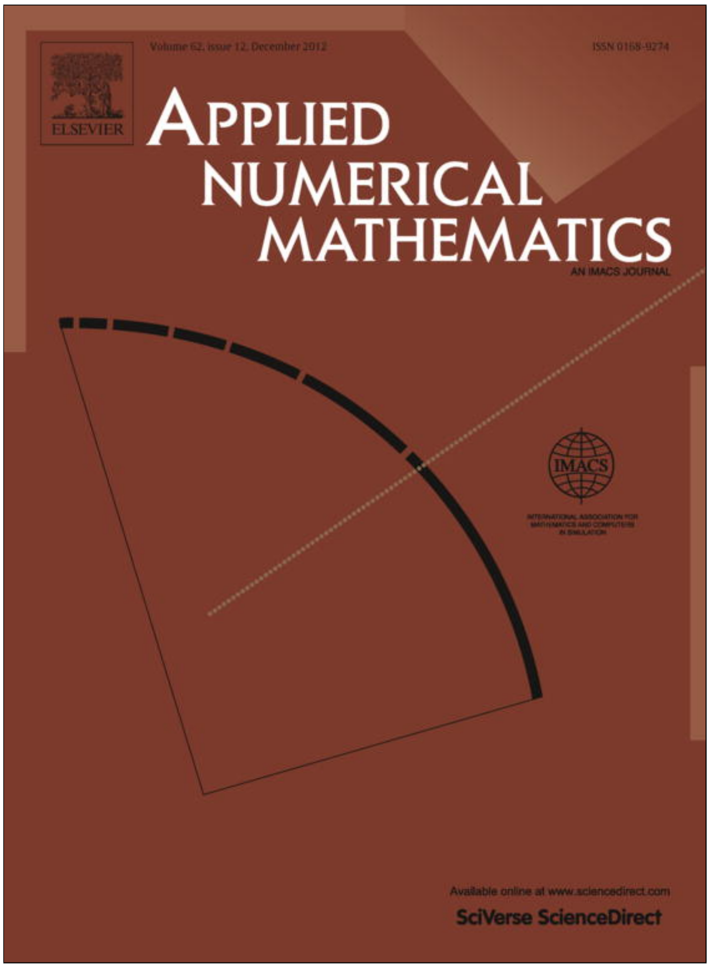
Volume 62, issue 12, December 2012
ISSN 0168-9274
ELSEVIER
APPLIED NUMERICAL MATHEMATICS
AN IMACS JOURNAL
IMACS
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Applied Numerical Mathematics 62 (2012) 1880–1892

Contents lists available at SciVerse ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Numerical convergence of a one step approximation of an integro-differential equation

Samir Kumar Bhowmik [1]

Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh

ARTICLE INFO

Article history:
Received 19 January 2010
Received in revised form 18 May 2011
Accepted 15 August 2012
Available online 17 August 2012

Keywords:
Stability
Convergence
Smoothness
Full discrete problem
Infinite domain
Periodic domain

ABSTRACT

We consider a linear partial integro-differential equation that arises in modeling various physical and biological processes. We study the problem in a spatial periodic domain. We analyze numerical stability and numerical convergence of a one step approximation of the problem with smooth and non-smooth initial functions.

1. Introduction

Many scientific problems have been modeled by reaction diffusion equations as well as by advection reaction diffusion equations. Many models consider nonlocal type diffusion operators [2–4,6–8,12,13] as well. In this article, we consider such a linear model (linear partial integro-differential equation (IDE))

$$u_t(x,t) = \varepsilon \int_{\Omega} J^{\infty}(x-y)\big(u(y,t)-u(x,t)\big)\,dy = \mathbb{L}u, \tag{1}$$

and analyze stability, accuracy and rate of convergence of a simple approximation in space and time. (1) is the linear part of the IDE [2,3,5,9,10,12]

$$u_t(x,t) = \varepsilon\left(\int_{\Omega} J^{\infty}(x-y)u(y,t)\,dy - u(x,t)\int_{\Omega} J^{\infty}(x-y)\,dy\right) + f(u). \tag{2}$$

Here the initial condition $u(x,0) = u_0(x)$, $x \in \Omega$ where $\Omega \subseteq \mathbb{R}$, $f(u)$ is a bistable nonlinearity for the associated ordinary differential equation $u_t = f(u)$ and $J^{\infty}(x)$ is a kernel function that measures interaction between particles at positions x and y. Here it is assumed that the effect of close neighbors x and y is greater than that from more distant ones; the spatial variation is incorporated in $J^{\infty}(x)$. We assume that $J^{\infty}(x)$ satisfies the following properties:

E-mail address: bhowmiksk@gmail.com.

[1] The author would like to thank Professor Dugald B. Duncan for his valuable advise and kind help.

0168-9274/$36.00 © 2012 Published by Elsevier B.V. on behalf of IMACS.
http://dx.doi.org/10.1016/j.apnum.2012.08.003

1. $J^\infty(x) \geqslant 0$;
2. $J^\infty(x)$ is normalized such that $\int_{-\infty}^{\infty} J^\infty(x)\,dx = 1$;
3. $J^\infty(x)$ is symmetric, i.e. $J^\infty(x) = J^\infty(-x)$, for all $x \in \mathbb{R}$.

In [6], the author studies stability and accuracy of an one step approximation of the IDE (1) considering infinite spatial domain. He first writes a discrete equivalent of the integral equation, then uses the forward Euler method for time discretization. Then he analyzes the accuracy and convergence of the scheme. He presents some numerical results to illustrate the rate of convergence considering both smooth and non-smooth initial functions. Whereas in [7] the author considers a linear partial IDE that contains a local and nonlocal diffusion operator. He analyzes accuracy of an explicit Euler approximation and a mixed Euler approximation. He presents some numerical results to demonstrate the accuracy of the schemes.

A brief discussion about numerical approximation of this type of models can also be found in [11]. The author considers several local and nonlocal operators that contain the operator acting on (1). He approximates the models using finite difference schemes and discusses some stability issues.

In [12], Duncan et al. consider (2) in a spatial periodic domain. They approximate the problem using piecewise constant basis functions with collocation and mid-point quadrature rule for space discretization, then they use some standard ODE solver for time integration. They present some numerical results to demonstrate their scheme.

We present the stability and convergence of the rescaled integro-differential equation model in a spatial periodic domain. Here we consider spatial $[0,1]$ periodic domain. If we choose spatially one-periodic initial data $u(x,0)$, then for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$

$$u(x,t) = u(x+1,t).$$

So, with any kernel function $J^\infty(x)$, (1) can be written as

$$\begin{aligned} u_t(x,t) &= \varepsilon \int_{\mathbb{R}} J^\infty(x-y)\big(u(y,t)-u(x,t)\big)\,dy \\ &= \varepsilon \int_0^1 J(x-z)\big(u(z,t)-u(x,t)\big)\,dz, \end{aligned} \tag{3}$$

where

$$J(x) = \sum_{r=-\infty}^{\infty} J^\infty(x-r), \tag{4}$$

and $x \in [0,1]$. Two sample kernels are shown in Fig. 1. Here it is assumed that the periodic kernel function satisfies the following conditions

A1. $J(x) \geqslant 0$,
A2. $J(x) = J(1-x)$,
A3. $\int_\Omega J(x)\,dx = 1$.

For simplicity of notations, from here J and u mean functions in the periodic domain, and J^∞ and u^∞ mean functions defined in the infinite domain. We consider $\varepsilon = 1$ only since it does not affect our analysis.

Here we study stability and accuracy of a simple scheme using the Fourier series and the discrete Fourier transform definitions. We organize our article in the following way. We present a space time discretization in Section 2 followed by stability analysis of the scheme in Section 3. We analyze accuracy of the full discrete scheme in Section 4 considering both smooth and non-smooth initial functions. We study accuracy of the semi-discrete time dependent scheme in Section 5. We finish our study in Section 6 with some numerical experiments and discussion.

2. Numerical approximation

We approximate (1) as

$$\frac{dU_j(t)}{dt} = h\sum_{k=0}^{N-1} J(x_j - x_k)(U_k - U_j) \tag{5}$$

for each $j = 0,1,2,\ldots,N-1$ where $U_j(t) \approx u(x_j,t)$ and $x_j = jh$. Applying Euler's method we can again approximate the semi-discrete problem (5) as

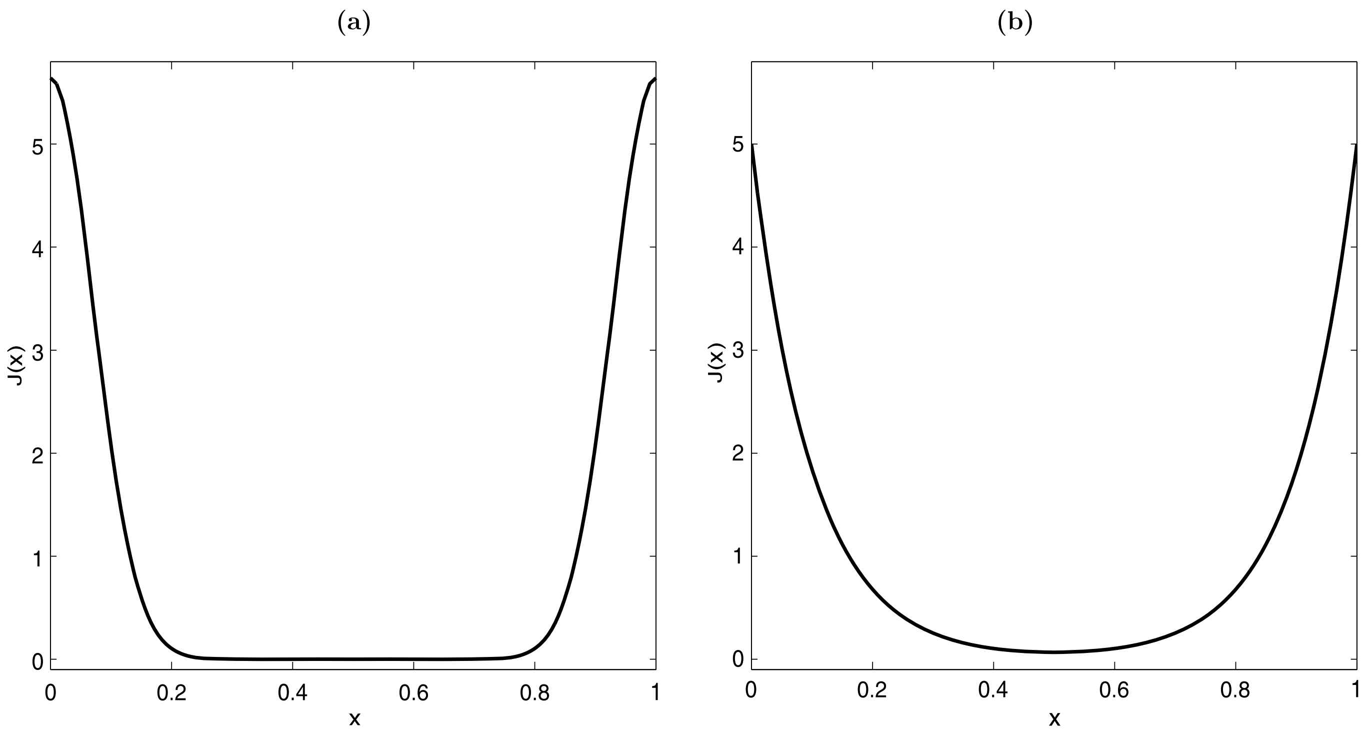

Fig. 1. Examples of kernel functions in $[0, 1]$ periodic domain defined by (4) where (a) $J^{\infty}(x) = \sqrt{\frac{100}{\pi}} e^{-100x^2}$, (b) $J^{\infty}(x) = \frac{c}{2} e^{-c|x|}$, $c = 10$.

$$U_j^{n+1} - U_j^n = h\Delta t \sum_{k=0}^{N-1} J(x_j - x_k)\big(U_k^n - U_j^n\big),$$

which gives

$$U_j^{n+1} = U_j^n\left(1 - h\Delta t \sum_{k=0}^{N-1} J(x_j - x_k)\right) + h\Delta t \sum_{k=0}^{N-1} J(x_j - x_k) U_k^n. \tag{6}$$

We use Fourier series and discrete Fourier transforms throughout. Before the main discussion let us introduce some necessary definitions and theorems. The Discrete Fourier Transform for a periodic function (DFT) can be defined as

$$\tilde{V}_k = h \sum_{j=0}^{N-1} v(x_j) e^{-i2\pi k x_j}$$

where $v(x, t)$ is a periodic function with period 1 and its inverse Fourier transform is defined as

$$v_j = \sum_{k=0}^{N-1} \tilde{V}_k e^{i2\pi k x_j} \quad \text{or} \quad v_j = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \tilde{V}_k e^{i2\pi k x_j}.$$

For any periodic function $f(x)$ with period 1, its Fourier series can be defined as

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{i2\pi n x} \quad \text{where } \hat{f}_n = \int_0^1 f(x) e^{-i2\pi n x}\, dx.$$

For any 1-periodic complex-valued function $v(x)$ the relation

$$\sum_{n=-\infty}^{\infty} |\hat{v}_n|^2 = \int_0^1 \big|v(x)\big|^2\, dx$$

where $\hat{v}_n$ are the Fourier coefficients of $v(x)$ and for a real-valued function $v(x)$

$$\frac{\hat{v}_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\hat{v}_{-n}^2 + \hat{v}_n^2\right) = \int_0^1 v^2(x)\,dx$$

is called Parseval's relation for the Fourier series where a_n, b_n are Fourier coefficients. For the Discrete Fourier Transform of $v(x)$, the relation

$$\sum_{j=0}^{N-1}|v_j|^2 = h\sum_{k=0}^{N-1}|\tilde{V}_k|^2 \quad \text{or} \quad h\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}|\tilde{V}_k|^2$$

is called Parseval's relation where $\tilde{V}_k$ is the DFT of v_j, both have the same length N.

Now we will discuss the relation between the DFT and the Fourier coefficients. From the DFT definition $\tilde{V}_k = h\sum_{j=0}^{N-1} v(jh)e^{-i2\pi k x_j}$, the Fourier coefficients are

$$\hat{v}_k = \int_0^1 v(x)e^{-i2\pi kx}\,dx$$

and the corresponding Fourier series is

$$v(x) = \sum_{k=-\infty}^{\infty}\hat{v}_k e^{i2\pi kx}.$$

Thus

$$\begin{aligned}\tilde{V}_k &= h\sum_{j=0}^{N-1}\sum_{r=-\infty}^{\infty}\hat{v}_r e^{i2\pi r x_j}e^{-i2\pi k x_j} \\ &= h\sum_{r=-\infty}^{\infty}\hat{v}_r N\delta_N(r-k) = \sum_{m=-\infty}^{\infty}\hat{v}_{k+mN} \qquad (7)\end{aligned}$$

where $r - k = mN$ and

$$\delta_N(j) = \begin{cases} 1 & j = mN \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \qquad (8)$$

The relation (7) is known as the discrete Poisson sum formula [15]. The Fourier coefficients of the kernel function in $[0,1]$ are given by

$$\hat{J}_j = \int_0^1 J(x)e^{-i2\pi jx}\,dx = \int_{-\infty}^{\infty} J^{\infty}(y)e^{-i2\pi jy}\,dy = \hat{J}^{\infty}(2\pi j), \qquad (9)$$

the relation between the Fourier coefficient of $J(x)$ defined in (4) and the continuous Fourier transform of $J^{\infty}(x)$.

Definition 1. (See [1, page 223].) For integer $k \geqslant 0$, $H^k(2\pi)$ is defined to be the closure of $C_p^k(2\pi)$ under the inner product norm

$$\|\varphi\|_{H^k} = \left[\sum_{j=0}^{k}\|\varphi^{(j)}\|_{L^2}^2\right]^{\frac{1}{2}}.$$

For arbitrary real $s \geqslant 0$, $H^s(2\pi)$ can also be obtained following [1, pages 219–223].

Theorem 1. *(See [1, page 223].) For $s \in \mathbb{R}$, $H^s(2\pi)$ is the set of all series*

$$\varphi(x) = \sum_{m=-\infty}^{\infty} a_m\psi_m(x)$$

for which

$$\|\varphi\|_{*,s}^2 = |a_0|^2 + \sum_{|m|>0} |m|^{2s} |a_m|^2 < \infty$$

where

$$\psi_m(x) = \frac{1}{\sqrt{2\pi}} e^{imx}, \quad m = 0, \pm 1, \pm 2, \ldots.$$

Moreover, the norm $\|\varphi\|_{*,s}$ is equivalent to the standard Sobolev norm $\|\varphi\|_{H^s}$ for $\varphi \in H^s(2\pi)$.

For exact details of Theorem 1 please see [1, page 223]. Here, to use norm definitions in a periodic domain $[0, 1]$, we use $\|\varphi\|_{H^s} = \|\varphi\|_{H^s(1)}$.

Now we return to the main discussion. Here we present approximate solutions in the Fourier domain. Multiplying both sides of (6) by $he^{-i2\pi kx_j}$ and summing over j

$$h \sum_{j=0}^{N-1} U_j^{n+1} e^{-i2\pi kx_j} = h \sum_{j=0}^{N-1} U_j^n e^{-i2\pi kx_j} \left(1 - h\Delta t \sum_{r=0}^{N-1} J(x_j - x_r)\right) + h \sum_{r=0}^{N-1} e^{-i2\pi kx_j} U_r^n \left(h\Delta t \sum_{j=0}^{N-1} J(x_j - x_r) e^{-i2\pi kx_j}\right).$$

That is,

$$\tilde{U}_k^{n+1} = \tilde{U}_k^n \left(1 - h\Delta t \sum_{r=0}^{N-1} J(x_j - x_r) + h\Delta t \sum_{j=0}^{N-1} J(x_j - x_r) e^{-i2\pi kx_j}\right) = g(h, \Delta t, k) \tilde{U}_k^n,$$

and clearly

$$\tilde{U}_k^n = g^n(h, \Delta t, k) \tilde{U}_k^0, \tag{10}$$

where

$$g(h, \Delta t, k) = 1 + h\Delta t \sum_{j=0}^{N-1} J(x_j - x_r)\big(e^{-i2\pi kx_j} - 1\big) = 1 + \Delta t(\tilde{J}_k - \tilde{J}_0)$$

since J is 1-periodic. Taking the inverse of the discrete Fourier transform in (10) we get the approximate solution of (1) as

$$U_j^n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} g^n(h, \Delta t, k) \tilde{U}_k^0 e^{ik2\pi x_j}. \tag{11}$$

Now the Fourier series of $u(x, t)$ be

$$u(x, t) = \sum_{j=-\infty}^{\infty} \hat{u}_j(t) e^{ij2\pi x}, \tag{12}$$

where

$$\hat{u}_j(t) = \int_0^1 u(x, t) e^{-ij2\pi x} \, dx,$$

and the Fourier series expansion of $J(x)$ be

$$J(x) = \sum_{j=-\infty}^{\infty} \hat{J}_j e^{ij2\pi x}, \tag{13}$$

where

$$\hat{J}_j = \int_0^1 J(x) e^{-ij2\pi x} \, dx.$$

Substituting (12)–(13) in (1)

$$\sum_{j=-\infty}^{\infty}\frac{d}{dt}\hat{u}_j(t)e^{ij2\pi x}=\left(\sum_{j=-\infty}^{\infty}\hat{u}_j(t)\hat{J}_je^{ij2\pi x}-\sum_{j=-\infty}^{\infty}\hat{u}_j(t)e^{ij2\pi x}\right)$$

gives

$$\frac{d}{dt}\hat{u}_j(t)=(\hat{J}_j-\hat{J}_0)\hat{u}_j(t)=\hat{q}_j\hat{u}_j(t),$$

with $\hat{q}_j=(\hat{J}_j-\hat{J}_0)$ where $\hat{J}_j$ is the j-th Fourier coefficient of the kernel function $J(x)$. Solving the above equation we have

$$\hat{u}_j(t)=e^{\hat{q}_jt}\hat{u}_j(0),$$

where

$$\hat{u}_j(0)=\int_0^1 u(x,0)e^{-ij2\pi x}\,dx.$$

Thus the exact solution of the IDE (1) can be written as

$$u(x,t)=\sum_{j=-\infty}^{\infty}\hat{u}_j(0)e^{\hat{q}_jt}e^{ij2\pi x}. \tag{14}$$

3. Stability analysis

In this section we establish stability properties of the scheme (6). To prove the stability of the scheme we need some reasonable restrictions on $J(x)$. We will examine properties of the DFT of $J(x)$ under some reasonable hypothesis on the function $J(x)$.

Theorem 2 *(Stability). If $J(x)$ is a periodic function in $[0,1]$, satisfies A1–A3, and, in addition,*

A4. $\frac{d}{dx}J(x)<0$ *for* $x\in(0,\frac{1}{2})$,
A5. $\hat{J}_k\geqslant 0$ *for* $-\frac{N}{2}+1\leqslant k\leqslant\frac{N}{2}$,

then there exists $\Delta t^*>0$ where $\Delta t^*=\frac{2}{C}$, $C>0$ such that $\|U^n\|_h\leqslant\|U^0\|_h$ for all $0<\Delta t\leqslant\Delta t^*$ and $n\geqslant 0$.

To prove the stability Theorem 2 we need to find a bound on $g(h,\Delta t,k)$. To this end we need the following lemma.

Lemma 1. *Assume that A1–A5 hold. Then $0\leqslant\tilde{J}_0-\tilde{J}_k\leqslant 2$, $\hat{J}_0-\hat{J}_k\leqslant 2$ where $-\frac{N}{2}+1\leqslant k\leqslant\frac{N}{2}$. Furthermore, $\tilde{J}_k\geqslant 0$.*

Proof. We have

$$\tilde{J}_k=h\sum_{r=0}^{N-1}J(x_r)e^{-ik2\pi x_r}\quad\text{and}\quad\tilde{J}_0=h\sum_{r=0}^{N-1}J(x_r).$$

So

$$\tilde{J}_0-\tilde{J}_k=h\sum_{r=0}^{N-1}J(x_r)\big(1-e^{-ik2\pi x_r}\big).$$

For simplicity from here we take N to be even. Using the symmetry of $J(x)$ in $[0,1]$

$$\tilde{J}_0-\tilde{J}_k=2h\sum_{r=0}^{\frac{N}{2}}J(x_r)\big(1-\cos(k2\pi x_r)\big)\geqslant 0$$

as $h>0$ and $J\geqslant 0$. Also $1-\cos(k2\pi x_r)\leqslant 2$. Thus

$$\tilde{J}_0-\tilde{J}_k=4h\sum_{r=0}^{\frac{N}{2}}J(x_r)\leqslant 4\int_0^{\frac{1}{2}}J(x)\,dx=2\int_0^1 J(x)\,dx=2.$$

So we conclude $0 \leqslant \tilde{J}_0 - \tilde{J}_k \leqslant 2$. The result $\hat{J}_0 - \hat{J}_k \leqslant 2$ follows from the definition of J and by using similar steps to those of [4]. □

Lemma 2. *If $J(x)$ satisfies* A1–A3 *then $1 \geqslant g(h, \Delta t, k) \geqslant 1 - C\Delta t$ for some $C > 0$ and if $J(x)$ satisfies* A4–A5 *from Lemma* 1 *as well then $C = 2$ and*

$$\left|g(h, \Delta t, k)\right| \leqslant 1 \quad \text{for all } 0 < \Delta t \leqslant \Delta t^* = \frac{2}{C}.$$

Proof. Proof of this lemma follows directly from Lemma 1. □

Proof of Theorem 2. We have

$$\begin{aligned}\|U^n\|_h^2 &= h\sum_{j=0}^{N-1}\left|U_j^n\right|^2 = h^2\sum_{j=0}^{N-1}\left|\tilde{U}_j^n\right|^2 = h^2\sum_{j=0}^{N-1}\left|g^n(h, \Delta t, j)\tilde{U}_j^0\right|^2\\ &\leqslant h^2\sum_{j=0}^{N-1}\left|\tilde{U}_j^0\right|^2 \leqslant h\sum_{j=0}^{N-1}\left|U_j^0\right|^2 = \|U_j^0\|_h^2,\end{aligned}$$

which gives the result. □

4. Convergence analysis of the fully discrete approximation

Here we analyze accuracy and rate of convergence of the fully discrete approximation presented above considering both smooth and non-smooth initial functions. Before giving the main convergence results let us introduce a lemma and a theorem with some reasonable restrictions on $J(x)$, which will be needed throughout this section.

Lemma 3. *If $J(x)$ satisfies* A1–A3 *in $[0, 1]$ then $\hat{q}_j \leqslant 0$ and $|\hat{J}_0 - \hat{J}_j| \leqslant 2$ where $\hat{q}_j = (\hat{J}_j - \hat{J}_0)$ with $\hat{J}_j = \int_0^1 J(x)e^{-ij2\pi x}\,dx$.*

Proof. The proof of Lemma 3 follows directly from the definition of $\hat{J}_j$, and assumptions on $J(x)$. □

Theorem 3 *(Accuracy). If $J(x) \in H^r(1)$, $r > \frac{1}{2}$ satisfies* A1–A5 *then there exist $C_1(h)$ and C_2 such that*

$$\left|e^{\hat{q}_j\Delta t} - g(h, \Delta t, k)\right| \leqslant \Delta t\big(C_1(h) + C_2\Delta t\big).$$

Proof. Here

$$\left|e^{\hat{q}_j\Delta t}\right| \leqslant 1 \quad \text{and} \quad \left|g(h, \Delta t, k)\right| \leqslant 1.$$

So

$$\left|e^{\hat{q}_k t_m} - g^m(h, \Delta t, k)\right| = \left|\left(e^{\hat{q}_k\Delta t}\right)^m - g^m(h, \Delta t, k)\right| \leqslant m\left|e^{\hat{q}_k\Delta t} - g(h, \Delta t, k)\right|.$$

Also,

$$\begin{aligned}e^{\hat{q}_k\Delta t} - g(h, \Delta t, k) &= e^{\Delta t(\hat{J}_k - \hat{J}_0)} - \big(1 + \Delta t(\tilde{J}_k - \tilde{J}_0)\big)\\ &= \Delta t\big((\hat{J}_k - \hat{J}_0) - (\tilde{J}_k - \tilde{J}_0)\big) + \sum_{j=2}^{\infty}\frac{\Delta t^j}{j!}(\hat{J}_k - \hat{J}_0)^j,\end{aligned} \tag{15}$$

and thus there exist $C_1(h)$ and C_2 (using (7) and (9)) such that

$$\left|e^{\hat{q}_j\Delta t} - g(h, \Delta t, k)\right| \leqslant \Delta t\big(C_1(h) + C_2\Delta t\big).$$

Here $C_1(h) \to 0$ as $h \to 0$, and C_2 is bounded, if $J(x)$ and $\hat{J}_j$ satisfy all the properties mentioned earlier in this article, see [4] for exact details. □

4.1. Convergence analysis for smooth initial data

Here we establish the main convergence result by the following theorem.

Theorem 4. *If the approximation* (6) *of the initial value problem* (1) *is stable,* J, $\hat{J}_k$, $-\frac{N}{2}+1 \leqslant k \leqslant \frac{N}{2}$ *satisfy* A1–A5 *and* $u_0 \in H^\sigma(1)$ *with* $\sigma > \frac{1}{2}$, *then there exist constants* $C_1(h)$, C_2, $C_3(\sigma)$ *such that*

$$\left\|u(x,t_m) - U_j^m(x)\right\| \leqslant \Delta t\big(C_1(h) + C_2\Delta t\big)\|u_0\| + C_3(\sigma)h^\sigma \|u_0\|_{H^\sigma(1)}.$$

Proof. From (14) and (11)

$$\begin{aligned} u(x_j,t_m) - U_j^m &= \sum_{k=-\infty}^{\infty} \hat{u}_k(0)e^{\hat{q}_k t_m} e^{i2\pi k x_j} - \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} g^m(h,\Delta t,k)\tilde{U}_k^0 e^{ik2\pi x_j} \\ &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \big(\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m(h,\Delta t,k)\tilde{U}_k^0\big)e^{i2\pi k x_j} \\ &\quad + \sum_{|k|>\frac{N}{2}} \hat{u}_k(0)e^{\hat{q}_k t_m} e^{ik2\pi x_j}. \end{aligned} \tag{16}$$

Now taking the inner product on (16) and applying Parseval's relation

$$\begin{aligned} \left\|u(x_j,t_m) - U_j^m\right\|_h^2 \leqslant{}& \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m(h,\Delta t,k)\tilde{U}_k^0\right|^2 \\ &+ \sum_{|k|>\frac{N}{2}} \left|\hat{u}_k(0)e^{\hat{q}_k t_m}\right|^2. \end{aligned} \tag{17}$$

The Poisson summation formula gives

$$\tilde{U}_k^m = \sum_{r=-\infty}^{\infty} \hat{u}_{k+rN}^m = \sum_{r=-\infty}^{\infty} \hat{u}_{k+rN}^{\infty,m}$$

where $\hat{u}_{k+rN}^m$ is the coefficient of the Fourier series at t_m for the periodic function u whereas $\hat{u}_{k+rN}^{\infty,m}$ represents the CFT for the nonperiodic (infinite-dimensional) case. A calculation similar to that leading to (9) gives

$$\hat{u}_k(0) = \int_0^1 u(x,0)e^{-ik2\pi x}dx = \hat{u}_k^0 = \hat{u}_0^\infty(2\pi k) = \hat{u}_k^{\infty,0}. \tag{18}$$

So the first part of the right-hand side of (17) can be written as

$$\begin{aligned} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\tilde{U}_k^0\right|^2 \leqslant{}& \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left|\big(e^{\hat{q}_k t_m} - g^m\big)\hat{u}_k^{\infty,0}\right|^2 \\ &+ \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left|g^m(h,\Delta t,k)\sum_{r\neq 0}\hat{u}_0^\infty\big(2\pi(k+rN)\big)\right|^2. \end{aligned} \tag{19}$$

Applying Theorem 3 on the first part of the right-hand side of (19) we get

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left|\big(e^{\hat{q}_j t_m} - g^m\big)\hat{u}_0^\infty\right|^2 \leqslant (m\Delta t)^2\big(C_1(h) + C_2\Delta t\big)^2\|u_0\|^2. \tag{20}$$

Now

$$\left|\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right| \leqslant \sum_{s\neq 0}\left|\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|$$
$$\leqslant \sqrt{\sum_{s\neq 0}\left|\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\big(2\pi(k+sN)\big)^{\sigma}\right|^2}\sqrt{\sum_{s\neq 0}\left|\big(2\pi(k+sN)\big)\right|^{-2\sigma}}$$

using the Cauchy–Schwartz inequality. Thus,

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\left|g^m(h,\Delta t,k)\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \leqslant \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\left|\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2$$
$$\leqslant \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\left(\sum_{s\neq 0}\left|\hat{u}_0\big(2\pi(k+sN)\big)\big(2\pi(k+sN)\big)^{\sigma}\right|^2\right)\left(\sum_{s\neq 0}\left|\big(2\pi(k+sN)\big)\right|^{-2\sigma}\right)$$

gives

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\left|g^m(h,\Delta t,k)\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \leqslant h^{2\sigma}C(\sigma)\sum_{|k|>\frac{N}{2}}|2\pi k|^{2\sigma}\left|\hat{u}_0(2\pi k)\right|^2$$
$$\leqslant C(\sigma)h^{2\sigma}\|u_0\|_{*,\sigma}^2 \equiv C(\sigma)h^{2\sigma}\|u_0\|_{H^{\sigma}(1)}^2 \tag{21}$$

for some $\sigma > \frac{1}{2}$. Thus applying above bounds, Lemma 3 and Theorem 1 in (17) one gets

$$\left\|u(x_j,t_m)-U_j^m\right\|_h^2 \leqslant \Delta t^2\big(C_1(h)+C_2\Delta t\big)^2\|u_0\|^2 + C(\sigma)h^{2\sigma}\|u_0\|_{H^{\sigma}(1)}^2 + \sum_{|k|>\frac{N}{2}}\left|\hat{u}_k(0)\right|^2$$
$$\leqslant \Delta t^2\big(C_1(h)+C_2\Delta t\big)^2\|u_0\|^2 + C(\sigma)h^{2\sigma}\|u_0\|_{H^{\sigma}(1)}^2$$
$$+\left(\frac{2}{N}\right)^{2\sigma}\sum_{|k|>\frac{N}{2}}|k|^{2\sigma}\left|\hat{u}_0(2k\pi)\right|^2$$
$$\leqslant \Delta t^2\big(C_1(h)+C_2\Delta t\big)^2\|u_0\|^2 + C_3(\sigma)h^{2\sigma}\|u_0\|_{H^{\sigma}(1)}^2, \tag{22}$$

for some $\sigma > \frac{1}{2}$. □

4.2. Convergence analysis for non-smooth initial data

Here we consider an initial function which is not smooth enough so that $\|u_0\|_{H^{\nu}(1)}$ is bounded when $\nu > \frac{1}{2}$, but there exists $\nu_1 < \nu$ such that $\|u_0\|_{H^{\nu_1}(1)} < \infty$ and there exists α such that

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\left|\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \leqslant \Delta x^{2\alpha}C(u_0). \tag{23}$$

For examples of such functions please see [4,15] and references therein.

Theorem 5. *If the initial value problem* (1) *with some non-smooth initial function* $u_0(x,0)=g(x)$ *is considered so that the condition* (23) *holds and is approximated by the stable one step finite difference formula* (6), J, $\hat{J}_k$, $-\frac{N}{2}+1 \leqslant k \leqslant \frac{N}{2}$ *satisfy* A1–A5, *then*

$$\left\|u(x_j,t_m)-U_j^m\right\| \leqslant t_m\big(C_1(h)+C_2\Delta t\big)\|u_0\| + C_2(\nu_1)\Delta x^{\nu_1}\|u_0\|_{H^{\nu_1}(1)} + \Delta x^{\alpha}C_3(u_0),$$

where $C_1(h)$, C_2 *and* C_3 *are constants.*

Proof. We start rewriting (17) as

$$\left\|u(x_j,t_m)-U_j^m\right\|_h^2 \leqslant \sum_{|k|\leqslant M}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\tilde{U}_k^0\right|^2$$
$$+\sum_{M\leqslant|k|\leqslant\frac{N}{2}}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\tilde{U}_k^0\right|^2 + \sum_{|k|>\frac{N}{2}}\left|\hat{u}_k(0)e^{\hat{q}_k t_m}\right|^2$$

for some $0 < M < \frac{N}{2}$. Now with the same algebraic operations as we performed for Theorem 4

$$\begin{aligned}\left\|u(x_j,t_m) - U_j^m\right\|_h^2 \leqslant \sum_{|k|\leqslant M}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\hat{u}_k^{\infty,0}\right|^2 + \sum_{M\leqslant|k|\leqslant\frac{N}{2}}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\hat{u}_k^{\infty}(0)\right|^2 \\ + \sum_{|k|>\frac{N}{2}}\left|\hat{u}_k(0)e^{\hat{q}_k t_m}\right|^2 + \sum_{|k|\leqslant M}\left|g^m\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \\ + \sum_{M\leqslant|k|\leqslant\frac{N}{2}}\left|g^m\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2,\end{aligned}$$

$$\sum_{|k|\leqslant M}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\hat{u}_k^{\infty}(0)\right|^2 \leqslant t_m^2\big(C_1(h) + C_2\Delta t\big)^2\|u_0\|^2,$$

and

$$\begin{aligned}&\sum_{M\leqslant|k|\leqslant\frac{N}{2}}\left|\hat{u}_k(0)e^{\hat{q}_k t_m} - g^m\hat{u}_k^{\infty}(0)\right|^2 + \sum_{|k|>\frac{N}{2}}\left|\hat{u}_k(0)e^{q t_m}\right|^2 \\ &= C\sum_{|k|>M}\left|\hat{u}_0^{\infty}\right|^2 \leqslant C\left(\frac{1}{M}\right)^{2\nu_1}\sum_{|k|>M}|k|^{2\nu_1}\left|\hat{u}^{\infty,0}\right|^2 \leqslant C_3(\nu_1)\Delta x^{2\nu_1}\|u_0\|_{H^{\nu_1}(1)}\end{aligned}$$

hold. Also following [4,15], there exists $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}&\sum_{|k|\leqslant M}\left|g^m\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 + \sum_{M\leqslant|k|\leqslant\frac{N}{2}}\left|g^m\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \\ &\leqslant \sum_{|k|\leqslant\frac{N}{2}-1}\left|\sum_{s\neq 0}\hat{u}_0^{\infty}\big(2\pi(k+sN)\big)\right|^2 \leqslant \Delta x^{2\alpha}C(u_0),\end{aligned}$$

which gives

$$\left\|u(x_j,t_m) - U_j^m\right\|^2 \leqslant t_m^2\big(C_1(h) + C_2\Delta t\big)^2\|u_0\|^2 + C_3(\nu_1)\Delta x^{2\nu_1}\|u_0\|_{H^{\nu_1}(1)} + \Delta x^{2\alpha}C(u_0). \qquad \square$$

5. Convergence of the semi-discrete approximation

In this section we analyze accuracy of the spatial approximation (5).

Theorem 6. *If the semi-discrete approximation (5) of the initial value problem (1) is stable, J, $\hat{J}_k$, $-\frac{N}{2}+1 \leqslant k \leqslant \frac{N}{2}$ satisfy* A1–A5 *and $u_0 \in H^{\sigma}(1)$ with $\sigma > \frac{1}{2}$, then there exist constants $C_1(h)$, $C_2(\sigma)$ such that*

$$\left\|u(x_j,t) - U(x_j,t)\right\| \leqslant tC_1(h)\|u_0\| + C_2(\sigma)h^{\sigma}\|u_0\|_{H^{\sigma}(1)}.$$

Proof. Applying the discrete Fourier transform to (5)

$$\frac{d}{dt}\tilde{U}_k = \tilde{q}_k\tilde{U}_k \tag{24}$$

where $\tilde{q}_k = (\tilde{J}_k - \tilde{J}_0)$ and its solution can be found as

$$\tilde{U}_k(t) = e^{\tilde{q}_k t}\tilde{U}_k(0). \tag{25}$$

Applying the inverse of the DFT in (25)

$$U(x_j,t) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{\tilde{q}_k t}\tilde{U}_k(0)e^{ik2\pi x_j}. \tag{26}$$

Thus comparing (14) and (26)

$$u(x_j,t)-U(x_j,t)=\sum_{k=-\infty}^{\infty}\hat{u}_k(0)e^{\hat{q}_k t}e^{ik2\pi x_j}-\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}e^{\tilde{q}_k t}e^{ik2\pi x_j}\tilde{U}_k(0)$$

$$=\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}e^{ik2\pi x_j}\big(\hat{u}_k(0)e^{\hat{q}_k t}-e^{\tilde{q}_k t}\tilde{U}_k(0)\big)+\sum_{|k|>\frac{N}{2}}\hat{u}_k(0)e^{\hat{q}_k t}e^{ik2\pi x_j}. \tag{27}$$

Applying Parseval's relation to (27)

$$\big|u(x_j,t)-U(x_j,t)\big|^2\leqslant\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\big|\hat{u}_k(0)e^{\hat{q}_k t}-e^{\tilde{q}_k t}\tilde{U}_k(0)\big|^2+\sum_{|k|>\frac{N}{2}}\big|\hat{u}_k(0)e^{\hat{q}_k t}e^{ik2\pi x_j}\big|^2$$

$$\leqslant\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\big|\hat{u}_k(0)e^{\hat{q}_k t}-e^{\tilde{q}_k t}\tilde{U}_k(0)\big|^2+\sum_{|k|>\frac{N}{2}}\big|\hat{u}_k(0)e^{\hat{q}_k t}\big|^2. \tag{28}$$

Now applying Poisson's formula to the first part of the right-hand side of (28),

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\big|\hat{u}_k(0)e^{\hat{q}_k t}-e^{\tilde{q}_k t}\tilde{U}_k(0)\big|^2=\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\big|e^{\hat{q}_k t}-e^{\tilde{q}_k t}\big|^2\big|\hat{u}_0^\infty(2\pi k)\big|^2$$

$$+\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\sum_{s\neq 0}e^{2\tilde{q}_k t}\big|\hat{u}_0^\infty\big(2\pi(k+sN)\big)\big|^2.$$

Now

$$\big|e^{\hat{q}_k t}-e^{\tilde{q}_k t}\big|^2=\big|e^{\hat{q}_k t}\big(1-e^{t(\tilde{q}_k-\hat{q}_k)}\big)\big|^2\leqslant t^2|\hat{q}_k-\tilde{q}_k|^2.$$

Now

$$\tilde{q}_k-\hat{q}_k=\left(\sum_{m=-\infty}^{\infty}\hat{q}_{k+mN}\right)-\hat{q}_k,\quad\text{using (7)}$$

$$=\sum_{m\neq 0}\hat{q}_{k+mN}=\sum_{m\neq 0}(\hat{J}_{k+mN}-\hat{J}_{mN}),\quad\text{since } \hat{q}_k=\hat{J}_k-\hat{J}_0,$$

$$=\sum_{m\neq 0}\big(\hat{J}^\infty\big(2\pi(k+mN)\big)-\hat{J}^\infty(2\pi mN)\big),\quad\text{using (9)}.$$

So from [4] it follows that

$$|\tilde{q}_k-\hat{q}_k|\leqslant 2\hat{J}^\infty\left(\frac{\pi}{h}\right)=C_1(h),$$

and $C_1(h)\to 0$ as $h\to 0$ when J^∞ is smooth enough (if $J^\infty\in L_2(\mathbb{R})$, then $|\hat{J}^\infty(\xi)|\to 0$ as $|\xi|\to\infty$ [14], [16, page 30]). Thus

$$\big|e^{\hat{q}_k t}-e^{\tilde{q}_k t}\big|^2\leqslant t^2C_1^2(h).$$

So applying the relation (18), Parseval's relation and Theorem 1

$$\sum_{|k|\leqslant\frac{N}{2}}\big|\big(e^{\hat{q}_k t}-e^{\tilde{q}_k t}\big)\hat{u}_0^\infty\big|^2\leqslant t^2C_1^2(h)\sum_{|k|\leqslant\frac{N}{2}}\big|\hat{u}_k(0)\big|^2$$

$$\leqslant t^2C_1^2(h)\left(\big|\hat{u}_0(0)\big|^2+\sum_{|k|>0}|k|^0\big|\hat{u}_k(0)\big|^2\right)$$

$$\leqslant t^2C_1^2(h)\|u_0\|_{*,0}^2\equiv t^2C_1^2(h)\|u_0\|_{H^0(1)}^2=t^2C_1^2(h)\|u_0\|^2.$$

Thus, similar to Section 4 there exists a constant $C_2(\sigma)$ such that

$$\big\|u(x_j,t)-U(x_j,t)\big\|^2\leqslant t^2C_1^2(h)\|u_0\|^2+C_2^2(\sigma)h^{2\sigma}\|u_0\|_{H^\sigma(1)}^2. \qquad\square$$

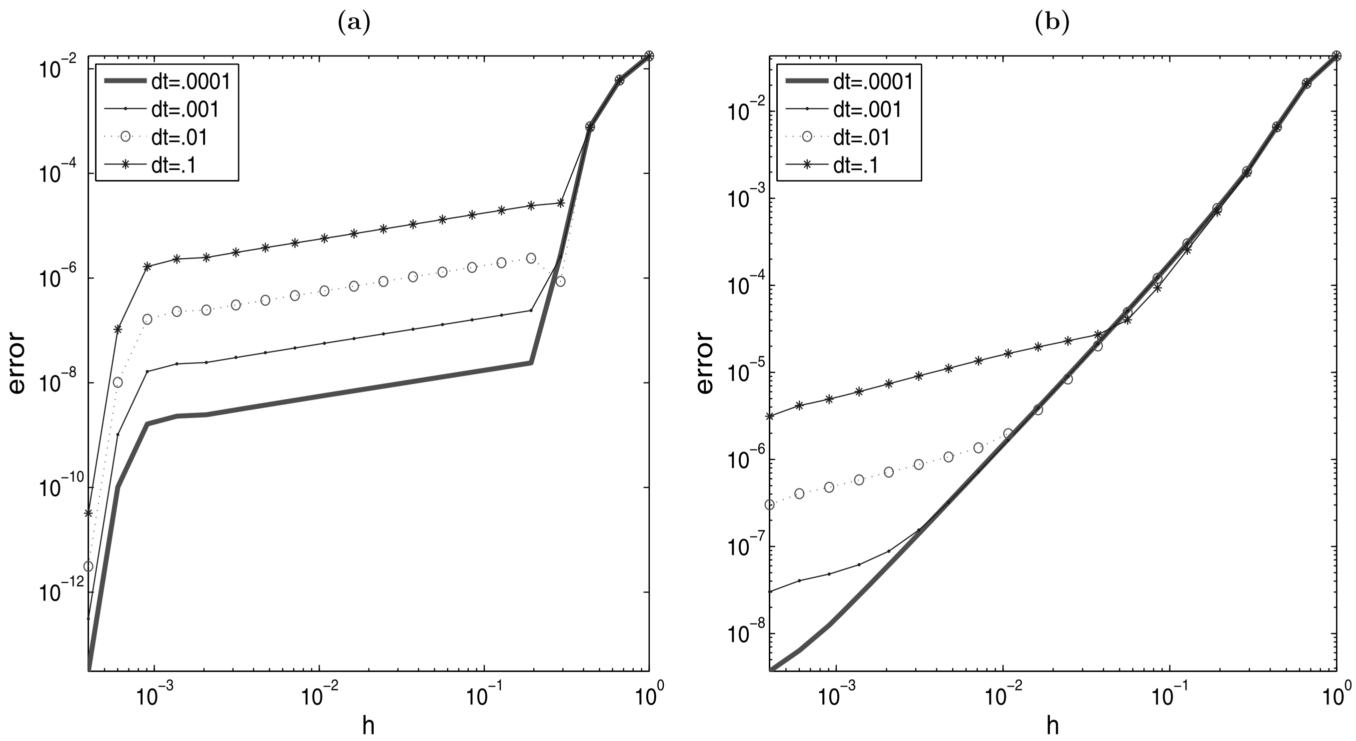

Fig. 2. Here we plot the error term $\|u(\cdot, t_n) - U^n(\cdot)\|_h$ at $t = 1$ by varying Δt and h with $J^\infty(x) = \sqrt{\frac{10}{\pi}} e^{-10x^2}$ and (a) $u_0(x) = \sqrt{\frac{1}{\pi}} e^{-(x-\frac{1}{2})^2}$ (left figure), (b) $u_0(x) = \frac{1}{2} e^{-|x-\frac{1}{2}|}$ (right figure).

6. Numerical experiments and discussion

Here we start by experimenting with numerical error in the approximation (6) of (1). We compute numerical error at $t = 1$. Fig. 2 shows the behavior of $\|u(\cdot, t_n) - U^n(\cdot)\|_h$ for various choices of h and Δt with smooth and non-smooth u_0 for all $x \in [0, 1]$ where $J^\infty(x) = \sqrt{\frac{10}{\pi}} \exp(-10x^2)$, $u(\cdot, t_n)$ is defined by (14) and $U^n(\cdot)$ is defined by (11). From Fig. 2, we observe that for smooth u_0 the rate of convergence of the solutions is faster than for the non-smooth u_0. Here we also notice that choices of h and Δt have an impact on the rate of convergence. This experiment agrees with our theoretical estimates (and also motivates us to investigate further the theoretical stability and convergence analysis of such a one step approximation). From this study we notice that the full discrete scheme is conditionally stable. The accuracy of the scheme depends on the smoothness of the initial function. We have some limitations in this study. We impose some reasonable restrictions on the kernel function to prove the stability and convergence results. The analysis of higher order schemes in one and multi-dimensions, which is of course more challenging, is left for future studies.

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