Stability and Convergence Analysis of a One Step Approximation of a Linear Partial Integro-Differential Equation

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Received 19 July 2009; accepted 21 January 2010
Published online in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num.20576

We study a linear partial integro-differential equation which arises in the modeling of various physical and biological sciences. We analyze numerical stability and numerical convergence of a one step approximation of the problem with smooth and non-smooth initial functions. Numer Methods Partial Differential Eq 000: 000–000, 2010

Keywords: convergence; Fourier transform; integro-differential equation; smoothness; stability

I. INTRODUCTION

Many problems from chemical, physical, mechanical, biological, and other systems have been modeled by reaction diffusion equation and advection reaction diffusion equations. Problems with a standard Laplacian-type diffusion operator (local problems) are well known [1–5]. A lot of problems contain a nonlocal [6–12] diffusion operator as well as both [13]. These type of models are typically complicated, interesting to the scientists, challenging to understand and to analyze.

Study in convolution model of phase transitions (initial value problem) is of ongoing interest and a lot about numerical analysis yet to be done. In this article, we analyze stability, accuracy, and rate of convergence of approximations of such a linear partial integro-differential equation (IDE)

$$u_t(x,t) = \varepsilon \int_{\Omega} J(x-y)(u(y,t) - u(x,t))dy = \mathbb{L}u,$$

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Contract grant sponsors: MACS studentship of Heriot-Watt University (Edinburgh, UK); Study leave grant from University of Dhaka (Dhaka, Bangladesh)

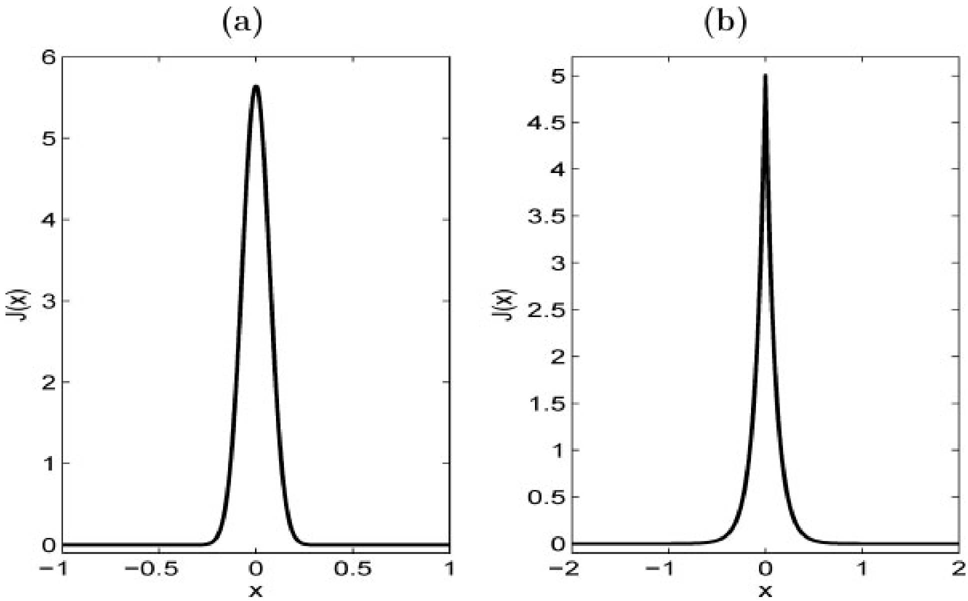

FIG. 1. Examples of kernel function: (a) $J(x) = \sqrt{\frac{100}{\pi}} e^{-100x^2}$, a smooth function and (b) $J(x) = \frac{c}{2} e^{-c|x|}$, $c = 10$ a non-smooth function at $x = 0$.

which is the linear part of the IDE [6, 7, 9, 10, 14–17]

$$u_t = \varepsilon \left(\int_\Omega J(x-y)u(y,t)dy - u(x,t) \int_\Omega J(x-y)dy \right) + f(u). \tag{1}$$

Here, the initial condition $u(x,0) = u_0(x), x \in \Omega$ where $\Omega \subseteq \mathbb{R}$, $f(u)$ is a bistable nonlinearity for the associated ordinary differential equation $u_t = f(u)$ [e.g., $f(u) = u - u^3$, see [8, 10] for more detail] and $J(x-y)$ is a kernel that measures interaction between particles at positions x and at y. Here, it is assumed that the effect of close neighbours x and y is greater than that from more distant ones; the spatial variation is incorporated in $J(x)$. We assume that J is a non-negative function satisfying smoothness, symmetry, and decay conditions. Figure 1 shows two sample kernel functions. u represents the density (concentration) at point x in Ω of a binary material and $\varepsilon \geq 0$ is a non-negative parameter.

Note that the convolution Eq. (1) is the L_2-gradient flow of the free energy functional [7, 10]

$$E(u) = \varepsilon \int_\Omega \int_\Omega J(x-y)(u(y,t) - u(x,t))^2 dx dy + \int_\Omega F(u(x,t))dx \tag{2}$$

where u and J are defined above and $f(u) = -F'(u)$. In Eq. (2), the first integral penalizes special variations in the solution, and when $f(u) = u - u^3$, the second integral penalizes states u for which $|u| \neq 1$.

This type of model also arises in firing rate models in neuronal networks [11, 12, 18–21]. For the one-dimensional case [11, 12]—and many other articles about neuronal models consider the equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{\Omega} w(x-y) f(u(y,t) - \text{th}) dy \tag{3}$$

where $\Omega \subseteq \mathbb{R}$, $w(x)$ represents the coupling between neurons, $u(x,t)$ represents synaptic input to a neuron at each point $x \in \Omega$, and f represents the firing rate of the neuron at $x \in \Omega$ with threshold th. Troy et al. [11] show that Eqs. (3) and (1) are similar.

In [7,22], existence, uniqueness, and regularity properties of the convolution problem [Eq. (1)] are established, these studies also include global asymptotic stability in the case of zero-velocity continuous waves. There it is shown that Eq. (1) is a direct analog of the more familiar bistable nonlinear diffusion equation (the Allan-Cahn equation), and shares many of its properties.

In [10,15,23], the authors study some important properties of the model [Eq. (1)]. Duncan et al. [10] inspect properties of stationary solutions, coarsening of solutions, stability, and numerical approximation of Eq. (1) by piecewise constant basis functions. Duncan [10, 15] shows results for the full nonlinear IDE model in detail as well as for the linear part, and also compares the model with the famous Allen-Cahn model. Then, he concentrates on approximation and accuracy analysis of such approximations using piecewise polynomial basis functions.

Motivation of this study comes from [10] where the authors present the approximation of the partly convolutional linear part of the model [Eq. (1)] using piecewise constant basis functions followed by mid-point rules for the integrals in each of the sub-domains. We consider the infinite domain $\Omega = \mathbb{R}$. We study stability, accuracy, and convergence of a finite difference method to approximate the linear part of Eq. (1). We show that the one step approximation is stable and present the rate of accuracy and convergence for the linear IDE considering smooth and non-smooth initial functions. We use the continuous Fourier transform [4, 24–26] and the discrete Fourier transform [4,25,26] for our analysis throughout this article.

The article is organized in the following way. In Section II, we present our problem and its semi-discrete and fully discrete approximations. We discuss stability of such a fully discrete approximation in Section III. The accuracy and convergence results for the fully discrete approximation are shown in Section IV whereas Section V contains convergence analysis for the spatial approximation (continuous in time) of the problem. We finish this study with some concluding remarks in Section VI.

II. SIMPLE APPROXIMATION IN AN INFINITE DOMAIN

Consider the linear integro-differential equation [10]

$$u_t(x,t) = \int_{-\infty}^{\infty} J(x-y)(u(y,t) - u(x,t)) dy \tag{4}$$

with $u(x,t_0) = u_0(x)$, $x \in \mathbb{R}$ [since $\varepsilon \geq 0$ is a constant, it does not affect our stability and accuracy analysis, and so we avoid multiplying ε in Eq. (4), we consider $\varepsilon = 1$]. This IVP can be approximated by a system of ordinary differential equations as follows:

$$\frac{dU_j(t)}{dt} = h \sum_{k=-\infty}^{\infty} J(x_j - x_k)(U_k - U_j) \tag{5}$$

for each $j \in \mathbb{Z}$ where $U_j(t) \approx u(x_j, t)$ and $x_j = jh$ where h is the spacing between the grid points x_j for all $j \in \mathbb{Z}$. Applying the explicit Euler approximation formula to Eq. (5) we obtain the fully discrete approximation

$$U_j^{n+1} - U_j^n = h\Delta t \sum_{k=-\infty}^{\infty} J(x_j - x_k)\big(U_k^n - U_j^n\big)$$

where $U_j^n = U(x_j, t_n)$. This is equivalent to

$$U_j^{n+1} = U_j^n \left(1 - h\Delta t \sum_{k=-\infty}^{\infty} J(x_j - x_k)\right) + h\Delta t \sum_{k=-\infty}^{\infty} J(x_j - x_k) U_k^n. \tag{6}$$

We need the following definitions and results to present the approximation in Fourier domain and to carry out the analysis of the approximations. The Discrete Fourier Transform (DFT) of the sequence $\{v_m : m \in \mathbb{Z}\}$ on the mesh points $\{x_m = mh : m \in \mathbb{Z}\}$ is defined by

$$\tilde{v}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-ihm\xi} v_m \tag{7}$$

if $v_m \in L_2(h\mathbb{Z})$ [4, page 198]. The inverse DFT is defined by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ihm\xi} \tilde{v}(\xi) d\xi \tag{8}$$

where $\xi \in [\frac{-\pi}{h}, \frac{\pi}{h}]$. If $u \in L_2(\mathbb{R})$, then the Continuous Fourier Transform (CFT) of $u(x)$ in space is defined as

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-ix\xi} dx \equiv (Fu)(\xi). \tag{9}$$

If $u, \hat{u} \in L_2(\mathbb{R})$, then the inverse Fourier transform is defined as

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{ix\xi} d\xi \equiv (F^{-1}\hat{u})(x).$$

Definition 1. [4, page 36] *For any function $u(x)$ with $x \in \mathbb{R}$, the H^r norm for any $r \in \mathbb{R}_+$ is defined by*

$$\|u\|_{H^r(\mathbb{R})} = \sqrt{\left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^r |\hat{u}(\xi)|^2 d\xi\right)}.$$

Multiplying Eq. (6) by $\frac{h}{\sqrt{2\pi}} e^{-ijh\xi}$ and summing over all j gives

$$\begin{aligned}\frac{h}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ijh\xi} U_j^{n+1} &= \frac{h}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ijh\xi} U_j^n \left(1 - h\Delta t \sum_{k=-\infty}^{\infty} J(x_j - x_k)\right) \\ &\quad + \frac{h}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ikh\xi} U_k^n \left[h\Delta t \sum_{j=-\infty}^{\infty} J(x_j - x_k) e^{-i(j-k)h\xi}\right].\end{aligned}$$

So,

$$\tilde{U}^{n+1}(\xi) = \left\{ 1 + h\Delta t \sum_{j=-\infty}^{\infty} J(x_j - x_k)(e^{i(k-j)h\xi} - 1) \right\} \tilde{U}^n(\xi).$$

Hence,

$$\tilde{U}^n(\xi) = (g(h\xi, \Delta t))^n \tilde{U}^0(\xi), \tag{10}$$

where

$$\begin{aligned} g(h\xi, \Delta t) &= 1 + \Delta t \left(h \sum_{r=-\infty}^{\infty} e^{-irh\xi} J(x_r) \right) - \Delta t \left(h \sum_{r=-\infty}^{\infty} e^{-irh0} J(x_r) \right) \\ &= 1 + \sqrt{2\pi}\, \Delta t (\tilde{J}(\xi) - \tilde{J}(0)), \end{aligned} \tag{11}$$

using the DFT definition [Eq. (7)].

III. STABILITY ANALYSIS

Now, we carry out a stability analysis of Eq. (6) following [4]. We examine properties of $\tilde{J}(\xi)$ and $\hat{J}(\xi)$ under reasonable hypotheses on $J(x)$. We use the bounds on $\tilde{J}(\xi)$ and $\hat{J}(\xi)$ obtained in the next Lemma to bound $g(h\xi, \Delta t)$.

Lemma 1. *Assume that $J(x) \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ satisfies*

H1 $J(x) \geq 0$;
H2 *$J(x)$ is normalized such that $\int_{-\infty}^{\infty} J(x)dx = 1$;*
H3 *$J(x)$ is symmetric, i.e., $J(x) = J(-x)$, for all $x \in \mathbb{R}$;*
H4 *$J(x)$ is decreasing on $(0, \infty)$;*
H5 $\hat{J}(\xi) \geq 0$.

Then, **H1**–**H4** *give the DFT results $0 \leq \tilde{J}(0)$ and $\tilde{J}(\xi) \leq \tilde{J}(0) \leq \sqrt{\frac{2}{\pi}} + \tilde{J}(\xi)$ for all $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$ and the CFT results $\hat{J}(\xi) \leq \hat{J}(0) \leq \sqrt{\frac{2}{\pi}} + \hat{J}(\xi)$. Further, if* **H5** *holds, then $\tilde{J}(\xi) \geq 0$ for $J \in H^r(\mathbb{R})$, $r > \frac{1}{2}$.*

Proof. Using **H1** and **H3** we get

$$\tilde{J}(0) \geq 0 \quad \text{and} \quad \tilde{J}(0) - \tilde{J}(\xi) = \sqrt{\frac{2}{\pi}} h \left[\sum_{m=1}^{\infty} (1 - \cos(mh\xi))\, J(mh) \right] \geq 0.$$

From the previous formula,

$$\tilde{J}(0) - \tilde{J}(\xi) \leq 2\sqrt{\frac{2}{\pi}} h \sum_{m=1}^{\infty} J(mh) \quad \text{since } 1 - \cos(mh\xi) \in [0, 2].$$

Conditions **H1–H4** ensure that

$$\sum_{m=1}^{\infty} hJ(mh) \leq \sum_{m=0}^{\infty} \int_{x_m}^{x_{m+1}} J(x)dx = \int_0^{\infty} J(x)dx = \frac{1}{2}$$

for all $m \geq 0$ with $x_m = mh$. So, we have $\tilde{J}(0) - \tilde{J}(\xi) \leq \sqrt{\frac{2}{\pi}}$. Hence, we conclude $\tilde{J}(\xi) \leq \tilde{J}(0) \leq \sqrt{\frac{2}{\pi}} + \tilde{J}(\xi)$. The result $\tilde{J}(\xi) \geq 0$ follows from Proposition 1 and **H5** if $J \in H^r$, $r > \frac{1}{2}$. The continuous Fourier transform of $J(x)$ and conditions **H1–H3** give

$$\hat{J}(0) - \hat{J}(\xi) \leq 2\sqrt{\frac{2}{\pi}} \int_0^{\infty} J(x)dx \leq \sqrt{\frac{2}{\pi}}.$$

Thus, we conclude $\hat{J}(\xi) \leq \hat{J}(0) \leq \hat{J}(\xi) + \sqrt{\frac{2}{\pi}}$. ∎

Lemma 2. *Assume that $J(x) \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ and* **H1** *and* **H3** *hold, then there exists $C > 0$ such that $1 - C\Delta t \leq g(h\xi, \Delta t) \leq 1$. If* **H2** *and* **H4** *also hold, then $C = 2$. In both cases, $|g(h\xi, \Delta t)| \leq 1$ for all $0 < \Delta t \leq \Delta t^* = \frac{2}{C}$.*

Proof. From Lemma 1 we observe that $\tilde{J}(0) - \tilde{J}(\xi)$ is bounded if $J \in L_2 \cap C(\mathbb{R})$. So, there is some constant $\frac{C}{\sqrt{2\pi}} > 0$ such that $\tilde{J}(0) - \tilde{J}(\xi) < \frac{C}{\sqrt{2\pi}}$. It then follows from

$$g(h\xi, \Delta t) = 1 + \sqrt{2\pi}\Delta t(\tilde{J}(\xi) - \tilde{J}(0))$$

and Lemma 1 that $1 - C\Delta t \leq g(h\xi, \Delta t) \leq 1$. Now, if **H1–H4** of Lemma 1 hold then we have

$$g(h\xi, \Delta t) = 1 + \sqrt{2\pi}\Delta t(\tilde{J}(\xi) - \tilde{J}(0)) \quad \text{gives} \quad 1 - 2\Delta t \leq g(h\xi, \Delta t) \leq 1$$

by Lemma 1. The last result follows by inspection. ∎

We now have the stability result. Let us recall Parseval's relations which are needed to prove the stability and accuracy results. The relations

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi \tag{12}$$

and

$$\|\tilde{v}\|_h^2 = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\tilde{v}(\xi)|^2 d\xi = \sum_{m=-\infty}^{\infty} h|v_m|^2 = \|v\|_h^2 \tag{13}$$

are the well known Parseval Formulae [4, 26].

Lemma 3. *If $J(x) = J(-x)$ and $J(x) \geq 0$ and $J \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ then there exists $\Delta t^* > 0$ given by Lemma 2 such that $\|U^n\|_h \leq \|U^0\|_h$ for all $0 < \Delta t \leq \Delta t^*$ and $n \geq 0$.*

Proof. We have

$$\|U^n\|_h^2 = h\sum_{m=-\infty}^{\infty}\left|U_m^n\right|^2 = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\tilde{U}^n(\xi)|^2 d\xi = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |g(h\xi)|^{2n}|\tilde{U}^0(\xi)|^2 d\xi$$

from Eq. (10). Using Lemma 2, we have $|g(h\xi,\Delta t)| \leq 1$ and so using Parseval's relation

$$\|U^n\|_h^2 \leq \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\tilde{U}^0(\xi)|^2 d\xi = h\sum_{m=-\infty}^{\infty}\left|U_m^0\right|^2 = \|U^0\|_h^2,$$

the result follows. ■

Hence, we conclude that the scheme [Eq. (6)] is stable in the discrete L_2 norm (Strikwerda [4, Definition 1.5.1]).

IV. ACCURACY AND CONVERGENCE

In this section, we investigate the accuracy and convergence of the scheme [Eq. (6)]. We find the rate at which the approximate solution converges to the exact solution of the IDE [Eq. (4)] for different choices of smooth and non-smooth initial conditions.

Applying the convolution property of the continuous Fourier transform to the exact IDE [Eq. (4)] we get

$$\hat{u}_t(\xi,t) = \hat{q}(\xi)\hat{u}(\xi,t), \tag{14}$$

where $\hat{q}(\xi) = \sqrt{2\pi}(\hat{J}(\xi) - \hat{J}(0))$. The solution of Eq. (14) is

$$\hat{u}(\xi,t) = e^{\hat{q}(\xi)t}\hat{u}_0(\xi). \tag{15}$$

Here, $\hat{q} \leq 0$ (from Lemma 1), gives the stability property $|\hat{u}(\xi,t)| \leq |\hat{u}_0(\xi,t)|$. Applying the inverse continuous Fourier transform to Eq. (15) we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}e^{\hat{q}(\xi)t}\hat{u}_0(\xi)d\xi,$$

which is the exact solution of Eq. (4).

Definition 2. [4, page 199] *Given $v \in L_2(h\mathbb{Z})$, then the interpolation operator $\mathcal{I}$ is defined by*

$$\mathcal{I}v(x) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi}\tilde{v}(\xi)d\xi$$

for each $x \in \mathbb{R}$. The continuous Fourier Transform of $\mathcal{I}v$ is

$$\widehat{\mathcal{I}v}(\xi) = \begin{cases} \tilde{v}(\xi) & \text{if } |\xi| \leq \dfrac{\pi}{h}, \\ 0 & \text{if } |\xi| > \dfrac{\pi}{h}. \end{cases}$$

The interpolation operator $\mathcal{I}$ maps functions in $L_2(h\mathbb{Z})$ to functions in $L_2(\mathbb{R})$.

Theorem 1. [27] *Let $u \in H^r(\mathbb{R})$ with $r > \frac{1}{2}$. The DFT and the CFT are linked by*

$$\tilde{u}(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}\left(\xi + \frac{2\pi j}{h}\right). \tag{16}$$

From Eqs. (8) and (10) we have the approximate solution

$$U_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} (g(h\xi, \Delta t))^n \tilde{u}_0(\xi) d\xi, \tag{17}$$

and the Fourier interpolant of that mesh function is

$$\mathcal{I}U^n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi} (g(h\xi, \Delta t))^n \tilde{u}_0(\xi) d\xi. \tag{18}$$

Now

$$\begin{aligned} u(x,t_n) - \mathcal{I}U^n(x) = &\frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} e^{ix\xi} (e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi) - (g(h\xi, \Delta t))^n \tilde{u}_0(\xi)) d\xi \\ &+ \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \frac{\pi}{h}} e^{ix\xi} e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi) d\xi. \end{aligned} \tag{19}$$

Using Parseval's relation [Eq. (12)] one obtains

$$\begin{aligned} \|u(x,t_n) - \mathcal{I}U^n(x)\|^2 &= \int_{-\infty}^{\infty} |u(x,t_n) - \mathcal{I}U^n(x)|^2 dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} |e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi) - (g(h\xi, \Delta t))^n \tilde{u}_0(\xi)|^2 d\xi \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \frac{\pi}{h}} |e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi)|^2 d\xi \\ &\le \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} |e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi) - (g(h\xi, \Delta t))^n \tilde{u}_0(\xi)|^2 d\xi \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \frac{\pi}{h}} |\hat{u}_0(\xi)|^2 d\xi \end{aligned} \tag{20}$$

using the stability property $\hat{q} \le 0$.

The right-hand side of Eq. (20) has a time-evolution error term and another term that is determined by the smoothness of the initial function u_0. We now bound the time dependent part. Let

$$a_1 = e^{\hat{q}(\xi)t_n} \hat{u}_0(\xi) - (g(h\xi, \Delta t))^n \hat{u}_0(\xi) \quad \text{and} \quad b_1 = (g(h\xi, \Delta t))^n \sum_{j \ne 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right),$$

noting that

$$\tilde{u}_0(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right) = \hat{u}_0(\xi) + \sum_{j\neq 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right).$$

Using the triangle inequality one obtains

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{|\xi|\leq \frac{\pi}{h}} |e^{\hat{q}(\xi)t_n}\hat{u}_0(\xi) - g(h\xi, \Delta t)^n \tilde{u}_0(\xi)|^2 d\xi &\leq \sqrt{\frac{2}{\pi}} \int_{|\xi|\leq \frac{\pi}{h}} |e^{\hat{q}(\xi)t_n} - g(h\xi, \Delta t)^n|^2 |\hat{u}_0(\xi)|^2 d\xi \\ &+ \sqrt{\frac{2}{\pi}} \int_{|\xi|\leq \frac{\pi}{h}} \left|\sum_{j\neq 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi,\end{aligned}$$

since $|g(h\xi, \Delta t)| \leq 1$. Now,

$$\left|\sum_{j\neq 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right| \leq \sqrt{\sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)^2 \left(\xi + \frac{2\pi j}{h}\right)^{2\sigma}\right|} \sqrt{\sum_{j\neq 0} \left|\left(\xi + \frac{2\pi j}{h}\right)\right|^{-2\sigma}},$$

by [4, page 204]. We assume that the initial function is smooth and there exists $\sigma > \frac{1}{2}$ such that $\|u_0\|_{H^\sigma(\mathbb{R})}$ is bounded. So, similar to [4],

$$\begin{aligned}&\sqrt{\frac{2}{\pi}} \int_{|\xi|\leq \frac{\pi}{h}} \left|\sum_{j\neq 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \\ &\quad \leq \sqrt{\frac{2}{\pi}} \int_{|\xi|\leq \frac{\pi}{h}} \left(\sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)^2 \left(\xi + \frac{2\pi j}{h}\right)^{2\sigma}\right|\right) \left(2\left(\frac{h}{\pi}\right)^{2\sigma} \sum_{j=1}^{\infty} (2j-1)^{-2\sigma}\right) d\xi,\end{aligned}$$

which gives

$$\begin{aligned}\sqrt{\frac{2}{\pi}} \int_{|\xi|\leq \frac{\pi}{h}} \left|\sum_{j\neq 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi &= h^{2\sigma} C(\sigma) \int_{|\xi|> \frac{\pi}{h}} |\hat{u}_0(\xi)|^2 |\xi|^{2\sigma} d\xi \\ &\leq C_1(\sigma) h^{2\sigma} \|u_0\|^2_{H^\sigma(\mathbb{R})}, \qquad (21)\end{aligned}$$

where $C_1(\sigma) = 2(\frac{1}{\pi})^{2\sigma} \sum_{j=1}^{\infty} (2j-1)^{-2\sigma}$. Then, following [4, page 203] for the second term in Eq. (20),

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{|\xi|> \frac{\pi}{h}} |\hat{u}_0(\xi)|^2 d\xi &\leq \frac{1}{\sqrt{2\pi}} \int_{|\xi|> \frac{\pi}{h}} \left(\frac{h}{\pi}\right)^{2\sigma} |\xi|^{2\sigma} |\hat{u}_0(\xi)|^2 d\xi \\ &\leq C_2(\sigma) h^{2\sigma} \|u_0\|^2_{H^\sigma(\mathbb{R})}. \qquad (22)\end{aligned}$$

Now, define $a = e^{\hat{q}(\xi)\Delta t}$ and note that $t_n = n\Delta t$. Thus,

$$e^{\hat{q}(\xi)t_n} - g(h\xi, \Delta t)^n = a^n - g^n = (a-g) \sum_{r=0}^{n-1} a^{n-r} g^r.$$

Because $\hat{q}(\xi) \le 0$ and $|g(h\xi, \Delta t)| \le 1$ we have $|a^n - g^n| \le n|a - g|$, or equivalently

$$|e^{\hat{q}(\xi)t_n} - g(h\xi, \Delta t)^n| \le n|e^{\hat{q}(\xi)\Delta t} - g(h\xi, \Delta t)|. \tag{23}$$

Now, for the scheme [Eq. (6)],

$$\begin{aligned} e^{\Delta t \hat{q}(\xi)} - g(h\xi, \Delta t) &= e^{\Delta t\sqrt{2\pi}(\hat{J}(\xi) - \hat{J}(0))} - (1 + \Delta t\sqrt{2\pi}(\tilde{J}(\xi) - \tilde{J}(0))) \\ &= \Delta t\sqrt{2\pi}(\hat{J}(\xi) - \hat{J}(0)) - \Delta t\sqrt{2\pi}(\tilde{J}(\xi) - \tilde{J}(0)) \\ &\quad + \sum_{j=2}^{\infty} \frac{\Delta t^j}{j!}(\sqrt{2\pi}(\hat{J}(\xi) - \hat{J}(0)))^j. \end{aligned} \tag{24}$$

Applying Theorem 1 and assuming that $J \in H^r(\mathbb{R})$ with $r > \frac{1}{2}$ one gets

$$\begin{aligned} \tilde{J}(\xi) - \tilde{J}(0) &= \sum_{j=-\infty}^{\infty} \left(\hat{J}\left(\xi + \frac{2\pi j}{h}\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right) \\ &= \hat{J}(\xi) - \hat{J}(0) + \sum_{j\neq 0} \left(\hat{J}\left(\xi + \frac{2\pi j}{h}\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right). \end{aligned} \tag{25}$$

Using Eq. (25), Eq. (24) becomes

$$\begin{aligned} e^{\Delta t \hat{q}(\xi)} - g(h\xi, \Delta t) &= -\Delta t\sqrt{2\pi} \sum_{j\neq 0} \left(\hat{J}\left(\xi + \frac{2\pi j}{h}\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right) \\ &\quad + \sum_{j=2}^{\infty} \frac{\Delta t^j}{j!}(\sqrt{2\pi}(\hat{J}(\xi) - \hat{J}(0)))^j. \end{aligned} \tag{26}$$

To obtain the main accuracy and convergence results we establish a bound for the first term in Eq. (26) under reasonable assumptions on $J(x)$.

Lemma 4. *Assume that* **H1**, **H3**, *and* **H5** *of Lemma 1 hold, and assume in addition*

H6 $\frac{d}{d\xi}\hat{J}(\xi) \le 0$ *for* $\xi \ge 0$.

Then, for all $|\xi| \le \frac{\pi}{h}$, $|\sum_{j\neq 0}(\hat{J}(\xi + \frac{2\pi j}{h}) - \hat{J}(\frac{2\pi j}{h}))| \le 2\hat{J}(\frac{\pi}{h})$.

Proof. For any $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$ and $j \in \mathbb{Z}_+$ condition **H6** ensures that

$$\hat{J}\left(\frac{\pi}{h}(2j+1)\right) \le \hat{J}\left(\xi + \frac{2\pi j}{h}\right) \le \hat{J}\left(\frac{\pi}{h}(2j-1)\right).$$

Thus, we have

$$\sum_{j=1}^{\infty} \left(\hat{J}\left(\xi + \frac{2\pi j}{h}\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right) \le \sum_{j=1}^{\infty} \left(\hat{J}\left(\frac{\pi}{h}(2j-1)\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right) \le \hat{J}_1 = \hat{J}\left(\frac{\pi}{h}\right)$$

and

$$\sum_{j=1}^{\infty}\left(\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right) \geq \sum_{j=1}^{\infty}\left(\hat{J}\left(\frac{\pi}{h}(2j+1)\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right) \geq -\hat{J}_2 \geq -\hat{J}\left(\frac{\pi}{h}\right)$$

as $\hat{J}_j \geq \hat{J}_{j+1}$ for all $j \geq 1$ where $\hat{J}_j = \hat{J}(\frac{2\pi j}{h})$. So, we have

$$\left|\sum_{j=1}^{\infty}\left(\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right)\right| \leq \hat{J}\left(\frac{\pi}{h}\right),$$

using **H5**. Hence, using **H1**, **H3**, and the definition of the Fourier transform we conclude

$$\left|\sum_{j\neq 0}\left(\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right)\right| \leq 2\hat{J}\left(\frac{\pi}{h}\right).$$

■

Example 1. Let us consider the kernel function $J(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$. The Fourier transform of $J(x)$ is $\hat{J}(\xi) = \frac{1}{\sqrt{2\pi}}e^{\frac{-\xi^2}{400}}$. For this $\hat{J}(\xi)$ we have (using Maple)

$$\left|\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right| \leq \frac{1}{\sqrt{2\pi}}e^{-\frac{\pi^2}{400h^2}(2|j|-1)}.$$

So,

$$\sum_{j\neq 0}\left|\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right| \leq \sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}e^{-\frac{\pi^2}{400h^2}(2|j|-1)}.$$

Thus, using Maple

$$\sum_{j\neq 0}\left|\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right| \leq \sqrt{\frac{2}{\pi}}\frac{e^{\frac{1}{400}\frac{\pi^2}{h^2}}}{e^{\frac{1}{200}\frac{\pi^2}{h^2}}-1} \leq Ce^{-\frac{1}{400}\frac{\pi^2}{h^2}},$$

where $C > \sqrt{\frac{2}{\pi}}$ and from Lemma 4,

$$\sum_{j\neq 0}\left|\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right| \leq Ce^{-\frac{\pi^2}{400h^2}}.$$

Example 2. Consider the kernel function $J(x) = \frac{c}{2}e^{-c|x|}$, then $\hat{J}(\xi) = \frac{c^2}{\sqrt{2\pi}(c^2+\xi^2)}$. Now for this $\hat{J}(\xi)$ we have for $j \geq 1$ (using Maple)

$$\sum_{j\neq 0}\left|\hat{J}\left(\xi+\frac{2\pi j}{h}\right)-\hat{J}\left(\frac{2\pi j}{h}\right)\right| \leq \frac{(\sinh(ch)-ch)}{\sqrt{2\pi}\sinh(ch)}.$$

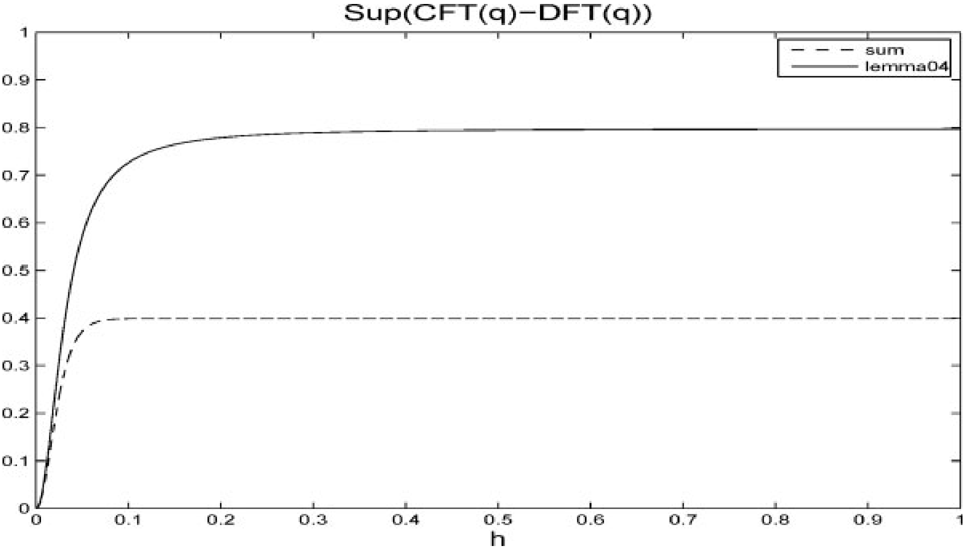

FIG. 2. The figure shows $\sum_{j\neq 0}|\hat{J}(\xi+\frac{2\pi j}{h})-\hat{J}(\frac{2\pi j}{h})|$ for both cases with $J(x)=\frac{c}{2}e^{-c|x|}$ with $c=100$.

The Fig. 2 plots the upper bound of the sum obtained from the Lemma 4 and from numerical observation. Here, we consider $J(x)=\frac{c}{2}e^{-c|x|}$. We observe that the infinite sum is bounded and hence prevails a bound for the numerical sum by Lemma 4.

We return to Eq. (26). We apply Lemma 1 and find

$$\sum_{j=2}^{\infty}\frac{\Delta t^j}{j!}(\sqrt{2\pi}(\hat{J}(\xi)-\hat{J}(0)))^j \leq \sum_{j=2}^{\infty}\frac{\Delta t^j}{j!}(2)^j=\exp(2\Delta t)-(1+2\Delta t)<C_2\Delta t^2. \tag{27}$$

Thus, applying Lemma 4 and Eq. (27), Eq. (26) implies

$$|e^{\Delta t\hat{q}(\xi)}-g(h\xi,\Delta t)|\leq \Delta t(C_1(h)+C_2\Delta t), \tag{28}$$

where $C_1(h)=2\sqrt{2\pi}\hat{J}(\frac{\pi}{h})$. If $J\in L_2(\mathbb{R})$, then $|\hat{J}(\xi)|\to 0$ as $|\xi|\to\infty$ [28], [29, page 30]. So, $C_1(h)\to 0$ as $h\to 0$. The rate of convergence determines the accuracy of the scheme. In the Example 1, one has $C_1(h)=\sqrt{\frac{2}{\pi}}e^{-\frac{\pi^2}{400h^2}}$, which decays exponentially fast as $h\to 0$. In the Example 2, $C_1(h)=\frac{c^2h^2}{\sqrt{2\pi}(c^2h^2+\pi^2)}$, which is $\mathcal{O}(h^2)$ as $h\to 0$. We have

$$\begin{aligned}
&\int_{|\xi|\leq\frac{\pi}{h}}|(e^{\hat{q}(\xi)t_n}-g(h\xi,\Delta t)^n)\hat{u}_0(\xi)|^2d\xi \\
&\leq \int_{|\xi|\leq\frac{\pi}{h}} n^2|e^{\hat{q}(\xi)\Delta t}-g(h\xi,\Delta t)|^2|\hat{u}_0(\xi)|^2d\xi, \quad \text{using Eq. (23)} \\
&\leq n^2\Delta t^2\int_{|\xi|\leq\frac{\pi}{h}}(C_1(h)+C_2\Delta t)^2|\hat{u}_0(\xi)|^2d\xi, \quad \text{using Eq. (28)} \\
&\leq t_n^2\int_{-\infty}^{\infty}(C_1(h)+C_2\Delta t)^2|\hat{u}_0(\xi)|^2d\xi \leq t_n^2(C_1(h)+C_2\Delta t)^2\|u_0\|^2.
\end{aligned} \tag{29}$$

Thus, applying Eqs. (21), (22), and (29), the estimate [Eq. (20)] takes the form

$$\|u(x,t_n) - \mathcal{I}U^n(x)\| \le t_n(C_1(h) + C_2\Delta t)\|u_0\| + C_3(\sigma)h^\sigma\|u_0\|_{H^\sigma(\mathbb{R})} \tag{30}$$

for all $u_0 \in H^\sigma(\mathbb{R})$ with $\sigma > \frac{1}{2}$. We end up with the following result.

Theorem 2. *If $J(x)$ and $\hat{J}(\xi)$ satisfy the assumptions* **H1–H6**, *then there exists $\Delta t^* > 0$ such that the approximation [Eq. (6)] of the IDE [Eq. (4)] is stable for all $0 < \Delta t \le \Delta t^*$ and, if in addition, $u_0 \in H^\sigma(\mathbb{R})$ with $\sigma > \frac{1}{2}$ then there exist constants $C_1(h)$, C_2, $C_3(\sigma)$ such that*

$$\|u(x,t_n) - \mathcal{I}U^n(x)\| \le t_n(C_1(h) + C_2\Delta t)\|u_0\| + C_3(\sigma)h^\sigma\|u_0\|_{H^\sigma(\mathbb{R})}.$$

A. Convergence Estimation for Piecewise Continuous Non-Differentiable Initial Data

Now, we consider continuous but not necessarily differentiable initial data. Here, we assume that the initial function is not smooth enough for $\|u_0\|_{H^\sigma(\mathbb{R})}$ to be bounded with $\sigma > \frac{1}{2}$. Instead we assume that there exists $\sigma_1 < \frac{1}{2}$ such that $\|u_0\|_{H^{\sigma_1}(\mathbb{R})}$ is bounded and, that for non-smooth initial functions there exists $\alpha > 0$ such that

$$\int_{|\xi|\le\frac{\pi}{h}} \sum_{j\ne 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \le h^{2\alpha}C(u_0). \tag{31}$$

Before we start our main discussion let us show the motivation with an example.

Example 3. Consider the initial function (similar to [4, page 217])

$$u_0(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ -e^{-x} & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{32}$$

The continuous Fourier transform of $u_0(x)$ is $\hat{u}_0(\xi) = \sqrt{\frac{2}{\pi}}\frac{i\xi}{1+\xi^2}$. Now, for any $\sigma_1 < \frac{1}{2}$ (using Maple)

$$\|u_0\|^2_{H^{\sigma_1}} = \int_{-\infty}^{\infty} (1+\xi^2)^{\sigma_1}|\hat{u}_0(\xi)|^2 d\xi \le C\Gamma\left(\frac{1}{2} - \sigma_1\right),$$

where $\Gamma(\cdot)$ denotes the Gamma function [25]. We evaluate the integral in Maple. Thus, $\|u_0\|_{H^{\sigma_1}} \le C^*$ for $\sigma_1 < \frac{1}{2}$. Also, for all $|\xi| \le \frac{\pi}{h}$ one has

$$\sum_{j\ne 0}\left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 = \frac{2}{\pi}\sum_{j\ne 0}\left(\frac{\left(\xi + \frac{2\pi j}{h}\right)^2}{\left(1 + \left(\xi + \frac{2\pi j}{h}\right)^2\right)^2}\right) \le \frac{h}{\pi}\tan h\left(\frac{1}{2}h\right).$$

Hence

$$\int_{|\xi|\le\frac{\pi}{h}} \sum_{j\ne 0}\left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \le \frac{1}{\pi}h\tan h\left(\frac{1}{2}h\right)\int_{|\xi|\le\frac{\pi}{h}} d\xi \le Ch$$

with $\alpha = \frac{1}{2}$, which is of the form Eq. (31).

Now, we return to the discussion of the accuracy of the approximation [Eq. (6)] of the IDE [Eq. (4)] with initial functions $u_0 \in H^\sigma(\mathbb{R})$, $\sigma < \frac{1}{2}$. We start by rewriting Eq. (20) as

$$\begin{aligned}
\|u(x,t_n) - \mathcal{I}U^n(x)\|^2 &= \int_{-\infty}^{\infty} |u(x,t_n) - \mathcal{I}U^n(x)|^2 dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{|\xi|\le\mu} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \tilde{u}_0(\xi)|^2 d\xi + \frac{1}{\sqrt{2\pi}} \int_{|\xi|>\frac{\pi}{h}} \left|e^{\hat{q}(\xi)t_n}\right|^2 |\hat{u}_0(\xi)|^2 d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mu<|\xi|\le\frac{\pi}{h}} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \tilde{u}_0(\xi)|^2 d\xi,
\end{aligned}$$

where $\mu = \frac{\pi}{h^\tau}$ for some $0 < \tau < 1$. Using results from the analysis of the continuous initial data problem we have

$$\begin{aligned}
&\|u(x,t_n) - \mathcal{I}U^n(x)\|^2 \\
&\le \frac{2}{\sqrt{2\pi}} \int_{|\xi|\le\mu} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \hat{u}_0(\xi)|^2 d\xi + \sqrt{\frac{2}{\pi}} \int_{|\xi|\ge\frac{\pi}{h}} |\hat{u}_0(\xi)|^2 d\xi \\
&\quad + \sqrt{\frac{2}{\pi}} \int_{\mu\le|\xi|\le\frac{\pi}{h}} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \hat{u}_0(\xi)|^2 d\xi \\
&\quad + \sqrt{\frac{2}{\pi}} \int_{|\xi|\le\mu} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi + \sqrt{\frac{2}{\pi}} \int_{\mu\le|\xi|\le\frac{\pi}{h}} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \\
&\le \sqrt{\frac{2}{\pi}} \int_{|\xi|\le\mu} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \hat{u}_0(\xi)|^2 d\xi + 4\sqrt{\frac{2}{\pi}} \int_{\mu\le|\xi|} |\hat{u}_0(\xi)|^2 d\xi \\
&\quad + \sqrt{\frac{2}{\pi}} \int_{|\xi|\le\mu} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi + \sqrt{\frac{2}{\pi}} \int_{\mu\le|\xi|\le\frac{\pi}{h}} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi.
\end{aligned}$$

Now, as in Eq. (29) we have

$$\sqrt{\frac{2}{\pi}} \int_{|\xi|\le\mu} |e^{\hat{q}(\xi)n\Delta t}\hat{u}_0(\xi) - (g(h\xi))^n \hat{u}_0(\xi)|^2 d\xi \le t_n^2 (C_1(h) + C_2\Delta t)^2 \|\hat{u}_0\|^2. \tag{33}$$

Using relation (31) one obtains

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}} \int_{|\xi|\le\mu} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi + \sqrt{\frac{2}{\pi}} \int_{\mu\le|\xi|\le\frac{\pi}{h}} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \\
&\qquad = \frac{2}{\sqrt{2\pi}} \int_{|\xi|\le\frac{\pi}{h}} \sum_{j\neq 0} \left|\hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi \le h^{2\alpha} C(u_0),
\end{aligned} \tag{34}$$

and finally

$$4\frac{2}{\sqrt{2\pi}}\int_{\mu\le|\xi|}|\hat{u}_0(\xi)|^2d\xi \le 4\frac{2}{\sqrt{2\pi}}\int_{\mu\le|\xi|}\left(\frac{h^\tau}{\pi}\right)^{2\sigma_1}|\xi|^{2\sigma_1}|\hat{u}_0(\xi)|^2d\xi$$
$$\le C(\sigma_1)h^{2\sigma_1\tau}\|\hat{u}_0(\xi)\|^2_{H^{\sigma_1}(\mathbb{R})}, \tag{35}$$

with τ and σ_1 defined above. From Eqs. (33)–(35), we conclude

$$\|u(x,t_n)-\mathcal{I}U^n(x)\| \le t_n(C_1(h)+C_2\Delta t)\|u_0\| + C_3(\sigma_1)h^{\sigma_1\tau}\|u_0\|_{H^{\sigma_1}(\mathbb{R})} + h^\alpha C(u_0) \tag{36}$$

for all $u_0 \in H^{\sigma_1}(\mathbb{R})$ with $\sigma_1 < \frac{1}{2}$. We summarize our results in the following theorem.

Theorem 3. *If $J(x)$ and $\hat{J}(\xi)$ satisfy the assumptions* **H1–H6**, *then there exists $\Delta t^* > 0$ such that the approximation [Eq. (6)] of the IDE [Eq. (4)] is stable for all $0 < \Delta t \le \Delta t^*$ and, if in addition, $\|u_0\|_{H^{\sigma_1}(\mathbb{R})}$ is bounded with some $\sigma_1 < \frac{1}{2}$, and Eq. (31) is satisfied, then there exist constants $C_1(h)$, C_2, $C_3(\sigma_1)$ and $C(u_0)$ such that*

$$\|u(x,t_n)-\mathcal{I}U^n(x)\| \le t_n(C_1(h)+C_2\Delta t)\|u_0\| + C_3(\sigma_1)h^{\sigma_1\tau}\|u_0\|_{H^{\sigma_1}(\mathbb{R})} + h^\alpha C(u_0)$$

for all $u_0 \in H^{\sigma_1}(\mathbb{R})$ with $\sigma_1 < \frac{1}{2}$.

B. Convergence in a Discrete Norm

Here, we estimate the accuracy of the scheme at the grid points $h\mathbb{Z}$ instead of the continuous approximation over $\mathbb{R}$ given by the Fourier interpolant of the approximation. We estimate the difference between $u(mh,t_n)$ and U^n_m at the grid points.

Definition 3 (Similar to [4]). *The truncation operator $T : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined by*

$$Tv(x) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi}\hat{v}(\xi)d\xi, \quad \textit{where} \quad v(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}\hat{v}(\xi)d\xi \in L_2(\mathbb{R}).$$

The Fourier Transform of Tv is

$$\widehat{Tv}(\xi) = \begin{cases} \hat{v}(\xi), & \textit{if } |\xi| \le \dfrac{\pi}{h} \\ 0, & \textit{if } |\xi| > \dfrac{\pi}{h}. \end{cases}$$

We have

$$\|u(\cdot,t_n)-U^n(\cdot)\|_h \le \|u(\cdot,t_n)-Tu(\cdot,t_n)\|_h + \|Tu(\cdot,t_n)-U^n(\cdot)\|_h. \tag{37}$$

The exact solution $u(x,t)$ can be written as

$$u(x,t) = \int_{-\infty}^{\infty} e^{ix\xi}e^{\hat{q}(\xi)t}\hat{u}_0(\xi)d\xi = \sum_{j=-\infty}^{\infty}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi}e^{\hat{q}\left(\xi+\frac{2\pi}{h}j\right)t}\hat{u}_0\left(\xi+\frac{2\pi}{h}j\right)d\xi.$$

Now, using Parseval's relation we estimate

$$\begin{aligned}\|u(\cdot,t)-Tu(\cdot,t)\|_h^2 &= \int_{|\xi|\le\frac{\pi}{h}}\left|\sum_{j\neq 0} e^{\hat{q}\left(\xi+\frac{2\pi}{h}j\right)t}\hat{u}_0\left(\xi+\frac{2\pi}{h}j\right)\right|^2 d\xi \\ &\le \int_{|\xi|\le\frac{\pi}{h}}\left|\sum_{j\neq 0}\hat{u}_0\left(\xi+\frac{2\pi}{h}j\right)\right|^2 d\xi \le C(\sigma)h^{2\sigma}\|u_0\|_{H^\sigma(\mathbb{R})}^2, \qquad (38)\end{aligned}$$

using the inequality (21) and $\hat{q}(\xi)\le 0$. From Eq. (15), one has

$$Tu(mh,t_n)=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}e^{\hat{q}(\xi)t_n}\hat{u}_0(\xi)d\xi,$$

and Eq. (17) gives an approximation at each grid point. We have

$$Tu(mh,t_n)-U_m^n=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}(e^{\hat{q}(\xi)t_n}\hat{u}_0(\xi)-(g(h\xi,\Delta t))^n\tilde{u}_0(\xi))d\xi.$$

Now using Parseval's relation [Eq. (13)] we have

$$\begin{aligned}\|(Tu)(\cdot,t_n)-U^n(\cdot)\|_h^2 &= h\sum_{j=-\infty}^{\infty}\left|(Tu)(jh,t_n)-U_j^n\right|^2 \\ &\le \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}|e^{\hat{q}(\xi)t_n}\hat{u}_0(\xi)-(g(h\xi,\Delta t))^n\tilde{u}_0(\xi)|^2d\xi. \qquad (39)\end{aligned}$$

Thus, applying Eqs. (21) and (29), the estimate [Eq. (39)] takes the form

$$\|Tu(\cdot,t_n)-U^n(\cdot)\|_h\le t_n(C_1(h)+C_2\Delta t)\|u_0\|+C_3(\sigma)h^\sigma\|u_0\|_{H^\sigma(\mathbb{R})} \qquad (40)$$

for all $u_0\in H^\sigma(\mathbb{R})$ with $\sigma>\frac{1}{2}$. Finally, combining Eqs. (40), (38), and (37) we get

$$\|u(.,t_n)-U^n(\cdot)\|_h\le t_n(C_1(h)+C_2\Delta t)\|u_0\|+C_3(\sigma)h^\sigma\|u_0\|_{H^\sigma(\mathbb{R})}, \qquad (41)$$

for all $u_0\in H^\sigma(\mathbb{R})$ with $\sigma>\frac{1}{2}$. This gives the following theorem.

Theorem 4. *If $J(x)$ and $\hat{J}(\xi)$ satisfy the assumptions* **H1–H6**, *then there exists $\Delta t^*>0$ such that the approximation [Eq. (6)] of the IDE [Eq. (4)] is stable for all $0<\Delta t\le\Delta t^*$ and, if in addition, $U(mh,0)=u_0(mh)$ where $u_0\in H^\sigma(\mathbb{R})$ with $\sigma>\frac{1}{2}$, then there exist constants $C_1(h)$, C_2, $C_3(\sigma)$ such that*

$$\|u(.,t_n)-U^n(\cdot)\|_h\le t_n(C_1(h)+C_2\Delta t)\|u_0\|+C_3(\sigma)h^\sigma\|u_0\|_{H^\sigma(\mathbb{R})}.$$

To show the computational error with some smooth and non-smooth initial functions, we compute the error in the approximation Eqs. (6) of (4). For the computation of the error term, we consider the functions $\hat{u}(\xi,t_n)$ defined in Eq. (15) in the domain $\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]$ and $\tilde{U}^n(\xi)$ defined in Eq. (10). We first compute the solutions in the Fourier domain, and find the difference

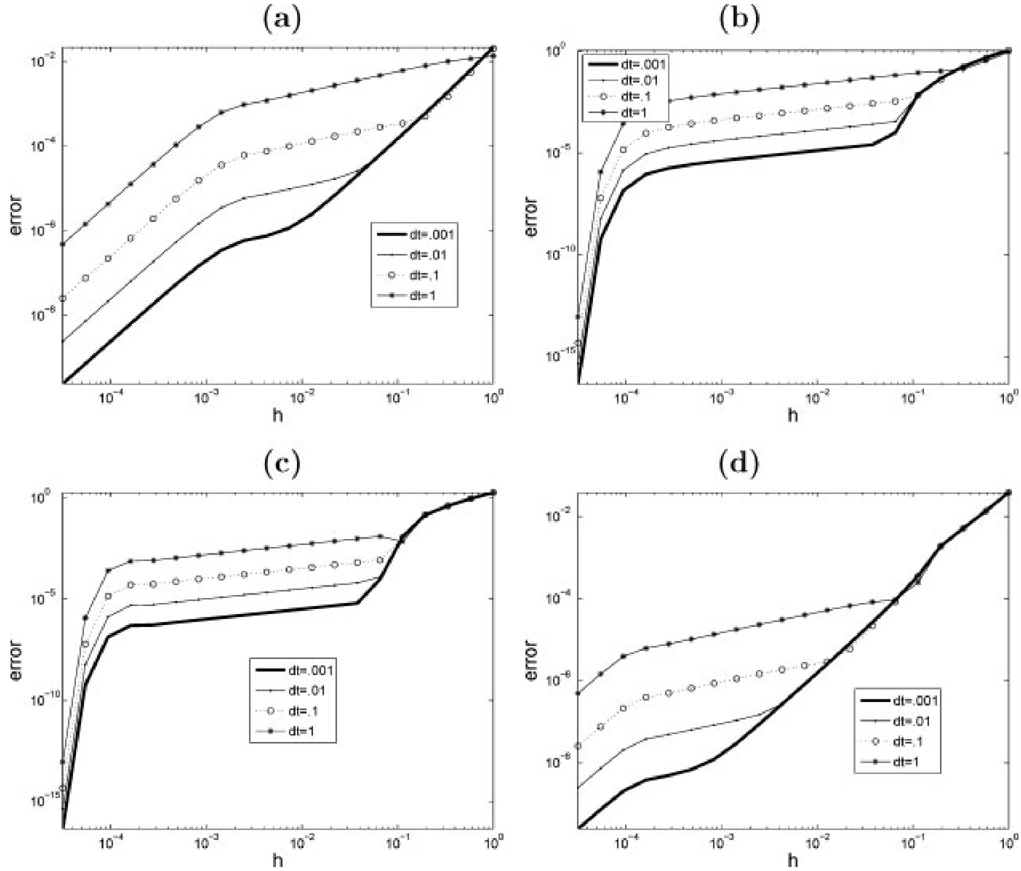

FIG. 3. Here, we plot the error term $\|u(\cdot,t_n) - U^n(\cdot)\|_h$ at $t = 1$ by varying Δt and h with (a) $J(x) = \frac{1}{2}e^{-|x|}$ and $u_0(x) = \frac{1}{2}e^{-|x|}$, (b) $J(x) = \frac{1}{2}e^{-|x|}$ and $u_0(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$, (c) $J(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$ and $u_0(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$, and (d) $J(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$ and $u_0(x) = \frac{1}{2}e^{-|x|}$.

$E(\cdot,t_n) = |\hat{u}(\cdot,t_n) - \tilde{U}^n(\cdot)|$. Then, we use MATLAB ifft(E) to get the error in the spatial domain and find the discrete L_2 norm of the error. Figure 3 shows the behavior of $\|u(\cdot,t_n) - U^n(\cdot)\|_h$ for various choices of h and Δt with smooth and non-smooth $J(x)$ and u_0 for all $x \in \mathbb{R}$ at $t = 1$. We observe from Fig. 3(b,c) that for smooth u_0 the rate of convergence of the solutions is faster than for the non-smooth u_0 shown in Figs. 3(a,d).

V. ACCURACY AND CONVERGENCE OF THE SEMIDISCRETE APPROXIMATION

Applying the discrete Fourier transform to the semidiscrete approximation [Eq. (5)] and using standard convolution results we get

$$\tilde{U}_t(\xi,t) = \tilde{q}(\xi)\tilde{U}(\xi,t) \tag{42}$$

where $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$ and $\tilde{q}(\xi) = \sqrt{2\pi}(\tilde{J}(\xi) - \tilde{J}(0))$. The solution of Eq. (42) is

$$\tilde{U}(\xi, t) = e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi). \tag{43}$$

Applying the inverse discrete Fourier transform to Eq. (43) we have

$$U_m(t) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi)d\xi. \tag{44}$$

Here, we return to the main discussion. Recall the Fourier interpolant (Definition 2)

$$\mathcal{I}U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix\xi} e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi)d\xi.$$

Now using Eq. (15)

$$\begin{aligned} u(x,t) - \mathcal{I}U(x,t) = {} & \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} e^{ix\xi}(e^{\hat{q}(\xi)t}\hat{u}_0(\xi) - e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi))d\xi \\ & + \frac{1}{\sqrt{2\pi}} \int_{|\xi| \ge \frac{\pi}{h}} e^{ix\xi} e^{\hat{q}(\xi)t}\hat{u}_0(\xi)d\xi. \end{aligned} \tag{45}$$

Using Parseval's relation [Eq. (12)] we have

$$\begin{aligned} \|u(\cdot,t) - \mathcal{I}U(\cdot,t)\|^2 &= \int_{-\infty}^{\infty} |u(\cdot,t) - \mathcal{I}U(\cdot,t)|^2 dx \\ &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left|\hat{u}(\xi,t) - \widetilde{\mathcal{I}U}(\xi,t)\right|^2 d\xi + \int_{|\xi| \ge \frac{\pi}{h}} |\hat{u}(\xi,t)|^2 d\xi \\ &\le \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} \left|e^{\hat{q}(\xi)t}\hat{u}_0(\xi) - e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi)\right|^2 d\xi + \frac{1}{\sqrt{2\pi}} \int_{|\xi| \ge \frac{\pi}{h}} |\hat{u}_0(\xi)|^2 d\xi \end{aligned} \tag{46}$$

since by Lemma 1 $\hat{q} \le 0$. Using Theorem 1 linking the DFT and the CFT, the Cauchy-Schwartz inequality [30, page 206], and the stability assumptions, the first part of the right-hand side of Eq. (46) can be written as

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} \left|e^{\hat{q}(\xi)t}\hat{u}_0(\xi) - e^{\tilde{q}(\xi)t}\tilde{U}_0(\xi)\right|^2 d\xi \\ & \qquad \le \frac{2}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} |e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t}|^2 |\hat{u}_0(\xi)|^2 d\xi + \frac{2}{\sqrt{2\pi}} \int_{|\xi| \le \frac{\pi}{h}} \left|\sum_{j \ne 0} \hat{u}_0\left(\xi + \frac{2\pi j}{h}\right)\right|^2 d\xi. \end{aligned}$$

Now,

$$e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t} = e^{\hat{q}(\xi)t}(1 - e^{(\tilde{q}(\xi) - \hat{q}(\xi))t}).$$

Applying the Mean Value Theorem [31, page 5] for the function $e^{(\tilde{q}(\xi)-\hat{q}(\xi))t}$ on the interval $(0, t)$, there exists $\tau \in (0, t)$ such that

$$e^{(\tilde{q}(\xi)-\hat{q}(\xi))t} = 1 + (\tilde{q}(\xi) - \hat{q}(\xi))te^{(\tilde{q}(\xi)-\hat{q}(\xi))\tau}.$$

So,

$$e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t} = t(\hat{q}(\xi) - \tilde{q}(\xi))e^{\hat{q}(\xi)(t-\tau)}e^{\tilde{q}(\xi)\tau}. \tag{47}$$

Thus, $|e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t}| \le t|\hat{q}(\xi) - \tilde{q}(\xi)|$ since $\tilde{q}(\xi) \le 0 \quad \text{and} \quad \hat{q}(\xi) \le 0$.

Hence,

$$\int_{|\xi|\le\frac{\pi}{h}} |e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t}|^2|\hat{u}_0(\xi)|^2 d\xi \le t^2 \int_{|\xi|\le\frac{\pi}{h}} |\hat{q}(\xi) - \tilde{q}(\xi)|^2|\hat{u}_0(\xi)|^2 d\xi.$$

Now apply Theorem 1 and Lemma 4 to obtain

$$|\hat{q}(\xi) - \tilde{q}(\xi)| = \sqrt{2\pi} \sum_{j=-\infty, j\ne 0}^{\infty} \left| \hat{J}\left(\xi + \frac{2\pi j}{h}\right) - \hat{J}\left(\frac{2\pi j}{h}\right)\right| \le C(h),$$

where $C(h) = 2\hat{J}(\frac{\pi}{h})$ goes to zero as h goes to zero (similar to (28)).

Hence,

$$\int_{|\xi|\le\frac{\pi}{h}} |e^{\hat{q}(\xi)t} - e^{\tilde{q}(\xi)t}|^2|\hat{u}_0(\xi)|^2 d\xi \le t^2 C(h)^2 \|u_0\|^2. \tag{48}$$

Thus, applying Eqs. (21), (22), and (48), Eq. (46) takes the form

$$\|u(x,t) - \mathcal{I}U(t)\| \le tC_1(h)\|u_0\| + C_3(\sigma)h^{\sigma}\|u_0\|_{H^{\sigma}(\mathbb{R})} \tag{49}$$

for all $u_0 \in H^{\sigma}(\mathbb{R})$ with $\sigma > \frac{1}{2}$. This gives the following result.

Theorem 5. *If Eq. (5) is a semidiscrete approximation to the IDE [Eq. (4)], $J, \hat{J}$ satisfy the assumptions* **H1–H6** *and $u_0 \in H^{\sigma}(\mathbb{R})$ with $\sigma > \frac{1}{2}$, then there exist constants $C_1(h)$, $C_3(\sigma)$ such that*

$$\|u(x,t) - \mathcal{I}U(t)\| \le tC_1(h)\|u_0\| + C_3(\sigma)h^{\sigma}\|u_0\|_{H^{\sigma}(\mathbb{R})}.$$

With similar computations as of Fig. 3, we compute the error $E(\cdot, t) = |\hat{u}(\cdot, t) - \tilde{U}(\cdot, t)|$ of the approximations (5) of (4). For the computation of the error term, we consider the functions $\hat{u}(\xi, t_n)$ defined in Eq. (15) in the domain $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$ and $\tilde{U}(\xi, t)$ defined in Eq. (43) in each grid point. We have plotted $\|u(\cdot, t) - U(\cdot, t)\|_h$ in Fig. 4 with smooth and non-smooth $J(x)$ and u_0 with various choices of space discretization at $t = 1$. We notice that the rates of convergence in Fig. 4(a,d) are very close to $\mathcal{O}(h^{1.8})$ where as Fig. 4(b,c) have very fast rate of convergence. Particularly from Fig. 3(b,c) and Fig. 4(b,c) we notice that approximate solutions with smooth $u_0(x)$ and $J(x)$ give a very fast convergence rate which also agree with our theoretical error estimates.

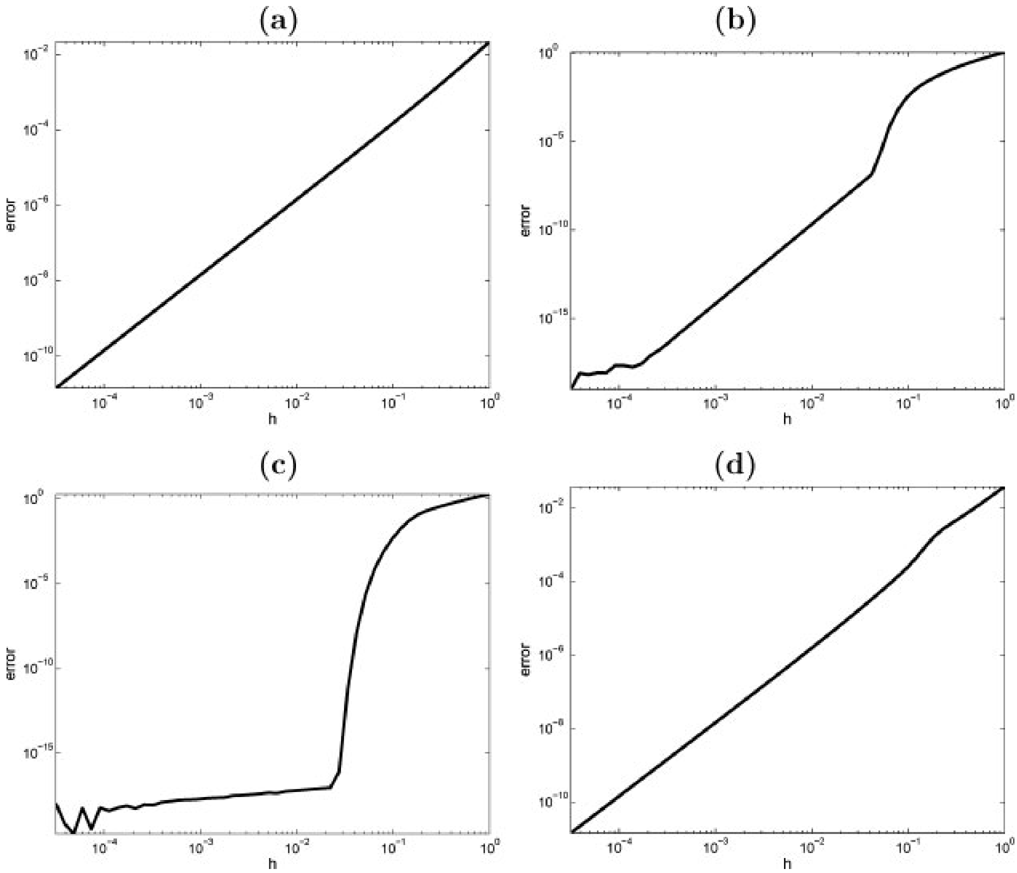

FIG. 4. Here, we plot the error in $\|u(\cdot,t) - U(\cdot,t)\|_h$ varying h at $t = 1$. We consider (a) $J(x) = \frac{1}{2}e^{-|x|}$ and $u_0(x) = \frac{1}{2}e^{-|x|}$, (b) $J(x) = \frac{1}{2}e^{-|x|}$ and $u_0(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$, (c) $J(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$ and $u_0(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$, and (d) $J(x) = \sqrt{\frac{100}{\pi}}e^{-100x^2}$ and $u_0(x) = \frac{1}{2}e^{-|x|}$.

A. Discrete Norm Convergence

Here, we follow the same steps as of Section B to estimate

$$\|u(\cdot,t) - U(\cdot,t)\|_h \leq \|u(\cdot,t) - Tu(\cdot,t)\|_h + \|Tu(\cdot,t) - U(\cdot,t)\|_h. \tag{50}$$

From (15),

$$Tu(mh,t) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} e^{\hat{q}(\xi)t} \hat{u}_0(\xi) d\xi,$$

and Eq. (44) gives approximation at each grid point. Thus, we have

$$Tu(mh,t) - U(mh,t) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} (e^{\hat{q}(\xi)t} \hat{u}_0(\xi) - e^{\tilde{q}(\xi)t} \tilde{u}_0(\xi)) d\xi.$$

Now, using Parseval's relation [Eq. (13)] we have

$$
\begin{aligned}
\|(Tu)(\cdot,t) - U(\cdot,t)\|_h^2 &= h \sum_{j=-\infty}^{\infty} \left|(Tu)(jh,t) - U_j^n\right|^2 \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |e^{\hat{q}(\xi)t}\hat{u}_0(\xi) - e^{\tilde{q}(\xi)t}\tilde{u}_0(\xi)|^2 d\xi. \qquad (51)
\end{aligned}
$$

Thus, applying Eqs. (21) and (48), Eq. (51) takes the form

$$
\|Tu(\cdot,t_n) - U(\cdot,t_n)\|_h \leq tC_1(h)\|u_0\| + C_3(\sigma)h^\sigma \|u_0\|_{H^\sigma(\mathbb{R})} \qquad (52)
$$

when $\sigma > \frac{1}{2}$. Finally, combining Eqs. (52), (38), and (50) we get

$$
\|u(\cdot,t) - U(\cdot,t)\|_h \leq tC_1(h)\|u_0\| + C_3(\sigma)h^\sigma \|u_0\|_{H^\sigma(\mathbb{R})}, \qquad (53)
$$

for all $u_0 \in H^\sigma(\mathbb{R})$ with $\sigma > \frac{1}{2}$. Thus, we get the following bound.

Theorem 6. *If Eq. (5) is a semidiscrete approximation to the IDE [Eq. (4)], $J, \hat{J}$ satisfy the assumptions* **H1–H6** *and $U(mh,0) = u(mh,0)$ where $u_0 \in H^\sigma(\mathbb{R})$ with $\sigma > \frac{1}{2}$, then there exist constants $C_1(h)$, $C_3(\sigma)$ such that*

$$
\|u(\cdot,t) - U(\cdot,t)\|_h \leq tC_1(h)\|u_0\| + C_3(\sigma)h^\sigma \|u_0\|_{H^\sigma(\mathbb{R})}.
$$

VI. CONCLUSIONS

A simple one step approximation for a linear integro-differential equation has been analyzed. We notice that the full discrete approximation is stable. The scheme is convergent in space and time. The rate of accuracy of the approximation depends on the choices of the initial function and the kernel function considered which have been presented theoretically as well as via numerical experiments. In this study, it is noticed that both the full discrete approximation and the semi-discrete approximation for the IDE give the same rate of convergence in $L_2(\mathbb{R})$ and $L_2(h\mathbb{Z})$.

The author is thankful to Professor Dugald B. Duncan for his valuable advise and kind help. He is thankful to Professor Gabriel Lord, Professor Andrew Lacey and Dr. David F. Griffiths for their valuable suggestions.

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