)=0 \\ u(\pi)=u''(\pi)=u^{iv}(\pi)=u^{vi}(\pi)=0 \end{cases} \tag{5.17b}$$

For exploiting Legendre polynomials we need to Change the boundary points from 0 to π into 0 to 1, we the equation (5.17a) by changing the variables $x=\pi t$.The transformed equation and the boundary conditions become:

$$\frac{1}{\pi^8}\frac{d^8 u}{dx^8} = \lambda u(x) \tag{5.18a}$$

$$\left.\begin{array}{l} u(0)=u''(0)=u^{iv}(0)=u^{vi}(0)=0 \\ u(1)=u''(1)=u^{iv}(1)=u^{vi}(1)=0 \end{array}\right\} \tag{5.18b}$$

The exact eigenvalues are given by $\lambda_k = k^6 \quad , \quad k=1,2,3,.........,n-1$ (5.19)

$$\sum_{i=1}^{n-1}\left[F_{i,j}-\lambda E_{i,j}\right]c_i=0 \qquad j=1,2,3,.........,n-1 \tag{5.20}$$

where,

$$F_{i,j}=\int_0^1 \frac{1}{\pi^8}\left\{\left[-\frac{d^7 B_j}{dx^7}\frac{dB_i}{dx}\right]dx+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=0}\right.$$

$$+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}+\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}$$

$$\left.-\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]\right\}_{x=0} \quad (5.20a)$$

$$E_{i,j}=\int_0^1\left[\ B_iB_jdx\right] \quad (5.20b)$$

The absolute error with modified ADM [Taher *et al* (2014)] and the first six eigenvalues using WRM exploiting Bernstein and Legendre polynomials have been displayed in tables 5.1 and 5.2 respectively. It is fairly clear that the Galerkin method based on Bernstein and Legendre polynomials provide the closest result and absolute errors are much smaller which proves the superiority over the existing other numerous techniques.

Example 5.2: The linear stability of the stationary solution in an electro hydrodynamic convection model in a layer situated between the walls $x = \pm 0.5$, against normal mode perturbations, is governed by the following eigenvalue problem from Dragomirescu and Gheorghiu (2010).

We consider

$$\left(D^2-a^2\right)^4 u - La^4u + Ra^2\left(D^2-a^2\right)u = 0 \quad x\in(-0.5,0.5) \quad (5.21a)$$

The boundary conditions containing even order derivatives are given by

$$u = D^2u = D^4u = D^6u = 0,\ \text{ at }\ x=\pm 0.5 \quad (5.21b)$$

Implementing the direct method the general solution can be obtained in terms of the roots of the characteristic equation and thus depends on the multiplicity of the roots λ_i of the characteristic equation associated with the eigenvalue problem.

Here the eigenfunction denoted by u characterizes the amplitude of the temperature field perturbation, the wave number represented by a and L is a parameter which measures the potential difference between the planes efficiently and R stands for

Rayleigh number. Besides, whenever the parameters a and L satisfy the "ellipticity" condition given by

$$a^4 - L \geq 0,$$

the problem reduces to a *minimization* one. The authors [Dragomirescu and Gheorghiu (2010)] applied direct analytical/numerical schemes and thus completed the investigation of this particular problem. It is evident from their exertion that the smallest eigenvalue is real and positive i.e., there exists a first Rayleigh number $R > 0$ which satisfies the eigenvalue problem (5.21). If *a, L, R*≠0, the discussions of multiplicity of the roots of the characteristic equations becomes difficult. So the use of direct analytical method is quite incomprehensible and alternative numerical method is sought.

Following the above facts, Dragomirescu and Gheorghiu (2010) applied two classes of methods to solve the formulation of the problem (5.21) explicitly by analytical method and Spectral methods (Galerkin, Tau and collocation) on utilizing Chebychev and Shifted Legendre polynomials. The authors used D^2 strategy worked out in the their previous article [Gheorghiu Dragomirescu (2009)].

Since Bernstein and Legendre polynomials have been utilized as trial functions which are derived over the interval [0, 1], so the equation (5.21) is to be converted to an equivalent problem on [0, 1]. We replaced x by $(\mu-\gamma)x+\gamma$ where $\gamma=-0.5$ and $\mu=0.5$ for the above problem. With this substitution the equation remains unchanged.

The residual equation using Galerkin WRM takes the form:

$$\sum_{i=1}^{n-1}\left[F_{i,j} - RE_{i,j}\right]c_i = 0 \qquad j=1,2,3,..........,n-1 \tag{5.22}$$

where,

$$F_{i,j} = \int_0^1 \left[-\frac{d^7 B_j}{dx^7}\frac{dB_i}{dx} - 4a^2 \frac{d^5 B_j}{dx^7}\frac{dB_i}{dx} + 6a^4 \frac{d^3 B_j}{dx^3}\frac{dB_i}{dx} - 4a^6 \frac{dB_j}{dx}\frac{dB_i}{dx} \right.$$

$$+\left(a^8 - La^4\right)B_i B_j\Big]dx$$

$$+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5 B_i}{dx^5}\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1} - \left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}$$

$$-4a^2\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3 B_i}{dx}\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3 B_i}{dx^3}\right]_{x=0}\right\}$$

$$-4a^2\left\{\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}\right\}$$

$$+6a^4\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}\right\} \quad (5.22a)$$

$$E_{i,j} = \int_0^1 \left[\, a^4\, B_i B_j\, dx\right] \quad (5.22b)$$

The numerical evaluations of first nine critical Rayleigh numbers for various Spectral methods (SCP, SLP, CC) as well as a Galerkin WRM and two sets of values of parameters a and L are exhibited in table 5.3. Here SCP, SLP, CC imply respectively the shifted Chebychev polynomial method, the shifted Legendre polynomial method and Chebychev collocation method. In table 5.3, relative errors have been depicted and are compared with the other numerical methods. We observed that the relative errors calculated by our current method are much smaller in magnitude than those of Chebychev and Legendre Spectral (SCP and SLP) methods. All the nine critical Rayleigh numbers computed by our proposed scheme converge to the analytical ones with fewer degree of polynomials (n=10). However, the parameter R achieved by collocation method is lowered severely in the CC method. from the analytical result.

From the comparison we conclude that our present scheme is much efficient and competes well with the other available techniques. Relative errors are presented graphically which is displayed in figure 5.1. It is also eminent that non-normality is responsible for a high spectral sensitivity. In this example non-normality ratio attains by our proposed Galerkin WRM executing the formula illustrated in chapter 1, is around 0.9 and does not depend upon the degree of polynomials in the range $10<n<40$. Figure 5.2 displays the curve $log_{10}(\text{cond }(A))$ versus $2<a<9$ for the degree of polynomials n= 20 and n=50 for example 5.2. It is noticed that condition numbers of the eigenvalues for different values of wave numbers using twenty polynomials varies between $O\left(10^{5}\right)$ to $O\left(10^{8}\right)$ and using fifty polynomials varies between $O\left(10^{35}\right)$ to $O\left(10^{37}\right)$ and are increasing for the increasing a.

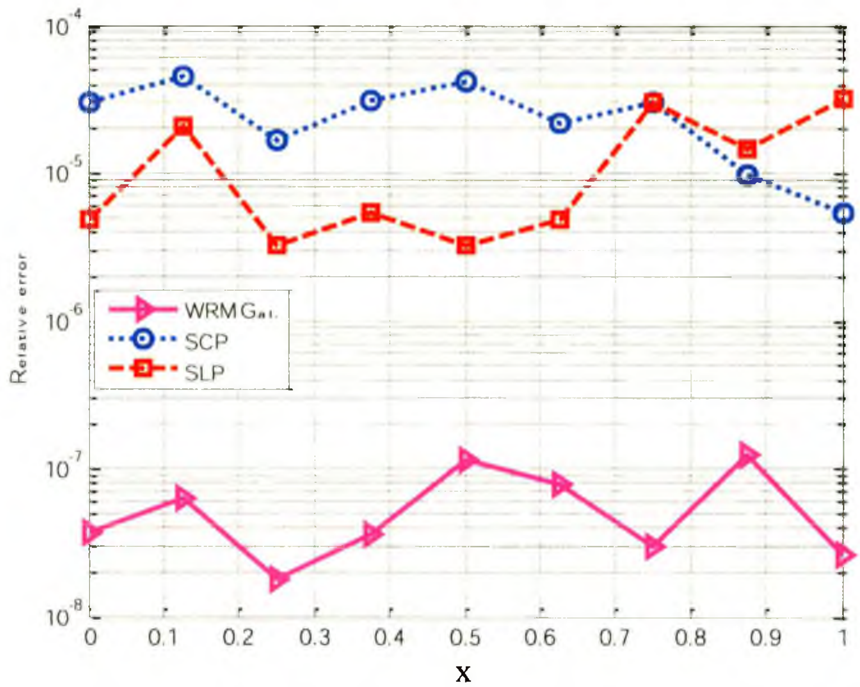

Figure 5.1: Relative errors using Galerkin WRM, Chebychev Spectral Galerkin and Legendre Spectral Galerkin for example 5.2.

Table 5.1: The first six estimated eigenvalues attained by Galerkin WRM exploiting Bernstein polynomials of example 5.1.

k	GAL WRM Bernst./Legn λ_k	Taher *et al* (2014) ADM λ_k	Absolute error $\lvert\lambda_{ADM}-\lambda_{GAL}\rvert$
1	1.00000000000000	1.000000000000000	0.0000000
2	256.0000000000000000	256.0000000000000000	0.0000000
3	6561.0000000000000000	6561.0000000000000000	0.0000000
4	65536.0000000000000000	65536.0000000000000000	0.0000000
5	390625.000000000000000	390625.000000000000000	0.0000000
6	1679616.0000000000000000	1679616.0000000000000000	1.94×10^{-9}

Example 5.3: We consider the eighth order ordinary differential equation worked out in [Twizel *et al,* 1990].

$$\left(D^8-j_1D^6+j_2D^4-j_3D^2+j_4\right)w(x)+i\mu\left(-j_5D^6+j_6D^4-j_7D^2+j_8\right)w(x)$$

$$+RA^2\left(D^2-A^2\right)w(x)-i\mu RA^2w(x)=0,\qquad 0<x<1 \qquad (5.23a)$$

The corresponding free-free boundary conditions in the book written by Chandrasekhar (1981) are to be imposed given by

$$w^{(2i)}(0)=w^{(2i)}(1)=0\ ,\quad i=0,1,2,3 \qquad (5.23b)$$

where,

$j_1=4A^2$, $j_2=6A^4-\mu^2(2p_1+1)+T$, $j_3=4A^6-2A^2\mu^2(2p_1+1)+A^2T$,

$j_4=A^8-A^4\mu^2(2p_1+1)$, $j_5=(p_1+2)$, $j_6=3A^2(p_1+2)$,

$j_7=3A^4(p_1+2)\mu^2p_1+p_1T$, $j_8=A^6(2p_1+2)-A^2\mu^2p_1$

$$\sum_{i=1}^{n-1}\left[F_{i,j}-R\,E_{i,j}\right]c_i=0 \qquad (5.24)$$

Table 5.2(a) The first six estimated eigenvalues attained by Galerkin WRM exploiting Bernstein polynomials of example 5.1

Bernstein polynomials						
n	λ_1	λ_2	λ_3	λ_3	λ_5	λ_6
16	1.00000000000000456	256.000000000000221	6561.000000000404011	65536.00005956873899856	390625.0089805654382166	1679671.2780190270422426
18	0.9999999999999280	255.999999999999982	6561.000000000402162	65536.0000000835243533	390625.0089805654140354	1679616.4873880163182179
19	1.0000000000011872	256.0000000000017249	6561.0000000000010882	65536.0000000835195493	390625.000261192226597	1679616.4873880191287340
20	1.0000000000254660	256.0000000000241267	6561.0000000000670183	65536.0000000003309337	390625.000026120005755	1679616.002538130481342
21	0.9999999997787389	255.9999999999555118	6560.9999999999002290	65535.9999999982809225	390625.0000000399743087	1679616.002538450388811
22	0.9999999997508210	55.99999999738771738	6560.9999999966856964	65535.99999998350093297	390625.000000061226846	1679616.000078773054768

Table 5.2(b) The first six estimated eigenvalues attained by Galerkin WRM exploiting Legendre polynomials of example 5.1

Legendre polynomials						
n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
18	1.000000000000000	256.0000000000000000	6561.0000000000404196	65536.0000000835285917	390625.0089805654413812	1679616.4873880172746142
19	1.0000000000000000000	256.0000000000000000	6561.0000000000000132	65536.0000000835285916	390625.0000261190436213	1679616.4873880172746068
20	1.0000000000000000000	256.0000000000000000	6561.000000000000013	65536.0000000000729972	390625.0000261190436218	1679616.0025381491233865
21	1.0000000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000729980	390625.0000000478114173	1679616.0025381491245036
22	0.99999999977873900	255.9999999999555112	6560.9999999999002290	65535.9999999982809224	390625.0000000399743087	1679616.0025384503888117
23	1.0000000000000000027	256.0000000000000000	6561.0000000000000029	65536.0000000000000752	390625.0000000000574540	1679616.000008380593335

Table 5.3: Numerical estimates for the Rayleigh number for various values of a and L applying various numerical techniques for example 5.2.

a	L	$R_{analytical}$	Bernstein *Gal.* *n=10*	Legendre *Gal.* *n=10*	R_{SCP} $N=6$	R_{SLP} $N=6$	CC $2^4 \le N \le 2^8$	Rel error Gal Bernstein	Rel error Gal Legendre	Rel error R_{SCP}	Rel error R_{SLP}
2	0	667.0098	667.0098	667.0098	667.030	667.013	667.0092	3.648×10^{-8}	2.999×10^{-8}	3.0284×10^{-5}	4.7975×10^{-6}
$\sqrt{4.92}$	0	657.5133	657.5133	657.5133	657.543	657.527	657.5133	6.345×10^{-8}	6.345×10^{-8}	4.5170×10^{-5}	2.0836×10^{-5}
3	0	746.5276	746.5276	746.5276	746.54	746.530	746.5276	1.802×10^{-8}	1.802×10^{-8}	1.6610×10^{-5}	3.2149×10^{-6}
2	1	666.7214	666.7214	666.7214	666.742	666.725	666.7214	3.584×10^{-8}	3.584×10^{-8}	3.0897×10^{-5}	5.3996×10^{-6}
$\sqrt{4.92}$	1	657.1806	657.1807	657.1807	657.208	657.192	657.1892	1.151×10^{-7}	1.151×10^{-7}	4.1693×10^{-5}	1.7347×10^{-5}
3	1	746.0506	746.0506	746.0507	746.067	746.053	746.0506	7.495×10^{-8}	7.495×10^{-8}	2.1982×10^{-5}	3.2169×10^{-6}
2	10	664.1258	664.1258	664.1258	664.146	664.129	664.1258	3.003×10^{-8}	1.234×10^{-7}	3.0416×10^{-5}	4.8184×10^{-6}
$\sqrt{4.92}$	10	654.1866	654.1866	654.1867	654.193	654.177	654.1866	1.234×10^{-7}	3.003×10^{-8}	9.7831×10^{-6}	1.4675×10^{-5}
2	16	662.3954	662.3954	662.39541	662.399	662.416	35.8873	2.613×10^{-8}	2.613×10^{-8}	5.4348×10^{-6}	3.1099×10^{-5}

where,

$$F_{i,j}=\int_0^1\left\{\left[\left[-\frac{d^7B_j}{dx^7}\frac{dB_i}{dx}+\left(j_1+i\mu j_5\right)\frac{d^5B_j}{dx^5}\frac{dB_i}{dx}-\left(j_2+i\mu j_6\right)\frac{d^3B_j}{dx^3}\frac{dB_i}{dx}\right.\right.\right.$$

$$\left.\left.+\left(j_3+i\mu j_7\right)\frac{dB_j}{dx}\frac{dB_i}{dx}+\left(j_4+i\mu j_8\right)B_iB_j\right]\right\}dx+\left(j_2+i\mu j_6\right)\left[\frac{d^4B_j}{dx^4}\frac{dB_i}{dx}\right]_{x=0}^1$$

$$+\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5B_i}{dx^5}\right]_{x=0}^1+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}^1+\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^1$$

$$-\left(j_1+i\mu j_5\right)\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}^1+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}^1\right\} \quad (5.24a)$$

$$E_{i,j}=-\left\{A^2\left[\frac{dB_i}{dx}B_j\right]_{x=0}^1+A^2\int_0^1\frac{dB_j}{dx}\frac{dB_i}{dx}dx+\left(A^4+i\mu A^2\right)B_iB_jdx\right\} \quad (5.24b)$$

The results are depicted in table 5.4 in comparison with other numerical techniques show that the present method is compatible and much efficient.

Example 5.4.: Consider the Sturm-Liouville problem worked out by Taher *et al* (2014).

$$\frac{d^{10}u}{dx^{10}}=-\lambda u(x) \quad (5.25)$$

subject to the boundary conditions

$$\begin{cases}u(0)=u^2(0)=u^4(0)=u^6(0)=u^8(0)=0\\ u(\pi)=u^2(\pi)=u^4(\pi)=u^6(\pi)=u^8(\pi)=0\end{cases} \quad (5.25a)$$

The first six eigenvalues utilizing WRM exploiting for equation (5.25) using Bernstein and Legendre polynomials along with their absolute error with ADM [Taher *et al* (2014)] are displayed in table 5.5.

Table 5.4: Estimated values of critical Rayleigh numbers utilizing Galerkin WRM for example 5.3.

n	T	A_{Gal}	A_{FDM}	A_C	μ_{Gal}	μ_{FDM}	μ_C	R_{Gal}	R_{FDM}	R_C
8	1.681×10^{4}	2.270	2.270	2.270	101.389	101.383	101.4	1388	1387.2	1388
8	1.68×10^{5}	2.594	2.594	2.594	307.901	307.881	307.9	1693.9	1693.9	1694
8	1.681×10^{6}	3.710	3.710	3.710	816.745	816.820	816.8	3436	3436.1	3436
8	1.681×10^{7}	5.698	5.698	5.698	1930.002	1930.237	1930.0	11023.1	11023.1	11020
8	1.681×10^{8}	8.626	8.626	8.626	4325.888	4326.506	4330.0	43679.9	43673.1	43680
CPU time								**2.023 seconds**	**2.82 seconds**	

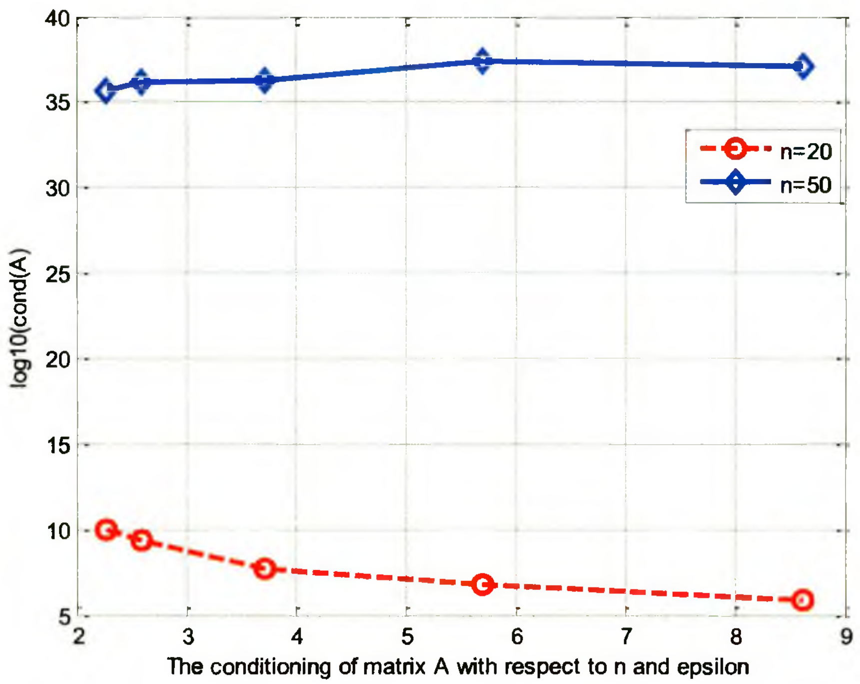

Figure 5.2: The conditioning of matrix *A* with respect to the wave number.

Table 5.5: The absolute errors for first six eigenvalues of example 5.4 attained by present (WRM) and ADM.

k	GAL WRM Bernst./Legn λ_k	Taher*et al* (2014) ADM λ_k	Absolute error $\lvert \lambda_{ADM} - \lambda_{GAL} \rvert$
1	1.000000000000000	1.0000000000000000	1.0e-00
2	1024.000000000000000	1024.0000000000000000	1.0e-00
3	59049.0000000000000000	59049.0000000000000000	1.0e-00
4	1048576.000000000000000	1048576.000000000000000	1.0e-00
5	9765625.000000000000000	9765625.000000000000000	1.460e-00
6	604666176.000000000000	604666175.99999945213	1.0492×10^{-8}

Example 5.5: We consider the tenth order eigenvalues studied by Twizell *et al* (1994)

$$\left(D^{10} - k_1 D^8 + k_2 D^6 - k_3 D^4 + k_4 D^2 - A^{10} \right) w(x) + RA^2 \left(D^4 + k_5 D^2 + A^4 \right) w(x) = 0, \quad (5.26)$$

$$0 < x < 1$$

$$k_1 = 5A^2 + 2Q \; k_2 = 10A^4 + 6A^2 Q + T + Q^2 \; k_3 = 10A^6 + 6A^4 Q + 2A^2 T + A^2 Q^2$$

$$k_4 = 5A^8 + 2A^6 Q + A^4 T, \quad k_5 = 2A^2 + Q$$

And the associated the free-free boundary conditions in Chandrasekhar (1981):

$$w^{(2i)}(0) = w^{(2i)}(1) = 0 \; ; i = 0,1,2,3,4 \quad (5.26a)$$

Equivalently, eigenvalues (Rayleigh numbers) for can be obtained by solving the system -of equation in matrix form as.

$$\sum_{i=1}^{n-1} \left[F_{i,j} + RE_{i,j} \right] c_i = 0 \quad (5.27)$$

where,

$$F_{i,j} = \int_0^1 \left[-\frac{d^9 B_j}{dx^9} \frac{dB_i}{dx} + k_1 \frac{d^7}{dx^7} \left[B_j(x) \right] \frac{dB_i}{dx} - k_2 \frac{d^5}{dx^5} \left[B_j(x) \right] \frac{dB_i}{dx} + k_3 \frac{d^3}{dx^3} \left[B_j(x) \right] \frac{dB_i}{dx} \right.$$

$$-k_4\frac{d}{dx}\left[B_j(x)\right]\frac{dB_i}{dx}-A^{10}B_i(x)B_j(x)\Bigg]dx+\left[\frac{d^2B_j}{dx^2}\frac{d^7B_i}{dx^7}\right]_{x=1}-\left[\frac{d^2B_j}{dx^2}\frac{d^7B_i}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^4B_j}{dx^4}\frac{d^5B_i}{dx^5}\right]_{x=1}-\left[\frac{d^4B_j}{dx^4}\frac{d^5B_i}{dx^5}\right]_{x=0}+\left[\frac{d^6B_j}{dx^6}\frac{d^3B_i}{dx^3}\right]_{x=1}-\left[\frac{d^6B_j}{dx^6}\frac{d^3B_i}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^8B_j}{dx^8}\frac{dB_i}{dx}\right]_{x=1}-\left[\frac{d^8B_j}{dx^8}\frac{dB_i}{dx}\right]_{x=0}-k_1\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5B_i}{dx^5}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^5B_i}{dx^5}\right]_{x=0}\right.$$

$$+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}+\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}$$

$$\left.-\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}+\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}-\left[\frac{d^6}{dx^6}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}\right\}$$

$$+k_2\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{d^3B_i}{dx^3}\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}\right.$$

$$\left.-\left[\frac{d^4}{dx^4}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[a_4(x)B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}\right\}$$

$$-k_3\left\{\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[B_j(x)\right]\frac{dB_i}{dx}\right]_{x=0}\right\} \qquad (5.27a)$$

$$E_{i,j}=A^2\int_0^1\left\{\left[\frac{d^3}{dx^3}\left[B_j(x)\right]\frac{dB_i}{dx}\right]-k_5\frac{d}{dx}\left[B_j(x)\right]\frac{dB_i}{dx}+A^4\left[B_iB_j\right]\right\}dx$$

$$+\left[\frac{d^2B_j}{dx^2}\frac{dB_i}{dx}\right]_{x=1} \qquad (5.27b)$$

The eigenvalue problem in equation (5.26) is solved using the formulation illustrated in this section. To compare the computed results demonstrated in Chandrasekhar (1981) and Twizell *et al* (1994), we can write the relationship between Q, Q_1 and T, T_1 as given below

$$Q_1 = \frac{Q}{\pi^2} \quad \text{and} \quad T_1 = \frac{T}{\pi^4}.$$

Numerical results are obtained using n=10, for the first six critical values of the problem 5.5 are depicted in figure 5.3. Computed results of R_{Gal} for the corresponding values of A obtained for T_1 =1000 and T_1=10000 are illustrated in tables 5.6(a) and 5.6(b).

Table 5.6(a): Estimated values of critical Rayleigh numbers employing WRM for T_1=1000 for example 5.5.

		Present method		Twizell (1990)		Chandrasekhar (1981)	
n	Q_1	R_{WRM}	A_1	R_1	A	R_C	A_C
10	10	2.016×10^4	7.90	2.016×10^4	7.90	2.016×10^4	7.90
10	50	1.605×10^4	4.50	1.604×10^4	4.50	1.605×10^4	4.50
10	100	1.952×10^4	5.23	1.951×10^4	5.22	1.952×10^4	5.23
10	500	6.378×10^4	7.47	6.377×10^4	7.46	6.380×10^4	7.47
10	1000	1.192×10^5	8.52	1.192×10^5	8.52	1.192×10^5	8.52
10	10000	1.065×10^6	12.80	1.065×10^6	12.80	1.065×10^6	12.80
10	50000	5.129×10^6	16.82	5.128×10^6	16.82	5.129×10^6	16.82
10	100000	1.005×10^7	18.94	1.015×10^7	18.94	1.015×10^7	18.94
		2.306 seconds		**1.94 seconds**			

The observed CPU time is also displayed. We have noticed from the results in table 5.6(b) that the actual minima depends on Q_1.

Two minima at A=3.51, minimum value of R =1713.60877 at A=3.54, minimum value of R =1713.6188.

It has been observed that for $Q_1 = 80$, $T_1 = 100000$, the minimum values of R and the corresponding values of A using n=10 are given below

R_{Gal}= 397721.411234870506 with A=18.3, R_{Gal}= 478101.133947 with A=3.38.

For $Q_1 = 100$, $T_1 = 100000$,

R_{Gal}=397672.43782 with A=18.2, R_{Gal} = 393296.2090005; A=3.37.

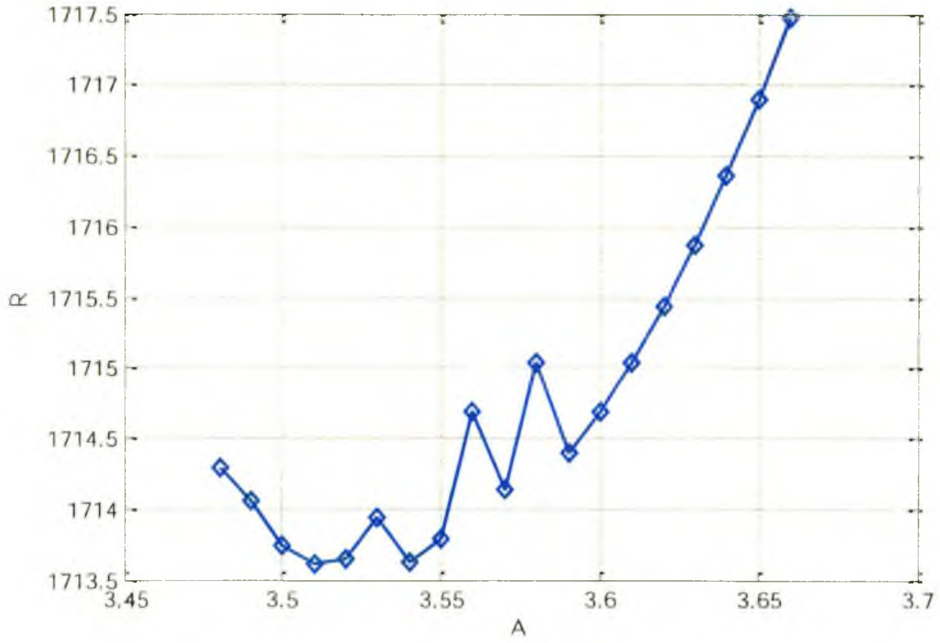

Figure 5.3: Estimated values of Critical Rayleigh numbers for example 5.5.

This situation is explained in Twizell *et al* (1994). For a given value of T_1, the cells appearing at marginal stability are elongated as the strength of the magnetic field increases. When the magnetic field reaches a certain value (100 for T_1=10000), cells of two different sizes will appear simultaneously. One set will be highly elongated and another set will relatively be highly flattened. If the strength of the magnetic field increases still further, the critical Rayleigh number will begin to decrease and pass through a minimum; eventually the inhibition due to the magnetic field will predominate.

Table 5.6(b): Estimated values of Critical Rayleigh numbers employing WRM for T_1=10000 for example 5.5.

Eigen index k	Degree of polyn. n	Q_1	Present method R_{WRM}	A	Twizell *et al* (1994) R_1	A_1	Chandrasekhar(1981) R_C	A_C
1	8	10	8.979×10^4	12.59	8.977×10^4	12.59	8.979×10^4	12.59
2	8	50	8.550×10^4	11.40	8.548×10^4	11.40	8.550×10^4	11.40
3	8	100	8.118×10^4	3.68	8.116×10^4	3.68	8.118×10^4	3.68
4	8	500	5.544×10^4	3.91	5.542×10^4	3.91	5.544×10^4	3.91
5	8	1000	7.545×10^4	6.50	7.543×10^4	6.51	7.545×10^4	6.50
6	8	10000	1.267×10^5	7.98	1.267×10^5	7.98	1.26×10^5	7.98
7	8	50000	1.067×10^6	12.80	1.066×10^6	12.77	1.067×10^6	12.80
8	8	100000	1.011×10^7	18.94	1.015×10^7	18.93	1.015×10^7	18.94

Example 5.6: We consider the following twelfth order eigenvalue problem illustrated in the article Twizell *et al* (1994)

$$\left(D^{12}-l_1D^{10}+l_2D^8-l_3D^6+l_4D^4-l_5D^2+l_6\right)w(x)$$
$$+i\left(-l_7D^{10}+l_8D^8-l_9D^6+l_{10}D^4-l_{11}D^2+l_{12}\right)w(x)$$
$$+i\,RA^2\left(-l_{16}D^4+l_{17}D^2-l_{18}\right)w(x)=0 \qquad (5.28)$$

The corresponding free-free boundary conditions illustrated in Chandrasekhar (1981):

$w^{(2i)}(0)=w^{(2i)}(1)=0$; $i=0,1,2,3,4,5$ are to be imposed. The coefficients l_i $(i=1,2,3,........,18)$ are given by

$$l_1=6A^2+2Q$$

$$l_2=15A^4+8QA^2+Q^2-\left[2p_2+2p_1(1+p_2)+(1+p_2)+(1+p_2)^2\right]\mu^2+T$$

$$l_3=20A^6+120QA^4-4\left[2p_2+2p_1(1+p_2)+(1+p_2)^2\right]\mu^2A^2+3TA^2+2Q^2A^2$$
$$-2(p_1+p_2+p_1p_2)\mu^2,$$

$$l_4=15A^8+8QA^6-6\left[2p_2+2p_1(1+p_2)+(1+p_2)^2\right]\mu^2A^2+3TA^2+2Q^2A^2,$$
$$-4(p_1+p_2+p_1p_2)\mu^2A^2+\left[p_2^2+2p_1p_2(1+p_2)\right]\mu^4-p_2(2p_1+p_2)T,$$

$$l_5=6A^{10}+2QA^8-4\left[2p_2+2p_1(1+p_2)+(1+p_2)^2\right]\mu^2A^4+2\left[p_2^2+2p_1p_2(1+p_2)\right]\mu^4A^2$$
$$-p_2(2p_1+p_2)T\mu^2A^2,$$

$$l_6=A^{12}-\left[2p_2+2p_1(1+p_2)+(1+p_2)^2\right]\mu^2A^8+\left[p_2^2+2p_1p_2(1+p_2)\right]\mu^4A^4$$
$$-p_2(2p_1+p_2)T\mu^2A^2,$$

$$l_7=(2+p_1+2p_2)\mu,$$

$$l_8=2(1+p_1+p_2)\mu Q+5(2+p_1+2p_2)\mu A^2,$$

$$l_9=10(2+p_1+2p_2)\mu A^4+6(1+p_1+p_2)\mu QA^2-\left[2p_1p_2+2p_2(1+p_2)+p_1(1+p_2)^2\right]\mu^3$$
$$+3(p_1+2p_2)T\mu+p_1\mu Q^2$$

$$l_{10}=10(2+p_1+2p_2)\mu A^6+6(1+p_1+p_2)\mu QA^4+2(p_1+2p_2)T\mu A^2$$

$$-3\left[2p_1p_2+2p_2(1+p_2)+p_1(1+p_2)^2\right]\mu^3A^2$$

$$l_{11}=5(2+p_1+2p_2)\mu A^8+2(1+p_1+p_2)\mu QA^6$$

$$-3\left[2p_1p_2+2p_2(1+p_2)+p_1(1+p_2)^2\right]\mu^3A^4$$

$$-3\left[2p_1p_2+2p_2(1+p_2)+p_1(1+p_2)^2\right]\mu^3A^2$$

$$+(p_1+2p_2)T\mu A^4+p_1p_2{}^2\mu^5-p_1p_2^2\mu^3T$$

$$l_{12}=(2+p_1+2p_2)\mu A^{10}-\left[2p_1p_2+2p_2(1+p_2)+p_1(1+p_2)^2\right]\mu^3A^6+p_1p_2^2\mu^5A^2$$

$$l_{13}=3A^2+Q,\ l_{14}=3A^4+QA^2-p_2(2+p_2)\mu^2,\ \ l_{15}=A^6-p_2(2+p_2)\mu^2A^2,$$

$$l_{16}=(1+2p_2)\mu$$

$$l_{17}=2(1+2p_2)\mu A^2+p_2\mu Q$$

$$l_{18}=(1+2p_2)\mu A^4-p_2^2\mu^3$$

Here $Q_1=\dfrac{Q}{\pi^2}$, $T_1=\dfrac{T}{\pi^4}$ and $\sigma=\dfrac{\sigma_1}{\pi^2}$

Numerical results are obtained applying Galerkin WRM for increasing Taylor number and same magnetic field intensity are illustrated in tables 5.7(a), 5.7(b) and 5.7(c), which also the corresponding results reported by Chandrasekhar (1981).

Comparisons with tables 5.7(a), 5.7(b), 5.7(c) reveals that our present methods attain very similar results except those in table 5.7(c) where the results are slightly deviated from the reported results. We conclude that for higher Taylor number, the computed results become slightly less accurate than those of Chandrasekhar (1981).

Table 5.7(a): Computed values of R, A_1 and σ for T_1 =1000 of example 5.6

		Computed results (present)			Twizell *et al* (1994)			Chandrasekhar (1981)		
n	Q_1	R	A	σ	R_1	A_1	σ_1	R_c	A_c	σ_c
16	10	7053	4.56	53.52i	7053	4.55	53.61i	7053	4.56	53.52i
38	100	35398	6.63	30.13i	35390	6.62	30.13i	35402	6.63	30.12i
18	500	50156	7.20	18.44i	50140	7.20	18.43i	50156	7.20	18.44i

Table 5.7(b) : Computed values of R, A_1 and σ , T_1 = 100000 of example 5.6

		Computed results present			Twizell *et al* (1994)			Chandrasekhar (1981)		
n	Q_1	R	A	σ	R_1	A_1	σ_1	R_c	A_c	σ_c
10	10	12839	6.03	140.80i	12839	6.02	53.61i	12840	6.03	140.80i
38	50	26772	7.01	12.43i	35390	7.05	120.43i	26790	7.05	120.40i
38	500	155675	9.84	61.95i	155635	9.88	61.45i	155700	9.88	61.95i

Table 5.7(c): Computed values of R, A_1 and σ for T_1 = 1000000 of example 5.6

		Computed results (Galerkin WRM)			Twizell *et al* (1994)			Chandrasekhar (1981)		
n	Q_1	R	A	σ	R_1	A_1	σ_1	R_c	A_c	σ_c
38	10	36495	8.35	342.45i	36390	8.35	341.5i	36500	8.35	342.1i
40	500	189238	11.50	236.14i	189209	11.55	236.14i	189300	11.54	236.3i
12	1000	3,26460	12.64	202.34i	326,666	12.64	202.45i	327100	12.64	202.5i

5.5 Conclusions

We observe that the absolute errors between the current method and the ADM are very negligible in tables 5.1 and 5.5. Our computed results in tables 5.2 obtained by using Galerkin WRM are illustrated in comparison with the analytical ones showing a very good agreement with the other existing methods. From tables 5.3, 5.4 and tables 5.6-5.7, it has been noticed that, estimated critical Rayleigh numbers corresponding to the fixed wave numbers are much closer to Chandrasekhar (1981) and Twizell *et al* (1994) which shows moderated performance in the sense of accuracy and applicability of our proposed method. Moreover, the cost of computations is much lesser which shows that our proposed Galerkin scheme is much economical. Although tables for twelfth order eigenvalue problems show little difference between the computed critical values of A and σ from those of Chandrashekhar (1981) but they show that computed values are lower and therefore predicts the onset of instability as overstability.

Appendix

ν	:	kinematic viscosity,
κ	:	thermometric conductivity,
μ	:	magnetic permeability,
σ	:	time constant (relative to dimensionless time and space coordinates),
η	:	$1/(4\pi\sigma\mu)$
H	:	uniform magnetic field,
d	:	depth of layer of fluid,
ρ	:	density of fluid,
Ω	:	angular velocity,
α	:	coefficient of volumetric expansion

β	:	adverse temperature gradient,
A	:	wave number,
g	:	acceleration due to gravity,
Q	=	$\mu H^2 d^2 / (4\pi \rho \nu \eta)$
T	=	$4\Omega^2 d^4 / \nu^2$: the Taylor number,
R	=	$g\alpha\beta d^4 / (\kappa\nu)$: Rayleigh number,
R_c	:	Critical Rayleigh number,
p_1	:	$\frac{\nu}{\kappa}$: Prandtl number,
p_2	=	$\frac{\eta}{\kappa}$,
x	:	dimensionless vertical co-ordinate,
w	=	$w(x)$: vertical co-ordinate.

Conclusions

Many physical systems lead to Sturm-Liouville problems and the numerical methods for solving these problems have several applications in physics, applied mathematics, engineering and applied sciences. Eigenvalue arises routinely in the linear stability analysis of 2-D incompressible flow. The main focus of this study is on the novel numerical methods and approaches which positively have potential value for researchers of Sturm-Liouville boundary value problems. Our aim is to compute eigenvalues of higher even order linear boundary value problems (from fourth up to twelfth order) in one dimension applying the technique of Galerkin weighted residual exploiting polynomials as basis function namely Bernstein and Legendre. Additionally, second, fourth and sixth order eigenvalue problems have been examined, analyzed and accomplished. Estimated eigenvalues applying weighted residual Galerkin, collocation and Spectral collocation methods have been compared with other available analytical/numerical methods.

In Chapter 1, we have conferred some definitions, theorems, lemmas, approximation theory on eigenvalues, existence and uniqueness of Sturm-Liouville problems etc. Furthermore, some important properties of Bernstein and Legendre polynomials, convergence of Bernstein polynomials, some familiar mathematical formulas crucial for the thesis have been demonstrated. We have exemplified different kinds of boundary conditions for both regular and singular SLEs which are vital for numerical applications as well.

Chapter 2 has dealt with eigenvalue calculation of second order Sturm-Liouville problems applying WRM Galerkin, WRM of collocation and Cheby-Legendre Spectral collocation technique. The technique is also extended for solving non-linear Bratu type problems.

The computed results have been presented in both tabular and graphical forms. Among the three said methods, Galerkin method estimates the eigenvalues of the lower spectrum efficiently whereas Spectral collocation computes all the eigenvalues competently with desired accuracy. The approximate eigenvalues converge to the exact solutions even with desired large significant digits with the increase of discretization points. Conversely, Bernstein collocation method is less convergent than the other two aforementioned methods for some particular problems. Since Bernstein polynomials vanish at the two end points of the interval which gives greater flexibility and is found to be more attractive for implementing weighted residual methods.

In Chapter 3 we have developed matrix formulations for the numerical computation of linear fourth order Sturm-Liouville problems employing two types of boundary conditions namely clamped and hinged. WRM Galerkin, Bernstein collocation and Spectral collocation methods have been utilized for these purposes. Among the three methods, WRM Galerkin gives much better accuracy even with fewer number of polynomials than the Bernstein collocation method for some problems. Still the method becomes less popular for huge number of eigenvalue computations and typically lost its accuracy for degree of polynomials greater than 40. It is apparent that computational time requires slightly more for Legendre Galerkin than those of Bernstein Galerkin technique for calculation of eigenvalues. Eigenvalues obtained by Bernstein collocation converges faster than Galerkin WRM. Numerical stability has been illustrated graphically by the condition number of differentiation matrices achieved by the two methods (Bernstein collocation and Spectral collocation). From graphical illustrations we revealed that WRM of collocation gives moderated size of condition number in comparison to Galerkin WRM and Spectral collocation method. The shortcoming of the Bernstein collocation method is that, in case of huge number of eigenvalues computation, higher eigenvalues are less convergent than the lower spectrum and with the increasing of the degree of polynomials the computational time highly increases, without leading to a significant improvement of the numerical values for some higher order problems. Although the slow convergent rate of Bernstein polynomials for some particular problems with complicated boundary conditions makes it less popular, still this drawback is to be compensated for achieving better accuracy on using fewer number of polynomials. Conversely numerical investigations exhibit the fact that Spectral collocation method is well suited for huge number of eigenvalues in higher spectrum with superior accuracy. From these comparisons we found that the eigenvalues obtained by the existing methods compete very well with other numerical methods studied by various authors.

Chapter 4 has been provided for approximating the eigenvalues of linear sixth order BVPs together with Benard convection problems employing the WRM Galerkin, collocation and Spectral collocation method using the polynomials as mentioned before. All the eigenvalues calculated by Galerkin method converge to the exact results and the relative errors are almost negligible due to machine precision. In the case of eigenvalue problem of circular ring structure, we have noticed that Galerkin WRM compensates for the difficulties that arise in the execution of the other discretization methods by giving desired results even with a few number of polynomials. Similar precision has been achieved applying Spectral collocation technique for the same problem with the increase of grid points. It has been noticed that the critical values of Benard Layer problem using Galerkin WRM method with Bernstein polynomials show better performance and the eigenvalues attained are smaller and fairly close to the analytical results. Moreover, the smallest Rayleigh number converges to large significant figure and relative errors obtained in tabular form demonstrate that the current methods are superior than the other numerical available studies. From the graph of differentiation matrices, it is clear that current Cheby-Legendre collocation method is more ill-conditioned than that of Bernstein collocation and the other Chebychev Spectral method.

Since the unknown coefficients in collocation method are expressed in terms of known coefficients of the boundary conditions and thus handling boundary conditions is much easier. Collocation method computes lower eigenvalues more efficiently and obtains relatively well–conditioned matrix than those of Spectral collocation method. This reveals that method is stable. We also observed that collocation matrix has much better conditioning than Galerkin matrix as the degree of polynomials increase.

Furthermore, this method with the aid of Matlab 13 code is well suited for both regular as well as singular Sturm-Liouville problems. In spite of these shortcomings, the present Spectral method achieves superior accuracy, is computationally efficient and much competent with the other earlier published works. Finally, the computational stable convergence for some eigenvalue problems is achieved.

We have introduced a novel means of incorporating boundary conditions in Legendre Spectral collocation method, which removes the ambiguity of transforming higher-order problems into lower order derivatives. We have tackled the longstanding issue with ill-conditioning of Spectral methods from a new perspective. Increasing the number of subintervals or the number of collocation points in subintervals results in improved accuracy. It has been found that the relative errors in absolute for some specific problems, but are better than other reported ones in the literature. The choice between a collocation method and a Galerkin method is problem dependent, and the advantage of collocation methods is more noticeable for problems with more complicated forms of governing equations.

The co-efficient matrix in Weighted residual Galerkin and collocation method is sparse and has symmetric banded matrix, which minimizes the computational effort and attains relatively smaller condition numbers. Although spectral methods attain high accuracy, and these eventually lead to ill-conditioned system. On the other hand, Spectral matrices are full and non-symmetric. Eigenvalues obtained by applying Galerkin method perform much better and more accurate than the collocation method for most of the numerical experiments. Furthermore, the current Spectral method requires less CPU time in comparison to Galerkin WRM and collocation method. The method can be considered an effective and reliable tool for solving eigenvalues sixth-order boundary value problems.

In our study, we have contributed a numerical method for computing of the approximate eigenvalues/critical numbers of regular eighth, tenth and twelfth order eigenvalue problems in Chapter 5. Bernstein and Legendre polynomials have been exploited as basis and all the derivative boundary conditions have been incorporated directly in the residual equation without reducing the order of the differentiation. Implementing the boundary conditions is much simpler and easier in these problems. Stability or instability of hydrodynamic and hydro magnetic system can be comprehended by a set of non-dimensional parameters. The present numerically computed critical Rayleigh numbers of the suggested method have been compared with the other existing methods. It has been observed that all the critical numbers are smaller than some given ranges of the critical numbers which predict the onset of instability as over stability.

The linear stability of electro-hydro dynamic (EHD) convection between two parallel walls was efficiently solved by exploiting Galerkin WRM. Galerkin matrix behave better than both Chebychev tau and collocation matrix since the Galerkin WRM produces more normal discretization matrix. Besides the obtained matrix in the later case is symmetric and their non-normality ratio agrees well with the other reported works. The main disadvantage of Chebychev Spectral, tau, Galerkin as well as collocation, comprises non-uniform weight associated with the Chebychev polynomials which introduce significant difficulties in the analysis of the Chebyshev spectral method. As opposed to the Chebyshev polynomials, the main advantage of Legendre polynomials is that they are mutually orthogonals in the standard L^2 inner product with unit weight, so the analysis of Legendre spectral methods is much easier than that of the Chebyshev Spectral method. The main shortcoming is that there is no practical fast discrete Legendre transform available. However, we have taken the advantage of both the Chebyshev and Legendre polynomials by constructing the so called Chebyshev-Legendre Spectral methods.

We intend to extend our future work on Bernstein collocation and Cheby-Legendre Spectral collocation methods from eighth up to twelfth order eigenvalue problems by changing the higher order BVPs into lower order BVPs. We also expect to develop some innovative numerical techniques, related theories, rigorous analysis of the differential eigenvalue problems and their convergence behavior, upper and lower bounds of the eigenvalues etc. in our future work. Furthermore, several boundary value problems which are frequently applied as physical problems in science and engineering for computing eigenvalues can be handled using the said techniques in multi-dimensions.

References

[1] Abbasbandy S, Shirazdi A., (2011)-A new application of the homotopy analysis method: Solving the Sturm-Liouville problems. *Communications in Nonlinear Science and Numerical Simulation*, **16**,112-126.

[2] Agarwal R. P., (1986)-Boundary value problems for higher order differential equations (World Scientific, Singapore).

[3] Akyuz Dascıoglu A., Isler N., (2013)-Bernstein collocation method for solving nonlinear differential equations *Mathematical Computational Applications*, **18** (3), 293–300.

[4] Allame M., Ghasemi H., Tavassoli Kajani M., (2015)-An Appropriate Method for Eigenvalues Approximation of Sixth-order Sturm-Liouville Problems by Using Integral Operation Matrix over the Chebyshev Polynomials. AIP Conference Proceedings 1684, 090001 doi: 10.1063/1.4934326.

[5] Akyuz-Dascıoglu A., Isler N., Guler C., (2014)-Bernstein Collocation method for solving nonlinear Fred Holm-Volterra integro-differential equations in the most general form, *Journal of Applied Mathematics*, **2014**, 1-8.

[6] Ahmad A-K., Saleh M., (2011)-Solutions of twelfth order boundary value problems using adomian decomposition method, *Journal of Applied Sciences Research,* **7**(6), 922-934.

[7] Al Mdalla Q.M., Syam M.I., (2014)-The Chebyshev collocation-path following method for solving sixth order Sturm-Liouville problems, *Appl. Math.Comput.,***232**, 391–398.

[8] Alipour M., Rostamy D., (2011)-Bernstein polynomials for solving Abel's integral equation, *The Journal of Mathematics and Computer Science*, **3**(4), 403 – 412.

[9] Alquran M.T., Al-Khaled K., (2010)–Approximation of Sturm-Liouville Eigenvalues Using Sinc Galerkin and Differential Transform Methods, *Application and Applied Mathematics*, **5**(1), 128-147.

[10] Amodio P., Settanni G., (2015)–Variable-step finite difference schemes for the solution of Sturm–Liouville problems, *Commun. Nonlinear Sci. Numer. Simulat,* **20**, 641–649.

[11] Attili B., Lesnic D., (2006)-An efficient method for computing eigen-elements of

Sturm Liouville fourth-order boundary value problems, *Applied Mathematics and Computation*, **182**:1247-1254.

[12] Aregbesola A. S., (2003)–Numerical solution of Bratu problem using the method of weighted residual, *Electronic Journal of Southern African Mathematical Sciences*, **3**, 1–7.

[13] Atkinson, Kendall E., (1989) –An Introduction to Numerical Analysis, John Wiley & Sons, NY, 2nd Edition,

[14] Auzinger W., Koch O., Karner E, Weinmuller E., (2006)–Collocation methods for solution of eigenvalue problems for singular ordinary differential equations, *Opuscula Mathematica*, **26**, 229-241.

[15] Bailey, P. B., Gordon, M. K., Shampine L. F., (1978)–Automatic solution of the Sturm-Liouville problem. *ACM Trans. Math. Softw*. **4**, 193–207.

[16] Bailey P. B., Everitt W. N., Zettl A., (1991)-Computing Eigenvalues of Singular Sturm-Liouville Problems, *Results in Mathematics*, **20,** 391-423.

[17] Baily, P. B., Everitt, P. B., and Zettl A., (2001)–The SLEIGN2 Sturm-Liouville code. *ACM Trans. Math. Software 21,* 143–192.

[18] Baldwin, P., (1987)-Asymptotic estimates of the eigenvalues of a sixth-order boundary-value problem obtained by using global phase-integral methods. Philos. Trans. Royal Soc. Lond. Ser. A **322**(1566), 281–305.

[19] Bhatti M I and Bracken P., (2007)–Solutions of differential equations in a Bernstein polynomial basis, J. Comput. Appl. Math., **20**, 272 – 280.

[20] Bhrawy A. H., (2009)–Legendre-Galerkin method for sixth-order boundary value problems, *Journal of the Egyptian Mathematical Society,* **17**(2), 173–188.

[21] Bratu G., (1914)-Sur les equations integrales non lineaires, *Bull. Soc. Math.* France **42**, 113–142.

[22] Bruce A. Finlayson (1972)–The method of Weighted residual and Variational principles with application in fluid mechanics, heat and mass transfer. Academic press, New York and London. Eigenvalue and Boundary Value Problems, Wiley India (P) Ltd., 2011.

[23] Caglar H., Caglar N., Özer M., (2010)–Valaristos A., Anagnostopoulos A. N., B-spline method for solving Bratu's problem, *International Journal of Computer*

Mathematics, **87**(8),1885-1891.

[24] Canosa J., Gomes De Oliveira J., (1970)-A new method for the solution of the Schr¨odinger equation, *Journal of Computational Physics*, **5**, 188-207.

[25] Celik I, Gokmen G., (2005) -Approximate solution of periodic Sturm-Liouville problems with Chebyshev collocation method. *Appl. Math. Comput*, **170**(1): 285–295- 10.1016/j.amc.2004.11.038.

[26] Celik I., (2005).-Approximation of eigenvalues with the method of weighted residuals collocation method, *Applied Mathematics and Computation*, **160**,401-410.

[27] Chanane, B., (1998) -Eigenvalues of Fourth Order Sturm-Liouville Problems, Using Fliess Series, *J. of Computational and Applied Mathematics*, **96,** 91-97

[28] Chanane B., (2002)-Fliess Series Approach to the Computation of the Eigenvalues of Fourth Order Sturm-Liouville Problems, *Applied Mathematics letters*, **15**,459-463.

[29] Chanane B., (2005)- Computation of the eigenvalues of Sturm-Liouville Problems with parameter dependent boundary conditions using the Regularized Sampling Method, *Mathematics of Computation*, **74** (252), 1793–1801.

[30] Chen C-K., and Ho S-H., (1996)-Application of Differential Transformation to Eigenvalue problems, *Applied Mathematics and Computation,* **79**, 173-188.

[31] Chanane B., (2007)-Computing the eigenvalues of singular Sturm-Liouville problems using the regularized sampling method. *Applied Mathematics and Computation*, **184**(2), 972-978.

[32] Chanane B., (2010)-Accurate Solutions of Fourth Order Sturm-Liouville Problems, *Journal of Computational and Applied Mathematics*, **234,** 3064-3071.

[33] Chanane B., Boucherif A., (2014)-Computation of the eigenpairs of two Parameter Sturm-Liouville (SL) problems using the Regularized Sampling Method, Hindawi Publishing Corporation, *Abstract and Applied Analysis*, **2014**, Article ID 695303, 1-6, http://dx.doi.org/10.1155/2014/695303.

[34] Chandrasekhar S., (1981)-Hydrodynamic and Hydro Magnetic Stability, Clarendon Press Oxford 1961(Reprinted: Dover Books, New York.

[35] Chawla M., (1983)-A New Fourth Order Finite Difference Method for Computing Eigenvalues of Fourth-Order Linear Boundary Value Problems, *IMA Journal of*

Numerical Analysis, **3**, 291-293.

[36] Chawla, M. M, Shivakumar P.N., (1993)–A symmetric finite difference method for computing eigenvalues of Sturm-Liouville problems, *Comput. Math. Appl.*, **26**,67-77.

[37] Chen H., Shizgal B. D., (2001) –A spectral solution of the Sturm–Liouville equation: comparison of classical and non-classical basis sets, *Journal of Computational and Applied Mathematics,* **136**, 17- 35.

[38] Collatz, L. (1966a) –Functional Analysis and Numerical Mathematics, Academic Press, New York.

[39] Courant, R., Hilbert, D., (1953) –Methods of Mathematical Physics, Vol. I. Wiley (Interscience), New York.

[40] Doha E.H., Bhrawy A.H., Saker M.A. (2011)–On the derivatives of Bernstein polynomials: an application for the solution of high even-order differential equations, *Boundary Value Problems*, **2011**, doi:10.1155/2011/829543.

[41] Drazin P. G., (1974) –On a **Model of Instability** of a Slowly-Varying Flow, *The Quarterly Journal of Mechanics and Applied Mathematics*, **27**(1), 69–86.

[42] Dragomirescu F.I., Gheorghiu C.I., (2010) –Analytical and numerical solutions to an electro-hydrodynamic stability problem, *Applied Mathematics and Computation*, **216**, 3718- 3727.

[43] Estep D., (2002) –Practical Analysis in One variable, **624**, ISBN:978-0-387-95484-4. Springer Verlag, Newyork.

[44] Farouki R., Rajan V.(1987)–On the numerical condition of polynomials in Bernstein form, Computer Aided Geometric Design, **4**(3) , 191–216.

[45] Gamel Md., Sameeh M., (2012)–An Efficient Technique for Finding the Eigenvalues of Fourth Order Sturm-Liouville Problems, *Applied Mathematics,* **3**,920-925.

[46] Gheorghiu C.I., Dragomirescu F.I., (2009)–Spectral methods in linear Stability. Application to thermal convection with variable gravity field, Appl. Numer. Math., **59**, 1290–1302.

[47] Gaudreau P., Slevinsky R.M., Safouhi H., (2016)–The Double Exponential Sinc Collocation Method for Singular Sturm-Liouville Problems, *Journal of*

Mathematical Physics, (043505):1-19.

[48] Gelfand I. M., (1963)- Some problems in the theory of quasi-linear equations, *Amer. Math. Soc. Transl. Ser.* **2**(29), 295–81.

[49] Gaudreau P., Slevinsky R. M. and Safouhi H., (2015)-Double exponential sinc-collocation method for solving the energy eigenvalues of harmonic oscillators, *Annals of Physics*, **360,** 520-538.

[50] Gutierrez R.H., Laura P.A. A., (1995)-Vibrations of non-uniform rings studied by means of differential quadrature method, *J. Sound Vib*., **185** (3), 507-513.

[51] Huang Y., Chen J., Luo Q., (2013)-A simple approach for determining the eigenvalues of the fourth-order Sturm–Liouville problem with variable coefficients, *Applied Mathematics Letters*, **26,** 729-734.

[52] Isık O. R., Sezer M., and Guney Z., (2013)-Bernstein series solution of a linear second-order partial differential equations with weakly with mixed conditions, *Mathematical Methods in the Applied Sciences*, **2014** (37), 609–619.

[53] Isik O., R., Sezer M.,(2013)- Bernstein series solution of a class of Lane-Emden Type Equations, *Mathematical Problems in Engineering,* **2013** Article ID 423797, 1-9.

[54] Islam M.S., Hossain M.B., (2015)-Numerical solutions of eighth order BVP by the Galerkin residual technique with Bernstein and Legendre polynomials, *Applied Mathematics and Computation,* **261**, 48-59.

[55] Islam M.S., Hossain M.B., Rahman M.A., (2015)-Numerical Approaches for Tenth and Twelfth Order Linear and Nonlinear Differential Equations, *British Journal of Mathematics and Computer Science*, **5** (5), 637 – 653.

[56] Jalilian R., (2010)- Non polynomial Spline method for solving Bratu's problem, *Computer Physics Communications*, **181**(11), 1868-1872.

[57] J¨uttler B., (1998)-The dual basis functions for the Bernstein polynomials, 8 *Advances in Computational Mathematics* **8**, 345–352.

[58] Jia G., Song Y., Li Q., (2005)-Calculation of eigenvalues for a class of crosswise vibration equation of the beam, *Applied Mathematics and Computation*, **165**,143-154.

[59] Kasi Viswanadham K.N.S., Ballem S., (2015)–Numerical Solution of Tenth Order Boundary Problems by Galerkin Method with Septic B-splines, *Int J. Appl. Sci. Eng.*, **13**, 3, 247-260.

[60] Kitzhofer G., O. Koch O., Pulverer G., Simon C., Weinmüller E., (2009)–Treatment of Singular BVPs: The New Matlab Code bvpsuite, Available at http://www.math.tuwien.ac.at/~ewa/.

[61] Khuri S. A., (2004) –A new approach to Bratu's problem, *Applied Mathematics and Computation,* **1479** (1), 131–136.

[62] Kreyszig E., (1979)–Bernstein polynomials and numerical integration, *Int. J. Numer. Methods Engrg,* **14**, 292 – 295.

[63] Kreyszig E., (1978) –Introduction to Functional Analysis with Applications, John Wiley and Sons Incorporated.

[64] Lewis P.E., Ward J.P., (1991)–The Finite Element Method, Principles and Applications, Addison -Wesley.

[65] Leonid Aukulenko D., Nesterov S.I. (2006) –High Precision method for eigenvalue Problem, Chapman Hall/CRC.

[66] Ledoux V., Van Daele M., Vanden Berghe GS., (2009) – Efficient computation of high index Sturm-Liouville eigenvalues for problems in physics, *Computer Physics Communications*, **180**, 241-250.

[67] Ledoux V., Van Daele M., (2010) –Solving Sturm-Liouville problems by Piecewise perturbation methods, *revisited Computer Physics Communications*, **181**, 1335-1345.

[68] Liao S., Tan Y., (2007) –A general approach to obtain series solutions of nonlinear differential Equations. *Studies in Applied Mathematics,* **119**(4), 297–354.

[69] Ledoux V., (2006-7) –Study of Special Algorithms for solving Sturm-Liouville and Schrodinger equations Universities Gent.

[70] Liu R., Wu T.Y., (2002)–Differential quadrature solutions of eighth-order boundary-value differential equations, *J. Comput. Appl. Math.* **145**(1), 223–235.

[71] Levasseur K. M., (1978)-A probabilistic proof of the Weierstrass approximation theorem, *The American Mathematical Monthly*, **91**(4), 249–250.

[72] Lui S.H., (2011) –Numerical analysis of partial differential equation, John Wiley &

Sons, Inc., Hoboken, New Jersey.

[73] Lorentz G. G., (1986) – Bernstein polynomials. Chelsea Publishing Co., NewYork.

[74] Marletta M. and Pryce, J. D., (1991) –A new multipurpose software package for Schrödinger and Sturm-Liouville computations. *Comput Phys. Commun., 62*,42–52.

[75] Marletta M., Pryce, J. D., (1992) –Automatic Solution of Sturm-Liouville problems using the Pruess method, *Journal of Computational and Applied Mathematics*, **39**, 57-78.

[76] Kumar M., Singh N., (2009) –A collection of computational techniques for solving singular boundary-value problem, *Advances in Engineering Software* 40, 288-297.

[77] Mikhlin, S. G., (1964) –Variational Methods in Mathematical Physics. Macmillan, NewYork.

[78] Mikhlin, S. G., and Smolitskiy, K. L. (1967) –Approximate Methods for Solution of Differential and Integral Equations, American Elsevier, New York.

[79] Min M. S., Gottileb D., (2005)–Domain Decomposition Spectral approximation for an Eigenvalue problem with piecewise constant co-efficient, Mathematics of Computation, *SIAM Journal of Numerical Analysis*, Society for Industrial and Applied Mathematics, **43**(1), 502–*520.*

[80] Orel B., Perne A., (2014) –Chebyshev-Fourier Spectral Methods for Non-periodic Boundary Value Problems, Hindawi Publishing Corporation, *Journal of Applied Mathematics*, **2014**, Article ID 572694, 1-10.

[81] Pirabaharan P., Chandrakumar R.D., (2016)–Computational method for solving a class of singular boundary value problems arising in science and engineering, *Egyptian journal of Basic and Applied Sciences*, **3** (4), 383-91.

[82] Pruess S., (1973)–Estimating the eigenvalues of Sturrn-Liouviile problems by approximating the differential equation, *SIAM J. Numer*. Anal., **10,** 55-68.

[83] Pruess S., (1975)–High order approximations to Sturm-Liouville eigenvalues, *Numer. Math.* **24**, 241-247.

[84] Pruess S., Fulton C., (1993)–Mathematical Software for Sturm-Liouville Problems ACM Transactions on Mathematical Software **19** (3), 360-376.

[85] Reedy J.N.,(1993)–An Introduction to Finite Element Method, McGraw-Hill, Mechanical Engineering.

[86] Reutskiy S. Yu., (2006)–The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains, *Engineering Analysis with Boundary Elements,* **30**, 150–159.

[87] Reutskiy S. Yu., (2010) –The method of external excitation for solving generalized Sturm–Liouville problems, *Journal of Computational and Applied Mathematics*, **233,** 2374–2386.

[88] Siddiqi S. S., Twizell E.H., (1997)–Spline solutions of linear twelfth order boundary value problems, *J. Comp. Appl. Maths.,* **78,** 371-390.

[89] Siddiqi S., Akram G., (2007) –Solution of 10th-order boundary value problem using non- polynomial spline technique, *Applied Mathematics and Computation*, **190**: 641– 651.

[90] Singh R., Kumar J., (2013)–Computation of Eigenvalues of Singular Sturm-Liouville Problems using Modified Adomian Decomposition Method, *International Journal Nonlinear Science,* **15**(3), 247-258.

[91] Shen J., Tang T., (1996)–Spectral and High Order Methods with Application, Mathematics Monograph Series 3.

[92] Shen Jie., (1996)–Efficient Chebyshev-Legendre Galerkin methods for elliptic problems. In A. V. Ilin and R. Scott, editors, *Proceedings of ICOSAHOM'95*, 233–240, Houston J. Math.

[93] Syam M., Siyyam H., (2009) –An Efficient Technique for Computing Eigenvalues of Fourth-Order Sturm-Liouville Problems, *Chaos, Solitons and Fractals,* **3**,659-665.

[94] Siyyam, H.I., Syam, M.I., (2011) –An efficient technique for finding the eigenvalue of sixth order Sturm–Liouville problems. *Appl. Math. Sci.*, **5**, 2425–2436.

[95] Shi Z. and Cao, Y., (2012) –Application of Haar Wavelet Method to Eigenvalue Problems of High Order Differential equations, *Applied Mathematical Modeling,***36**, 4020-4026.

[96] Straughan B., (2003) –The Energy Method, Stability, and Nonlinear Convection, Springer, Berlin.

[97] Taher, A.H.S, Malek, A., Masuleh, S.H.M., (2013)–Chebychev differentiation matrices for efficient Computation of the eigenvalues of fourth order Sturm-

Liouville problems, *Applied Mathematical Modeling*, **37**, 4634-4642.

[98] Taher A. H. S, Malek A., Thabet A. S. A., (2014) –Semi-analytical Approximation for Solving High-order Sturm-Liouville Problem, *British Journal of Mathematics &Computer Science,* **4**(23), 3345-3357.

[99] Trefethen L. N., (2000) –*Spectral Methods in MATLAB*. SIAM, Philadelphia, PA.

[100] Twizell E. and Matar S., (1992)-Numerical Methods for Computing the eigenvalues of Linear Fourth-Order Boundary Value Problems, *Journal of Computational and Applied Mathematics* ,**40**, 115- 125.

[101] Twizell E.H., (1988) –Numerical methods for sixth-order boundary value problems, *International Series of Numerical Mathematics Basel*: Berkhauser.**86**, 495-506.

[102] Twizell, E.H., Boutayeb B., (1990) – Numerical methods for the solution of special and general sixth-order boundary-value problems, with applications to Benard-Layer eigenvalue problems. Proc. R. Soc. A **431**, 433–450.

[103] Twizell E.H., Boutayeb, B., Djidjeli K., (1994)–Numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability problems, *Advances in Computational Mathematics,* **2**, 407-436.

[104] Wu T.Y., Liu G.R. (2001)-The generalized differential quadrature rule for fourth-order differential equations, *Internat. J. Numer. Methods Eng.,* **50**, 1907–1929.

[105] Weideman J.A.C., Reddy S. C., (2000) –A MATLAB Differentiation matrix suite, ACM Trans. Math. Softw.**26**: 465–511.

[106] Weikang Qi., Riedel M. D., Rosenberg I.,(2011)–Uniform approximation and Bernstein polynomials with coefficients in the unit interval, *European Journal of Combinatorics,***32**, 448–463.

[107] Wu T.Y., Liu G. R., (2000) –Application of generalized differential quadrature rule to sixth order differential equation, Commun. Numer. Methods Eng., **16**, 777–784.

[108] Wang Y., Zhao Y.B., Wei G.W., (2003)–A note on numerical solution of high order differential equations, *J. Comput. Appl. Math.***,** **159**, 387-398.

[109] Yucel U., (2006)–Approximations of Sturm-Liouville eigenvalues using differential quadrature (DQ) method, *Journal of Computational and Applied Mathematics*, **192**(2), 310–319.

[110] Weidman J., (1987)–Spectral theory of ordinary differential operators, Lecture

Notes in Mathematics 1258 (Springer-Verlag, Heidelberg).

[111] Wong P.J.Y., Agarwal R.P., (1996)–Eigen values of boundary value problems for higher order Differential equations, *Mathematical Problems in Engineering* **2**, 401-434.

[112] Ycel, U., Boubaker, K., (2012)–Differential Quadrature Method (DQM) and Boubaker Polynomials Expansion Scheme (BPES) for Efficient Computation of the Eigenvalues of Fourth Order Sturm-Liouville Problems, *Applied Mathematical Modeling,* **36**, 158- 167.

[113] Yousefi S. A., Behroozifar M., Dehghan M., (2011)–The operational matrices of Bernstein polynomials for solving the parabolic equation subject to specification of mass, *Journal of Computational and Applied Mathematics*, **235**, 5272-5283.

[114] Zarebnial M., Sarvari Z., (2012)–Parametric Spline Method for Solving Bratu's Problem, *International Journal of Nonlinear Science,* **149**(1), 3-10.

[115] Zhang C., Liu W., Wang L-L., (2016)–A New Collocation Scheme Using Non polynomial Basis Functions, J Sci Comput, (2017), **70**: 793-818.