Numerical Solutions of Higher Order Boundary Value Problems Using Piecewise Polynomial Bases

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of

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by

Md. Bellal Hossain

Registration No. 05, Session 2010-2011

Under the supervision of

Dr. Md. Shafiqul Islam

Professor, Department of Mathematics

University of Dhaka, Dhaka-1000

June 2014

This work is dedicated to the sacred memory of

my late father

CERTIFICATE

This is to certify that the thesis entitled "**Numerical Solutions of Higher Order Boundary Value Problems Using Piecewise Polynomial Bases"** submitted by Mr. Md. Bellal Hossain to the University of Dhaka, Dhaka-1000, is a record of bonafide research work carried out under my supervision and I consider it to be worthy of consideration for the award of the degree of Doctor of Philosophy of the University.

(Dr. Md. Shafiqul Islam)
Supervisor and Professor
Department of Mathematics
University of Dhaka,
Dhaka-1000, Bangladesh.

CANDIDATE'S DECLARATION

I here by declare that this work which is being presented in this thesis entitled **"Numerical Solutions of Higher Order Boundary Value Problems Using Piecewise Polynomial Bases"** submitted in partial fulfillment of the requirement for the award of the degree of Ph.D in Mathematics under the University of Dhaka, Dhaka-1000, Bangladesh is an authentic record of my own work.

It has not been submitted elsewhere (Universities or Institutions) for the award of any other degree.

(Md. Bellal Hossain)

Date: 22 June 2014

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Contents

Abstract

The Boundary Value Problems (BVPs), either the linear or nonlinear, arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Many studies concerned with solving second order boundary value problems using several numerical methods. But few studies concerned with especial cases of higher order BVPs have been solved applying several numerical techniques. In our thesis, we have used the Galerkin technique for solving higher order linear and nonlinear BVPs (from order four up to order twelve). The well known Bernstein and Legendre polynomials are exploited as basis functions in the technique.

The main steps, in this thesis, depend on:

1. To use the Bernstein and Legendre polynomials we need to satisfy the corresponding homogeneous form of the boundary conditions and modification is thus needed.
2. A rigorous matrix formulation is developed by the Galerkin method for linear and nonlinear systems and solved it using Bernstein and Legendre polynomials.
3. Using the Newton's iterative method for nonlinear problems to obtain more accurate results.

The numerical results for the Galerkin method that appear in this thesis are good, but the errors in the method increase when the order of the differential equations become high. Also the accuracy of the method depends on the boundary conditions as well as the changes on the order of boundary conditions. In addition, this method requires long computing time when the order of the BVPs increase.

The thesis entitled "**Numerical solutions of higher order BVPs using piecewise polynomial bases**" contains ten chapters, among them the first chapter is confined as "**Introduction**". In this chapter we discuss some mathematical preliminaries

which are important to study the problems investigated in this thesis, such as, some theorems, corollaries which are used in the subsequent chapters, Bernstein polynomials and its properties; Legendre polynomials and its properties, etc. We also include in this chapter, the objectives and scope and a layout of the thesis.

Chapter 2 is devoted to find the **numerical solutions of the fourth order linear and nonlinear differential equations** using Bernstein and Legendre polynomials as basis functions. We derive rigorous matrix Formulations: Formulation I, Formulation II and Formulation III, by Galerkin method for two different types of boundary conditions.

(1). Formulation I: $a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b$

subject to the boundary conditions

$u(a) = A_0,\ u(b) = B_0, u'(a) = A_1, u'(b) = B_1.$

(2) Formulation II: $a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b$

subject to the boundary conditions

$u(a) = A_0, u(b) = B_0, u''(a) = A_2, u''(b) = B_2.$

(3) Formulation III: $\frac{d^2}{dx^2}\left(p(x)\frac{d^2u}{dx^2} \right) + r(x)u = s(x), a \le x \le b$

subject to the boundary conditions

$u(a) = A_0, u(b) = B_0, u''(a) = A_2, u''(b) = B_2.$

For the numerical verification of the proposed formulations we consider four linear and two nonlinear BVPs. It is observed that the approximate solutions converge to the exact solutions even with desired large significant digits.

The **numerical solutions of fifth order BVPs** is studied in chapter 3. In this chapter, we first derive the matrix formulation for solving linear fifth order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials as trial functions and then we extend our idea for solving nonlinear differential equations. Two linear and two nonlinear BVPs are considered to verify the reliability and efficiency of the proposed method. The computed results are presented in tabular form and also graphically. It is noted that the present method is quite efficient and yields better results when compared with the existing methods.

Chapter 4 is dealt with the **numerical solutions of sixth order BVPs** and is devoted to find the numerical solutions of linear and nonlinear differential equations using Formulation I and Formulation II which are derived for two types of boundary conditions by the Galerkin method. In this method the basis functions are modified into a new set of basis functions which must satisfy the corresponding homogeneous form of Dirichlet boundary conditions. Numerical verification of the method is performed by considering four linear and two nonlinear BVPs. It is found that the obtained results are superior to other existing methods.

The **computations of seventh order linear and nonlinear BVPs** are provided in chapter 5 by the Galerkin method using Bernstein and Legendre polynomials as basis functions. The basis functions are transformed into a new set of basis functions to satisfy the corresponding homogeneous form of boundary conditions where the essential types of boundary conditions are mentioned. The method is formulated as a rigorous matrix form which is tested on three linear and one nonlinear BVPs. The numerical results are shown both in tabular form and also by the depicted of graphs, and it is observed that the results are better than other existing methods.

The **numerical solutions of eighth order BVPs** are illustrated in chapter 6. In this chapter, five linear and two nonlinear differential equations are solved numerically by Galerkin method with Bernstein and Legendre polynomials as basis functions using Formulation I and Formulation II for two different types of boundary conditions. The numerical results of the proposed method are compared with both the exact solutions and the results of the other methods available in the literature. The comparison shows that the present method is of great accuracy and convenient.

Chapter 7 is devoted to find the **numerical solutions of ninth order BVPs**. The aim of this chapter is to apply Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions. The method is formulated as a rigorous matrix form. Then only one numerical example of linear BVP is considered, which is available in all existing literatures, to verify the proposed formulation and the solution is thus compared with the existing methods.

Chapter 8 is entitled as **Tenth Order BVPs.** In this chapter, we consider Formulation I and Formulation II for two different kinds of boundary conditions by the Galerkin method. Then we solve five linear and two nonlinear BVPs using these two formulations and we get better results than the previous results.

In chapter 9, we consider an application of Galerkin method for **the numerical solutions of linear and nonlinear eleventh order BVPs** with Bernstein and Legendre polynomials as basis functions. Results of two linear and one nonlinear BVPs are tabulated to compare the errors with those methods developed previously.

The last chapter entitled **Twelfth Order BVPs** is devoted to find the numerical solutions of linear and nonlinear differential equations by the Galerkin method using Formulation I and Formulation II for two types of boundary conditions. In this chapter we have solved four linear and two nonlinear BVPs and get superior results to other existing methods.

All problems of this thesis have been solved by using the software MATLAB, which is used to perform scientific computations and visualization. Finally, the conclusion and a list of references are appended towards the last of this thesis.

List of chapter wise publications

1. **Md. Shafiqul Islam** and **Md. Bellal Hossain** (2013) – On the Use of Piecewise Standard Polynomials in the Numerical Solutions of Fourth Order Boundary Value Problems, *GANIT Journal of Bangladesh Mathematical Society*, 33, 53-64. (Chapter 2)
2. **Md. Bellal Hossain** and **Md. Shafiqul Islam** (2014) – Numerical Solutions of General Fourth Order Two point Boundary Value Problems by Galerkin Method with Legendre Polynomials, *The Dhaka University Journal of Science*, Accepted for publication, Vol. 62(2). (Chapter 2)
3. **Md. Bellal Hossain** and **Md. Shafiqul Islam** (2014) – A Novel Numerical Approach for Odd Higher Order Boundary Value Problems, *Mathematical Theory and Modeling*, 4(5), 1-11. (Chapters 3, 5 & 7)
4. **Md. Bellal Hossain** and **Md. Shafiqul Islam** (2014) – Numerical Solutions of Sixth Order Linear and Nonlinear Boundary Value Problems, *Journal of Advances in Mathematics*, 7(2), 1180 – 1190. (Chapter 4)
5. **Md. Bellal Hossain** and **Md. Shafiqul Islam** – The Numerical Solution of Eighth Order Boundary Value Problems Using Piecewise Polynomials by the Galerkin Method, Submitted (Chapter 6)
6. **Md. Shafiqul Islam, Md. Bellal Hossain** – Approximate Solutions of Tenth and Twelfth Order Boundary Value Problems using Bernstein Polynomials, *International Journal of Applied Mathematics and Computation*, Submitted. (Chapters 8 & 10)
7. **Md. Bellal Hossain**, **Md. Shafiqul Islam** (2014)– Numerical Solutions of Eleventh Order Boundary Value Problems Using Piecewise Polynomials, Submitted.(Chapter 9)

CHAPTER 1

Introduction

In the real life phenomena, the boundary value problems (BVPs) either the linear or nonlinear problems have many scientific applications. These problems occur in many branches of applied mathematics, theoretical physics, and engineering, the most significant among them being the boundary layer theory, the study of stellar interiors and control and optimization theory. Problems involving the wave equation, such as the determination of normal modes, are often stated as BVPs. A large class of important BVPs is the Sturm-Liouville problems. Much theoretical work in the field of partial differential equations is devoted to proving that BVPs arising from scientific and engineering applications are in fact well-posed. Higher order BVPs occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering and applied physics. The BVPs of higher order have been examined due to their mathematical importance and applications in diversified applied sciences. The BVPs for singularly perturbed differential difference equations arise in various practical problems in biomechanics and physics such as in variation problems in control theory and depolarization in Stein's model. The depolarization in Stein's model is continuous time, continuous state space, Markov process whose sample paths have discontinuities of first kind. The mathematical modeling of the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general BVP for singularly perturbed differential difference equations with small shifts. Many phenomena in applied mathematics and other sciences can be described very successfully by models using mathematical tools from ordinary differential equations.

1.1 Objectives and Scope of the Thesis

The BVPs for fourth order differential equations arise in variety of areas of applied mathematics, physics and variational problems of control theory. For example $u^4(t) = f(t, u(t))$ subject to boundary value conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ can be used to model the deflection of elastic beams simply supported at the endpoints. The fourth order problem is applied as an advanced version of the Bolzano's theorem. Many approaches such as the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, fixed point theorems and coincidence degree theory are used to acquire the existence and multiplicity results. The fourth order BVP is the problem of bending a rectangular clamped beam of length l resting on an elastic foundation. The vertical deflection ω of the beam satisfies the system

$$\left[L + \left(\frac{k}{D}\right)\right]\omega = D^{-1}q(x)$$

$$\omega(0) = \omega(l) = \omega'(0) = \omega'(l) = 0.$$

where $L = \frac{d^4}{dx^4}, D$ is the flexural rigidity of the beam, k is the spring constant of the elastic foundation and the load $q(x)$ acts vertically downwards per unit length of the beam.

The induction motor behavior is represented by a fifth order differential equation model. Addition of a torque correction factor to the model accurately reproduces the transient torques and instantaneous real and reactive power flows of the full seventh order differential equation model. Singular fifth order BVPs arise in the fields of gas dynamics, Newtonian fluid mechanics, fluid mechanics, fluid dynamics, elasticity, reaction-diffusion processes, chemical kinetics and other branches of applied mathematics. These problems generally arise in the mathematical modeling of viscoelastic flows and other branches of mathematical, physical and engineering sciences. In general, it is not possible to obtain the analytical solution of the fifth order BVPs. Consequently, we usually resort to some numerical methods for obtaining an approximate solution of the fifth order BVPs.

Many mathematical models arising in various applications can be written as BVPs. One such problem is the sixth order BVP which plays an important role in astrophysics and the narrow convecting layers bounded by stable layers which are believed to surround A-type stars. It may also note that dynamo action in some stars may be modeled by such equation.

The seventh order BVPs generally arise in modeling induction motors with two rotor circuits. The induction motor behavior is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage but it does not hold up during the transient conditions, when rotor frequency is not a single value. So the behavior of such models shows up in the seventh order.

Eighth order BVPs govern the physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability sets in as over stability, it is modeled by an eighth order ordinary differential equation. Eighth order differential equations are also modeled while considering the motion of a cylindrical shell. Equations for the equilibrium in terms of displacement components for an orthotropic thin circular cylindrical shell subjected to a load that is not symmetric about the axis of the shell, which resulted in eighth order differential equations. We would like to point out that the eighth order BVPs arise in the torsional vibration of uniform beam.

The BVPs of ninth order have been presented due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability whereas the eleventh order BVPs have been developed due to their mathematical importance and the potential for applications in numerous fields of science and engineering. If a uniform magnetic field is applied across the fluid in the same direction so that of gravity, then the instability may be sets in as over stability which can be modeled by a twelfth or eighth order BVP; whereas the instability which occur as ordinary convection can be modeled by a tenth order BVP. Twelfth order differential equations have several important applications in engineering. Such problems arise in geophysics when studying core fluid adjacent to the core mantle boundary.

Therefore, we have attempted to solve numerically higher order (fourth to twelfth order) BVPs by applying the following steps to get high accuracy:

- To use the Bernstein and Legendre polynomials we need to satisfy the corresponding homogeneous form of boundary conditions and modification is thus needed.
- A rigorous matrix formulation is developed by the Galerkin method for linear and nonlinear systems using Bernstein and Legendre polynomials as trial functions.
- Using the Newton's iterative method for nonlinear problems to obtain more accurate results.

The computed results, in respect of all the above mentioned BVPs, are represented through tables and graphs. For doing this, all calculations are performed by the appropriate and widely used software – *MATLAB*.

1.2 Plan of the Thesis

To achieve the objectives set out in the previous section, the plan of the thesis comprises the following ten chapters:

- Chapter 1 contains some important definitions which are related to our thesis and will be used in the subsequent chapters. This chapter also highlights in detail the properties of Bernstein and Legendre polynomials which are used as basis functions in the Galerkin method to study the problems in this thesis. Objectives and scope of the thesis are also given in this chapter.
- Chapter 2 is dealt with the numerical solutions of the fourth order BVPs where Bernstein and Legendre polynomials are used as basis functions. We derive matrix Formulation I, Formulation II and Formulation III by applying the Galerkin method with two different types of boundary conditions for solving these problems.
- Chapter 3 is devoted to find the numerical solutions of fifth order BVPs. In this chapter we first derive the matrix formulation for solving linear fifth order BVP and then we extend our idea for solving nonlinear differential equations.
- The numerical solutions of sixth order BVPs using Formulation I and Formulation II for two types of boundary conditions by the Galerkin method is discussed in Chapter 4.
- In Chapter 5, the numerical solutions of seventh order BVPs by the Galerkin method are evaluated. The method is formulated as a rigorous matrix form which is tested on several linear and nonlinear BVPs to compare the results with the existing methods.
- We provide the numerical solutions of eighth order BVPs by applying Formulation I and Formulation II for two different types of boundary conditions in Chapter 6. The numerical results of the proposed method show that the present method is of high precision, efficient and convenient.
- In Chapter 7, we illustrate the numerical solutions of ninth order BVPs. The aim of this chapter is to apply Galerkin weighted residual method with

Bernstein and Legendre polynomials as basis functions for numerical solution of a linear BVP.

- Chapter 8 is concentrated on the numerical solutions of tenth order BVPs. In this chapter we consider Formulation I and Formulation II for two different kinds of boundary conditions. Several numerical examples both linear and nonlinear BVPs are solved using these two formulations and we get better results than the previous results obtained so far.
- In Chapter 9, we consider an application of Galerkin method for the numerical solutions of eleventh order BVPs with Bernstein and Legendre polynomials as basis functions. Numerical results of several linear and nonlinear BVPs are tabulated to compare the errors with those developed so far.
- Numerical solutions of twelfth order BVPs are investigated in Chapter 10 using Formulation I and Formulation II for two types of boundary conditions. Some numerical examples are considered to verify the proposed method.
- Conclusions and references are given towards the last of the thesis.

1.3 Some Mathematical Preliminaries

In this section we discuss some important definitions those are related to our thesis and the properties of Bernstein and Legendre polynomials which are used as basis functions in the Galerkin method.

1.3.1 Definitions

In this section we write down some definitions which will be used in subsequent chapters of our thesis.

Solution of a differential Equation:

Consider the *n*th-order ordinary differential equation

$$f\left[x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right] = 0 \tag{1.1}$$

where *f* is a real function of its (*n*+*2*) arguments $x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}$.

Then, we may say that a solution of the differential equation (1.1) is a relation between independent and dependent variables, not containing derivatives, which identically satisfies the differential equation (1.1).

Example: The function f defined for all real x by

$f(x) = \sin x$ is a solution of the differential equation $\frac{d^2 f}{dx^2} + f = 0$, that

$-\sin(x) + \sin(x) = 0$ which holds for all real x.

Analytical solution:

An exact solution to a problem that can be calculated symbolically by manipulating equations. Manipulating means to work, operate or treat with hand.

Numerical solution:

Numerical analysis is a technique to do higher mathematics problems on a computer and also widely used by scientists and engineers to solve their problems. An important advantage for numerical analysis is that a numerical solution can be obtained even when a problem has no "analytical" solution. It is important to realize that a numerical solution is always numeric but analytical methods usually give a result in terms of mathematical functions that can be evaluated for specific instances. However, numerical results can be plotted to show some of the behavior of the solution. Another important distinction is that the result from numerical analysis is an approximation, but results can be made as accurate as desired.

Initial value problem: Constraints that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems.

Boundary value problem:

Constraints that are specified at the boundary points, generally space points, are called boundary conditions. The number of boundary conditions is usually equal to the order of the ODE. Problems with specified boundary conditions are called boundary value problems.

Boundary value problems (BVPs) for ordinary linear or nonlinear differential equations arise in many branches of applied mathematics, theoretical physics and engineering, the most important among them being the boundary layer theory, the study of stellar interiors and control and optimization theory.

Existence and Uniqueness of Solution:

Here, we shall consider the following *n*th order differential equation

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1} y}{dx^{n-1}}\right) \tag{1.2}$$

with boundary conditions

$$\left.\begin{array}{c} y(a_i) = A_{1,i}, y'(a_i) = A_{2,i} \cdots, y^{(k_i)}(a_i) = A_{k_i+1,i} \\ a \le a_1 < a_2 < \cdots < a_r \le b, 0 \le k_i \end{array}\right\} \tag{1.3}$$

and

$$\left.\begin{array}{l} y^{(i)}(a_1) = A_i, i = 0,1,\cdots,n-2 \\ y^{(p)}(a_r) = B_p, (0 \le p \le n-1) \end{array}\right\} \tag{1.4}$$

also

$$\left.\begin{array}{l} y^{(p)}(a_1) = A_p, (0 \le p \le n-1) \\ y^{(i)}(a_r) = B_i, i = 0,1,\cdots,n-2 \end{array}\right\} \tag{1.5}$$

The function $y(x)$ can be written equivalently the integral equation as

$$y(x) = l_j(x) + \int_{a_1}^{a_r} G_j(x,s) f(s, y(s), \cdots, y^{(q)}(s)) ds \tag{1.6}$$

where

$l_j(x)$ is polynomial of degree (*n-1*) satisfying eqn. (1.3) for *j=1*, (1.4) for *j=2* and (1.5) for *j=3* and also

$G_1(x,s) = g(x,s), G_2(x,s) = -h_1(x,s), G_3(x,s) = -h_2(x,s)$

The function *f* we shall assume continuous on $[a,b] \times R^{q+1}$ throughout without mention.

Theorem 1.1 [8]: Let $k_i > 0, i = 0,1,\cdots,q$ be given real numbers and let Q be the maximum of $\left|f(t,u_0,u_1,\cdots,u_q)\right|$ on the compact set

$\{(t,u_0,u_1,\cdots,u_q), a \le t \le b, \left|u_i\right| \le 2k_i, i = 0,1,\cdots,q\}$

then if

Case1:

$$\max_{a \le t \le b} \left|l_1^{(i)}(t)\right| \le k_i \text{ and } (b-a) \le \left(\frac{k_i}{QC_{n,i}}\right)^{\frac{1}{n-i}}, i = 0,1,\cdots,q \tag{1.7}$$

the boundary value problem (1.2) with boundary conditions (1.3) has a solution.

Case2: $a_1 = a, a_r = b$

$$\max_{a_1 \le t \le a_r} \left|l_1^{(i)}(t)\right| \le k_i \text{ and } (a_r - a_1) \le \left(\frac{k_i}{QC^{**}_{n,i}}\right)^{\frac{1}{n-i}}, i = 0,1,\cdots,q \tag{1.8}$$

the boundary value problem (1.2) with boundary conditions (1.3) has a solution.

Case3: $a_1 = a, a_r = b$

$$\max_{a_1 \le t \le a_r} \left|l_j^{(i)}(t)\right| \le k_i \text{ and } (a_r - a_1) \le \left(\frac{k_i}{Q\alpha_{n,i}}\right)^{\frac{1}{n-i}}, i = 0,1,\cdots,q, j = 2,3 \tag{1.9}$$

the BVP (1.2) with boundary conditions (1.4) or (1.5) has a solution

Theorem 1.2 [8]: Let $f(t,u_0,u_1,\cdots,u_q)$ satisfy the condition

$$\left|f(t,u_0,u_1,\cdots,u_q)\right| \le L + \sum_{j=0}^{q} L_j \left|u_j\right| \tag{1.10}$$

For all $(t,u_0,u_1,\cdots,u_q) \in [a_1,a_r] \times R^{q+1}$ where L is any number and let $L_j\,(j = 0,1,\cdots,q)$ satisfy the inequality

$\theta = \sum_{i=0}^{q} C^{**}_{n,i} L_i (a_r - a_1)^{n-i} < 1$. Then the BVP (1.2) with boundary conditions (1.3) has at least one solution for any $A_{i,k}$

Theorem 1.3 [8]: Let $f(t,u_0,u_1,\cdots,u_q)$ satisfy the Lipschitz condition

$$\left|f(t,u_0,u_1,\cdots,u_q)-f(t,v_0,v_1,\cdots,v_q)\right|\le\sum_{i=0}^{q}L_i\left|u_i-v_i\right|$$

Then

1. if $\theta<1$, the BVP (1.2) with boundary conditions (1.3) has a unique solution for any $A_{i,k}$

2. if $\alpha=\sum_{i=0}^{q}\alpha_{n,i}L_i(a_r-a_1)^{n-i}<1$ (1.11)

each of the BVPs (1.2) with boundary conditions (1.4); (1.2) with boundary conditions (1.5) has a unique solution for any A_i and B_i

Residual Function:

To obtain the residual function [10] we first collect all the terms in the differential equation on the left hand side. The exact solution will produce an answer which is identically zero for all values of x in the problem domain when substituted into the left hand side. But an approximate solution will not produce an identically zero function but a function say, $R(x)$ which is called residual function.

Example: If $\dfrac{d^4u}{dx^4}+4u=1$ is a differential equation. Then $R(x)=\dfrac{d^4\tilde{u}}{dx^4}+4\tilde{u}-1$ is the Residual Function of the differential equation where $\tilde{u}(x)$ is the approximate solution.

Galerkin Method:

A more commonly used technique involving residuals, particularly in connection with finite elements, is Galerkin's method [10]. Our first step is to look for an approximate solution of the form: $\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}a_i\theta_i(x)$

where

(a) the function $\theta_0(x)$ is chosen to satisfy the given boundary conditions of the problem.

(b) The functions $\theta_i(x), i = 1,2,\cdots,n$ must each satisfy the corresponding homogenous form of the boundary conditions. In this method we determine the *n* unknown parameters by selecting *n* weighting functions which are multiplied by unknown parameters. Galerkin's method then involves determining these parameters by solving the *n* weighted residual equations:

$$\int_a^b R(x)\theta_i(x)dx = 0, i = 1,2,\cdots,n$$

Modified Galerkin Method:

If we continue to use the Galerkin technique in conjunction with piecewise linear coordinate functions then second derivative terms in the differential equation would make no contribution to the approximation leading to poor results. Hence it is desirable to use an alternative weighted residual technique which involves only first derivative terms. The new technique is obtainable using integration by parts from the standard Galerkin approach and is known as the modified Galerkin method [10].

Also in the modified Galerkin technique we shall demand of the trial solution still taken in the form

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} a_i\theta_i(x) \tag{1.12}$$

where $\theta_0(x)$ satisfies any essential boundary condition present and $\theta_i(x), i = 1,\cdots,n$ should satisfy the corresponding homogeneous form of any such essential boundary condition.

It is important to realize that boundary conditions are of two basic types, referred to as essential and suppressible. For second-order differential equations a boundary condition containing a derivative term is called suppressible; otherwise it is referred to as essential. For example consider a second-order differential equation:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = r(x), \quad a \le x \le b \tag{1.13}$$

with boundary conditions

$$\left.\begin{aligned}\alpha_0 u(a)+\alpha_1 u'(a)=c_1\\ \beta_0 u(b)+\beta_1 u'(b)=c_2\end{aligned}\right\} \tag{1.14}$$

where $p(x), q(x)$ and $r(x)$ are specified continuous functions and $\alpha_0, \alpha_1, \beta_0, \beta_1$, c_1, c_2 are given constants. Here if α_1, β_1 are both non-zero then both boundary conditions are suppressible. If $\alpha_1 = 0$ and $\beta_1 \neq 0$ then the first condition is essential, the second is suppressible and so on.

Now using eqn. (1.12) into eqn. (1.13) we obtain weighted residual equations of the form

$$\int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{d\tilde{u}}{dx}\right)+q(x)\tilde{u}-r(x)\right]\theta_j(x)dx = 0,\ j = 1,2,3,\cdots,n \tag{1.15}$$

Integrating first term by parts of eqn. (1.15) and using eqn. (1.12) and after minor simplification, we can obtain

$$\int_a^b \left\{p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx}+q(x)\theta_0\theta_j - r(x)\theta_j + \sum_{i=1}^{n}\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i\theta_j\right]a_i\right\}dx$$

$$-p(b)\tilde{u}'(b)\theta_j(b)+p(a)\tilde{u}'(a)\theta_j(a)=0 \tag{1.16}$$

Now we consider the following boundary conditions:

Case 1: *Robin* (mixed) boundary conditions $(i.e., \alpha_0 \neq 0, \alpha_1 \neq 0, \beta_0 \neq 0, \beta_1 \neq 0)$

From eqn. (1.14) we have

$$\tilde{u}'(a) = \frac{c_1 - \alpha_0\tilde{u}(a)}{\alpha_1} \text{ and } \tilde{u}'(b) = \frac{c_2 - \beta_0\tilde{u}(b)}{\beta_1}$$

Hence eqn. (1.16) becomes

$$\sum_{i=0}^{n}\left[\int_a^b\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i\theta_j\right]dx + \frac{\beta_0 p(b)\theta_i(b)\theta_j(b)}{\beta_1} - \frac{\alpha_0 p(a)\theta_i(a)\theta_j(a)}{\alpha_1}\right]a_i$$

$$= \int_a^b\left[r(x)\theta_j(x) - p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx} - q(x)\theta_0\theta_j\right]dx + \frac{c_2 p(b)\theta_j(b)}{\beta_1} - \frac{c_1 p(a)\theta_j(a)}{\alpha_1}$$

$$-\frac{\beta_0 p(b)\theta_j(b)\theta_0(b)}{\beta_1} + \frac{\alpha_0 p(a)\theta_j(a)\theta_0(a)}{\alpha_1}$$

Or, equivalently in matrix form

$$\sum_{i=1}^{n} D_{i,j}\ a_i = F_j\ ,\quad j = 1,2,\cdots,n$$

where

$$D_{i,j} = \int_a^b \left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx} + q(x)\theta_i\theta_j \right] dx + \frac{\beta_0 p(b)\theta_i(b)\theta_j(b)}{\beta_1} - \frac{\alpha_0 p(a)\theta_i(a)\theta_j(a)}{\alpha_1}$$

$$F_j = \int_a^b \left[r(x)\theta_j(x) - p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx} - q(x)\theta_0\theta_j \right] dx + \frac{c_2 p(b)\theta_j(b)}{\beta_1} - \frac{c_1 p(a)\theta_j(a)}{\alpha_1}$$

$$-\frac{\beta_0 p(b)\theta_j(b)\theta_0(b)}{\beta_1} + \frac{\alpha_0 p(a)\theta_j(a)\theta_0(a)}{\alpha_1}$$

Case 2: ***Dirichlet*** boundary conditions $(i.e., \alpha_0 \neq 0, \alpha_1 = 0, \beta_0 \neq 0, \beta_1 = 0)$

Here the boundary terms vanish because the boundary conditions imply $\theta_j(a) = 0$ and $\theta_j(b) = 0$.

Hence from eqn. (1.16), we obtain

$$\sum_{i=0}^{n} \left[\int_a^b \left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx} + q(x)\theta_i\theta_j \right] \right] a_i dx = \int_a^b \left[r(x)\theta_j(x) - p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx} - q(x)\theta_0\theta_j \right] dx$$

where

$$D_{i,j} = \int_a^b \left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx} + q(x)\theta_i\theta_j \right] dx,\quad i,j = 1,2,\cdots,n$$

$$F_j \int_a^b \left[r(x)\theta_j(x) - p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx} - q(x)\theta_0\theta_j \right] dx$$

Case 3: ***Neumann*** boundary conditions $(i.e., \alpha_0 = 0, \alpha_1 \neq 0, \beta_0 = 0, \beta_1 \neq 0)$

From eqn. (1.14) we have

$\tilde{u}'(a) = \dfrac{c_1}{\alpha_1}$ and $\tilde{u}'(b) = \dfrac{c_2}{\beta_1}$

Hence eqn. (1.16) reduces to

$$\sum_{i=0}^{n}\left[\int_a^b\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i\theta_j\right]\right]a_i dx=\int_a^b\left[r(x)\theta_j(x)-p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx}-q(x)\theta_0\theta_j\right]dx$$

$$+\frac{c_2 p(b)\theta_j(b)}{\beta_1}-\frac{c_1 p(a)\theta_j(a)}{\alpha_1}$$

where

$$D_{i,j}=\int_a^b\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i\theta_j\right]dx,\quad i,j=1,2,\cdots,n$$

$$F_j=\int_a^b\left[r(x)\theta_j(x)-p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx}-q(x)\theta_0\theta_j\right]dx+\frac{c_2 p(b)\theta_j(b)}{\beta_1}-\frac{c_1 p(a)\theta_j(a)}{\alpha_1}$$

Case 4: *Cauchy* boundary conditions:

***(i)*:** when $\alpha_1\neq 0,\beta_1=0$

Here we obtain from eqn. (1.14) that

$\tilde{u}'(a)=\dfrac{c_1-\alpha_0\tilde{u}(a)}{\alpha_1}$ and $\beta_0\tilde{u}(b)=c_2$

It also follows that $\theta_j(b)=0$

Therefore eqn. (1.16) becomes is this case

$$\sum_{i=0}^{n}\left[\int_a^b\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i\theta_j\right]dx-\frac{\alpha_0 p(a)\theta_i(a)\theta_j(a)}{\alpha_1}\right]a_i$$

$$=\int_a^b\left[r(x)\theta_j(x)-p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx}-q(x)\theta_0\theta_j\right]dx-\frac{c_1 p(a)\theta_j(a)}{\alpha_1}+\frac{\alpha_0 p(a)\theta_j(a)\theta_0(a)}{\alpha_1}$$

where

$$D_{i,j}=\int_a^b\left[p(x)\frac{d\theta_i}{dx}\frac{d\theta_j}{dx}+q(x)\theta_i(x)\theta_j(x)\right]dx-\frac{\alpha_0 p(a)\theta_i(a)\theta_j(a)}{\alpha_1}$$

$$F_j=\int_0^1\left[r(x)\theta_j-p(x)\frac{d\theta_0}{dx}\frac{d\theta_j}{dx}-q(x)\theta_0\theta_j\right]dx-\frac{c_1 p(a)\theta_j(a)}{\alpha_1}+\frac{\alpha_0 p(a)\theta_0(a)\theta_j(a)}{\alpha_1}$$

***(ii)*:** when $\alpha_1=0,\beta_1\neq 0$

Here we obtain from eqn. (1.14) that

$\alpha_0 \tilde{u}(a) = c_1$ and $\tilde{u}'(b) = \dfrac{c_2 - \beta_0 \tilde{u}(b)}{\beta_1}$

It also follows that $\theta_j(a) = 0$

Therefore eqn. (1.16) becomes is this case

$$\sum_{i=0}^{n} \left[\int_a^b \left[p(x) \frac{d\theta_i}{dx} \frac{d\theta_j}{dx} + q(x)\theta_i \theta_j \right] dx + \frac{\beta_0 p(b)\theta_i(b)\theta_j(b)}{\beta_1} \right] a_i$$
$$= \int_a^b \left[r(x)\theta_j(x) - p(x) \frac{d\theta_0}{dx} \frac{d\theta_j}{dx} - q(x)\theta_0 \theta_j \right] dx + \frac{c_2 p(b)\theta_j(b)}{\beta_1} - \frac{\beta_0 p(b)\theta_j(b)\theta_0(b)}{\beta_1}$$

where

$$D_{i,j} = \int_a^b \left[p(x) \frac{d\theta_i}{dx} \frac{d\theta_j}{dx} + q(x)\theta_i(x)\theta_j(x) \right] dx + \frac{\beta_0 p(b)\theta_i(b)\theta_j(b)}{\beta_1}$$

$$F_j = \int_0^1 \left[r(x)\theta_j - p(x) \frac{d\theta_0}{dx} \frac{d\theta_j}{dx} - q(x)\theta_0 \theta_j \right] dx + \frac{c_2 p(b)\theta_j(b)}{\beta_1} - \frac{\beta_0 p(b)\theta_j(b)\theta_0(b)}{\beta_1}$$

Piecewise polynomials or basis functions:

A piecewise polynomial or basis function is a function defined on $[a,b]$ by $p(x) = p_i(x),\ x_i \le x \le x_{i+1},\ i = 0,1,\cdots,n-1$ where for $i = 0,1,\cdots,n-1$ each function $p_i(x)$ is a polynomial defined on $[x_i, x_{i+1}]$. The degree of $p(x)$ is the maximum degree of each polynomial $p_i(x)$ for $i = 0,1,\cdots,n-1$

Dual basis:

In linear algebra, given a vector space *V* with a basis *B* of vectors indexed by an index set *l* (the cardinality of *l* is the dimensionality of *V*), its dual set is a set B^* of vectors in the dual space V^* with the same index set *l* such that *B* and B^* form a biorthogonal system. The dual set is always linearly independent but does not necessarily span V^*. If it does span V^*, then B^* is called the dual basis for the basis *B*. Denoting the indexed vector sets as $B = \{v_i\}_{i \in I}$ and $B^* = \{v^i\}_{i \in I}$, being biortho- gonal means that the elements pair to 1 if the indexes are equal and to zero otherwise. Symbolically, evaluating a dual vector in V^* on a vector in the original space V:

$$v^i\left(v_j\right) = \delta^i_j = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

where δ^i_j is the kronecker delta symbol.

1.3.2 Bernstein polynomials and its properties:

The Bernstein polynomials are used in approximations of functions as well as in other fields such as smoothing in statistics, in numerical analysis, constructing the Bezier curves. The Bernstein polynomials are also used to solve differential equations. According to Farouki [106], the Bernstein polynomial basis was introduced 100 years ago (Bernstein, 1912) as a means to constructively prove the ability of polynomials to approximate any continuous function, to any desired accuracy, over a prescribed interval. Their slow convergence rate and the lack of digital computers to efficiently construct them caused the Bernstein polynomials to lie dormant in the theory rather than practice of approximation for the better part of a century. The Bernstein coefficients of a polynomial provide valuable insight into its behavior over a given finite interval, yielding many useful properties and elegant algorithms that are now being increasingly adopted in other application domains.

The general form of the Bernstein polynomials [2–4] of nth degree over the interval $[a,b]$ is defined by

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, a \leq x \leq b, i = 0,1,2,\ldots,n$$

The Bernstein polynomials of nth degree form a complete basis over $[0,1]$ and they are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, 0 \leq i \leq n$$

where the binomial coefficients are given by $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$.

For example, the first 11 Bernstein polynomials of degree 10 over the interval $[0,\ 1]$ are given bellow:

$B_0(x) = (1-x)^{10}$ $\quad B_4(x) = 210(1-x)^6 x^4$ $\quad B_8(x) = 45(1-x)^2 x^8$

$B_1(x) = 10(1-x)^9 x$ $\quad B_5(x) = 252(1-x)^5 x^5$ $\quad B_9(x) = 10(1-x)x^9$

$B_2(x) = 45(1-x)^8 x^2$ $\quad B_6(x) = 210(1-x)^4 x^6$ $\quad B_{10}(x) = x^{10}$

$B_3(x) = 120(1-x)^7 x^3$ $\quad B_7(x) = 120(1-x)^3 x^7$

Note that each of these n+1 polynomials having degree n satisfies the following properties:

(i) For convenience we set, $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$ (1.17)

(ii) It can be readily shown that each of the Bernstein polynomials is positive and also the sum of all the Berstein polynomials is unity for all real x belonging to the interval $[0,\ 1]$ that is, $\sum_{i=0}^{n} B_{i,n}(x) = 1$ (1.18)

(iii) It can be easily shown that any given polynomial of degree n can be expanded in terms of a linear combination of the basis functions, that is,

$$p(x) = \sum_{i=0}^{n} C_i B_{i,n}(x), n \geq 1 \tag{1.19}$$

(iv) without first and last Bernstein polynomials, all Bernstein polynomials becomes zero at the points 0 and 1, that is,

$$B_{i,n}(0) = B_{i,n}(1) = 0, \quad i = 1,2,\ldots,n-1 \tag{1.20}$$

(v) The derivatives of the nth degree Bernstein polynomials are polynomials of degree (n-1) and are given by

$$D(B_{i,n}(x)) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x)), D \equiv \frac{d}{dx} \tag{1.21}$$

(vi) The multiplication of two Bernstein bases is

$$B_{i,j}(x) B_{k,m}(x) = \frac{\binom{j}{i}\binom{m}{k}}{\binom{j+m}{i+k}} B_{i+k,j+m}(x) \tag{1.22}$$

and the moments of Bernstein basis are

$$x^m B_{i,n}(x) = \frac{\binom{n}{i}}{\binom{n+m}{i+m}} B_{i+m,n+m}(x) \tag{1.23}$$

(vii) Like any basis of the space $\prod_n$, the Bernstein polynomials have a unique dual basis $(D_{0,n}, D_{1,n}, \cdots, D_{n,n})$ which consists of the $n+1$ dual basis functions

$$D_{i,n}(x) = \sum_{j=0}^{n} c_{i,j} B_{j,n}(x),\ (j = 0,1,\cdots,n) \tag{1.24}$$

where

$$c_{i,j} = \frac{(-1)^{i+j}}{\binom{n}{i}\binom{n}{j}} \sum_{k=0}^{\min(i,j)} (2k+1)\binom{n+k+1}{n-i}\binom{n-k}{n-i}\binom{n+k+1}{n-j}\binom{n-k}{n-j},\ (i,j = 0,1,\cdots,n) \tag{1.25}$$

The dual basis functions must satisfy the relation of duality

$$\int_0^1 B_{i,n}(x) D_{k,n}(x) = \delta_{i,k} \tag{1.26}$$

$(viii)$ Indefinite integral of Bernstein basis is given by

$$\int B_{i,n}(x)dx = \frac{1}{n+1} \sum_{j=i+1}^{n+1} B_{j,n+1}(x) \tag{1.27}$$

and all Bernstein basis function of the same order have the same definite integral over the interval $[0,\ 1]$ namely

$$\int_0^1 B_{i,n}(x)dx = \frac{1}{n+1} \tag{1.28}$$

For these properties, Bernstein polynomials are used as the trail functions satisfying the corresponding homogeneous form of the *essential* boundary conditions in the Galerkin method to solve a BVP.

Derivatives of Bernstein Polynomials

The main objective of this section is to prove the following two theorems for the derivatives of Bernstein polynomials and Bernstein coefficients of the qth derivative of $f(x)$.

Theorem 1.4 [107]:

$$D^p B_{i,n}(x) = \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k,n-p}(x) \tag{1.29}$$

Proof: For $p=1$, (1.29) leads us to go back to (1.21)

If we use the induction method on p, letting that (1.29) holds, we want to show that

$$D^{p+1} B_{i,n}(x) = \frac{n!}{(n-p-1)!} \sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)} (-1)^{k+p+1} \binom{p+1}{k} B_{i-k,n-p-1}(x)$$

If we differentiate (1.29), then we have (using eqn. (1.21))

$$D^{p+1} B_{i,n}(x) = D(D^p (B_{i,n}(x)))$$

$$= D\left(\frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k,n-p}(x) \right)$$

$$= \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} D(B_{i-k,n-p}(x))$$

$$= \frac{n!(n-p)}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} (B_{i-k-1,n-p-1}(x) - B_{i-k,n-p-1}(x))$$

$$= \frac{n!}{(n-p-1)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k-1,n-p-1}(x)$$

$$- \frac{n!}{(n-p-1)!} \sum_{k=\max(0,i+p+1-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k,n-p-1}(x) \tag{1.30}$$

Set $k = k-1$ in the first term of the right hand side of eqn. (1.30), we obtain

$$D^{p+1}B_{i,n}(x)=\frac{n!}{(n-p-1)!}\sum_{k=\max(1,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\binom{p}{k-1}B_{i-k,n-p-1}(x)$$

$$+\frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p)}(-1)^{k+p+1}\binom{p}{k}B_{i-k,n-p-1}(x) \tag{1.31}$$

It can be easily shown that

$$D^{p+1}B_{i,n}(x)=\frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\left(\binom{p}{k-1}+\binom{p}{k}\right)B_{i-k,n-p-1}(x)$$

$$=\frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\binom{p+1}{k}B_{i-k,n-p-1}(x) \tag{1.32}$$

This completes the induction and proves the theorem.

Lemma 1.5 [107]

$$B_{k,n}(x)=\sum_{j=k}^{k+p}\frac{\binom{n}{k}\binom{p}{j-k}}{\binom{n+p}{j}}B_{j,n+p}(x) \tag{1.33}$$

Let $f(x)$ be a differentiable function of degree n defined on the interval $[0, 1]$, then we can write

$$f(x)=\sum_{i=0}^{n}a_{i,n}B_{i,n}(x) \tag{1.34}$$

Further, let $a_{i,n}^{(q)}$ denote the Bernstein coefficients of the qth derivative of $f(x)$, that is,

$$f^{(q)}(x)=\frac{d^q f(x)}{dx^q}=\sum_{i=0}^{n}a_{i,n}D^{(q)}B_{i,n}(x),\ a_{i,n}^{(0)}=a_{i,n} \tag{1.35}$$

Then, we can state and prove the following theorem.

Theorem 1.6 [107]

$$a_{i,n}^{(q)}=\sum_{k=-q}^{q}C_k(i,n,q)a_{i-k,n} \tag{1.36}$$

where

$$C_k(i,n,q) = q!\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-m-k} \tag{1.37}$$

Proof: Since

$$f(x) = \sum_{i=0}^{n} a_{i,n} B_{i,n}(x)$$

$$f^{(q)}(x) = \frac{d^q f(x)}{dx^q} = \sum_{i=0}^{n} a_{i,n} D^{(q)} B_{i,n}(x) \tag{1.38}$$

Then by the **theorem 1.4** immediately yields

$$f^{(q)}(x) = \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \sum_{k=\max(0,i+q-n)}^{\min(i,q)} (-1)^{k+q}\binom{q}{k} B_{i-k,n-q}(x)$$

$$f^{(q)}(x) = \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \sum_{k=\max(0,i+q-n)}^{\min(i,q)} (-1)^{k+q}\binom{q}{k} B_{i-k,n-q}(x) \tag{1.39}$$

If we change the degree of Bernstein polynomials using (1.33), then we obtain

$$f^{(q)}(x) = \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \sum_{k=0}^{q} (-1)^{k+q}\binom{q}{k} \sum_{m=0}^{q} \frac{\binom{n-q}{i-k}\binom{q}{m}}{\binom{n}{i-k+m}} B_{i+m-k,n}(x)$$

$$= \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \sum_{k=0}^{q} (-1)^{k+q}\binom{q}{k} \sum_{m=0}^{q} \frac{\binom{n-q}{i-k}\binom{q}{m}}{\binom{n}{i-k+m}} B_{i+m-k,n}(x)$$

$$= \frac{n!}{(n-q)!} \sum_{i=0}^{n} a_{i,n} \left[\sum_{k=0}^{q} (-1)^{k+q}\binom{q}{k}\binom{n-q}{i-k} \sum_{m=0}^{q} \frac{\binom{q}{m}}{\binom{n}{i-k+m}} B_{i+m-k,n}(x) \right] \tag{1.40}$$

Expanding the two summations $\sum_{k=0}^{q}\ \sum_{m=0}^{q}$ and rearranging the coefficients of $B_{i+k,n}$ from $-q \le k \le q$, we have

$$f^{q}(x)=\frac{n!}{(n-q)!}\sum_{i=0}^{n}a_{i,n}\left[\sum_{k=-q}^{q}\frac{1}{\binom{n}{i+k}}B_{i+k,n}(x)\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{n-q}{i-m}\binom{q}{m+k}\right]$$

$$=\frac{n!}{(n-q)!}\sum_{i=k}^{n+k}a_{i-k,n}\sum_{k=-q}^{q}\frac{1}{\binom{n}{i}}B_{i,n}(x)\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{n-q}{i-k-m}\binom{q}{m+k}$$

$$=\frac{n!}{(n-q)!}\sum_{i=0}^{n}a_{i-k,n}\sum_{k=-q}^{q}\frac{1}{\binom{n}{i}}B_{i,n}(x)\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{n-q}{i-k-m}\binom{q}{m+k}$$

$$=\sum_{i=0}^{n}\left[\frac{n!}{(n-q)!}\sum_{k=-q}^{q}\frac{1}{\binom{n}{i}}\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{n-q}{i-k-m}\binom{q}{m+k}a_{i-k,n}\right]B_{i,n}(x)$$

$$=\sum_{i=0}^{n}\left[q!\sum_{k=-q}^{q}\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-m-k}a_{i-k,n}\right]B_{i,n}(x)$$

$$=\sum_{i=0}^{n}a_{i,n}^{(q)}B_{i,n}(x) \tag{1.41}$$

and this completes the proof of **Theorem 1.6**

The following two corollaries will be of fundamental importance in what follows.

Corollary 1.7 [107]

$$\int_{0}^{1}B_{i,n}^{(p)}(x)B_{j,n}(x)dx=\frac{n!\binom{n}{j}}{(2n-p+1)(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\frac{\binom{p}{k}\binom{n-p}{i-k}}{\binom{2n-p}{i+j-k}} \tag{1.42}$$

Proof: We can express explicitly the pth derivatives of Bernstein polynomials from theorem 1.4 to obtain

$$\int_{0}^{1}B_{i,n}^{(p)}(x)B_{j,n}(x)dx=\int_{0}^{1}\frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k,n-p}(x)B_{j,n}(x)dx$$

$$= \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} \int_0^1 B_{i-k,n-p}(x) B_{j,n}(x) dx \tag{1.43}$$

Now, eqn. (1.42) can be easily derived by applying (1.17) and (1.23), we get

$$\int_0^1 B_{i,n}^{(p)}(x) B_{j,n}(x) dx = \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} \int_0^1 \frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}} B_{i+j-k,2n-p}(x) dx$$

$$= \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} \frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}} \int_0^1 B_{i+j-k,2n-p}(x) dx$$

$$= \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} \frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}} \frac{1}{2n-p+1} \tag{1.44}$$

Corollary1. 8 [107]

$$\int_0^1 B_{i,n}^{(p)}(x) D_{j,n}(x) dx = \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \frac{\binom{p}{k}\binom{n-p}{i-k}\binom{p}{j-i+k}}{\binom{n}{j}} \tag{1.45}$$

Proof: Applying theorem 1.4, we obtain

$$\int_0^1 B_{i,n}^{(p)}(x) D_{j,n}(x) dx = \int_0^1 \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} B_{i-k,n-p}(x) D_{j,n}(x) dx$$

(1.46)

It follows immediately from (1.34) and (1.21) that

$$\int_0^1 B_{i,n}^{(p)}(x) D_{j,n}(x) dx = \frac{n!}{(n-p)!} \sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p} \binom{p}{k} \sum_{q=i-k}^{i-k+p} \frac{\binom{n-p}{i-k}\binom{p}{q-i+k}}{\binom{n}{q}}$$

$$\times \int_0^1 B_{q,n}(x) D_{j,n}(x) dx$$

$$= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k}\sum_{q=i-k}^{i-k+p} \frac{\binom{n-p}{i-k}\binom{p}{q-i+k}}{\binom{n}{q}}\delta_{q,j}$$

$$= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k}\frac{\binom{n-p}{i-k}\binom{p}{j-i+k}}{\binom{n}{j}} \tag{1.47}$$

1.3.3 Legendre polynomials and its properties:

The Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre as the coefficients in the expansion of the Newtonian potential

$$\frac{1}{|X-X'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}} p_l(\cos\gamma)$$

where r and r' are the lengths of the vectors X and X' respectively and γ is the angle between those vectors. The series converges when $r > r'$. The expression gives the gravitational potential associated to a point charge. The expansion using Legendre polynomials might be useful, for instance, where integrating this expression over a continuous mass or charge distribution. Legendre polynomials occur in the solution of Laplace eqn. of the potential $\nabla^2\phi(x) = 0$ in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry.

Now we introduce Legendre polynomials through the generating function

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} p_n(x)t^n \tag{1.48}$$

The importance of Legendre polynomials in physics is that they satisfy the following differential equation (Legendre's equation)

$$(1-x^2)p_n''(x) - 2xp_n'(x) + n(n+1)p_n(x) = 0 \tag{1.49}$$

which arises in the solution of many partial differential equations, particularly in the boundary value problems for spheres.

The solution of the Legendre's equation (1.49) is called the Legendre polynomial of degree n and is denoted by $p_n(x)$. The general form of the Legendre polynomials [5, 6] over the interval [-1, 1] is defined by

$$p_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \tag{1.50}$$

where $N = \frac{n}{2}$ for n even and $N = \frac{n-1}{2}$ for n odd.

The first 10 Legendre polynomials are given below:

$$p_1(x) = x$$

$$p_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$p_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$p_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$p_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$p_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

$$p_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

$$p_9(x) = \frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$$

$$p_{10}(x) = \frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$$

Rodrigue's Formula:

We have obtained the Legendre polynomials as solutions of the Legendre's equation. They can also be represented using the following Rodrigue's formula

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n \tag{1.51}$$

Shifted Legendre Polynomials:

The shifted Legendre polynomials are defined as $\widetilde{p}_n(x) = p_n(2x-1)$. Here shifting function $x \to 2x-1$ (in fact, it is an affine transformation) is chosen such that it bijectively maps the interval [0, 1] to the interval [-1, 1], implying that the polynomials $\widetilde{p}_n(x)$ are orthogonal on [0, 1]:

$$\int_0^1 \widetilde{p}_m(x)\widetilde{p}_n(x)dx = \frac{1}{2n+1}\delta_{mn}.$$

An explicit expression for the shifted Legendre polynomials is given by

$$\widetilde{p}_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k \tag{1.52}$$

and the analogue of Rodrigue's formula for the shifted Legendre polynomials over [0, 1] is given by

$$\widetilde{p}_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n \tag{1.53}$$

Now we **modify** the above shifted Legendre polynomials given in eqn. (1.48) to satisfy the conditions $\widetilde{p}_n(0) = \widetilde{p}_n(1) = 0, n \ge 1$, so that they can be used as set of basis functions satisfying the corresponding homogeneous form of the *Dirichlet* boundary conditions to derive the matrix formulation for solving a BVP in the Galerkin method in the following form

$$\widetilde{p}_n(x) = \left[\frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n - (-1)^n\right] \times (x-1) \tag{1.54}$$

We write first 10 modified Legendre polynomials over the interval [0, 1]:

$\widetilde{p}_1(x) = 2x(x-1)$

$\widetilde{p}_2(x) = 6x(x-1)^2$

$\widetilde{p}_3(x) = 2x(x-1)(10x^2 - 15x + 6)$

$\widetilde{p}_4(x) = 20x - 110x^2 + 230x^3 - 210x^4 + 70x^5$

$\widetilde{p}_5(x) = -30x + 240x^2 - 770x^3 + 1190x^4 - 882x^5 + 252x^6$

$\widetilde{p}_6(x) = 42x - 462x^2 + 2100x^3 - 4830x^4 + 5922x^5 - 3696x^6 + 924x^7$

$\widetilde{p}_7(x) = -56x + 812x^2 - 4956x^3 + 15750x^4 - 28182x^5 + 28644x^6 - 15444x^7 + 3432x^8$

$\widetilde{p}_8(x) = 72x - 1332x^2 + 10500x^3 - 43890x^4 + 106722x^5 - 156156x^6 + 135564x^7 - 64350x^8 + 12870x^9$

$\widetilde{p}_9(x) = -90x + 2070x^2 - 20460x^3 + 108570x^4 - 342342x^5 + 672672x^6 - 832260x^7 + 630630x^8 - 267410x^9 + 48620x^{10}$

$\widetilde{p}_{10}(x) = 110x - 3080x^2 + 37290x^3 - 244530x^4 + 966966x^5 - 2438436x^6 + 4015440x^7 - 4302870x^8 + 2892890x^9 - 1108536x^{10} + 184756x^{11}$

Orthogonality of Legendre polynomials:

The differential equation and the boundary conditions those are satisfied by the Legendre polynomials form a Sturm-Liouville system. They should therefore satisfy the orthogonality relation

$$\int_{-1}^{1} p_m(x)p_n(x)dx = \begin{cases} 0, m \neq n \\ \dfrac{2}{2n+1}, m = n \end{cases} \tag{1.55}$$

Recurrence Relations:

We start by generating a recurrence relation between Legendre polynomials of different order

(i) $(n+1)p_{n+1}(x) = (2n+1)xp_n(x) - np_{n-1}(x)$ (1.56)

This is the promised recurrence relation between Legendre polynomials of different order.

(ii) $p'_{n+1}(x) + p'_{n-1}(x) = 2xp'_n(x) + p_n(x)$ (1.57)

(iii) $(n+1)p'_{n+1}(x) + np'_{n-1}(x) = (2n+1)[p_n(x) + xp'_n(x)]$ (1.58)

Eqns. (1.57) and (1.58) can be used to obtain Legendre's equation.

CHAPTER 2

Fourth Order Boundary Value Problems

2.1 Introduction

Fourth order linear and nonlinear BVPs arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plates deflection theory, beam element theory and many more applications of engineering and applied mathematics. These BVPs are solved either analytically or numerically. A major advantage for numerical analysis is that a numerical answer can be obtained even when a problem has no "analytical" solution .For this, many authors have attempted to solve fourth order BVPs to obtain high accuracy rapidly by using a numerous methods, such as least square method, finite difference method, Sinc-Galerkin method, and also some other methods using polynomial and nonpolynomial spline functions. Since the piecewise polynomials can be differentiated and integrated easily, and can be approximated any function to any accuracy desired. So Bernstein polynomials have been studied by many authors [2 – 4], spline functions [11 – 15] have been studied extensively for solving only linear BVP. Recently Loghmani and Alavizadeh [12] has attempted to solve both linear and nonlinear BVP using least square method with B-splines. Special nonlinear BVPs have been studied by Twizell and Tirmizi [18] using multiderivatives with Pade` approximation method, also by El-Gamel *et al* [16] and only linear BVP by Smith *et al* [17] by the technique of Sinc-Galerkin methods. Usmani [13] and Usmani and Warsi [19] developed and analyzed second order and fourth order convergent methods for the solution of linear fourth order two-point BVP using quartic, quintic and sextic polynomial spline functions, respectively. Al-Said and Noor [20] and Al-Said *et al* [21] demonstrated second order convergent method based on cubic and quartic polynomial spline functions for the solution of fourth order obstacle problems. Also Rashidinia and Golbabaee [14] and Siddiqi and Akram [19] generated a difference scheme via quintic spline functions for this problem. Loghmani and Alavizadeh [12] converted this problem into an optimal control problem and then constructed the approximate solution as a combination of quartic B-splines. Van Daele *et al* [24] introduced a new second order method for solving the BVPs with the boundary conditions involving first derivatives based on nonpolynomial spline function. Very recently Kasi *et al* [25] used Quintic B-splines for solving fourth order BVP by the Galerkin method. Besides spline functions and Bernstein polynomials, there is another type of piecewise continuous polynomial, namely Legendre polynomial [5, 6].

This chapter is devoted to find the numerical solutions of the fourth order linear and nonlinear differential equations using piecewise continuous and differentiable polynomials such as Bernstein and Legendre polynomials with two types of boundary conditions. We derive rigorous matrix formulations for solving linear and nonlinear fourth order BVP and special care is taken about how the polynomials satisfy the given boundary conditions. The linear combinations of each polynomial are exploited in the Galerkin weighted residual approximation. The derived formulation is illustrated through various numerical examples. Our approximate solutions are compared with the exact solutions, and also with the solutions of the existing methods. The approximate solutions converge to the exact solutions monotonically even with desired large significant digits. Then we discuss in section 2.2, the formulation for solving linear fourth order BVP by Galerkin weighted residual method [1], using Bernstein and Legendre polynomials as basis functions in the approximation, in details. Then we deduce similar formulation for nonlinear problems in the next section. Numerical examples, for both linear and nonlinear BVPs, are considered to verify the proposed formulation, and the obtained results are compared as well. Finally we have given the conclusions of this chapter.

2.2 Galerkin Weighted Residual Formulation

In this section we first obtain the rigorous matrix formulation for fourth order linear BVP and then we extend our idea for solving nonlinear BVP. For this, we consider a linear fourth order differential equation

$$a_4 \frac{d^4 u}{dx^4} + a_3 \frac{d^3 u}{dx^3} + a_2 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b \tag{2.1a}$$

subject to the following two types of boundary conditions

Type I: $u(a) = A_0, \quad u(b) = B_0, \quad u'(a) = A_1, \quad u'(b) = B_1$ (2.1b)

Type II: $u(a) = A_0, \quad u(b) = B_0, \quad u''(a) = A_2, \quad u''(b) = B_2$ (2.1c)

where $A_i, B_i, i = 0,1,2$ are finite real constants and $a_i, i = 0,1,\cdots 4$ and r are all continuous and differentiable functions of x defined on the interval $[a, b]$.

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (2.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$c_4 \frac{d^4 u}{dx^4} + c_3 \frac{d^3 u}{dx^3} + c_2 \frac{d^2 u}{dx^2} + c_1 \frac{du}{dx} + c_0 u = s,\ 0 < x < 1 \tag{2.2a}$$

$$u(0) = A_0, \qquad \frac{1}{b-a} u'(0) = A_1,$$

$$u(1) = B_0, \qquad \frac{1}{b-a} u'(1) = B_1 \tag{2.2b}$$

and

$$u(0) = A_0, \qquad \frac{1}{(b-a)^2} u''(0) = A_2,$$

$$u(1) = B_0, \qquad \frac{1}{(b-a)^2} u''(1) = B_2 \tag{2.2c}$$

where

$$c_4 = \frac{1}{(b-a)^4} a_4((b-a)x+a), \qquad c_3 = \frac{1}{(b-a)^3} a_3((b-a)x+a),$$

$$c_2 = \frac{1}{(b-a)^2} a_2((b-a)x+a), \qquad c_1 = \frac{1}{b-a} a_1((b-a)x+a),$$

$$c_0 = a_0((b-a)x+a), \qquad s = r((b-a)x+a)$$

We approximate the solution of the differential equation (2.2a) as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{2.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0$, for each $i = 1,2,3,\ldots,n.$

Using eqn. (2.3) into eq. (2.2a), the Galerkin weighted residual equations are:

$$\int_0^1 \left[c_4 \frac{d^4\tilde{u}}{dx^4} + c_3 \frac{d^3\tilde{u}}{dx^3} + c_2 \frac{d^2\tilde{u}}{dx^2} + c_1 \frac{d\,\tilde{u}}{dx} + c_0\,\tilde{u} - s \right] N_{j,n}(x)dx = 0 \qquad (2.4)$$

2.2.1 Formulation I

In this section, we formulate the matrix form with boundary conditions of type I. Integrating by parts the terms up to second derivative on the left hand side of (2.4), we have

$$\int_0^1 c_4 \frac{d^4\tilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx \quad [\text{Since}\, N_{j,n}(0) = N_{j,n}(1) = 0\,]$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (2.5)$$

$$\int_0^1 c_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x) \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (2.6)$$

$$\int_0^1 c_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x) \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (2.7)$$

Substituting eqns. (2.5), (2.6) and (2.7) into eqn. (2.4) and using approximation for $\widetilde{u}(x)$ given in equation (2.3) and after applying the conditions given in type I, eqn. (2.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\; j = 1,2,\ldots,n \tag{2.8a}$$

where

$$\begin{aligned} D_{i,j} = \int_0^1 \Bigg\{ \Bigg[& -\frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \\ & + c_1 N_{j,n}(x)\Bigg] \frac{dN_{i,n}(x)}{dx} + c_0 N_{i,n}(x) N_{j,n}(x) \Bigg\} dx - \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=0} \\ & + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=1} \end{aligned} \tag{2.8b}$$

$$\begin{aligned} F_j = \int_0^1 \Bigg\{ & s N_{j,n}(x) + \left[\frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] - \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] + \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] - c_1 N_{j,n}(x) \right] \frac{d\theta_0}{dx} \\ & - c_0 N_{j,n}(x)\theta_0 \Bigg\} dx + \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right] - \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \right]_{x=1} (b-a)\times B_1 \\ & - \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right] - \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \right]_{x=0} (b-a)\times A_1 + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\theta_0}{dx^2} \right]_{x=1} \\ & - \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\theta_0}{dx^2} \right]_{x=0} \end{aligned} \tag{2.8c}$$

Solving the system (2.8a), we find the values of the parameters α_i and then substituting these parameters into eqn. (2.3), we get the approximate solution of the BVP (2.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (2.1).

2.2.2 Formulation II

In this section, we obtain the matrix formulation by using the boundary conditions of type II.

In the same way of section (2.2.1), integrating by parts the terms consisting fourth, third, and second derivatives on the left hand side of (2.4), and applying the conditions prescribed in type II, eqn. (2.2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \tag{2.9a}$$

where

$$\begin{aligned} D_{i,j} = \int_0^1 \Bigg\{ \Bigg[& -\frac{d^3}{dx^3}\big[c_4 N_{j,n}(x)\big] + \frac{d^2}{dx^2}\big[c_3 N_{j,n}(x)\big] - \frac{d}{dx}\big[c_2 N_{j,n}(x)\big] \\ & + c_1 N_{j,n}(x)\Bigg]\frac{dN_{i,n}(x)}{dx} + c_0 N_{i,n}(x) N_{j,n}(x)\Bigg\} dx - \left[\frac{d}{dx}\big[c_3 N_{j,n}(x)\big]\frac{dN_{i,n}(x)}{dx}\right]_{x=1} \\ & + \left[\frac{d}{dx}\big[c_3 N_{j,n}(x)\big]\frac{dN_{i,n}(x)}{dx}\right]_{x=0} + \left[\frac{d^2}{dx^2}\big[c_4 N_{j,n}(x)\big]\frac{dN_{i,n}(x)}{dx}\right]_{x=1} \\ & - \left[\frac{d^2}{dx^2}\big[c_4 N_{j,n}(x)\big]\frac{dN_{i,n}(x)}{dx}\right]_{x=0} \end{aligned} \tag{2.9b}$$

$$\begin{aligned} F_j = \int_0^1 \Bigg\{ & s N_{j,n}(x) + \left[\frac{d^3}{dx^3}\big[c_4 N_{j,n}(x)\big] - \frac{d^2}{dx^2}\big[c_3 N_{j,n}(x)\big] + \frac{d}{dx}\big[c_2 N_{j,n}(x)\big] - c_1 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \\ & - c_0 N_{j,n}(x)\theta_0 \Bigg\} dx - \left[\frac{d^2}{dx^2}\big[c_4 N_{j,n}(x)\big]\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^2}{dx^2}\big[c_4 N_{j,n}(x)\big]\frac{d\theta_0}{dx}\right]_{x=0} \\ & + \left[\frac{d}{dx}\big[c_3 N_{j,n}(x)\big]\frac{d\theta_0}{dx}\right]_{x=1} - \left[\frac{d}{dx}\big[c_3 N_{j,n}(x)\big]\frac{d\theta_0}{dx}\right]_{x=0} + \left[\frac{d}{dx}\big[c_4 N_{j,n}(x)\big]\right]_{x=1} \\ & \times (b-a)^2 \times B_2 - \left[\frac{d}{dx}\big[c_4 N_{j,n}(x)\big]\right]_{x=0} \times (b-a)^2 \times A_2 \end{aligned} \tag{2.9c}$$

Solving the system (2.9a), we find the values of the parameters α_i and then substituting these parameters into eqn. (2.3), we get the approximate solution of the BVP (2.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (2.1).

2.2.3 Formulation III

In this portion, we obtain the matrix formulation by applying the boundary conditions of type II.

Here we consider a linear fourth order differential equation given by

$$\frac{d^2}{dx^2}\left(p(x)\frac{d^2u}{dx^2}\right)+r(x)u=s(x), a\le x\le b \tag{2.10}$$

where $p(x)$, $s(x)$ and $r(x)$ are specified continuous functions. We want to solve the BVP (2.10) by the Galerkin method using Bernstein and Legendre polynomials as trial functions.

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (2.10) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$\frac{d^2}{dx^2}\left(\widetilde{p}(x)\frac{d^2u}{dx^2}\right)+\widetilde{r}(x)u=\widetilde{s}(x), 0\le x\le 1 \tag{2.11}$$

where

$$\widetilde{p}(x)=\frac{1}{(b-a)^4}p((b-a)x+a), \widetilde{r}(x)=r((b-a)x+a), \widetilde{s}(x)=s((b-a)x+a)$$

We approximate the solution of the differential equation (2.11) as

$$\widetilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\ge 1 \tag{2.12}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0)=N_{i,n}(1)=0$, for each $i=1,2,3,\ldots,n$.

Using eqn. (2.12) into eqn. (2.11), the Galerkin weighted residual equations are:

$$\int_0^1\left[\frac{d^2}{dx^2}\left(\widetilde{p}(x)\frac{d^2\widetilde{u}}{dx^2}\right)+\widetilde{r}(x)\widetilde{u}-\widetilde{s}(x)\right]N_{j,n}(x)dx=0,\ j=1,2,\ldots,n \tag{2.13}$$

Now integrating the first term of (2.13) by parts, we have

$$\int_0^1 \left[\frac{d^2}{dx^2}\left(\tilde{p}(x)\frac{d^2\tilde{u}}{dx^2} \right) \right] N_{j,n}(x)dx = \left[\frac{d}{dx}\left(\tilde{p}(x)\frac{d^2\tilde{u}}{dx^2} \right) N_{j,n}(x) \right]_0^1 - \int_0^1 \frac{d}{dx}\left(\tilde{p}(x)\frac{d^2\tilde{u}}{dx^2} \right) \frac{dN_{j,n}(x)}{dx} dx$$

$$= -\left[\tilde{p}(x)\frac{d^2\tilde{u}}{dx^2}\frac{dN_{j,n}(x)}{dx} \right]_0^1 + \int_0^1 \tilde{p}(x)\frac{d^2\tilde{u}}{dx^2}\frac{d^2N_{j,n}(x)}{dx^2} dx \tag{2.14}$$

Substituting eqn. (2.14) into eqn. (2.4) and using approximation for $\tilde{u}(x)$ given in equation (2.12) and after using the boundary conditions given in type II, eqn (2.2c) and rearranging the terms for the resulting equations, we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\ \alpha_i = F_j, \tag{2.15a}$$

where

$$D_{i,j} = \int_0^1 \left[\tilde{p}(x)\frac{d^2N_{i,n}(x)}{dx^2}\frac{d^2N_{j,n}(x)}{dx^2} + \tilde{r}(x)N_{i,n}(x)N_{j,n}(x) \right] dx \tag{2.15b}$$

$$F_j = \int_a^b \left[\tilde{s}(x)N_{j,n}(x) - \tilde{p}(x)\frac{d^2\theta_0}{dx^2}\frac{d^2N_{j,n}(x)}{dx^2} - \tilde{r}(x)\theta_0 N_{j,n}(x) \right] dx$$

$$+ \left[\tilde{p}(x)\frac{dN_{j,n}(x)}{dx} \right]_{x=1} \times (b-a)^2 B_2 - \left[\tilde{p}(x)\frac{dN_{j,n}(x)}{dx} \right]_{x=0} \times (b-a)^2 A_2 \tag{2.15c}$$

Solving the system (2.15a), we find the values of the parameters α_i, and then substituting into eqn. (2.12), we get the approximate solution of the BVP (2.11). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (2.10).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (2.8) and (2.9) and (2.15). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. These formulations are described through the numerical examples in the next section.

2.3. Numerical examples and results

In this section, we consider four linear and two nonlinear BVPs to verify the proposed formulations in sections (2.2.1), (2.2.2) and (2.2.3). For this, we give the results for linear problems in brief depending on prescribed boundary conditions, but the nonlinear problem is illustrated in details. All computations are performed by *MATLAB 10*. Since the convergence of linear BVP is calculated by

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using n polynomials and δ (depends on the problem) which is less than 10^{-12}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^{N}\right| < \delta$$

where δ is less than 10^{-10} and N is the Newton's iteration number.

Example 1: Consider the linear differential equation [7]

$$\frac{d^4u}{dx^4} + 2\frac{d^3u}{dx^3} + \frac{d^2u}{dx^2} + 8\frac{du}{dx} - 12u = 12\sin x - e^{-x}, \; 0 < x < 1 \quad (2.16a)$$

subject to the boundary conditions of type I in eqn. (2.2b):

$$u(0) = u(1) = 0, \; u'(0) = 0, u'(1) = 1 \, . \quad (2.16b)$$

whose exact solution is

$$u(x) = 0.4518\, e^{x} - 0.3173\, e^{-3x} - 0.2769\, \sin 2x + 0.2155\, \cos 2x - 0.8 \sin x$$

$$- 0.4\cos x + 0.05 e^{-x} \, .$$

Using the method illustrated in (2.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \ge 1 \quad (2.17)$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (2.16b). Now the parameters $\alpha_i \, (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \quad (2.18a)$$

where

$$D_{i,j} = \int_0^1 \left\{ \left[-\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] + \frac{d^2}{dx^2}[2N_{j,n}(x)] - \frac{d}{dx}[N_{j,n}(x)] + 8N_{j,n}(x) \right] \frac{dN_{i,n}(x)}{dx} \right.$$

$$\left. -12N_{i,n}(x)N_{j,n}(x)dx \right\} dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=0} \qquad (2.18b)$$

$$F_j = \int_0^1 (12\sin x - e^{-x}) N_{j,n}(x)dx - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[2N_{j,n}(x)\right] \right]_{x=1} \qquad (2.18c)$$

Solving the system (2.18a), we obtain the values of the parameters and then substituting these parameters into eqn. (2.17), we get the approximate solution of the BVP (2.16) for different values of n.

The numerical results for this problem are shown in **Table 1**.

Table 1: Maximum absolute errors for the example 1.

x	Exact Results	13, Bernstein Polynomials		13, Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	1.2759373E-000
0.1	-0.0121694898	-0.0121694898	7.8062556E-017	-0.0121694898	2.5376055E-013
0.2	-0.0416565792	-0.0416565792	3.9551695E-016	-0.0416565792	4.2368192E-013
0.3	-0.0791115756	-0.0791115756	1.5265567E-016	-0.0791115756	6.0500216E-013
0.4	-0.1164681503	-0.1164681503	4.4408921E-016	-0.1164681503	4.5652371E-013
0.5	-0.1466834463	-0.1466834463	2.2204460E-016	-0.1466834463	1.5656920E-013
0.6	-0.1635744681	-0.1635744681	4.4408921E-016	-0.1635744681	7.8939633E-013
0.7	-0.1617212698	-0.1617212698	1.9428903E-016	-0.1617212698	9.0033536E-013
0.8	-0.1364140352	-0.1364140352	4.4408921E-016	-0.1364140352	6.0421113E-013
0.9	-0.0836262301	-0.0836262301	4.7184479E-016	-0.0836262301	4.3842707E-013
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 13$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-11}.

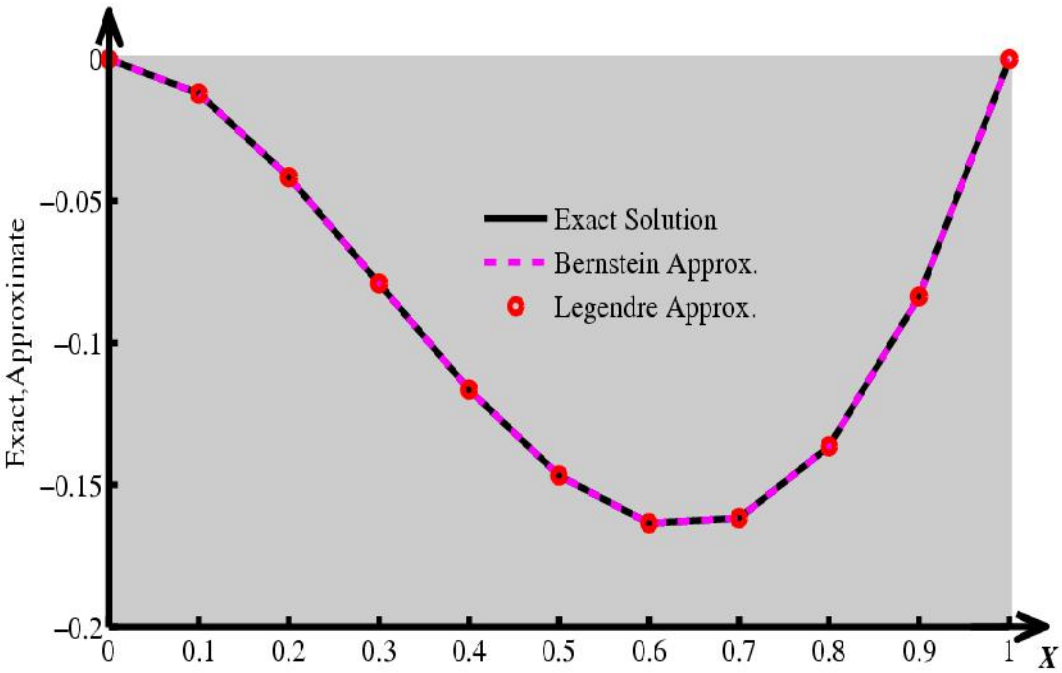


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 13 polynomials.

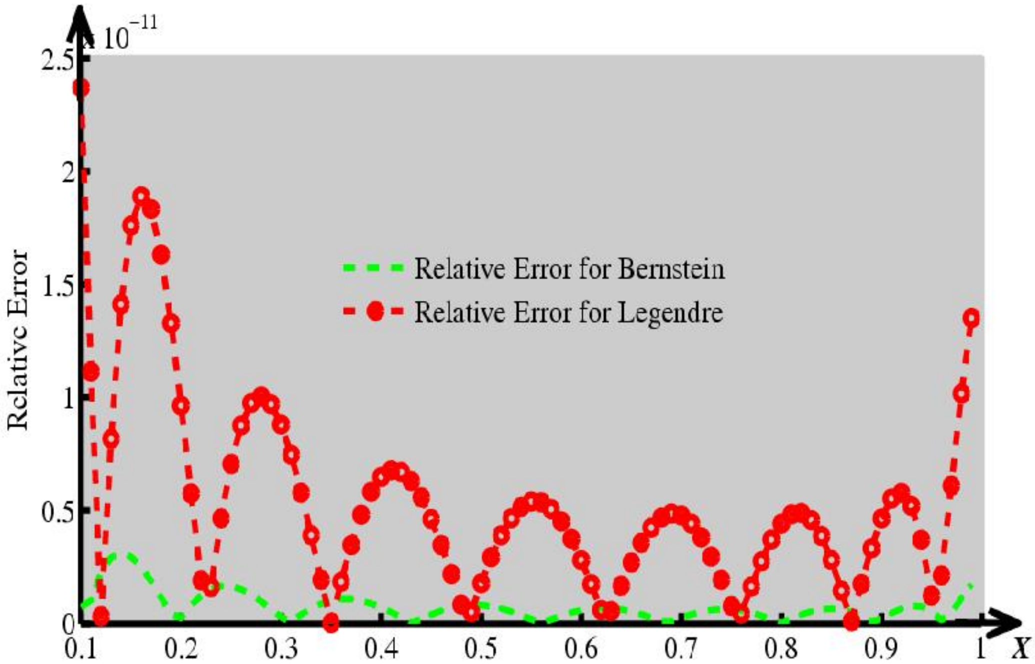


Fig. 1(b): Graphical representation of relative error of example 1 using 13 polynomials.

Example 2: Consider the linear differential equation [14, 19, 25]

$$\frac{d^4u}{dx^4} + xu = -(8 + 7x + x^3)e^x, \quad 0 < x < 1 \tag{2.19a}$$

subject to the boundary conditions of type I in eqn. (2.1b):

$$u(0)=u(1)=0,\ u'(0)=1, u'(1)=-e\,. \quad (2.19b)$$

whose exact solution is, $u(x)=x(1-x)e^{x}$.

Applying the method illustrated in (2.1.1), we approximate $u(x)$ in a form

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i\,N_{i,n}(x),\ n\geq 1 \quad (2.20)$$

Here $\theta_0=0$ is specified by the *Dirichlet* boundary conditions of equation (2.19b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n}D_{i,j}\,\alpha_i=F_j\,, j=1,2,\ldots,n \quad (2.21a)$$

where

$$D_{i,j}=\int_0^1\left\{\left[-\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]\frac{dN_{i,n}(x)}{dx}+x\,N_{i,n}(x)N_{j,n}(x)\right\}dx-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1}$$
$$+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=0} \quad (2.21b)$$

$$F_j=-\int_0^1(8+7x+x^3)e^xN_{j,n}(x)dx+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\right]_{x=1}\times e+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\right]_{x=0} \quad (2.21c)$$

Solving the system (2.21a) we obtain the values of the parameters and then substituting these parameters into eqn. (2.20), we get the approximate solution of the BVP (2.19) for different values of *n*.

The numerical results obtained by our method are given in **Table 2**.

Table 2: Maximum absolute errors for the example 2.

x	Exact Results	11, Bernstein Polynomials		11, Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	6.2404583E-026
0.1	0.0994653826	0.0994653826	1.8735014E-015	0.0994653826	4.6490589E-014
0.2	0.1954244413	0.1954244413	2.7755576E-015	0.1954244413	1.2767565E-013
0.3	0.2834703496	0.2834703496	4.5519144E-015	0.2834703496	1.7824631E-013
0.4	0.3580379274	0.3580379274	3.7747583E-015	0.3580379274	2.8310687E-015
0.5	0.4121803177	0.4121803177	1.6653345E-016	0.4121803177	2.5018876E-013
0.6	0.4373085121	0.4373085121	3.9412917E-015	0.4373085121	1.8657298E-013
0.7	0.4228880686	0.4228880686	4.6074256E-015	0.4228880686	8.5431662E-014
0.8	0.3560865486	0.3560865486	2.8310687E-015	0.3560865486	2.0078383E-013
0.9	0.2213642800	0.2213642800	1.8873791E-015	0.2213642800	1.4432899E-013
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand the maximum absolute errors have been found by Rashidinia and Golbabaee [14], Al-Said *et al* [19] and Kasi *et al* [25] are $5.37\times10^{-6}, 2.36\times10^{-7}$ and 5.99×10^{-6} respectively.

We have shown the exact and approximate solutions in Fig. 2(a) and the relative errors in Fig. 2(b) of example 2 for $n=11$. It is found from Fig. 2(b) that the error is of the order 10^{-12}

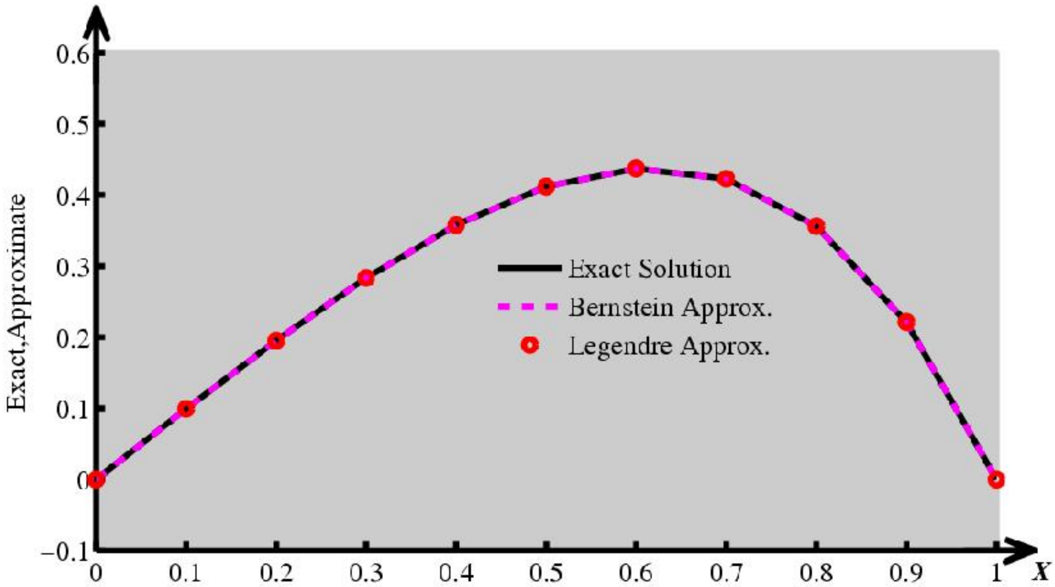


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 11 polynomials.

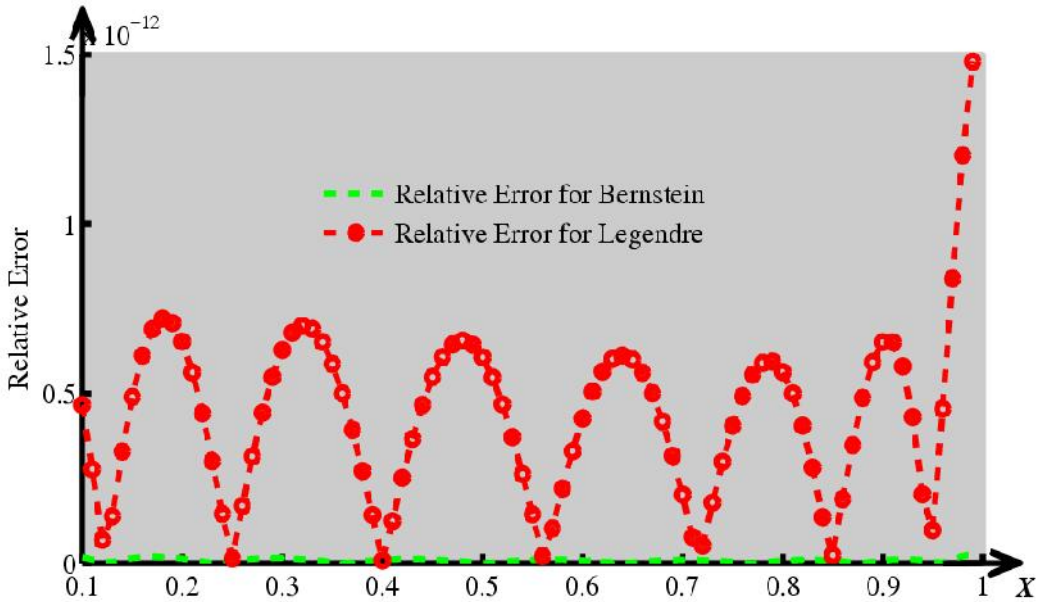


Fig. 2(b): Graphical representation of relative error of example 2 using 11 polynomials.

Example 3: We consider the linear BVP [11]:

$$\frac{d^4u}{dx^4} - u = -4(2x\cos x + 3\sin x),\ 0 < x < 1 \tag{2.22a}$$

subject to the boundary conditions of type II in eqn. (2.2c):

$$u(0) = u(1) = 0, u''(0) = 0,\ \ u''(1) = 2\sin 1 + 4\cos 1. \tag{2.22b}$$

whose exact solution is $u(x) = (x^2 - 1)\sin x$.

Using the method illustrated in section (2.2.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{2.23}$$

Here $\theta_0(x) = 0$ as specified by the essential boundary conditions of eqn. (2.22b). Now the parameters $\alpha_i\,(i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\alpha_i = F_j,\ j = 1,2,\ldots,n \tag{2.24a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^3N_{j,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx} - N_{i,n}(x)N_{j,n}(x) \right]dx + \left[\frac{d^2N_{j,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx} \right]_{x=1} - \left[\frac{d^2N_{j,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx} \right]_{x=0} \tag{2.24b}$$

$$F_j = \int_0^1 -4(2x\cos x + 3\sin x)N_{j,n}(x)dx + (2\sin 1 + 4\cos 1) \times \left[\frac{dN_{j,n}(x)}{dx} \right]_{x=1} \tag{2.24c}$$

Solving the system (2.24a), we find the values of the parameters and then substituting these parameters into eqn. (2.23), we get the approximate solution of the BVP (2.22) for different values of n.

The numerical results for this problem are tabulated in **Table 3**.

The maximum absolute error has been found by Ramadan *et al* [11] is 1.417×10^{-11}

Table 3: Maximum absolute errors for the example 3.

x	Exact Results	12, Bernstein Polynomials		12, Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	2.2411893E-026
0.1	-0.0988350825	-0.0988350825	6.9388939E-017	-0.0988350825	5.4053984E-014
0.2	-0.1907225576	-0.1907225576	0.0000000E+000	-0.1907225576	4.3104409E-014
0.3	-0.2689233881	-0.2689233881	5.5511151E-017	-0.2689233881	3.4527936E-014
0.4	-0.3271114075	-0.3271114075	5.5511151E-017	-0.3271114075	8.3932861E-014
0.5	-0.3595691540	-0.3595691540	5.5511151E-017	-0.3595691540	3.3861802E-014
0.6	-0.3613711830	-0.3613711830	0.0000000E+000	-0.3613711830	3.0253577E-014
0.7	-0.3285510205	-0.3285510205	0.0000000E+000	-0.3285510205	3.5527137E-014
0.8	-0.2582481927	-0.2582481927	5.5511151E-017	-0.2582481927	1.8540725E-014
0.9	-0.1488321128	-0.1488321128	0.0000000E+000	-0.1488321128	1.2018164E-014
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

We depict the exact and approximate solutions in Fig. 3(a) and a plot of relative errors in Fig. 3(b) of example 3 for $n = 12$. From Fig. 3(b) we observe that the error is nearly the order 10^{-13}

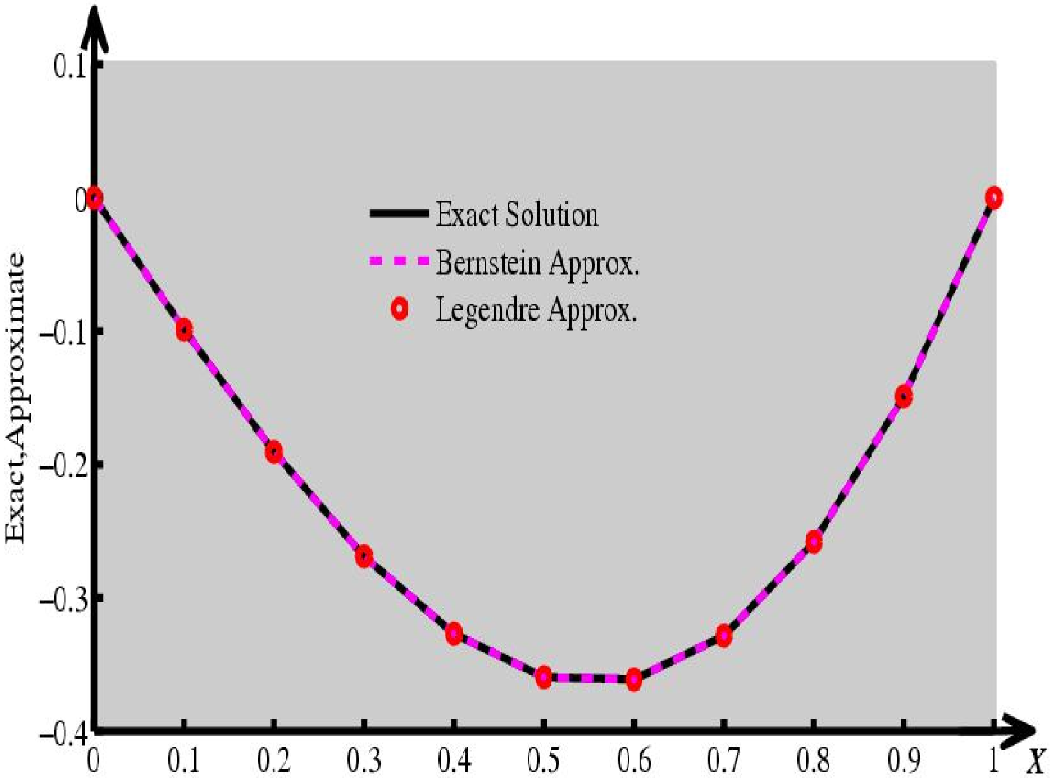


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 12 polynomials.

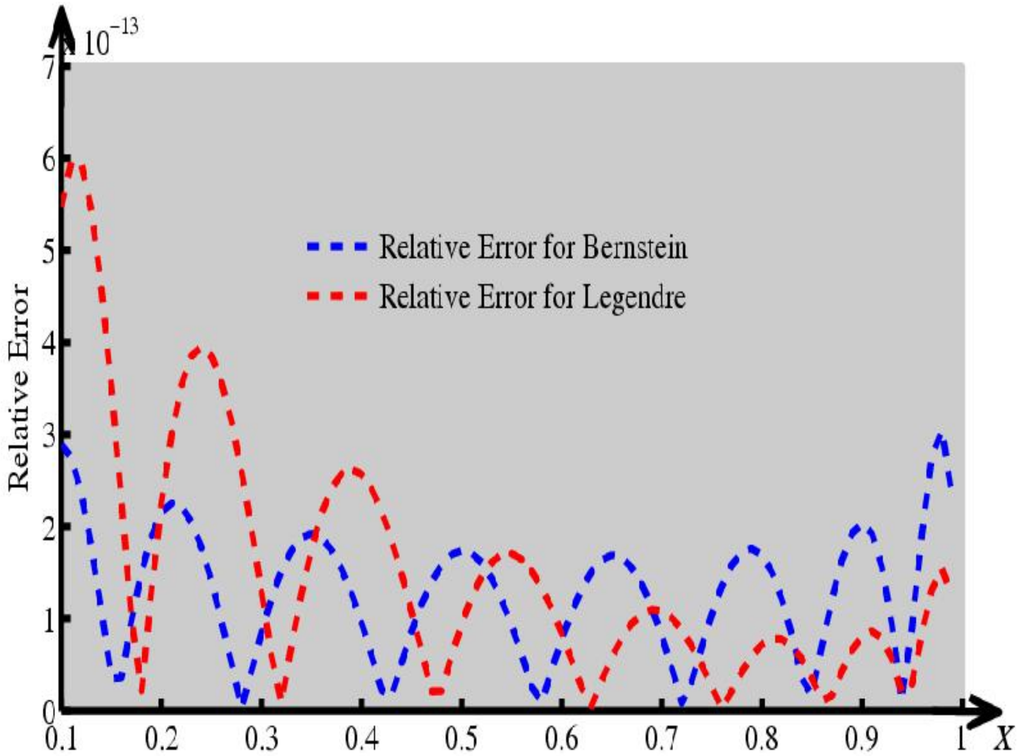


Fig. 3(b): Graphical representation of relative error of example 3 using 12 polynomials.

Example 4: We consider another linear differential equation [13]

$$\frac{d^4u}{dx^4} - xu = -(11 + 9x + x^2 - x^3)e^x, -1 \le x \le 1 \qquad (2.25a)$$

subject to the boundary conditions of type II, eqn (2.1c)

$$u(-1) = u(1) = 0, u''(-1) = \frac{2}{e}, u''(1) = -6e. \qquad (2.25b)$$

Exact solution of this BVP is $u(x) = (1 - x^2)e^x$.

The equivalent BVP over [0, 1] to the BVP (2.25) is,

$$\frac{1}{2^4}\frac{d^4u}{dx^4} - xu = -(11 + 9x + x^2 - x^3)e^x, \; 0 < x < 1 \qquad (2.26a)$$

$$u(0) = u(1) = 0, \; \frac{1}{4}u''(0) = \frac{2}{e}, \frac{1}{4}u''(1) = -6e \qquad (2.26b)$$

Applying the method illustrated in section (2.2.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \ge 1 \qquad (2.27)$$

Here $\theta_0(x) = 0$ as specified by the essential boundary conditions of eqn. (2.26b). Now the parameters $\alpha_i (i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\alpha_i = F_j, j = 1,2,\ldots,n \tag{2.28a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^3 N_{j,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx} - 16x\, N_{i,n}(x) N_{j,n}(x) \right] dx + \left[\frac{d^2 N_{j,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx} \right]_{x=1} - \left[\frac{d^2 N_{j,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx} \right]_{x=0} \tag{2.28b}$$

$$F_j = \int_0^1 -(11+9x+x^2-x^3)e^x N_{j,n}(x)dx + \left[\frac{dN_{j,n}(x)}{dx}\right]_{x=1} \times (-24e) - \left[\frac{dN_{j,n}(x)}{dx}\right]_{x=0} \times \left(\frac{8}{e}\right) \tag{2.28c}$$

Solving the system (2.28a) we obtain the values of the parameters and then substituting these parameters into eqn. (2.27), we get the approximate solution of the BVP (2.26) for different values of n. Replacing $x = \frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (2.25).

The numerical results for this problem are shown in **Table 4**

Table 4: Maximum absolute errors for the example 4.

x	Exact Results	12, Bernstein Polynomials		12, Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
-1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	4.3341268E-023
-0.8	0.1617584271	0.1617584271	1.5840107E-013	0.1617584271	4.4986237E-012
-0.6	0.3512394471	0.3512394471	6.4837025E-014	0.3512394471	6.6550099E-012
-0.4	0.5630688387	0.5630688387	2.8876901E-013	0.5630688387	9.5666808E-012
-0.2	0.7859815230	0.7859815230	6.7412742E-013	0.7859815229	8.3194562E-012
0.0	1.0000000000	1.0000000000	8.5709218E-013	1.0000000000	3.9990233E-013
0.2	1.1725466478	1.1725466478	6.4681593E-013	1.1725466478	8.5980112E-012
0.4	1.2531327460	1.2531327460	2.4802382E-013	1.2531327460	1.1270762E-011
0.6	1.1661560322	1.1661560322	3.0642155E-014	1.1661560323	8.3508755E-012
0.8	0.8011947343	0.8011947343	1.5110135E-013	0.8011947343	5.7696070E-012
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

The maximum absolute error has been found by Usmani [13] is 1.84×10^{-5}

In Figs. 4(a) and 4(b), the exact and approximate solutions, and the relative errors of example 4 for $n=12$ are depicted respectively. We see from Fig. 4(b) that the error is nearly the order 10^{-11}

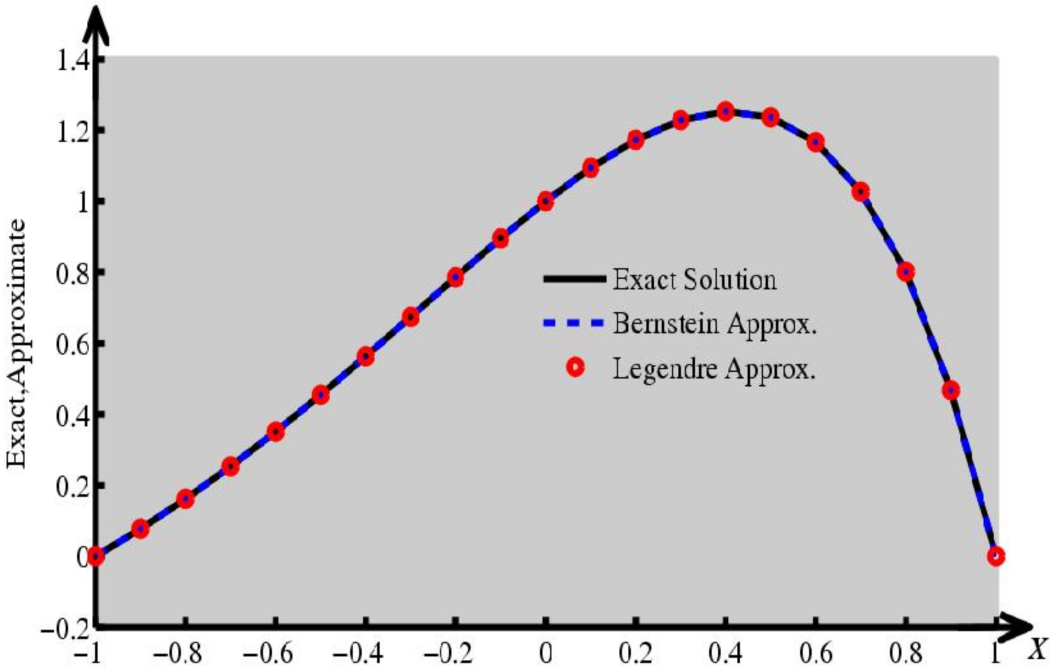


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 12 polynomials.

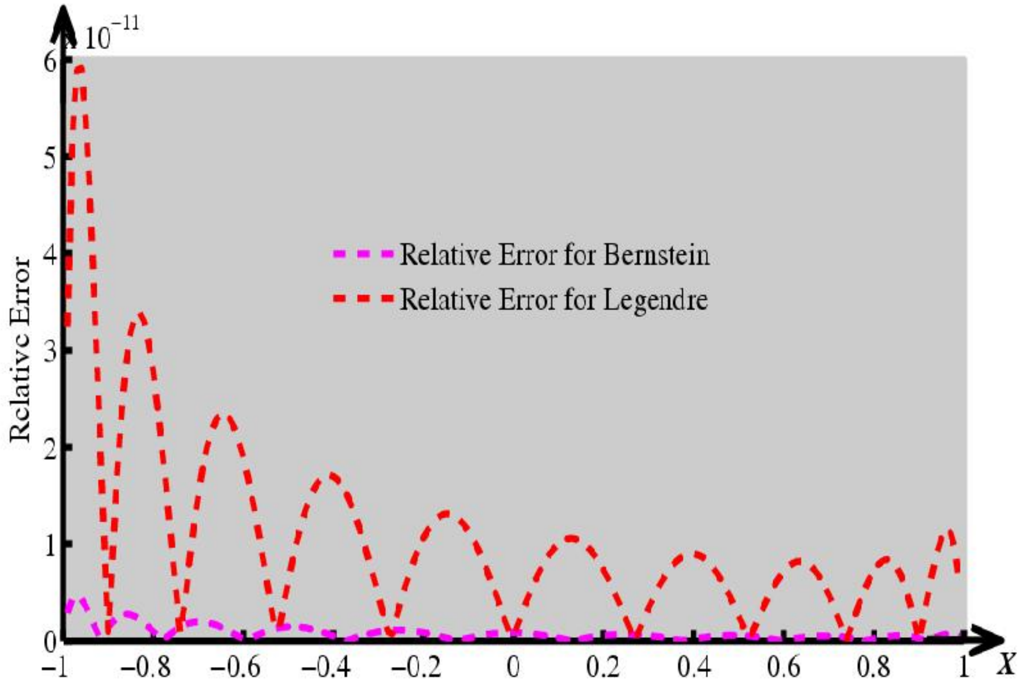


Fig. 4(b): Graphical representation of relative error of example 4 using 12 polynomials.

Example 5: Consider the **nonlinear** differential equation [25]

$$\frac{d^4u}{dx^4} = \sin x + \sin^2 x - \left(\frac{d^2u}{dx^2}\right)^2, \; 0 < x < 1 \tag{2.29a}$$

subject to the boundary conditions of type I in eqn. (2.2b):

$$u(0) = 0, u(1) = \sin 1, \; u'(0) = 1, u'(1) = \cos 1. \tag{2.29b}$$

The exact solution of this BVP is $u(x) = \sin x$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \geq 1 \tag{2.30}$$

Here $\theta_0(x) = x\sin 1$ is specified by the essential boundary conditions in eqn.(2.29b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Using eqn. (2.30) into eqn. (2.29a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^4\tilde{u}}{dx^4} + \left(\frac{d^2\tilde{u}}{dx^2}\right)^2 - \sin x - \sin^2 x\right] N_{k,n} dx = 0 \tag{2.31}$$

Integrating first term of eqn. (2.31) by parts we have

$$\int_0^1 \frac{d^4\tilde{u}}{dx^4} N_{k,n}(x)\, dx = \left[N_{k,n}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{dN_{k,n}(x)}{dx} \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2N_{k,n}(x)}{dx^2}\frac{d^2\tilde{u}}{dx^2} dx \; [\text{Since } N_{k,n}(1) = N_{k,n}(0) = 0]$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^3N_{k,n}(x)}{dx^3}\frac{d\tilde{u}}{dx} dx \tag{2.32}$$

Putting eqn. (2.32) into equation (2.31) and using approximation for $\tilde{u}(x)$ given in equation (2.30) and after applying the conditions given in eqn. (2.29b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1 \left[-\frac{d^3N_{k,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx} + 2\frac{d^2\theta_0}{dx^2}\frac{d^2N_{i,n}(x)}{dx^2} N_{k,n} + \sum_{j=1}^{n}\alpha_j\left(\frac{d^2N_{i,n}(x)}{dx^2}\frac{d^2N_{j,n}(x)}{dx^2} N_{k,n}(x)\right)\right] dx\right.$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=0}\Bigg]\alpha_i=\int_0^1\Big[(\sin x+\sin^2 x)N_{k,n}(x)$$

$$+\frac{d^3N_{k,n}(x)}{dx^3}\frac{d\theta_0}{dx}-\left(\frac{d^2\theta_0}{dx^2}\right)^2N_{k,n}(x)\Bigg]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\theta_0}{dx^2}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\theta_0}{dx^2}\right]_{x=0}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=1}\times\cos 1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0} \tag{2.33}$$

The above equation (2.33) is equivalent to matrix form

$$(D+B)A=G \tag{2.34a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{d^3N_{k,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx}+2\frac{d^2\theta_0(x)}{dx^2}\frac{d^2N_{i,n}(x)}{dx^2}N_{k,n}(x)\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=0} \tag{2.34b}$$

$$b_{i,k}=\sum_{j=1}^{n}\alpha_j\int_0^1\left(\frac{d^2N_{i,n}(x)}{dx^2}\frac{d^2N_{j,n}(x)}{dx^2}N_{k,n}(x)\right)dx \tag{2.34c}$$

$$g_k=\int_0^1\left[(\sin x+\sin^2 x)N_{k,n}(x)+\frac{d^3N_{k,n}(x)}{dx^3}\frac{d\theta_0}{dx}-\left(\frac{d^2\theta_0}{dx^2}\right)^2N_{k,n}(x)\right]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\theta_0}{dx^2}\right]_{x=1}$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\theta_0}{dx^2}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=1}\times\cos 1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0} \tag{2.34d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (2.29a). That is, to find initial coefficients we solve the system

$$DA=G \tag{2.35a}$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1\left[-\frac{d^3N_{k,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx}\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=0}$$

(2.35b)

$$g_k = \int_0^1 \left[(\sin x + \sin^2 x) N_{k,n}(x) + \frac{d^3 N_{k,n}(x)}{dx^3} \frac{d\theta_0}{dx} \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^2\theta_0}{dx^2} \right]_{x=1}$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^2\theta_0}{dx^2} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=1} \times \cos 1 + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=0} \quad (2.35c)$$

Once the initial values of the coefficients α_i are obtained from eqn. (2.35a), they are substituted into eqn. (2.34a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (2.30), we obtain an approximate solution of the BVP (2.29).

The numerical results for this problem are presented in T**able 5**.

Table 5: Maximum absolute errors for the example 5 with 5 iterations

x	Exact Results	12, Bernstein Polynomials		12, Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.000000E-000	0.0000000000	0.000000E-000
0.1	0.0998334166	0.0998334163	1.573325E-011	0.0998334162	3.341089E-010
0.2	0.1986693308	0.1986693308	1.140406E-011	0.1986693307	5.051098E-010
0.3	0.2955202067	0.2955202066	1.492606E-011	0.2955202060	5.717921E-011
0.4	0.3894183423	0.3894183423	9.988721E-012	0.3894183423	2.823010E-011
0.5	0.4794255386	0.4794255386	6.413814E-012	0.4794255386	9.905154E-010
0.6	0.5646424734	0.5646424734	7.907730E-012	0.5646424733	1.433638E-011
0.7	0.6442176872	0.6442176872	6.589806E-012	0.6442176870	1.854461E-010
0.8	0.7173560909	0.7173560909	2.166500E-012	0.7173560909	8.770040E-011
0.9	0.7833269096	0.7833269092	1.818679E-011	0.7833269090	4.991119E-011
1.0	0.8414709848	0.8414709848	0.000000E-000	0.8414709848	0.000000E-000

The maximum absolute error has been obtained by Kasi *et al* [25] is 1.359×10^{-5}.

Example 6: We consider the **nonlinear** differential equation [16, 18]

$$\frac{d^4 u}{dx^4} - 6e^{-4u} = -12(1+x)^{-4},\ 0 < x < 1 \quad (2.36a)$$

subject to the boundary conditions of type II, eqn (2.2c)

$$u(0) = 0, u(1) = \ln 2, u''(0) = -1, u''(1) = -\frac{1}{4}. \quad (2.36b)$$

The exact solution of this BVP is $u(x) = \ln(1+x)$.

In Figs. 5(a) and 5(b) we have given the exact and approximate solutions, and the relative errors of example 5 for $n = 12$. From Fig. 5(b) we observed that the error is nearly the order 10^{-7}.

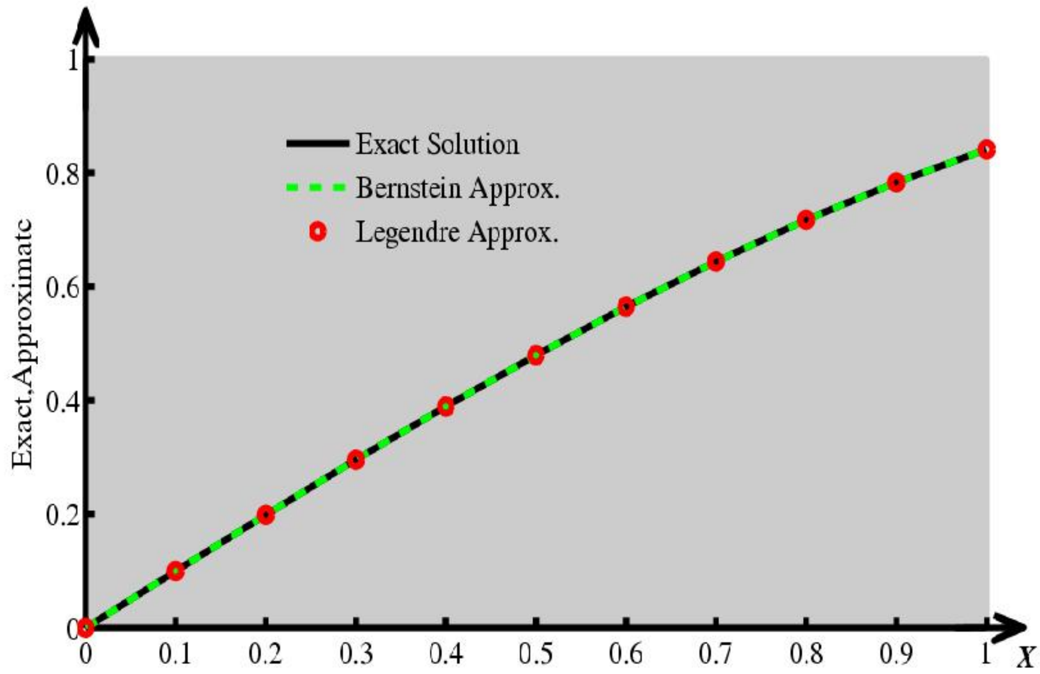


Fig. 5(a): Graphical representation of exact and approximate solutions of example 5 using 12 polynomials.

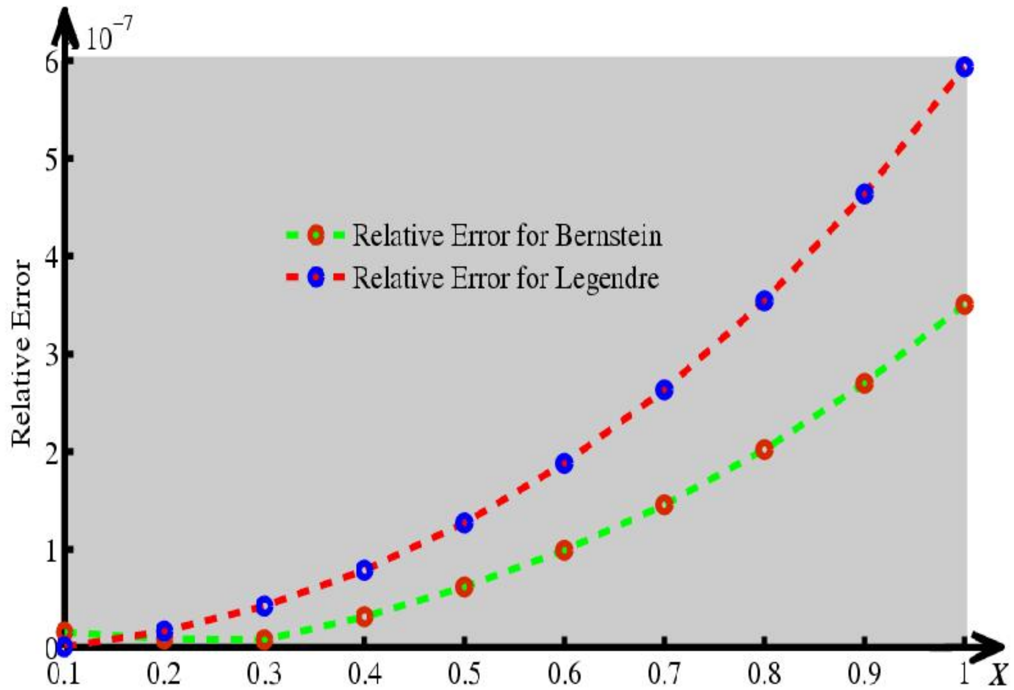


Fig. 5(b): Graphical representation of relative error of example 5 using 12 polynomials.

Consider the approximate solution of $u(x)$ as

$$\widetilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i\, N_{i,n}(x),\ \ n\geq 1 \tag{2.37}$$

Here $\theta_0(x)=x\ln 2$ is specified by the essential boundary conditions in (2.36b). Also $N_{i,n}(0)=N_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Substituting eqn. (2.37) into eqn. (2.36a), the Galerkin weighted residual eqns. are

$$\int_0^1\left[\frac{d^4\widetilde{u}}{dx^4}-6e^{-4\widetilde{u}}+12(1+x)^{-4}\right]N_{k,n}(x)dx=0 \tag{2.38}$$

In the same way of example 5, integrating first term of (2.38) by parts we get

$$\int_0^1\frac{d^4\widetilde{u}}{dx^4}N_{k,n}(x)dx=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^3N_{k,n}(x)}{dx^3}\frac{d\widetilde{u}}{dx}dx \tag{2.39}$$

Putting eqn. (2.39) into equation (2.38) and using approximation for $\widetilde{u}(x)$ given in equation (2.37) and after applying the boundary conditions given in eqn. (2.36b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^3N_{k,n}(x)}{dx^3}\frac{dN_{i,n}(x)}{dx}\right]dx+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx}\right]_{x=1}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{dN_{i,n}(x)}{dx}\right]_{x=0}\right]\alpha_i$$

$$=6\int_0^1\left[e^{-4\left[\theta_0+\sum_{j=1}^{n}\alpha_jN_{j,n}(x)\right]}\right]N_{k,n}(x)dx+\int_0^1\frac{d^3N_{k,n}(x)}{dx^3}\frac{d\theta_0}{dx}dx$$

$$-12\int_0^1(1+x)^{-4}N_{k,n}(x)dx-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-\frac{1}{4})-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}\times(-1) \tag{2.40}$$

The above equation (2.40) is equivalent to matrix form

$$DA=B+G \tag{2.41a}$$

where the elements of the square matrix D and the column matrices B and G are given by

$$d_{i,k} = \int_0^1 \left[-\frac{d^3 N_{k,n}(x)}{dx^3} \frac{dN_{i,n}(x)}{dx} \right] dx + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{dN_{i,n}(x)}{dx} \right]_{x=1}$$

$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{dN_{i,n}(x)}{dx} \right]_{x=0} \tag{2.41b}$$

$$b_k = 6 \int_0^1 \left[e^{-4\left[\theta_0 + \sum_{j=1}^{n} \alpha_j N_{j,n}(x)\right]} \right] N_{k,n}(x) dx \tag{2.41c}$$

$$g_k = \int_0^1 \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d\theta_0}{dx} - 12(1+x)^{-4} N_{k,n}(x) \right] dx - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d\theta_0}{dx} \right]_{x=1}$$

$$+ \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d\theta_0}{dx} \right]_{x=0} + \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} \times (-\frac{1}{4}) - \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0} \times (-1) \tag{2.41d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (2.36a). That is, to find initial coefficients we solve the system

$$DA = G \tag{2.42a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left[-\frac{d^3 N_{k,n}(x)}{dx^3} \frac{dN_{i,n}(x)}{dx} \right] dx + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{dN_{i,n}(x)}{dx} \right]_{x=1}$$

$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{dN_{i,n}(x)}{dx} \right]_{x=0} \tag{2.42b}$$

$$g_k = \int_0^1 \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d\theta_0}{dx} - 12(1+x)^{-4} N_{k,n}(x) \right] dx - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d\theta_0}{dx} \right]_{x=1}$$

$$+ \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d\theta_0}{dx} \right]_{x=0} + \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} \times (-\frac{1}{4}) - \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0} \times (-1) \tag{2.42c}$$

Once the initial values of the coefficients α_i are obtained from eqn. (2.42a), they are substituted into eqn. (2.41a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (2.37), we obtain an approximate solution of the BVP (2.36).

The maximum absolute errors for different number of polynomials are shown in **Table 6** with 5 iterations to compare with the results obtained so far

Table 6: Maximum absolute errors for the example 6 with 5 iterations.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
7	1.150×10^{-7}	3.170×10^{-7}	
8	8.370×10^{-8}	1.707×10^{-8}	2.2×10^{-8} (El-Gamel *et al* [16])
9	9.710×10^{-9}	8.640×10^{-9}	6.5×10^{-5} (Twizell and Tirmizi [18])
10	9.740×10^{-10}	5.840×10^{-10}	

The exact and approximate solutions are depicted in Fig. 6(a) and a plot of the relative errors is shown in Fig. 6(b) of example 6 for $n=10$. We observed from Fig. 6(b) that the error is of the order 10^{-6}.

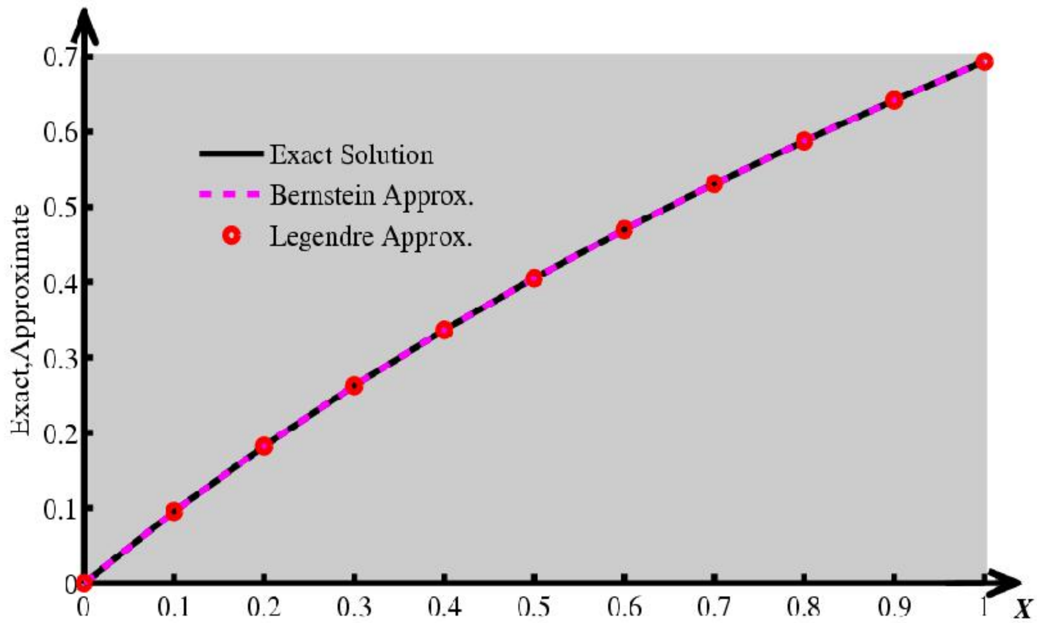


Fig. 6(a): Graphical representation of exact and approximate solutions of example 6 using 10 polynomials.

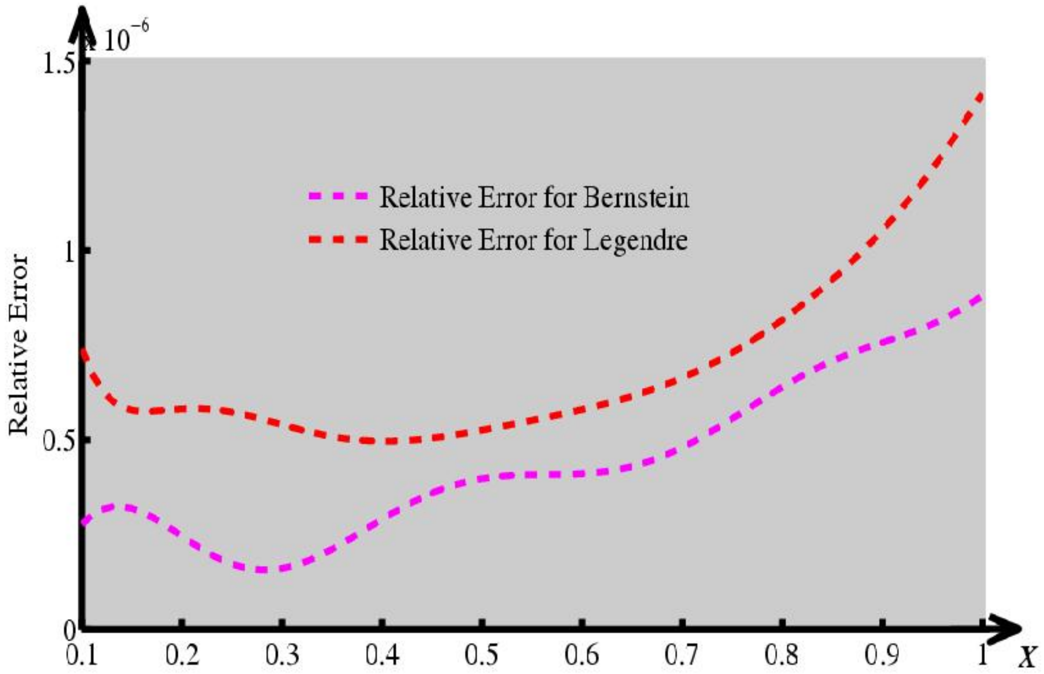


Fig. 6(b): Graphical representation of relative error of example 6 using 10 polynomials.

2.4 Conclusions

In this chapter, we have considered an application of Galerkin method for the numerical solution of fourth order BVPs with Bernstein and Legendre polynomials as basis functions for two different cases of boundary conditions. The exact solutions, numerical results, absolute errors and relative errors are given for each example. From the tables we see that the numerical results obtained by our method are superior to other existing methods. In addition, Bernstein polynomials yield the better results than the Legendre polynomials.

CHAPTER 3

Fifth Order Boundary Value Problems

3.1 Introduction

In the literature of numerical analysis, we observe that the fifth order BVPs arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences specially in the mathematical modeling of viscoelastic flows [26, 27]. Agarwal [8] has discussed extensively the existence and uniqueness theorem of solutions of such BVPs in his book without any numerical examples.

There are few numerical techniques available to solve fifth order BVPs. Such as Caglar *et al* [28] used sixth degree B-spline functions for the numerical solution of fifth order BVPs where their approach is divergent and unexpected situation is found near the boundaries of the interval. The spline methods have been discussed for the solution of higher order BVPs. Kasi *et al* [33] presented the numerical solution of fifth order BVPs by collocation method with sixth order B-splines. Lamnii *et al* [34] derived sextic spline solution of fifth order BVPs. The numerical solution of fifth order BVPs by the decomposition method was developed by Wazwaz [35] while differential transformation method was used by Erturk [36] for solving nonlinear problems.

In this chapter, we present Galerkin weighted residual method for constructing the numerical solution of fifth order linear and nonlinear BVPs with two point boundary conditions. To obtain accurate result by the Galerkin technique with Bernstein and Legendre polynomials, we use the transformation of the original polynomials into to a new set of basis functions to satisfy the corresponding homogeneous form of boundary conditions where the essential types of boundary conditions are given. The method is formulated as a rigorous matrix form.

In the present chapter, first we derive the matrix formulation for solving linear fifth order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions. Then we extend our idea for solving nonlinear differential equations. Few numerical examples of both linear and nonlinear BVPs, available in the literature, are presented to illustrate the reliability and efficiency of the proposed method. In section 3.4, we mention the conclusion of this chapter.

3.2 Galerkin Method for Matrix formulation

In this portion, we first obtain the rigorous matrix formulation for fifth order linear BVP and then we extend our idea for solving nonlinear BVP. For the numerical solution we consider a general fifth order linear boundary value problem of the form:

$$a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b \tag{3.1}$$

subject to the following boundary conditions

$$u(a) = A_0, \qquad u(b) = B_0, \qquad u'(a) = A_1, \qquad u'(b) = B_1,$$
$$u''(a) = A_2 \tag{3.2}$$

where $A_i, i = 0,1,2$ and $B_j, j = 0,1$ are finite real constants and $c_i, i = 0,1,\cdots 5$ and r are all continuous functions defined on the interval [a, b]. The boundary value problem (3.1) is solved with the boundary conditions of eqn (3.2).

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (3.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x + a$, and thus we have:

$$c_5 \frac{d^5u}{dx^5} + c_4 \frac{d^4u}{dx^4} + c_3 \frac{d^3u}{dx^3} + c_2 \frac{d^2u}{dx^2} + c_1 \frac{du}{dx} + c_0 u = s,\ 0 < x < 1 \tag{3.3}$$

$$u(0) = A_0, \qquad \frac{1}{b-a} u'(0) = A_1, \qquad \frac{1}{(b-a)^2} u''(0) = A_2,$$

$$u(1) = B_0, \qquad \frac{1}{b-a} u'(1) = B_1 \tag{3.4}$$

where

$$c_5 = \frac{1}{(b-a)^5} a_5((b-a)x + a), \qquad c_4 = \frac{1}{(b-a)^4} a_4((b-a)x + a),$$

$$c_3 = \frac{1}{(b-a)^3} a_3((b-a)x + a), \qquad c_2 = \frac{1}{(b-a)^2} a_2((b-a)x + a),$$

$c_1 = \frac{1}{b-a} a_1((b-a)x + a)$, $\qquad c_0 = a_0((b-a)x + a)$,

$s = r((b-a)x + a)$

We approximate the solution of the boundary value problem (3.3) as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{3.5}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0$, for each $i = 1,2,3,\ldots,n.$

Using eqn. (3.5) into eqn. (3.3), the weighted residual equations are

$$\int_0^1 \left[c_5 \frac{d^5\tilde{u}}{dx^5} + c_4 \frac{d^4\tilde{u}}{dx^4} + c_3 \frac{d^3\tilde{u}}{dx^3} + c_2 \frac{d^2\tilde{u}}{dx^2} + c_1 \frac{d\tilde{u}}{dx} + c_0\tilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{3.6}$$

Integrating by parts the terms up to second derivative on the left hand side of (3.6), we get

$$\int_0^1 c_5 \frac{d^5\tilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x) \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx \ \ [\text{Since}\, N_{j,n}(0) = N_{j,n}(1) = 0]$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{3.7}$$

$$\int_0^1 c_4 \frac{d^4\tilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.8}$$

$$\int_0^1 c_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.9}$$

$$\int_0^1 c_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x)\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{3.10}$$

Substituting eqns. (3.7) – (3.10) into eqn. (3.6) and using approximation for $\tilde{u}(x)$ given in equation (3.5) and after applying the boundary conditions given in eqn. (3.4) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \tag{3.11a}$$

where

$$D_{i,j} = \int_0^1 \left\{\left[\frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\right.\right.$$
$$\left.\left. - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] + c_1 N_{j,n}(x)\right]\frac{d}{dx}[N_{i,n}(x)] + c_0 N_{i,n}(x) N_{j,n}(x)\right\}dx$$
$$- \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)]\right]_{x=1} + \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2}{dx^2}[N_{i,n}(x)]\right]_{x=1}-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^2}{dx^2}[N_{i,n}(x)]\right]_{x=1} \quad (3.11b)$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[-\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2N_{j,n}(x)\right]\right.\right.$$

$$\left.\left.-c_1N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0N_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2\theta_0}{dx^2}\right]_{x=1}+\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^2\theta_0}{dx^2}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2+\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1 \quad (3.11c)$$

Solving the system (3.11a), we find the values of the parameters α_i, and then substituting these parameters into eqn. (3.5), we get the approximate solution of the BVP (3.4). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (3.1).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (3.11). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

3.3 Numerical examples and results

To test the applicability of the proposed method, we consider two linear and two nonlinear problems. For all examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB* 10**. The convergence of linear BVP is calculated by

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-12}.

In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^{N}\right| < \delta$$

where $\delta < 10^{-10}$ and N is the Newton's iteration number

Example 1: Consider the linear differential equation [33]

$$\frac{d^5u}{dx^5} + xu = 19xcox + 2x^3\cos x + 41\sin x - 2x^2\sin x,\ -1 \le x \le 1 \tag{3.12a}$$

subject to the boundary conditions

$$u(-1) = u(1) = \cos 1,\ u'(-1) = -u'(1) = -4\cos 1 + \sin 1, u''(-1) = 3\cos 1 - 8\sin 1. \tag{3.12b}$$

The analytic solution of the above system is, $u(x) = (2x^2 - 1)\cos x$.

The equivalent BVP over [0, 1] to the BVP (3.12) is,

$$\frac{1}{2^5}\frac{d^5u}{dx^5} + (2x-1)u = 19(2x-1)\cos(2x-1) + 2(2x-1)^3 cox(2x-1) + 41\sin(2x-1)$$

$$-2(2x-1)^2\sin(2x-1),\ 0 < x < 1 \tag{3.13a}$$

$$u(0) = u(1) = \cos 1, \frac{1}{2}u'(0) = -\frac{1}{2}u'(1) = -4\cos 1 + \sin 1, \frac{1}{4}u''(0) = 3\cos 1 - 8\sin 1 \tag{3.13b}$$

Applying the method illustrated in section (3.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n \ge 1 \tag{3.14}$$

Here $\theta_0(x) = (1-2x)^2 \cos 1$ is specified by the essential boundary conditions of eqn. (3.13b). Now the parameters $\alpha_i \, (i = 1,2,\dots\dots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\alpha_i = F_j, \; j = 1,2,\dots,n \tag{3.15a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^4 N_{j,n}(x)}{dx^4} \frac{dN_{i,n}(x)}{dx} + 32(2x-1) N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{dN_{j,n}(x)}{dx} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=1}$$
$$+ \left[\frac{dN_{j,n}(x)}{dx} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^2 N_{j,n}(x)}{dx^2} \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=1} \tag{3.15b}$$

$$F_j = \int_0^1 \Big\{ \big[19(2x-1)\cos(2x-1) + 2(2x-1)^3 cox(2x-1) + 41\sin(2x-1)$$
$$- 2(2x-1)^2 \sin(2x-1) \big] 32 N_{j,n}(x) - \frac{d^4 N_{j,n}(x)}{dx^4} \frac{d\theta_0}{dx} - (2x-1)\theta_0 N_{j,n}(x) \Big\} dx$$
$$+ \left[\frac{dN_{j,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{dN_{j,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^2 N_{j,n}(x)}{dx^2} \frac{d^2\theta_0}{dx^2} \right]_{x=1}$$
$$+ \left[\frac{d^2 N_{j,n}(x)}{dx^2} \right]_{x=1} (12\cos 1 - 32\sin 1) + \left[\frac{d^3 N_{j,n}(x)}{dx^3} \right]_{x=1} (8\cos 1 - 2\sin 1)$$
$$- \left[\frac{d^3 N_{j,n}(x)}{dx^3} \right]_{x=0} (-8\cos 1 + 2\sin 1) \tag{3.15c}$$

Solving the system (3.15a), we obtain the values of the parameters and then substituting these parameters into eqn. (3.14), we get the approximate solution of the BVP (3.13) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (3.12).

In **Table 1**, we list the maximum absolute errors to compare with the existing methods.

Table 1: Maximum absolute errors for the example 1.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
10	2.150×10^{-11}	2.478×10^{-9}	4.5162×10^{-4} (Kasi *et al* [13])
11	6.487×10^{-12}	6.244×10^{-12}	
12	4.256×10^{-12}	3.584×10^{-12}	
13	7.994×10^{-15}	6.912×10^{-13}	

Example 2: Consider the linear differential equation [28, 33, 34, 35]

$$\frac{d^5u}{dx^5} = u - 15e^x - 10xe^x,\ 0 \le x \le 1 \tag{3.16a}$$

subject to the boundary conditions

$$u(0) = u(1) = 0,\ u'(0) = 1, u'(1) = -e, u''(0) = 0\,. \tag{3.16b}$$

The analytic solution of the above system is, $u(x) = x(1-x)e^x$.

Using the method mentioned in section (3.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{3.17}$$

Here $\theta_0(x) = 0$ as specified by the essential boundary conditions of eqn. (3.16b). Now the parameters $\alpha_i\,(i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\alpha_i = F_j,\ j = 1,2,\ldots,n \tag{3.18a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^4N_{j,n}(x)}{dx^4}\frac{dN_{i,n}(x)}{dx} - N_{i,n}(x)N_{j,n}(x) \right] dx - \left[\frac{dN_{j,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3} \right]_{x=1}$$

$$+ \left[\frac{dN_{j,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^2N_{j,n}(x)}{dx^2}\frac{d^2N_{i,n}(x)}{dx^2} \right]_{x=1} \tag{3.18b}$$

$$F_j = \int_0^1 (-15e^x - 10xe^x) N_{j,n}(x)dx + \left[\frac{d^3N_{j,n}(x)}{dx^3} \right]_{x=1} (-e) - \left[\frac{d^3N_{j,n}(x)}{dx^3} \right]_{x=0} \tag{3.18c}$$

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 13$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-11}.

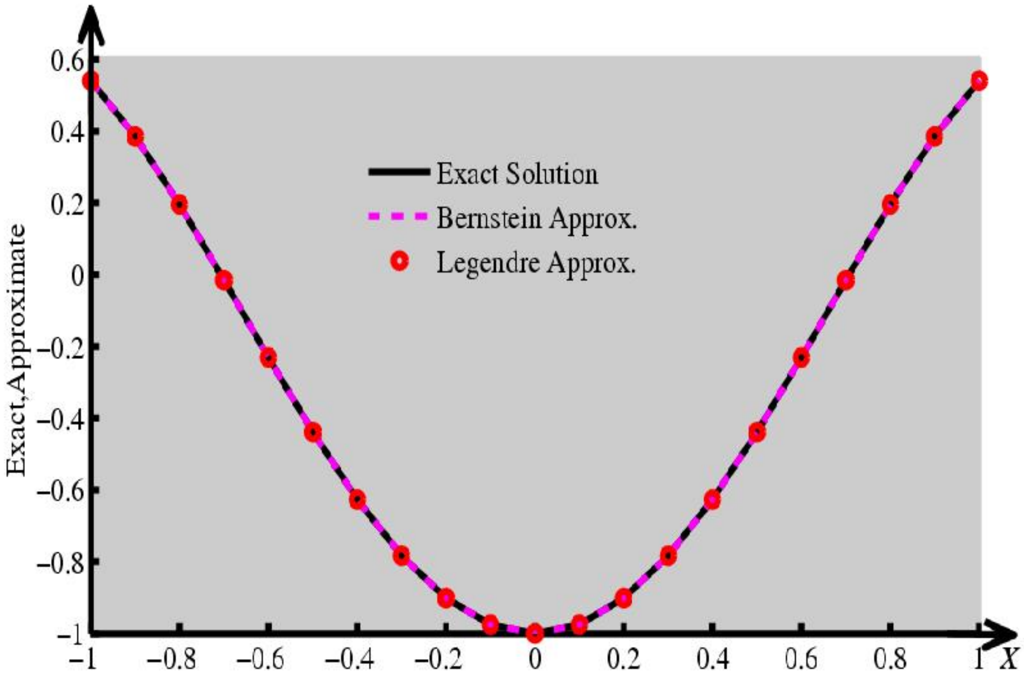


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 13 polynomials.

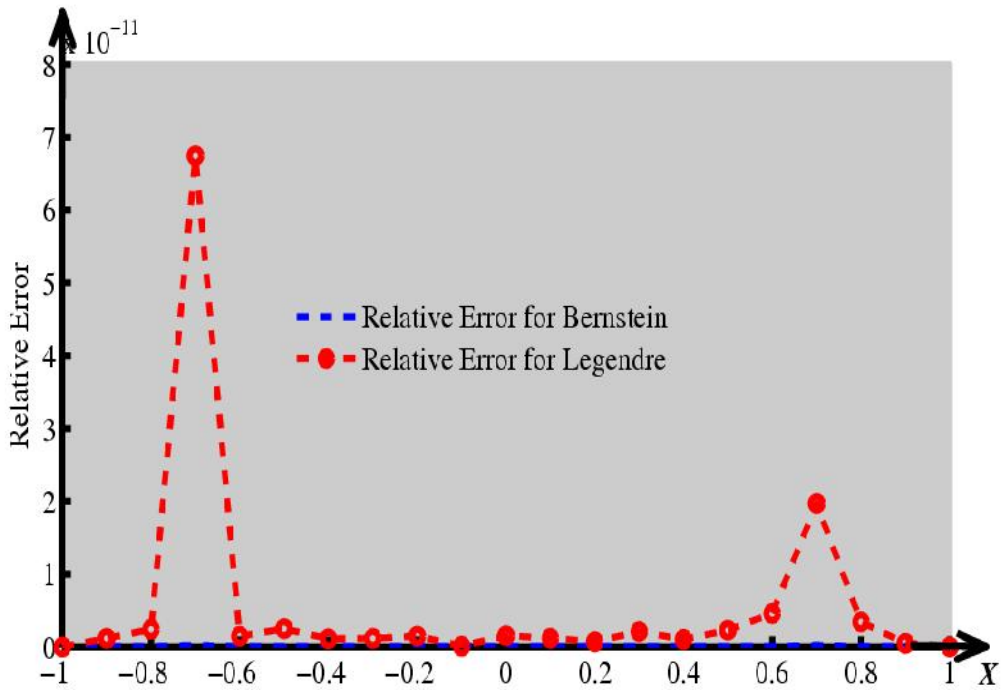


Fig. 1(b): Graphical representation of relative error of example 1 using 13 polynomials.

Solving the system (3.18a), we obtain the values of the parameters and then substituting these parameters into eqn. (3.17), we get the approximate solution of the desired BVP (3.16).

The numerical results for this problem are summarized in **Table 2**.

Table 2: Maximum absolute errors for the example 2.

x	Exact Results	11 Bernstein Polynomials		11 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	6.2656048E-026
0.1	0.0994653826	0.0994653826	6.7168493E-015	0.0994653826	4.6283810E-013
0.2	0.1954244413	0.1954244413	9.8809849E-015	0.1954244413	4.5449755E-013
0.3	0.2834703496	0.2834703496	1.0158541E-014	0.2834703496	9.0760732E-014
0.4	0.3580379274	0.3580379274	1.3877788E-015	0.3580379274	6.1872729E-013
0.5	0.4121803177	0.4121803177	8.7707619E-015	0.4121803177	2.9387603E-013
0.6	0.4373085121	0.4373085121	8.8262730E-015	0.4373085121	3.4067194E-013
0.7	0.4228880686	0.4228880686	1.2767565E-015	0.4228880686	3.6154413E-013
0.8	0.3560865486	0.3560865486	3.0531133E-015	0.3560865486	3.7581049E-014
0.9	0.2213642800	0.2213642800	2.3869795E-015	0.2213642800	7.2025719E-014
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand the maximum absolute errors have been found by Caglar *et al* [28], Kasi *et al* [33], Lamnii *et al* [34] and Wazwaz [35] are up to $1.570\times 10^{-2}, 1.5228\times 10^{-5}, 8.6115\times 10^{-7}$ and 2.2×10^{-9} respectively.

Example 3: Consider the **nonlinear** differential equation [35, 36]

$$\frac{d^5u}{dx^5} = u^2 e^{-x},\ 0 \le x \le 1 \tag{3.19a}$$

subject to the following boundary conditions

$$u(0) = 1,\ u(1) = e,\ u'(0) = 1, u'(1) = e, u''(0) = 1. \tag{3.19b}$$

The exact solution of this BVP is $u(x) = e^x$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{3.20}$$

Here $\theta_0(x) = 1 - x(1-e)$ is specified by the essential boundary conditions in (3.19b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

We have shown the exact and approximate solutions in Fig. 2(a) and the relative errors in Fig. 2(b) of example 2 for $n = 11$. It is found from Fig. 2(b) that the error is of the order 10^{-12}.

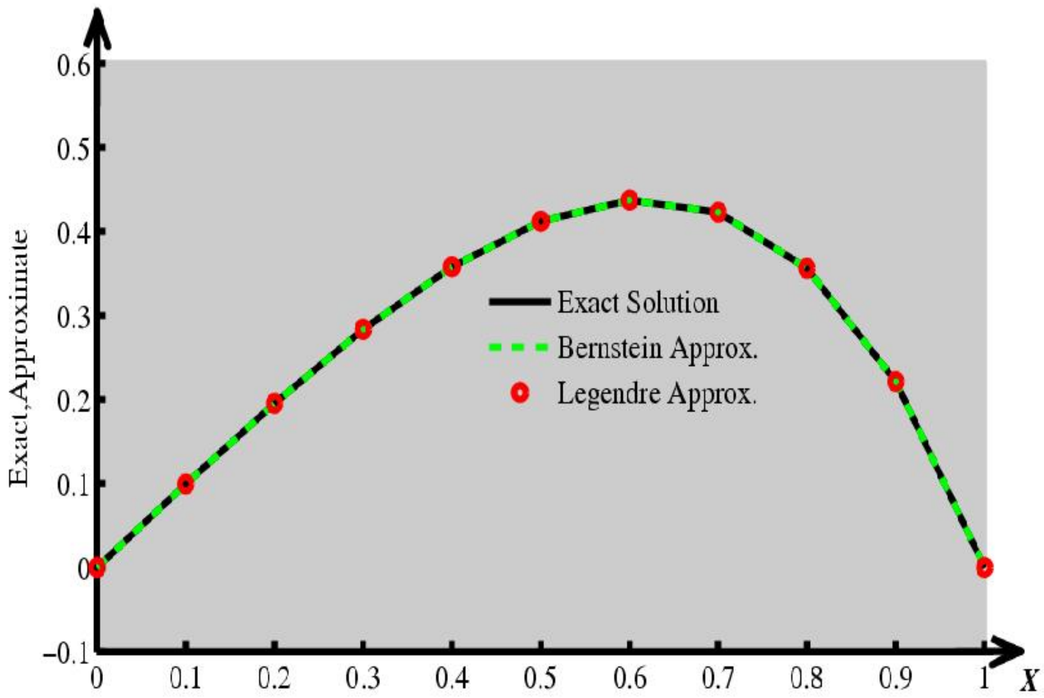


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 11 polynomials.

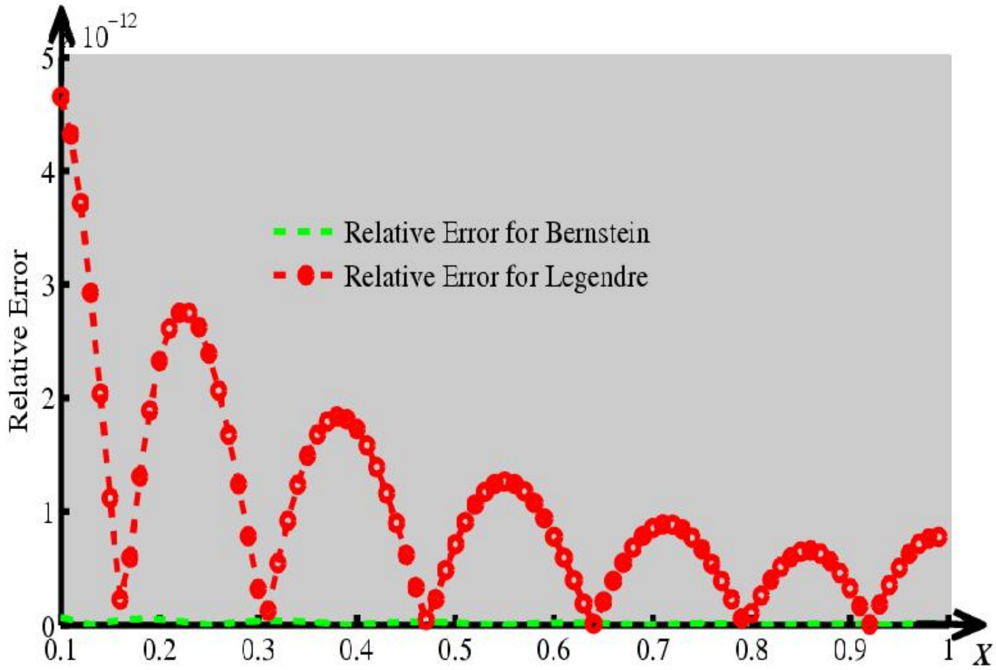


Fig. 2(b): Graphical representation of relative error of example 2 using 11 polynomials.

Using eqn. (3.20) into eqn. (3.19a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^5 \tilde{u}}{dx^5} - \tilde{u}^2 e^{-x} \right] N_{k,n}(x) dx = 0, k = 1,2,\cdots,n \tag{3.21}$$

Integrating first term of (3.21) by parts, we obtain

$$\int_0^1 \frac{d^5 \tilde{u}}{dx^5} N_{k,n}(x)\, dx = \left[N_{k,n}(x) \frac{d^4 \tilde{u}}{dx^4} \right]_0^1 - \int_0^1 \frac{dN_{k,n}(x)}{dx} \frac{d^4 \tilde{u}}{dx^4} dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 + \int_0^1 \frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^3 \tilde{u}}{dx^3} dx \text{ [Since } N_{k,n}(0) = N_{k,n}(1) = 0 \text{]}$$

$$= -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^2 \tilde{u}}{dx^2} dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^4 N_{k,n}(x)}{dx^4} \frac{d\tilde{u}}{dx} dx \tag{3.22}$$

Putting eqn. (3.22) into eqn. (3.21) and using approximation for $\tilde{u}(x)$ given in eqn. (3.20) and after applying the boundary conditions given in eqn. (3.19b) and rearranging the terms for the resulting equations, we obtain

$$\sum_{i=1}^{n} \left[\int_0^1 \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{dN_{i,n}(x)}{dx} - 2\theta_0 e^{-x} N_{i,n}(x) N_{k,n}(x) - \sum_{j=1}^{n} \alpha_j (N_{i,n}(x) N_{j,n}(x) N_{k,n}(x)) e^{-x} \right] dx \right.$$

$$\left. - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^2 N_{i,n}(x)}{dx^2} \right]_{x=1} \right] \alpha_i$$

$$= \int_0^1 \left[-\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} + \theta_0^2 e^{-x} N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 \theta_0}{dx^3} \right]_{x=1} - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3 \theta_0}{dx^3} \right]_{x=0}$$

$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^2 \theta_0}{dx^2} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=0} + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times e - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0} \tag{3.23}$$

The above equation (3.23) is equivalent to matrix form as

$$(D+B)A=G \tag{3.24a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{dN_{i,n}(x)}{dx}-2\theta_0e^{-x}N_{i,n}(x)N_{k,n}(x)\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1} \tag{3.24b}$$

$$b_{i,k}=-\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\,dx \tag{3.24c}$$

$$g_k=\int_0^1\left[-\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}+\theta_0^{\,2}e^{-x}N_{k,n}(x)\right]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^2\theta_0}{dx^2}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times e$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} \tag{3.24d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (3.19a). That is, to find initial coefficients we solve the system

$$DA=G \tag{3.25a}$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1\frac{d^4N_{k,n}(x)}{dx^4}\frac{dN_{i,n}(x)}{dx}dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^2N_{i,n}(x)}{dx^2}\right]_{x=1} \tag{3.25b}$$

$$g_k=\int_0^1-\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^2\theta_0}{dx^2}\right]_{x=1}+\left[\frac{d^2 N_{k,n}(x)}{dx^2}\right]_{x=0}+\left[\frac{d^3 N_{k,n}(x)}{dx^3}\right]_{x=1}\times e$$
$$-\left[\frac{d^3 N_{k,n}(x)}{dx^3}\right]_{x=0} \quad (3.25c)$$

Once the initial values of α_i are obtained from eqn. (3.25a), they are substituted into eqn. (3.24a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (3.20), we obtain an approximate solution of the BVP (3.19).

The maximum absolute errors for this problem using Bernstein and Legendre polynomials are shown in **Table 3** with 6 iterations.

Table 3: Maximum absolute errors of example 3 using 6 iterations.

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	2.5376055E-013	1.1051709181	2.9918024E-012
0.2	1.2214027582	1.2214027585	4.2368192E-012	1.2214027582	3.4379395E-011
0.3	1.3498588076	1.3498588076	6.0500216E-012	1.3498588076	1.1878055E-012
0.4	1.4918246976	1.4918246979	4.5652371E-013	1.4918246976	2.3980112E-011
0.5	1.6487212707	1.6487212707	1.5656920E-012	1.6487212707	3.4213951E-012
0.6	1.8221188004	1.8221188004	7.8939633E-013	1.8221188004	3.6519182E-012
0.7	2.0137527075	2.0137527075	9.0033536E-013	2.0137527075	2.8570975E-012
0.8	2.2255409285	2.2255409287	6.0421113E-012	2.2255409285	1.4395748E-011
0.9	2.4596031112	2.4596031110	4.3842707E-013	2.4596031112	2.8618543E-011
1.0	2.7182818285	2.7182818285	0.0000000E+000	2.7182818285	1.2759373E-000

On the other hand, the maximum absolute errors have been obtained by Wazwaz [35] and Erturk [36] are 4.1×10^{-8} and 1.52×10^{-10} respectively.

Example 4: Consider the **nonlinear** differential equation [28, 33]

$$\frac{d^5u}{dx^5}+24e^{-5u}=48(1+x)^{-5}, 0\le x\le 1 \quad 3.26a)$$

subject to the following boundary conditions

$$u(0)=0u(1)=\ln 2, u'(0)=1, u'(1)=0.5, u''(0)=-1\,. \quad (3.26b)$$

We depict the exact and approximate solutions in Fig. 3(a) and a plot of relative errors in Fig. 3(b) of example 3 for $n = 12$. From Fig. 3(b) we observe that the error is nearly the order 10^{-8}

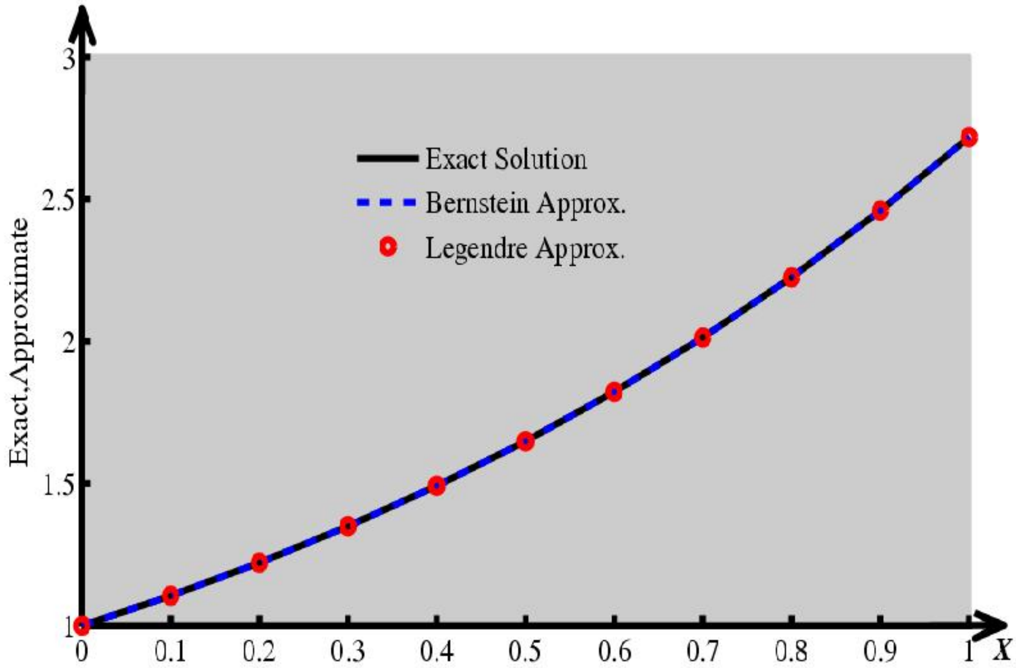


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 12 polynomials.

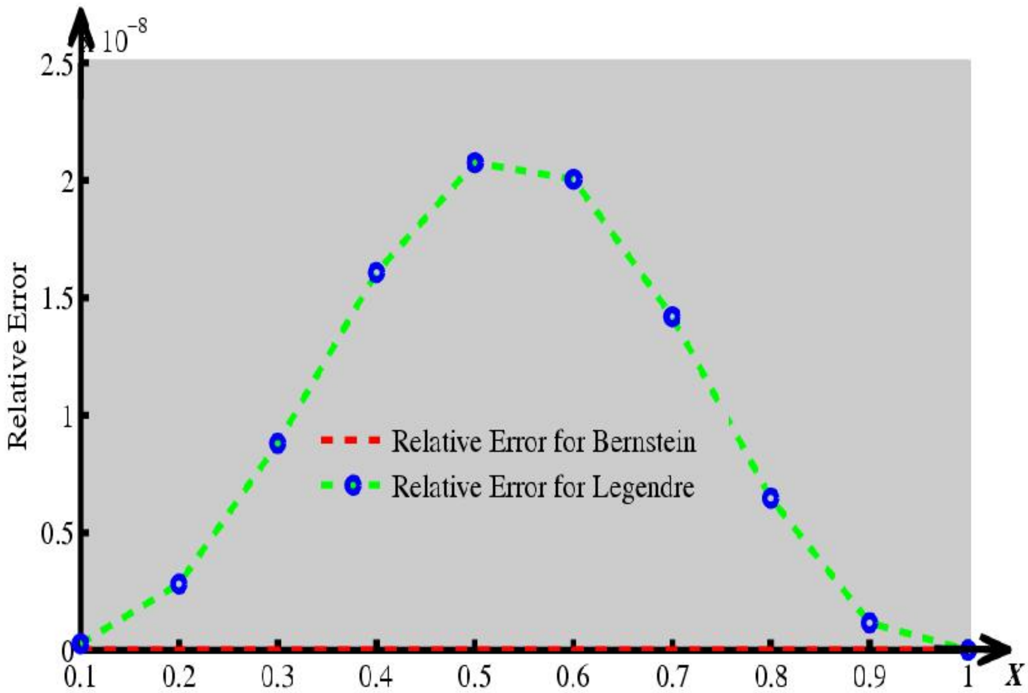


Fig. 3(b): Graphical representation of relative error of example 3 using 12 polynomials.

The exact solution of this BVP is $u(x) = \ln(1+x)$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ n \geq 1 \tag{3.27}$$

Here $\theta_0(x) = x\ln 2$ is specified by the essential boundary conditions in (3.26b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Putting eqn. (3.27) into eqn. (3.26a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^5\tilde{u}}{dx^5} + 24e^{-5\tilde{u}} - 48(1+x)^{-5} \right] N_{k,n}(x)dx = 0, k = 1,2,\ldots,n \tag{3.28}$$

In the same way of example 3, integrating first term of (3.28) by parts we obtain

$$\int_0^1 \frac{d^5\tilde{u}}{dx^5} N_{k,n}(x)dx = -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d\tilde{u}}{dx} \right]_0^1$$

$$+ \int_0^1 \frac{d^4N_{k,n}(x)}{dx^4} \frac{d\tilde{u}}{dx} dx \tag{3.29}$$

Using eqn. (3.29) into eqn. (3.28) and using approximation for $\tilde{u}(x)$ given in eqn. (3.27) and after applying the boundary conditions given in eqn. (3.26b) and rearranging the terms for the resulting equations, we obtain

$$\sum_{i=1}^{n} \left[\int_0^1 \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{dN_{i,n}(x)}{dx} \right] dx - \left[\frac{dN_{i,n}(x)}{dx} \frac{d^3N_{k,n}(x)}{dx^3} \right]_{x=1} + \left[\frac{dN_{i,n}(x)}{dx} \frac{d^3N_{k,n}(x)}{dx^3} \right]_{x=0} \right.$$

$$\left. + \left[\frac{d^2N_{i,n}(x)}{dx^2} \frac{d^2N_{k,n}(x)}{dx^2} \right]_{x=0} \right] \alpha_i = -24 \int_0^1 \left[e^{-5\left[\theta_0 + \sum_{j=1}^{n} \alpha_j N_{j,n}(x) \right]} \right] N_{k,n}(x)dx$$

$$- \int_0^1 \frac{d^4N_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} dx + 48 \int_0^1 (1+x)^{-5} N_{k,n}(x)dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=1}$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^2\theta_0}{dx^2} \right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2} \right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(0.5)-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} \tag{3.30}$$

The above equation (3.30) is equivalent to matrix form

$$DA = B + G \tag{3.31a}$$

where the elements of the square matrix D and the column matrices B and G are given by

$$d_{i,k} = \int_0^1\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{dN_{i,n}(x)}{dx}\right]dx - \left[\frac{dN_{i,n}(x)}{dx}\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1} + \left[\frac{dN_{i,n}(x)}{dx}\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^2N_{i,n}(x)}{dx^2}\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0} \tag{3.31b}$$

$$b_k = -24\int_0^1\left[e^{-5\left[\theta_0+\sum_{j=1}^{n}\alpha_j N_{j,n}(x)\right]}\right]N_{k,n}(x)dx \tag{3.31c}$$

$$g_k = \int_0^1\left[-\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}dx + 48(1+x)^{-5}N_{k,n}(x)\right]dx + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=0} - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^2\theta_0}{dx^2}\right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(0.5)-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} \tag{3.31d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (3.26a). That is, to find initial coefficients we solve the system

$$DA = G \tag{3.32a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{dN_{i,n}(x)}{dx} \right] dx - \left[\frac{dN_{i,n}(x)}{dx} \frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} + \left[\frac{dN_{i,n}(x)}{dx} \frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0}$$

$$+ \left[\frac{d^2 N_{i,n}(x)}{dx^2} \frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=0} \tag{3.32b}$$

$$g_k = \int_0^1 \left[-\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} dx + 48(1+x)^{-5} N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=1}$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^2\theta_0}{dx^2} \right]_{x=1} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=0}$$

$$+ \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times (0.5) - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0} \tag{3.32c}$$

Once the initial values of the coefficients α_i are obtained from eqn. (3.32a), they are substituted into eqn. (3.31a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (3.27), we obtain an approximate solution of the BVP (3.26).

The maximum absolute errors, using different number of polynomials, by the present method with 5 iterations and the previous results obtained so far, are summarized in **Table 4.**

Table 4: Maximum absolute errors for the example 4 with 5 iterations.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
7	2.150×10^{-7}	7.980×10^{-7}	4.6×10^{-2} (Caglar *et al* [28]) 5.3197×10^{-5} (Kasi *et al* [33])
8	5.370×10^{-8}	1.770×10^{-8}	
9	7.515×10^{-9}	5.440×10^{-9}	
10	9.750×10^{-10}	8.940×10^{-10}	

In Figs. 4(a) and 4(b), the exact and approximate solutions, and the relative errors of example 4 for $n = 10$ are depicted respectively. We see from Fig. 4(b) that the error is nearly the order 10^{-6}

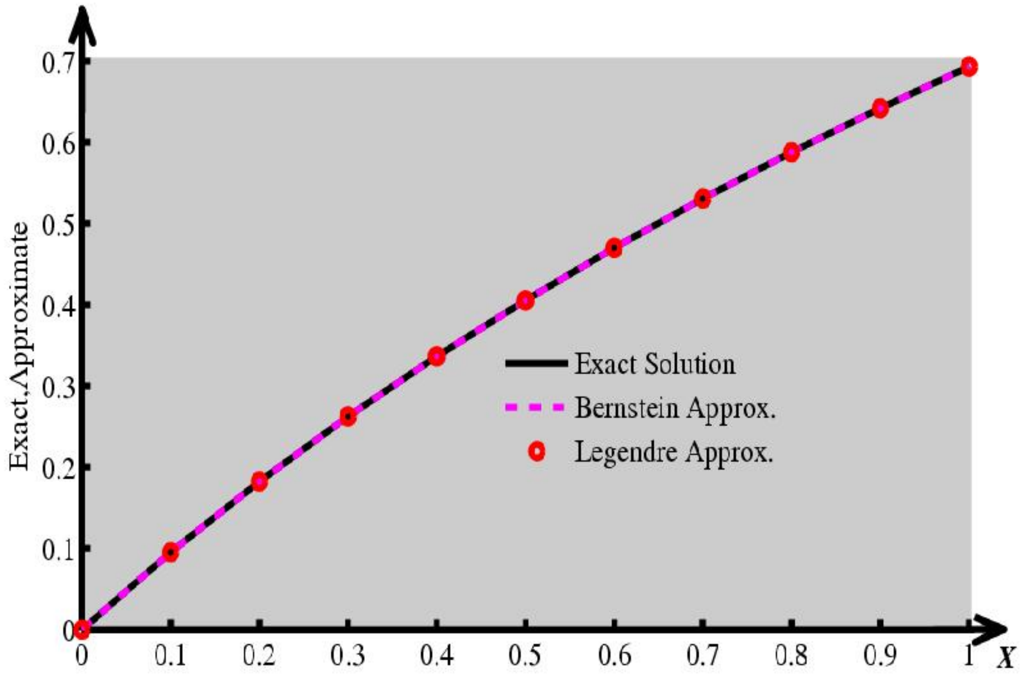


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 10 polynomials.

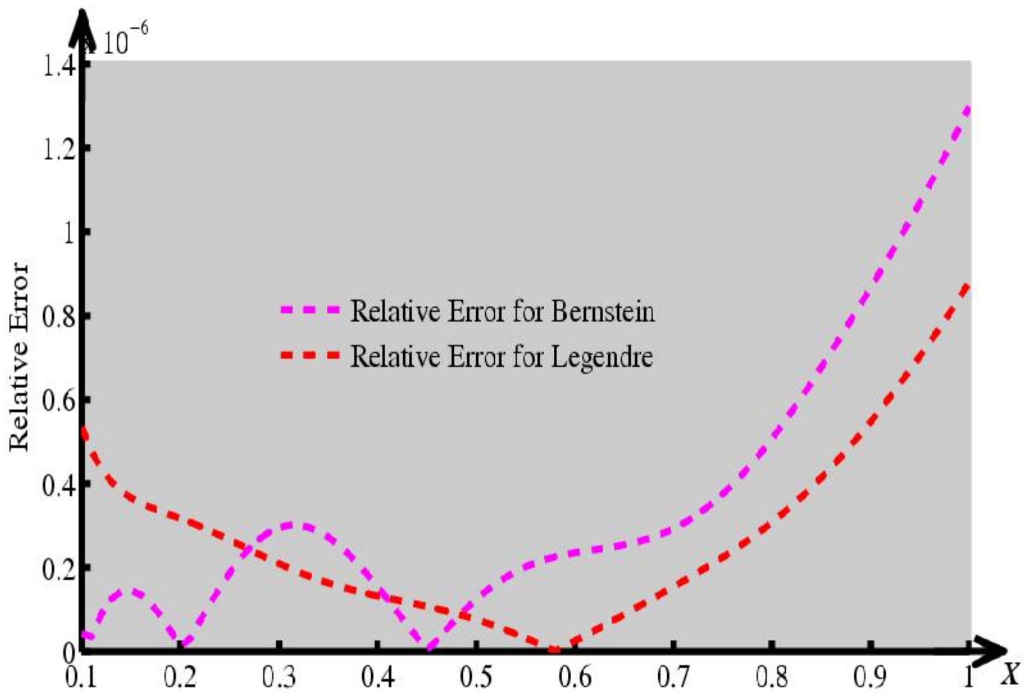


Fig. 4(b): Graphical representation of relative error of example 4 using 10 polynomials.

3.4 Conclusions

In this chapter we have used Bernstein and Legendre piecewise continuous and differentiable polynomials as basis functions for the numerical solutions of fifth order linear and nonlinear BVPs in the Galerkin method. We see from the tables that the numerical results obtained by our method are better than other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. It may also notice that the numerical solutions coincide with the exact solution even lower order Bernstein and Legendre polynomials are used in the approximation and are shown in Figs. [1-4]. Thus the present method is quite efficient.

CHAPTER 4

Sixth Order Boundary Value Problems

4.1 Introduction

Sixth order BVPs arise in many real life phenomena, for example the vibration behavior of ring structures is governed by a sixth order ordinary differential equation and also in the mathematical modeling of astrophysics; the narrow convecting layers which are believed to surround A-type stars [39]. Moreover, Chandrasekhar [9] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order. Agarwal [8] has discussed the theorems of the conditions for the existence and uniqueness of solutions of the sixth order BVPs thoroughly in a book, but no numerical methods are contained there in. Non-numerical techniques were developed by Baldwin [37, 38] for solving such BVPs.

There are many researchers have attempted to solve sixth order BVPs numerically. For example, Boutayeb and Twizell [40] developed a family of numerical methods for the solution of special and general nonlinear sixth order BVPs. Numerical methods for the solution of special and general sixth order BVPs with application to Benard layer eigenvalue problem were introduced by Twizell and Boutayeb [41]. Glatzmaier [42] also noticed that dynamo action in some stars may be modeled by such BVPs. Siddiqi *et al* [43] presented the Quintic spline solution of linear sixth order BVPs. Siraj-ul-Islam *et al* [44] used nonpolynomial splines approach to the solution of sixth order BVPs. Siddiqi and Akram [45] developed septic spline solutions of sixth order BVPs. On the other hand, Chawla and Katti [46] presented numerical methods of solutions implicitly, although the authors concentrated their attention on fourth order BVPs. A second order method was introduced in [47] for solving special and general sixth order BVPs and in later work Twizell and Boutayeb [41] developed finite difference methods of order two, four, six and eight for solving such problems. Gamel *et al* [48] used Sinc-Galerkin method for the solution of sixth order BVPs. Wazwaz

[49] developed decomposition and modified domain decomposition methods to find the solution of the sixth order BVPs. Siddiqi and Twizell [32] solved the sixth order BVPs using polynomial splines of degree six where spline values at the mid knots of the interpolation interval and the corresponding values of the even order derivatives were related through consistency relations. Recently, Khan and Sultana [50] used parametric quintic spline solution for sixth order two point BVPs. Fazal-i-Haq *et al* [51] developed the solution of sixth order BVPs by collocation method using Haar wavelets. Akram and Siddiqi [52] presented the solution of sixth order BVPs using non-polynomial spline technique. Logmani and Ahmadinia [53] derived numerical solution of sixth order BVPs with sixth degree B-spline functions.

In this chapter, Galerkin method with Bernstein and Legendre polynomials as basis functions is devoted to find the numerical solutions of the sixth order linear and nonlinear differential equations for two different cases of boundary conditions. In this method, the basis functions are transformed into a new set of basis functions, which must satisfy the homogeneous form of Dirichlet boundary conditions.

However, the formulation for solving linear sixth order BVP by Galerkin weighted residual method is described in the portion 4.2. Two formulations are described considering two types of boundary conditions which are presented in sections 4.2.1 and 4.2.2 respectively. Then we deduce similar formulation for nonlinear problems in the next section. In section 4.3, numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation. Finally conclusions of this chapter are given in section 4.4.

4.2 Matrix Formulation

In this section we first derive the matrix formulation for sixth order linear BVP and then we extend our idea for solving nonlinear BVP. For the numerical solution, we consider a general sixth order linear BVP given by

$$a_6 \frac{d^6u}{dx^6} + a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r \ , \ a < x < b \qquad (4.1a)$$

subject to the following two types of boundary conditions

Type I: $u(a) = A_0$, $u(b) = B_0$, $u'(a) = A_1$, $u'(b) = B_1$,

$$u''(a) = A_2, \quad u''(b) = B_2 \tag{4.1b}$$

Type II: $u(a) = A_0$, $u(b) = B_0$, $u''(a) = A_2$, $u''(b) = B_2$,

$$u^{(iv)}(a) = A_4, \quad u^{(iv)}(b) = B_4 \tag{4.1c}$$

where $A_i, B_i, i = 0,1,2,4$ are finite real constants and $a_i, i = 0,1,\cdots 6$ and r are all continuous and differentiable functions of x defined on the interval [a, b]. The BVP (4.1) is to be solved with both the boundary conditions of type I and type II. Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (4.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$c_6 \frac{d^6u}{dx^6} + c_5 \frac{d^5u}{dx^5} + c_4 \frac{d^4u}{dx^4} + c_3 \frac{d^3u}{dx^3} + c_2 \frac{d^2u}{dx^2} + c_1 \frac{du}{dx} + c_0 u = s, \; 0 < x < 1 \tag{4.2a}$$

$$u(0) = A_0, \quad \frac{1}{b-a} u'(0) = A_1, \quad \frac{1}{(b-a)^2} u''(0) = A_2,$$

$$u(1) = B_0, \quad \frac{1}{b-a} u'(1) = B_1, \quad \frac{1}{(b-a)^2} u''(1) = B_2 \tag{4.2b}$$

and

$$u(0) = A_0, \quad \frac{1}{(b-a)^2} u''(0) = A_2, \quad \frac{1}{(b-a)^4} u^{(iv)}(0) = A_4,$$

$$u(1) = B_0, \quad \frac{1}{(b-a)^2} u''(1) = B_2, \quad \frac{1}{(b-a)^4} u^{(iv)}(1) = B_4 \tag{4.2c}$$

where

$$c_6 = \frac{1}{(b-a)^6} a_6((b-a)x+a), \quad c_5 = \frac{1}{(b-a)^5} a_5((b-a)x+a),$$

$$c_4 = \frac{1}{(b-a)^4} a_4((b-a)x+a), \quad c_3 = \frac{1}{(b-a)^3} a_3((b-a)x+a),$$

$$c_2 = \frac{1}{(b-a)^2} a_2((b-a)x+a), \quad c_1 = \frac{1}{b-a} a_1((b-a)x+a),$$

$$c_0 = a_0((b-a)x+a), \quad s = r((b-a)x+a)$$

To solve the boundary value problem (4.2) by the Galerkin method we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \geq 1 \tag{4.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0,$ for each $i = 1,2,3,\ldots,n.$

Using eqn. (4.3) into eqn (4.2a), the Galerkin weighted residual equations are

$$\int_0^1 \left[c_6 \frac{d^6\widetilde{u}}{dx^6} + c_5 \frac{d^5\widetilde{u}}{dx^5} + c_4 \frac{d^4\widetilde{u}}{dx^4} + c_3 \frac{d^3\widetilde{u}}{dx^3} + c_2 \frac{d^2\widetilde{u}}{dx^2} + c_1 \frac{d\widetilde{u}}{dx} + c_0\widetilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{4.4}$$

4.2.1 Formulation I

In this portion, we have derived the matrix formulation by applying the boundary conditions of type I.

Integrating by parts the terms up to second derivative on the left hand side of (4.4), we get

$$\int_0^1 c_6 \frac{d^6\widetilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x) \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} dx \ \ [\text{Since}\, N_{j,n}(0) = N_{j,n}(1) = 0\,]$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} \right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{4.5}$$

$$\int_0^1 c_5\frac{d^5\widetilde{u}}{dx^5}N_{j,n}(x)dx=\left[c_5N_{j,n}(x)\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{4.6}$$

$$\int_0^1 c_4\frac{d^4\widetilde{u}}{dx^4}N_{j,n}(x)dx=\left[c_4N_{j,n}(x)\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{4.7}$$

$$\int_0^1 c_3\frac{d^3\widetilde{u}}{dx^3}N_{j,n}(x)dx=\left[c_3N_{j,n}(x)\frac{d^2\widetilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{4.8}$$

$$\int_0^1 c_2\frac{d^2\widetilde{u}}{dx^2}N_{j,n}(x)dx=\left[c_2N_{j,n}(x)\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_2N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\,dx \qquad (4.9)$$

Substituting eqns. (4.5) to (4.9) into eqn. (4.4) and using approximation for $\tilde{u}(x)$ given in eqn. (4.3) and after applying the boundary conditions given in type I, eqn. (4.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (4.10a)$$

where

$$D_{i,j} = \int_0^1 \left\{\left[-\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\right.\right.$$

$$\left.\left. - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] + c_1 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x)\right\}dx$$

$$- \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$- \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (4.10b)$$

$$F_j = \int_0^1 \left\{ s N_{j,n}(x) + \left[\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] - \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] + \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] - \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\right.\right.$$

$$\left.\left. + \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] - c_1 N_{j,n}(x)\right]\frac{d\theta}{dx} - c_0 \theta_0 N_{j,n}(x)\right\}dx + \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\theta}{dx^4}\right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\theta}{dx^4}\right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=0}$$

$$+ \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=1} - \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$
$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 - \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$
$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$
$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$
$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$
$$+\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$
$$-\left[\frac{d}{dx}[c_3 N_{j,n}(x)]\right]_{x=0} \times (b-a) A_1 \qquad (4.10c)$$

Solving the system (4.10a), we find the values of the parameters α_i, and then substituting these parameters into eqn. (4.3), we get the approximate solution of the BVP (4.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (4.1).

4.2.2 Formulation II

In this section, we have used the boundary conditions of type II for obtaining the matrix form.

In the same way of portion (4.2.1), integrating by parts the terms consisting sixth, fifth, fourth, third, and second derivatives on the left hand side of (4.4), and applying the boundary conditions prescribed in type II, eqn (4.2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j, j = 1,2,\ldots,n \qquad (4.11a)$$

where

$$D_{i,j}=\int_0^1\Bigg\{\Bigg[-\frac{d^5}{dx^5}\big[c_6N_{j,n}(x)\big]+\frac{d^4}{dx^4}\big[c_5N_{j,n}(x)\big]-\frac{d^3}{dx^3}\big[c_4N_{j,n}(x)\big]+\frac{d^2}{dx^2}\big[c_3N_{j,n}(x)\big]$$

$$-\frac{d}{dx}\big[c_2N_{j,n}(x)\big]+c_1N_{j,n}(x)\Bigg]\frac{d}{dx}\big[N_{i,n}(x)\big]+c_0N_{i,n}(x)N_{j,n}(x)\Bigg\}dx$$

$$+\left[\frac{d^2}{dx^2}\big[c_6N_{j,n}(x)\big]\frac{d^3}{dx^3}\big[N_{i,n}(x)\big]\right]_{x=1}-\left[\frac{d^2}{dx^2}\big[c_6N_{j,n}(x)\big]\frac{d^3}{dx^3}\big[N_{i,n}(x)\big]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\big[c_6N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=1}-\left[\frac{d^4}{dx^4}\big[c_6N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\big[c_5N_{j,n}(x)\big]\frac{d^3}{dx^3}\big[N_{i,n}(x)\big]\right]_{x=1}+\left[\frac{d}{dx}\big[c_5N_{j,n}(x)\big]\frac{d^3}{dx^3}\big[N_{i,n}(x)\big]\right]_{x=0}$$

$$+\left[\frac{d^3}{dx^3}\big[c_5N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=0}+\left[\frac{d^2}{dx^2}\big[c_4N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\big[c_4N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=0}-\left[\frac{d}{dx}\big[c_3N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\big[c_3N_{j,n}(x)\big]\frac{d}{dx}\big[N_{i,n}(x)\big]\right]_{x=0} \qquad (4.11b)$$

$$F_j=\int_0^1\Bigg\{sN_{j,n}(x)+\Bigg[\frac{d^5}{dx^5}\big[c_6N_{j,n}(x)\big]-\frac{d^4}{dx^4}\big[c_5N_{j,n}(x)\big]+\frac{d^3}{dx^3}\big[c_4N_{j,n}(x)\big]-\frac{d^2}{dx^2}\big[c_3N_{j,n}(x)\big]$$

$$+\frac{d}{dx}\big[c_2N_{j,n}(x)\big]-c_1N_{j,n}(x)\Bigg]\frac{d\theta}{dx}-c_0\theta_0N_{j,n}(x)\Bigg\}dx+\left[\frac{d}{dx}\big[c_5N_{j,n}(x)\big]\frac{d^3\theta}{dx^3}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\big[c_5N_{j,n}(x)\big]\frac{d^3\theta}{dx^3}\right]_{x=0}-\left[\frac{d^3}{dx^3}\big[c_5N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=1}+\left[\frac{d^3}{dx^3}\big[c_5N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=0}$$

$$-\left[\frac{d^2}{dx^2}\big[c_4N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=1}+\left[\frac{d^2}{dx^2}\big[c_4N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=0}+\left[\frac{d}{dx}\big[c_3N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\big[c_3N_{j,n}(x)\big]\frac{d\theta}{dx}\right]_{x=0}+\left[\frac{d}{dx}\big[c_6N_{j,n}(x)\big]\right]_{x=1}\times(b-a)^4B_4$$

$$-\left[\frac{d}{dx}\big[c_6N_{j,n}(x)\big]\right]_{x=0}\times(b-a)^4A_4+\left[\frac{d^3}{dx^3}\big[c_6N_{j,n}(x)\big]\right]_{x=1}\times(b-a)^2B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$
$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$
$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 \qquad (4.11c)$$

Solving the system (4.11a), we find the values of the parameters α_i and then substituting these parameters into eqn. (4.3), we get the approximate solution of the BVP (4.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (4.1).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (4.10) and (4.11). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

4.3. Numerical examples and results

To test the applicability of the proposed method, we consider four linear and two nonlinear problems consisting of both types of boundary conditions. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB*** **10**. The convergence of linear BVP is calculated by

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-12}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^{N}\right| < \delta$$

where $\delta < 10^{-10}$ and N is the Newton's iteration number.

Example 1: Consider the linear differential equation [43, 45, 50, 51, 52]

$$\frac{d^6u}{dx^6} - u = -6e^x,\ 0 \le x \le 1 \tag{4.12a}$$

subject to the boundary conditions of type I in eqn. (4.2b):

$$u(0) = 1,\ u(1) = 0,\ u'(0) = 0, u'(1) = -e, u''(0) = -1, u''(1) = -2e\ . \tag{4.12b}$$

The analytic solution of the above problem is, $u(x) = (1-x)e^x$.

Using the method illustrated in (4.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), n \ge 1 \tag{4.13}$$

Here $\theta_0(x) = 1 - x$ as specified by the essential boundary conditions of eqn. (4.12b). Now the parameters $\alpha_i (i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\ \alpha_i = F_j, j = 1,2,\ldots,n \tag{4.14a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - N_{i,n}(x)N_{j,n}(x) \right] dx$$
$$- \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=0}$$
$$+ \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=0} \tag{4.14b}$$

$$F_j = \int_0^1 \left\{ -6e^x N_{j,n}(x) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d\theta}{dx} \right] + \theta_0 N_{j,n}(x) \right\} dx + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4\theta}{dx^4} \right]_{x=1}$$
$$- \left[\frac{d}{dx}[N_{j,n}(x)]\frac{d^4\theta}{dx^4} \right]_{x=0} - \left[\frac{d^2}{dx^2}[N_{j,n}(x)]\frac{d^3\theta}{dx^3} \right]_{x=1} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-1)$$
$$+ \left[\frac{d^2}{dx^2}[N_{j,n}(x)]\frac{d^3\theta}{dx^3} \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-2e) - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-e)$$

(4.14c)

Solving the system (4.14a), we obtain the values of the parameters and then substituting these parameters into eqn. (4.13), we get the approximate solution of the BVP (4.12) for different values of n.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 1.**

Table 1: Maximum absolute errors of example 1.

x	Exact Results	11 Bernstein Polynomials		11 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	1.1102230E-016	0.9946538263	1.0880186E-014
0.2	0.9771222065	0.9771222065	1.5543122E-015	0.9771222065	3.0531133E-014
0.3	0.9449011653	0.9449011653	1.9984014E-015	0.9449011653	5.0848215E-014
0.4	0.8950948186	0.8950948186	1.9984014E-015	0.8950948186	2.1316282E-014
0.5	0.8243606354	0.8243606354	0.0000000E+000	0.8243606354	5.9507954E-014
0.6	0.7288475202	0.7288475202	2.3314684E-015	0.7288475202	1.0658141E-014
0.7	0.6041258122	0.6041258122	1.3322676E-015	0.6041258122	4.7961635E-014
0.8	0.4451081857	0.4451081857	4.9960036E-016	0.4451081857	9.8254738E-015
0.9	0.2459603111	0.2459603111	8.8817842E-016	0.2459603111	1.3655743E-014
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand Siddiqi *et al* [43], Siddiqi and Akram [45], Akram and Siddiqi [52] have found the accuracy nearly the order 10^{-9} and Khan and Sultana [50], Fazal-i-Haq *et al* [51] have found the accuracy nearly the order 10^{-11}.

Example 2: Consider the linear differential equation [45]

$$\frac{d^6u}{dx^6} + (5x+1)u = (185x - 25x^2 + 10x^4)\cos x + (270 - 36x^2)\sin x, \; -1 \le x \le 1 \tag{4.15a}$$

subject to the boundary conditions of type I in eqn. (4.1b):

$$u(-1) = 4\cos 1,\; u(1) = -2\cos 1,\; u'(-1) = \cos 1 + 4\sin 1, u'(1) = \cos 1 + 2\sin 1$$
$$u''(-1) = -16\cos 1 + 2\sin 1, u''(1) = 14\cos 1 - 2\sin 1. \tag{4.15b}$$

The analytic solution of the above BVP is $u(x) = (2x^3 - 5x + 1)\cos x$.

The equivalent BVP over [0, 1] to the BVP (4.15) is,

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 11$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-13}.

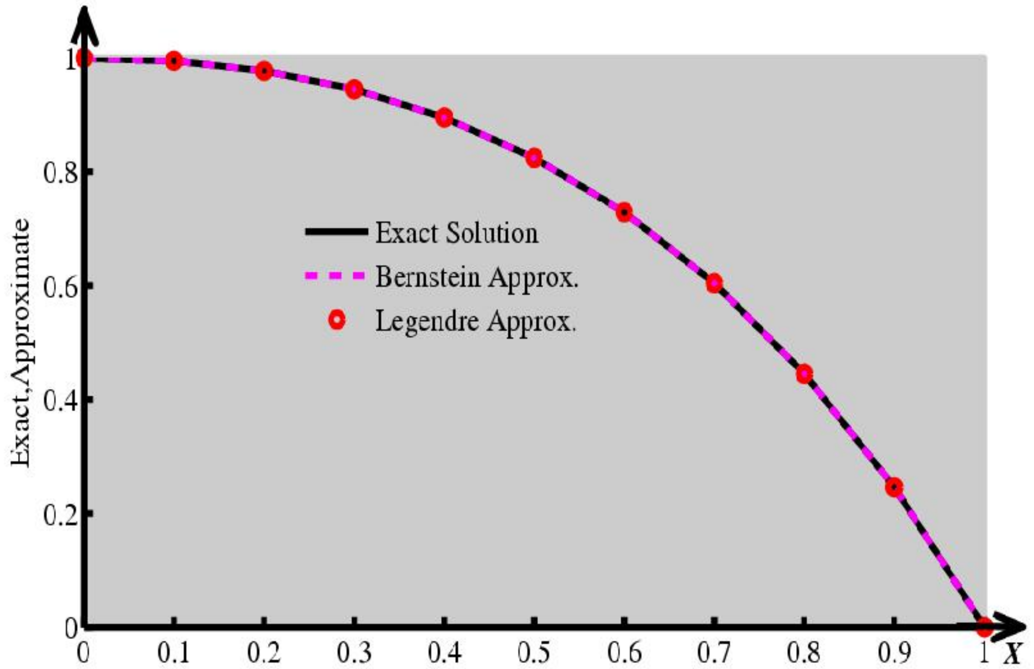


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 11 polynomials.

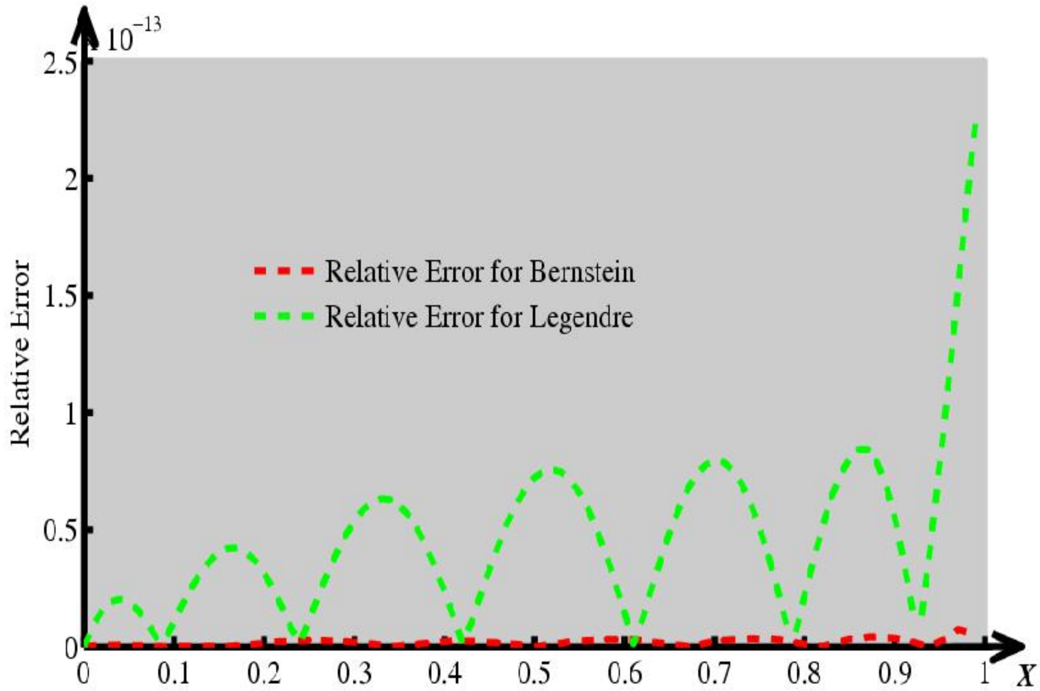


Fig. 1(b): Graphical representation of relative error of example 1 using 11 polynomials.

$$\frac{1}{2^6}\frac{d^6u}{dx^6}+(5(2x-1)+1)u=(185(2x-1)-25(2x-1)^2+10(2x-1)^4)\cos(2x-1)$$

$$+(270-36(2x-1)^2)\sin(2x-1),\ \ 0<x<1 \tag{4.16a}$$

$$u(0)=4\cos 1, u(1)=-2\cos 1,\ \frac{1}{2}u'(0)=\cos 1+4\sin 1, \frac{1}{2}u'(1)=\cos 1+2\sin 1,$$

$$\frac{1}{4}u''(0)=-16\cos 1+2\sin 1, \frac{1}{4}u''(1)=14\cos 1-2\sin 1. \tag{4.16b}$$

Using the method illustrated in (4.2.1), we approximate $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ \ n\ge 1 \tag{4.17}$$

Here $\theta_0(x)=(4-6x)\cos 1$ is specified by the essential boundary conditions of equation (4.16b). Now the parameters $\alpha_i\ (i=1,2,\dots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j\,, j=1,2,\dots,n \tag{4.18a}$$

where

$$D_{i,j}=\int_0^1\left[-\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]+64((10x-5)+1)N_{i,n}(x)N_{j,n}(x)\right]dx$$

$$-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{4.18b}$$

$$F_j=\int_0^1\Big\{\Big[(185(2x-1)-25(2x-1)^2+10(2x-1)^4)\cos(2x-1)$$

$$+(270-36(2x-1)^2)\sin(2x-1)\Big]64N_{j,n}(x)+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d\theta}{dx}\right]$$

$$-64((10x-5)+1)\theta_0 N_{j,n}(x)\Big\}dx+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4\theta}{dx^4}\right]_{x=1}-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^4\theta}{dx^4}\right]_{x=0}$$

$$-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3\theta}{dx^3}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1}$$

$$\times(56\cos 1-8\sin 1)-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0}\times(-64\cos 1+8\sin 1)-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=1}$$

$$\times(2\cos 1+4\sin 1)+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=0}\times(2\cos 1+8\sin 1) \qquad (4.18c)$$

Solving the system (4.18a) we obtain the values of the parameters and then substituting these parameters into eqn. (4.17), we get the approximate solution of the BVP (4.16) for different values of n. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (4.15).

The numerical results for this problem are given in **Table 2**.

Table 2: Maximum absolute errors of example 2.

x	Exact Results	14 Bernstein Polynomials		14 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
-1.0	2.1612092235	2.1612092235	0.0000000E+000	2.1612092235	0.0000000E+000
-0.8	2.7701058764	2.7701058764	4.4408921E-016	2.7701058764	1.1102230E-013
-0.6	2.9447974740	2.9447974740	8.8817842E-016	2.9447974740	8.1046281E-013
-0.4	2.6452871748	2.6452871748	2.6645353E-015	2.6452871748	4.7823967E-012
-0.2	1.9444520904	1.9444520904	1.7763568E-015	1.9444520904	8.9346308E-012
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	6.2458927E-012
0.2	0.0156810652	0.0156810652	2.1024849E-015	0.0156810653	2.8920269E-012
0.4	-0.8031651868	-0.8031651868	2.3314684E-015	-0.8031651868	9.1874286E-012
0.6	-1.2941262442	-1.2941262442	1.3322676E-015	-1.2941262442	8.3202334E-012
0.8	-1.3766924577	-1.3766924577	1.1102230E-015	-1.3766924577	5.0157656E-012
1.0	-1.0806046117	-1.0806046117	0.0000000E+000	-1.0806046117	0.0000000E+000

It is observed that the accuracy is found nearly the order 10^{-7} by Siddiqi and Akram in [45].

In Fig. 2(a), the exact and approximate solutions are given and a plot of relative errors are shown in Fig. 2(b) of example 2 for $n=14$. It is observed from Fig. 2(b) that the error is nearly the order 10^{-12}.

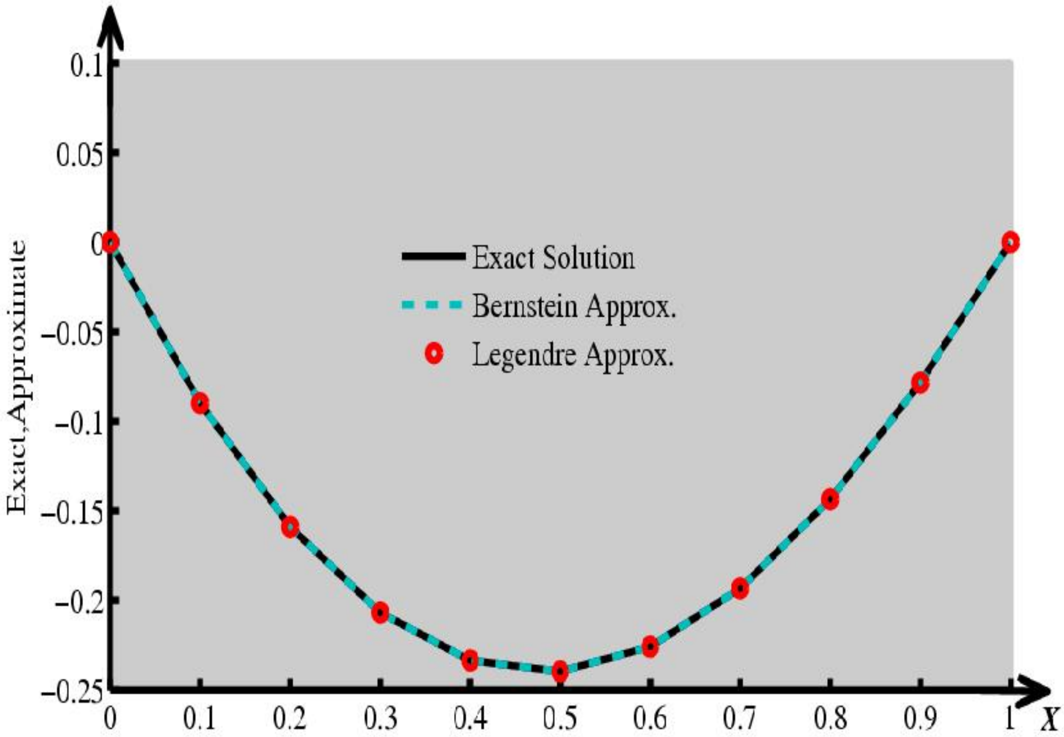


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 14 polynomials.

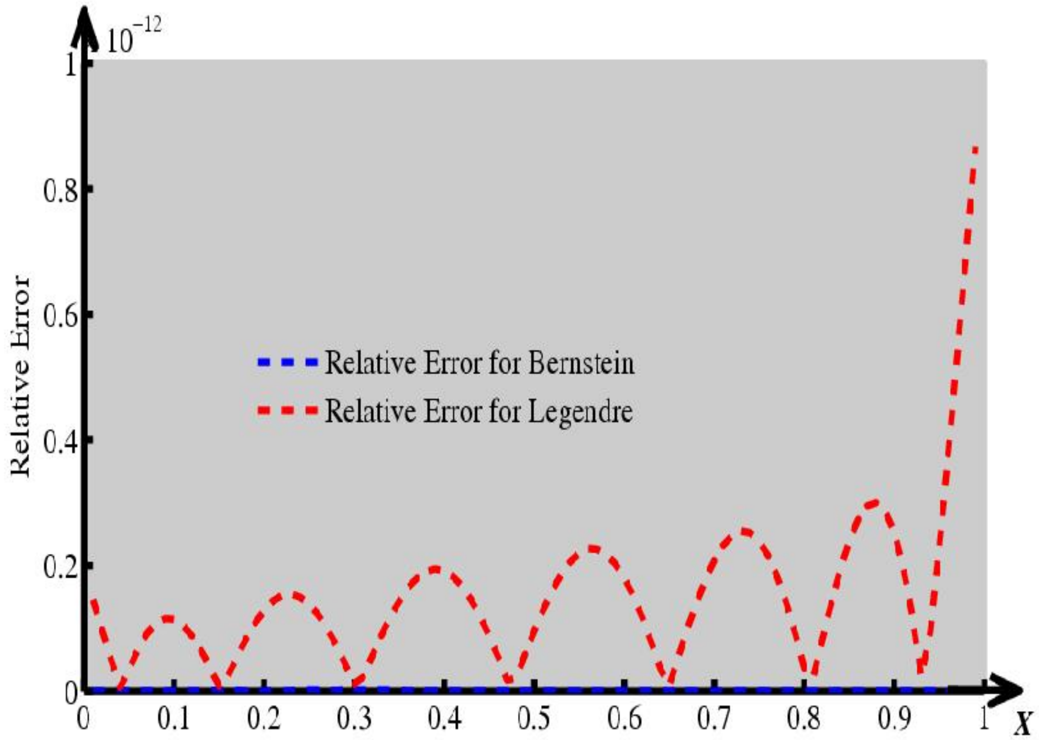


Fig. 2(b): Graphical representation of relative error of example 2 using 14 polynomials.

Example 3: Consider the linear differential equation [44]

$$\frac{d^6u}{dx^6} + xu = -(24 + 11x + x^3), \;\; 0 \le x \le 1 \tag{4.19a}$$

subject to the boundary conditions of type II in eqn. (4.2c):

$$u(0) = u(1) = 0,\; u''(0) = 0,\; u''(1) = -4e,\; u^{(iv)}(0) = -8,\; u^{(iv)}(1) = -16e\;. \tag{4.19b}$$

The analytic solution of the above problem is, $u(x) = x(1-x)e^x$.

Applying the method illustrated in (4.2.2), we approximate $u(x)$ in a form

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i\, N_{i,n}(x),\;\; n \ge 1 \tag{4.20}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (4.19b). Now the parameters $\alpha_i\; (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j\,, j = 1,2,\ldots,n \tag{4.21a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + xN_{i,n}(x)N_{j,n}(x) \right] dx$$

$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{4.21b}$$

$$F_j = \int_0^1 -(24 + 11x + x^3)e^x N_{j,n}(x)dx + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=1} \times (-16\,e)$$

$$-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=0} \times (-8) + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1} \times (-4e) \tag{4.21c}$$

Solving the system (4.21a) we obtain the values of the parameters and then substituting these parameters into eqn. (4.20), we get the approximate solution of the BVP (4.19) for different values of *n*.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are tabulated in **Table 3.**

Table 3: Maximum absolute errors for the example 3.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
9	2.880×10^{-11}	2.880×10^{-11}	5.682×10^{-11}, (Siraj-ul-Islam *et al* [44])
10	6.723×10^{-13}	6.725×10^{-13}	
11	9.881×10^{-15}	3.092×10^{-14}	
12	6.106×10^{-16}	8.316×10^{-14}	

We depict the exact and approximate solutions in Fig. 3(a) and a plot of relative errors in Fig. 3(b) of example 3 for $n=12$. From Fig. 3(b) we observe that the error is nearly the order 10^{-13}

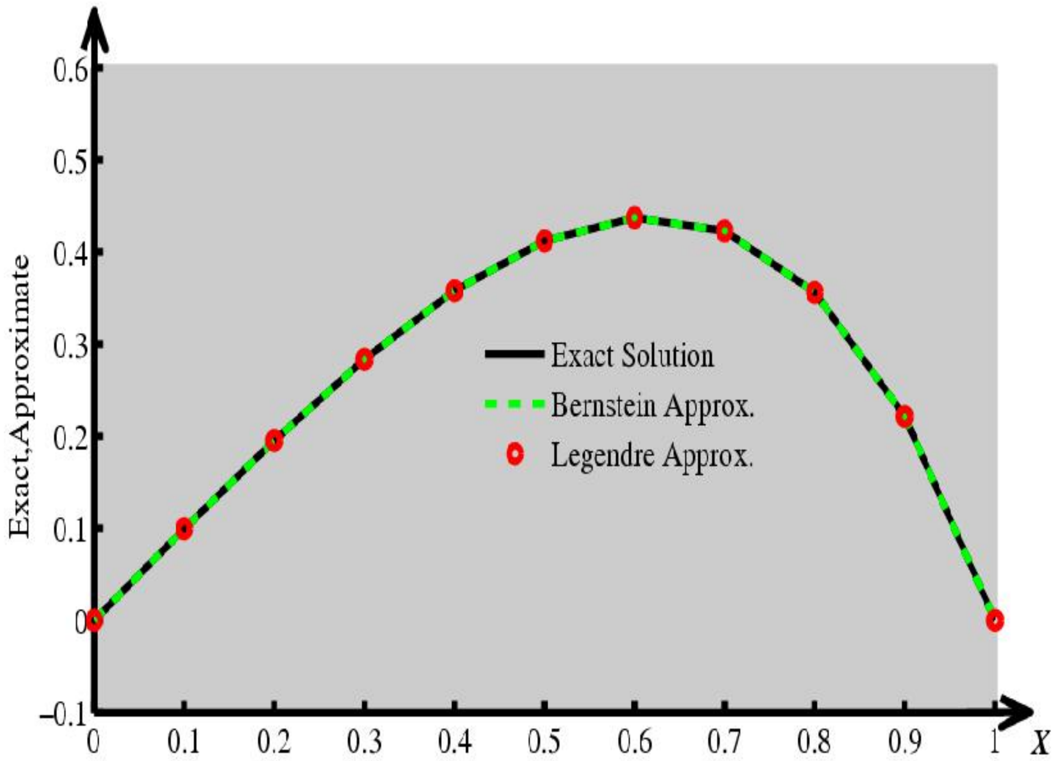


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 12 polynomials.

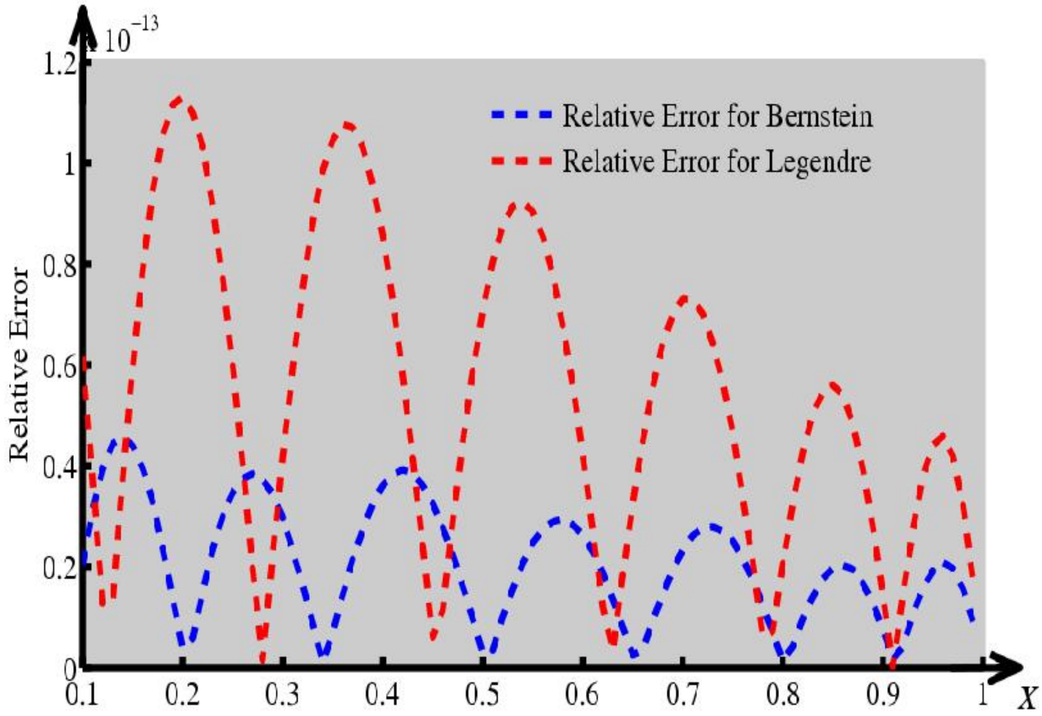


Fig. 3(b): Graphical representation of relative error of example 3 using 12 polynomials.

Example 4: Consider the linear BVP [44, 50, 53]

$$\frac{d^6u}{dx^6} + u = 6(2x\cos x + 5\sin x),\ -1 \le x \le 1 \tag{4.22a}$$

$u(-1) = u(1) = 0,\ u''(-1) = -4\cos(-1) + 2\sin(-1),\ u''(1) = 4\cos 1 + 2\sin 1$

$$u^{(iv)}(-1) = 8\cos(-1) - 12\sin(-1),\ u^{(iv)}(1) = -8\cos 1 - 12\sin 1\ . \tag{4.22b}$$

The analytic solution of the above system is, $u(x) = (x^2 - 1)\sin x$.

The equivalent BVP over [0, 1] to the BVP (4.22) is,

$$\frac{1}{2^6}\frac{d^6u}{dx^6} + u = 6\left[2(2x-1)\cos(2x-1) + 5\sin(2x-1)\right], 0 < x < 1 \tag{4.23a}$$

$u(0) = u(1) = 0,\ \frac{1}{4}u''(0) = -\frac{1}{4}u''(1) = -4\cos 1 - 2\sin 1, \frac{1}{16}, u^{(iv)}(0) = 8\cos 1 + 12\sin 1$

$$\frac{1}{16}u^{(iv)}(1) = -(8\cos 1 + 12\sin 1)\,. \tag{4.23b}$$

Employing the method illustrated in (4.2.2), we approximate $u(x)$ in a form

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \ge 1 \tag{4.24}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (4.23b). Now the parameters $\alpha_i \; (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, \, j = 1,2,\ldots,n \tag{4.25a}$$

where

$$\begin{aligned} D_{i,j} = & \int_0^1 \left[-\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + 64\, N_{i,n}(x) N_{j,n}(x) \right] dx \\ & + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=0} \\ & + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=0} \end{aligned} \tag{4.25b}$$

$$\begin{aligned} F_j = & \int_0^1 \left[2(2x-1)\cos(2x-1) + 5\sin(2x-1) \right] 384\, N_{j,n}(x)\, dx \\ & + \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-128\cos 1 - 192\sin 1) - \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=0} \times (128\cos 1 + 192\sin 1) \\ & + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=1} \times (16\cos 1 + 8\sin 1) - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-16\cos 1 - 8\sin 1) \end{aligned} \tag{4.25c}$$

Solving the system (4.25a) we obtain the values of the parameters and then substituting these parameters into eqn. (4.24), we get the approximate solution of the BVP (4.23) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (4.22).

In **Table 4**, we list the maximum absolute errors for different values of *n* by applying the present method mentioned in (4.2.2) to compare with existing results obtained so far.

Table 4: Maximum absolute errors of example 4

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
9	5.905×10^{-11}	7.596×10^{-09}	9.45×10^{-11} (Siraj-ul-Islam *et al* [44])
10	7.655×10^{-14}	9.343×10^{-11}	3.47×10^{-9} (Khan and Sultana [50])
11	7.661×10^{-14}	9.344×10^{-12}	5.80×10^{-5} (Loghmani and Ahmadinia [53])
12	2.776×10^{-16}	6.928×10^{-14}	

Example 5: Consider the **nonlinear** differential equation [49]

$$\frac{d^6u}{dx^6} = u^2e^x,\ 0 \le x \le 1 \tag{4.26a}$$

consisting of boundary conditions of type I defined in eqn. (4.2b)

$$u(0) = 1,\ u(1) = e^{-1},\ u'(0) = -1, u'(1) = -e^{-1}, u''(0) = 1, u''(1) = e^{-1}\ . \tag{4.26b}$$

The exact solution of this BVP is, $u(x) = e^{-x}$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ \ n \ge 1 \tag{4.27}$$

Here $\theta_0(x) = 1 - x(1-e^{-1})$ is specified by the essential boundary conditions in (4.26b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\dots,n$.

Using eqn. (4.27) into eqn. (4.26a), the Galerkin weighted residual eqns. are

$$\int_0^1 \left[\frac{d^6\tilde{u}}{dx^6} - \tilde{u}^2e^x\right] N_{k,n}(x)dx = 0, k = 1,2,\cdots,n \tag{4.28}$$

Integrating first term of (4.28) by parts, we obtain

$$\int_0^1 \frac{d^6\tilde{u}}{dx^6} N_{k,n}(x)\,dx = \left[N_{k,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{dN_{k,n}(x)}{dx}\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\tilde{u}}{dx^4}dx\ \ [\text{Since } N_{k,n}(1) = N_{k,n}(0) = 0\,]$$

In Figs. 4(a) and 4(b), the exact and approximate solutions, and the relative errors of example 4 for $n = 12$ are depicted respectively. We see from Fig. 4(b) that the error is nearly the order 10^{-13}

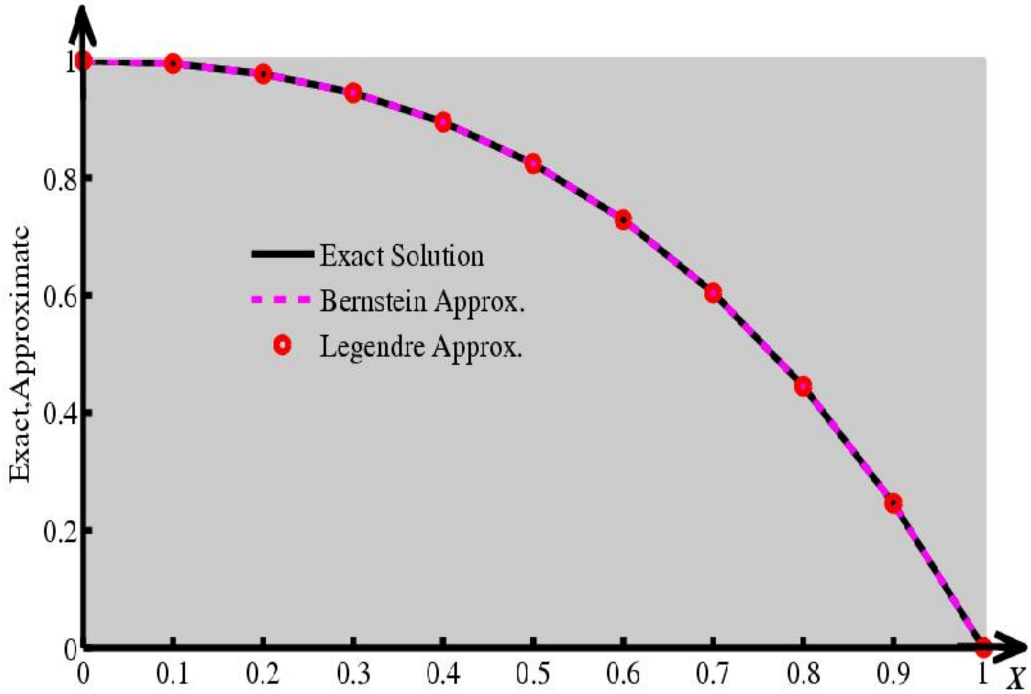


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 12 polynomials.

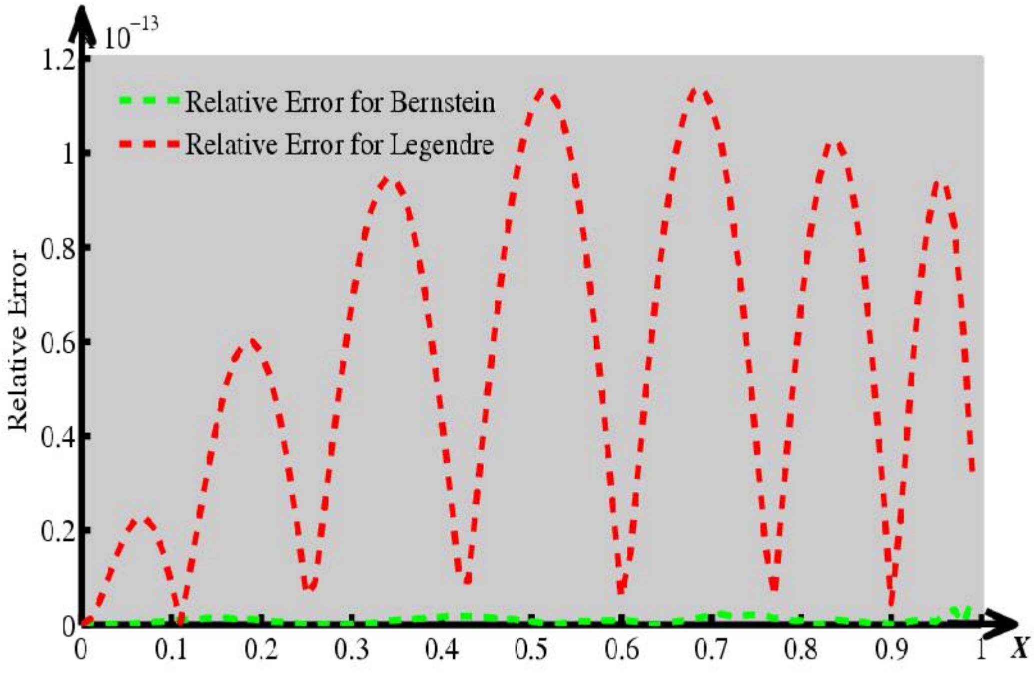


Fig. 4(b): Graphical representation of relative error of example 4 using 12 polynomials.

$$=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$-\int_0^1\frac{d^5N_{k,n}(x)}{dx^5}\frac{d\widetilde{u}}{dx}dx \tag{4.29}$$

Using eqn. (4.29) into equation (4.28) and using approximation for $\widetilde{u}(x)$ given in equation (4.27) and after applying the conditions given in eqn. (4.26b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^5N_{k,n}(x)}{dx^5}\frac{dN_{i,n}(x)}{dx}-2\theta_0e^x N_{i,n}(x)N_{k,n}(x)-\sum_{j=1}^{n}\alpha_j(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^x\right]dx\right.$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3N_{i,n}(x)}{dx^4}\right]_{x=1}$$

$$\left.-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}\right]\alpha_i=\int_0^1\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx}+\theta_0{}^2e^xN_{k,n}(x)\right]dx$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times e^{-1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}\times e^{-1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0} \tag{4.30}$$

The above equation (4.40) is equivalent to matrix form

$$(D+B)A=G \tag{4.31a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k} = \int_0^1 \left[-\frac{d^5 N_{k,n}(x)}{dx^5}\frac{dN_{i,n}(x)}{dx} - 2\theta_0 e^x N_{i,n}(x) N_{k,n}(x) \right] dx - \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=1}$$
$$+ \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=1}$$
$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=0} \tag{4.31b}$$

$$b_{i,k} = -\sum_{j=1}^{n} \alpha_j \int_0^1 (N_{i,n}(x) N_{j,n}(x) N_{k,n}(x)) e^x \, dx \tag{4.31c}$$

$$g_k = \int_0^1 \left[\frac{d^5 N_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx} + \theta_0{}^2 e^x N_{k,n}(x) \right] dx + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times e^{-1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0}$$
$$+ \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=1} \times e^{-1} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 \theta_0}{dx^4} \right]_{x=1}$$
$$- \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 \theta_0}{dx^4} \right]_{x=0} + \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=1} \times e^{-1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^3 \theta_0}{dx^3} \right]_{x=0} \tag{4.31d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (4.26a). That is, to find initial coefficients we solve the system

$$DA = G \tag{4.32a}$$

where the matrices are constructed from

$$d_{i,k} = \int_0^1 \left[-\frac{d^5 N_{k,n}(x)}{dx^5}\frac{dN_{i,n}(x)}{dx} \right] dx - \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=0}$$
$$+ \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=1} - \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=0} \tag{4.32b}$$

$$g_k = \int_0^1 \left[\frac{d^5 N_{k,n}(x)}{dx^5} \frac{d\theta_0}{dx} + \theta_0^2 e^x N_{k,n}(x) \right] dx + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times e^{-1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0}$$
$$+ \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=1} \times e^{-1} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^4\theta_0}{dx^4} \right]_{x=1}$$
$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^4\theta_0}{dx^4} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^3\theta_0}{dx^3} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^3\theta_0}{dx^3} \right]_{x=0} \qquad (4.32c)$$

Once the initial values of the α_i are obtained from eqn. (4.32a), they are substituted into eqn. (4.31a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (4.27), we obtain an approximate solution of the BVP (4.26).

The numerical results with 7 iterations for this problem are presented in **Table 5**.

Table 5: Maximum absolute errors of example 5 using 7 iterations.

x	Exact Results	11 Bernstein Polynomial		11 Legendre Polynomial	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.000000E-000	1.0000000000	0.000000E-000
0.1	0.9048374180	0.9048374180	6.991551E-012	0.9048374111	5.303428E-010
0.2	0.8187307531	0.8187307532	9.528275E-011	0.8187307539	3.085906E-010
0.3	0.7408182207	0.7408182207	7.833204E-012	0.7408182272	3.344482E-010
0.4	0.6703200460	0.6703200460	2.975769E-012	0.6703200545	9.157376E-010
0.5	0.6065306597	0.6065306596	3.409741E-011	0.6065306563	1.033783E-009
0.6	0.5488116361	0.5488116365	9.459007E-011	0.5488116390	5.706066E-010
0.7	0.4965853038	0.4965853006	1.301415E-010	0.4965853081	2.427534E-010
0.8	0.4493289641	0.4493289641	1.432544E-012	0.4493289565	9.238728E-010
0.9	0.4065696597	0.4065696587	9.5354360E011	0.4065696597	9.542350E-011
1.0	0.3678794412	0.3678794412	0.000000E-000	0.3678794412	0.000000E-000

On the other hand the maximum absolute error obtained by Wazwaz [49] is 1.389×10^{-6}

Fig.5 (a) depicts the exact and approximate solutions and the relative errors are shown in Fig. 5(b) of example 8 for $n = 11$. We see from Fig. 5(b) that the error is nearly the order 10^{-11}.

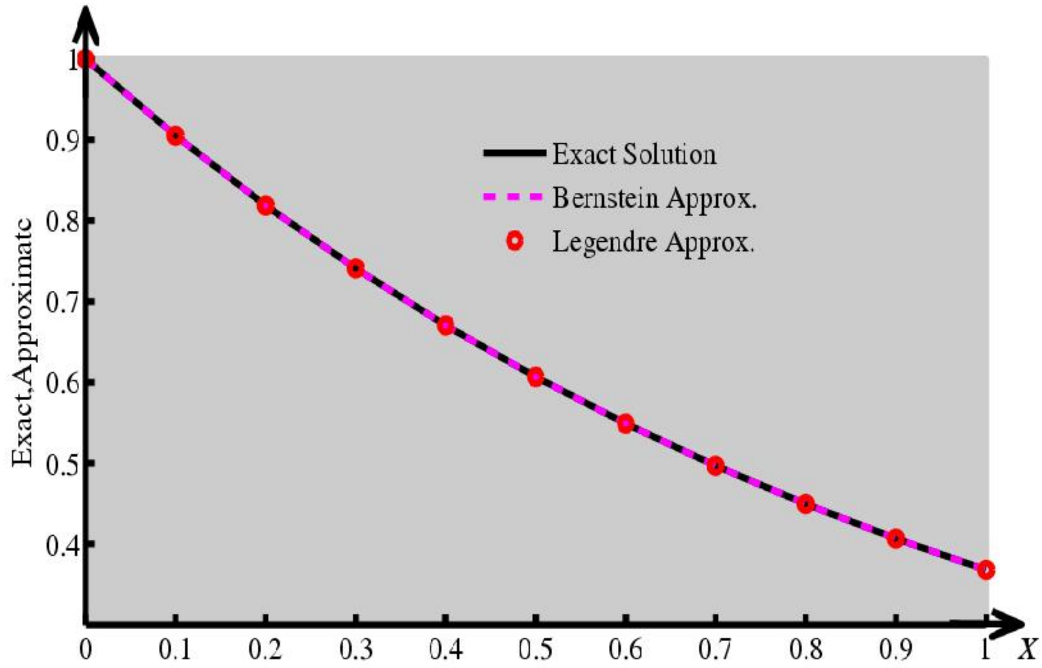


Fig. 5(a): Graphical representation of exact and approximate solutions of example 5 using 11 polynomials.

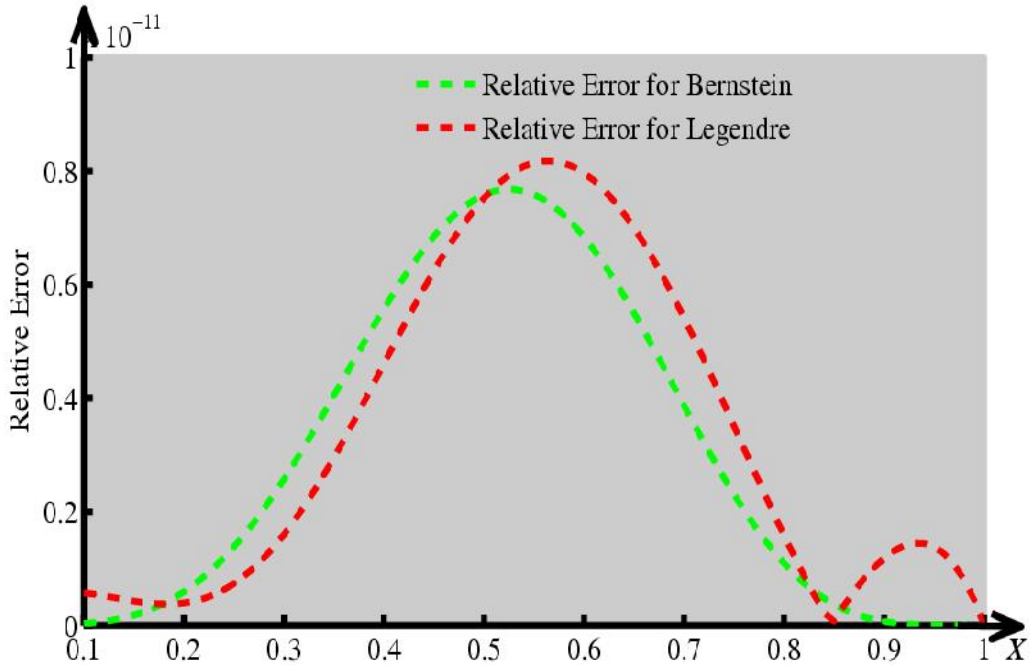


Fig. 5(b): Graphical representation of relative error of example 5 using 11 polynomials.

Example 6: Consider the **nonlinear** differential equation [44]

$$\frac{d^6u}{dx^6} = 20\,e^{-36u} - 40(1+x)^{-6},\ 0 \le x \le 1 \tag{4.33a}$$

with boundary conditions type II, defined in eqn. (4.2c)

$$u(0)=0,\,u(1)=\frac{1}{6}\ln 2,\,u''(0)=-\frac{1}{6},u''(1)=-\frac{1}{24},u^{(iv)}(0)=-1,u^{(iv)}(1)=-\frac{1}{16}\ . \quad (4.33b)$$

The exact solution of this BVP is, $u(x)=\frac{1}{6}\ln(1+x)$.

Consider the approximate solution of $u(x)$ as

$$\widetilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\geq 1 \quad (4.34)$$

Here $\theta_0(x)=\frac{1}{6}x\ln 2$ is specified by the essential boundary conditions in eqn. (4.33b). Also $N_{i,n}(0)=N_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Applying eqn. (4.34) into eqn. (4.33a), the Galerkin weighted residual eqns. are

$$\int_0^1\left[\frac{d^6\widetilde{u}}{dx^6}-20e^{-36\widetilde{u}}+40(1+x)^{-6}\right]N_{k,n}(x)dx=0,k=1,2,\ldots,n \quad (4.35)$$

Similarly of example 5, integrating first term of (4.35) by parts we obtain

$$\int_0^1\frac{d^6\widetilde{u}}{dx^6}N_{k,n}(x)dx=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$
$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^5N_{k,n}(x)}{dx^5}\frac{d\widetilde{u}}{dx}dx \quad (4.36)$$

Putting eqn. (4.36) into eqn. (4.35) and using approximation for $\widetilde{u}(x)$ given in eqn. (4.34) and after applying the boundary conditions given in eqn. (4.33b) and rearranging the terms for the resulting eqns. we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^5N_{k,n}(x)}{dx^5}\frac{dN_{i,n}(x)}{dx}\right]dx+\left[\frac{d^3N_{i,n}(x)}{dx^3}\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=1}-\left[\frac{d^3N_{i,n}(x)}{dx^3}\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0}\right.$$
$$\left.+\left[\frac{dN_{i,n}(x)}{dx}\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}-\left[\frac{dN_{i,n}(x)}{dx}\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0}\right]\alpha_i=\int_0^1\frac{d^5N_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx}dx$$
$$+20\int_0^1\left[e^{-36\left[\theta_0+\sum_{j=1}^{n}\alpha_j N_{j,n}(x)\right]}\right]N_{k,n}(x)dx-40\int_0^1(1+x)^{-6}N_{k,n}(x)dx-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=0} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-\frac{1}{16}) - \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-1) + \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(-\frac{1}{24})$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times(-\frac{1}{6}) \tag{4.37}$$

The above equation (4.37) is equivalent to matrix form

$$DA = B + G \tag{4.38a}$$

where the elements of the square matrix D and the column matrices B and G are given by

$$d_{i,k} = \int_0^1 \left[-\frac{d^5N_{k,n}(x)}{dx^5}\frac{dN_{i,n}(x)}{dx}\right]dx + \left[\frac{d^3N_{i,n}(x)}{dx^3}\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=1} - \left[\frac{d^3N_{i,n}(x)}{dx^3}\frac{d^2N_{k,n}(x)}{dx^2}\right]_{x=0}$$

$$+\left[\frac{dN_{i,n}(x)}{dx}\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1} - \left[\frac{dN_{i,n}(x)}{dx}\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0} \tag{4.38b}$$

$$b_k = 20\int_0^1 \left[e^{-36\left[\theta_0 + \sum_{j=1}^{n} \alpha_j N_{j,n}(x)\right]} \right] N_{k,n}(x)dx \tag{4.38c}$$

$$g_k = \int_0^1 \left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx} - 40(1+x)^{-6}N_{k,n}(x)\right]dx - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=0} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-\frac{1}{16}) - \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-1) + \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(-\frac{1}{24})$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times(-\frac{1}{6}) \tag{4.38d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in eqn. (4.33a). That is, to find initial coefficients we solve the system

$$DA = G \tag{4.39a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left[-\frac{d^5 N_{k,n}(x)}{dx^5} \frac{dN_{i,n}(x)}{dx} \right] dx + \left[\frac{d^3 N_{i,n}(x)}{dx^3} \frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=1} - \left[\frac{d^3 N_{i,n}(x)}{dx^3} \frac{d^2 N_{k,n}(x)}{dx^2} \right]_{x=0}$$
$$+ \left[\frac{dN_{i,n}(x)}{dx} \frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=1} - \left[\frac{dN_{i,n}(x)}{dx} \frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=0} \tag{4.39b}$$

$$g_k = \int_0^1 \left[\frac{d^5 N_{k,n}(x)}{dx^5} \frac{d\theta_0}{dx} - 40(1+x)^{-6} N_{k,n}(x) \right] dx - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^3 \theta_0}{dx^3} \right]_{x=1}$$
$$+ \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^3 \theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} \right]_{x=1} + \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} \right]_{x=0}$$
$$+ \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} \times (-\frac{1}{16}) - \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0} (-1) + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times (-\frac{1}{24})$$
$$- \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0} \times (-\frac{1}{6}) \tag{4.39c}$$

Once the initial values of the coefficients α_i are obtained from eqn. (4.39a), they are substituted into eqn. (4.38a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (4.34), we obtain an approximate solution of the BVP (4.33).

The maximum absolute errors, using different number of polynomials by the present method with 5 iterations and the previous results obtained so far, are summarized in **Table 6**.

In Figs. 6(a) and 6(b) we have given the exact and approximate solutions, and the relative errors of example 6 for $n = 10$. From Fig. 6(b) we observed that the error is nearly the order 10^{-6}.

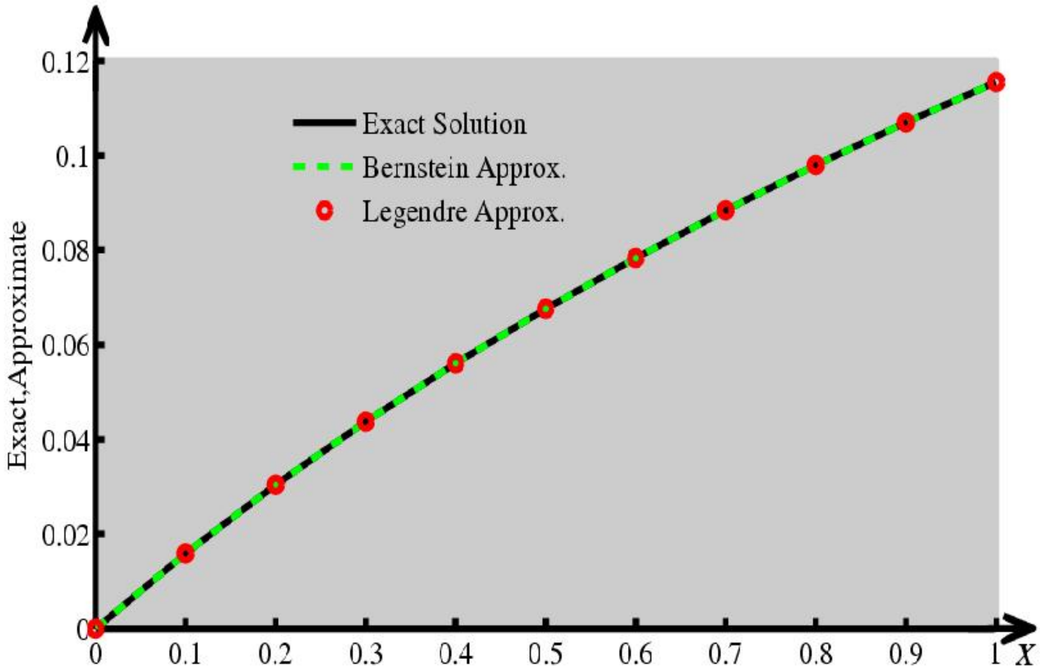


Fig. 6(a): Graphical representation of exact and approximate solutions of example 6 using 10 polynomials.

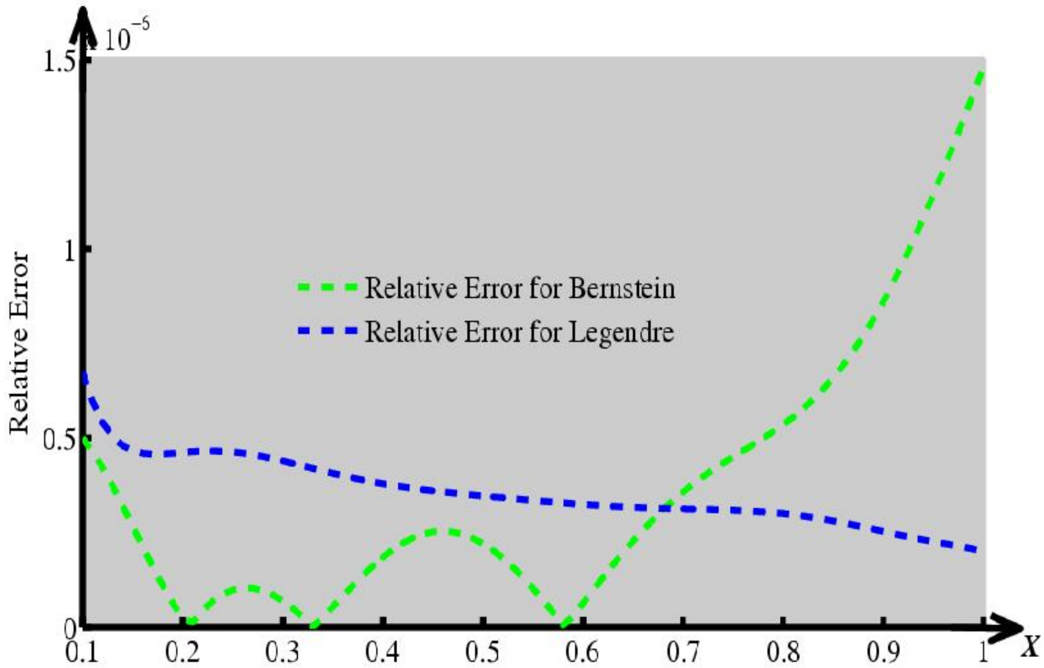


Fig. 6(b): Graphical representation of relative error of example 6 using 10 polynomials.

Table 6: Maximum absolute errors for the example 6 with 5 iterations.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
7	8.915×10^{-7}	8.915×10^{-7}	2.68×10^{-11} (Siraj-ul-Islam *et al* [44])
8	8.908×10^{-8}	8.905×10^{-8}	
9	9.870×10^{-9}	9.864×10^{-9}	
10	4.790×10^{-11}	4.794×10^{-10}	

4.4 Conclusions

In this chapter, we have used Bernstein and Legendre, the piecewise continuous and differentiable polynomials as basis functions for the numerical solution of sixth order linear and nonlinear BVPs in the Galerkin method for two different types of boundary conditions. We see from the tables that the numerical results obtained by our method are superior to other existing methods. Also, we get better results for Bernstein polynomials than the Legendre polynomials. It may also note that the numerical solutions identical with the exact solution even lower order Bernstein and Legendre polynomials are used in the approximation which are shown in Figs. [1-6].

CHAPTER 5

Seventh Order Boundary Value Problems

5.1 Introduction

From the literature on the numerical solutions of BVPs, it is observed that the higher order differential equations arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Particularly, the seventh order BVPs arise in modeling induction motors with two rotor circuits. The performance of the induction motor behavior is modeled by a fifth order differential equation. This model is constructed with two stator state variables, two rotor state variables and one shaft speed. Generally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. For neglecting the computational burden of additional state variables when additional rotor circuits are needed, the model is often bounded to the fifth order. The rotor impedance is done under the assumption that the frequency of rotor currents depends on rotor speed. This process is appropriate for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when the rotor frequency is not a single value. So this behavior is modeled in the seventh order differential equation [54]. Presently, the literature on the numerical solutions of seventh order BVPs and associated eigen value problems is not too much available. The existence and uniqueness theorem of solutions of BVPs was presented in a book written by Agarwal [8] which does not contain any numerical examples. Siddiqi and Twizell [29 – 31] used eighth, tenth and twelfth degree splines for solving eighth, tenth and twelfth order BVPs respectively. Siddiqi and Ghazala [55 – 61] solved fifth, sixth, eighth, tenth and twelfth order BVPs using polynomial and nonpolynomial spline techniques. Siddiqi *et al* [62, 63] presented the solutions of seventh order BVPs by the differential transformation method and variational iteration technique, respectively. Very recently the solution of seventh order BVPs was developed by Siddiqi and Muzammal [64] using variation of parameters method.

However, the aim of this chapter is to apply Galerkin weighted residual method for solving seventh order linear and nonlinear BVPs. In this method, we exploit Bernstein and Legendre polynomials as basis functions which are modified into to a new set of basis functions to satisfy the corresponding homogeneous boundary conditions where the essential types of boundary conditions are mentioned. The method is formulated as a rigorous matrix form. Moreover, the formulation for solving linear seventh order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials is presented in section 5.2. Then numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation and the solutions are compared with the existing methods in section 5.3 and conclusions of this chapter are illustrated in section 5.4.

5.2 Description of the Galerkin Method

In this section, we first derive the matrix formulation for seventh order linear BVP and then we extend our idea for solving nonlinear BVP. In the present chapter, we have used Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions for the numerical solution of a general seventh order linear boundary value problem of the following form:

$$a_7 \frac{d^7u}{dx^7} + a_6 \frac{d^6u}{dx^6} + a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b \tag{5.1a}$$

subject to the following boundary conditions

$$u(a) = A_0, \quad u(b) = B_0, \quad u'(a) = A_1, \quad u'(b) = B_1,$$

$$u''(a) = A_2, \quad u''(b) = B_2, \quad u'''(a) = A_3 \tag{5.1b}$$

where $A_i, i = 0,1,2,3$ and $B_j, j = 0,1,2$ are finite real constants and $a_i, i = 0,1,\cdots 7$ and r are all continuous functions defined on the interval $[a, b]$. The boundary value problem (5.1a) is solved with the boundary conditions of eqn. (5.1b).

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (5.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$ and thus we have:

$$c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=s,\ 0<x<1 \tag{5.2a}$$

$$u(0)=A_0,\qquad \frac{1}{b-a}u'(0)=A_1,\qquad \frac{1}{(b-a)^2}u''(0)=A_2,$$

$$u(1)=B_0,\qquad \frac{1}{b-a}u'(1)=B_1,\qquad \frac{1}{(b-a)^2}u''(1)=B_2,$$

$$\frac{1}{(b-a)^3}u'''(0)=A_3 \tag{5.2b}$$

where

$$c_7=\frac{1}{(b-a)^7}a_7((b-a)x+a),\qquad c_6=\frac{1}{(b-a)^6}a_6((b-a)x+a),$$

$$c_5=\frac{1}{(b-a)^5}a_5((b-a)x+a),\qquad c_4=\frac{1}{(b-a)^4}a_4((b-a)x+a),$$

$$c_3=\frac{1}{(b-a)^3}a_3((b-a)x+a),\qquad c_2=\frac{1}{(b-a)^2}a_2((b-a)x+a),$$

$$c_1=\frac{1}{b-a}a_1((b-a)x+a),\qquad c_0=a_0((b-a)x+a),$$

$$s=r((b-a)x+a)$$

To solve the boundary value problem (5.2) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_iN_{i,n}(x),\ n\geq 1 \tag{5.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0)=N_{i,n}(1)=0$, for each $i=1,2,3,\ldots,n.$

Using eqn. (5.3) into eqn. (5.2a), the weighted residual equations are

$$\int_0^1\left[c_7\frac{d^7\tilde{u}}{dx^7}+c_6\frac{d^6\tilde{u}}{dx^6}+c_5\frac{d^5\tilde{u}}{dx^5}+c_4\frac{d^4\tilde{u}}{dx^4}+c_3\frac{d^3\tilde{u}}{dx^3}+c_2\frac{d^2\tilde{u}}{dx^2}+c_1\frac{d\tilde{u}}{dx}+c_0\tilde{u}-s\right]N_{j,n}(x)dx=0 \tag{5.4}$$

Integrating by parts the terms up to second derivative on the left hand side of (5.4), we get

$$\int_0^1 c_7 \frac{d^7\tilde{u}}{dx^7} N_{j,n}(x)dx = \left[c_7 N_{j,n}(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx \text{ [Since } N_{j,n}(0) = N_{j,n}(1) = 0 \text{]}$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.5)$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x) \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{5.6}$$

$$\int_0^1 c_5\frac{d^5\tilde{u}}{dx^5}N_{j,n}(x)dx=\left[c_5N_{j,n}(x)\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{5.7}$$

$$\int_0^1 c_4\frac{d^4\tilde{u}}{dx^4}N_{j,n}(x)dx=\left[c_4N_{j,n}(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.8)$$

$$\int_0^1 c_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.9)$$

$$\int_0^1 c_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x)\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.10)$$

Substituting eqns. (5.5) – (5.10) into eqn. (5.4) and using approximation for $\tilde{u}(x)$ given in eqn. (5.3) and after applying the boundary conditions given in eqn. (5.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (5.11a)$$

where

$$D_{i,j} = \int_0^1 \left\{\left[\frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\right.\right.$$

$$\left.\left. + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] + c_1 N_{j,n}(x)\right]\frac{d}{dx}[N_{i,n}(x)] + c_0 N_{i,n}(x)N_{j,n}(x)\right\}dx$$

$$-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}[N_{i,n}(x)]\right]_{x=1} + \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}[N_{i,n}(x)]\right]_{x=0}$$

$$-\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4}{dx^4}[N_{i,n}(x)]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4}{dx^4}[N_{i,n}(x)]\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4}{dx^4}[N_{i,n}(x)]\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)]\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)]\right]_{x=1}+\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4}{dx^4}[N_{i,n}(x)]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)]\right]_{x=1} \qquad (5.11b)$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[-\frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\right.\right.$$

$$\left.\left.+\frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2 N_{j,n}(x)\right]-c_1 N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0 N_{j,n}(x)\right\}dx$$

$$+\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3$$

$$+\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1$$

$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1+\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) B_1$$

$$+\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1$$

$$-\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1 \qquad (5.11c)$$

Solving the system (5.11a), we find the values of the parameters α_i and then substituting these parameters into eqn. (5.3), we get the approximate solution of the BVP (5.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (5.1).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (5.11). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

5.3. Numerical examples and results

To test the applicability of the proposed method, we consider three linear and one nonlinear problems. For all examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB*** **10**. The convergence of linear BVP is calculated by

$$E=\left|\widetilde{u}_{n+1}(x)-\widetilde{u}_n(x)\right|<\delta$$

where $\widetilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-13}.

In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\widetilde{u}_n^{N+1} - \widetilde{u}_n^{N}\right| < \delta$$

where $\delta < 10^{-11}$ and N is the Newton's iteration number.

Example 1: Consider the linear boundary value problem [62]

$$\frac{d^7u}{dx^7} = xu + e^x(x^2 - 2x - 6), 0 \le x \le 1 \qquad (5.12a)$$

$$u(0) = 1, u(1) = 0,\ u'(0) = 0, u'(1) = -e, u''(0) = -1, u''(1) = -2e, u'''(0) = -2. \qquad (5.12b)$$

The analytic solution of the above system is, $u(x) = (1-x)e^x$.

Using the method illustrated in the portion (5.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \qquad (5.13)$$

Here $\theta_0(x) = 1 - x$ as specified by the essential boundary conditions of eqn. (5.12b). Now the parameters $\alpha_i (i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \qquad (5.14a)$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - xN_{i,n}(x)N_{j,n}(x) \right] dx$$

$$- \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0}$$

$$+ \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=0}$$

$$- \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=1} \qquad (5.14b)$$

$$F_j = \int_0^1 \left[(x^2 - 2x - 6)e^x N_{j,n}(x) - \frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d\theta}{dx} + x\theta_0 N_{j,n}(x) \right] dx$$

$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1} \times(-2) - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=1} \times(-2e)$$

$$+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=0} \times(-1) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1} \times(-e) \quad (5.14c)$$

Solving the system (5.14a), we obtain the values of the parameters and then substituting these parameters into eqn. (5.13), we get the approximate solution of the BVP (5.12) for different values of n.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 1.**

Table 1: Maximum absolute errors of example 1.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
8	4.312×10^{-11}	3.846×10^{-9}	8.922×10^{-11}(Siddiqi *et al* [62])
9	5.359×10^{-13}	4.312×10^{-11}	
10	9.215×10^{-15}	5.416×10^{-13}	
11	6.661×10^{-16}	9.548×10^{-15}	

Example 2: Consider the linear differential equation [64]

$$\frac{d^7u}{dx^7} = -u - e^x(2x^2 + 12x + 35),\ 0 \le x \le 1 \quad (5.15a)$$

subject to the boundary conditions

$$u(0) = u(1) = 0,\ u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u'''(0) = -3. \quad (5.15b)$$

The analytic solution of the above system is, $u(x) = x(1-x)e^x$.

Applying the method mentioned in section (5.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \quad (5.16)$$

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 11$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-14}.

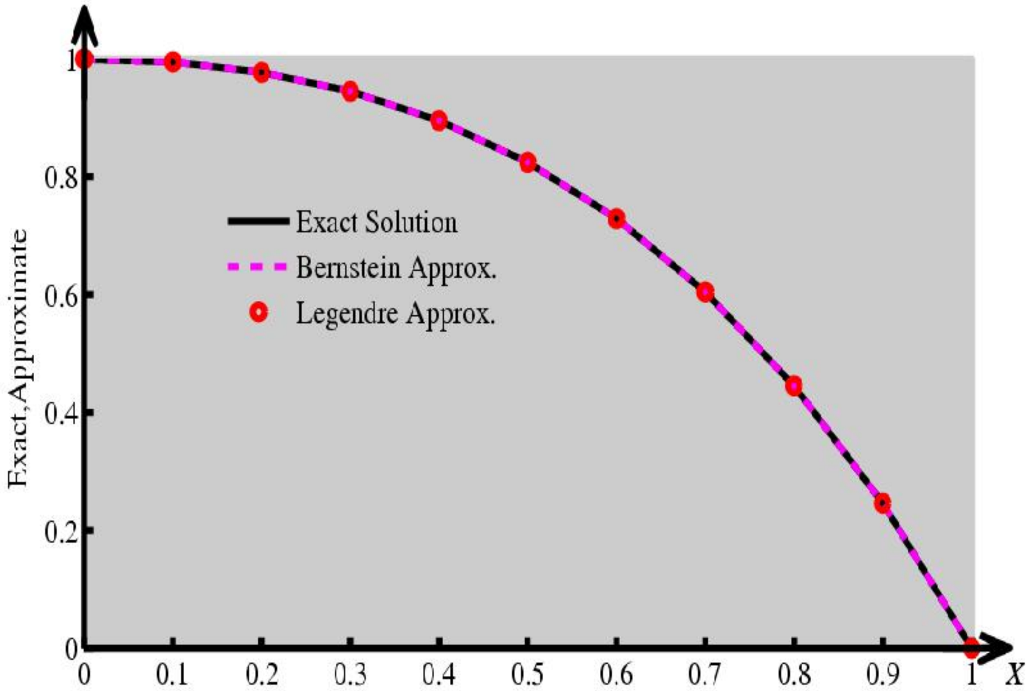


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 11 polynomials.

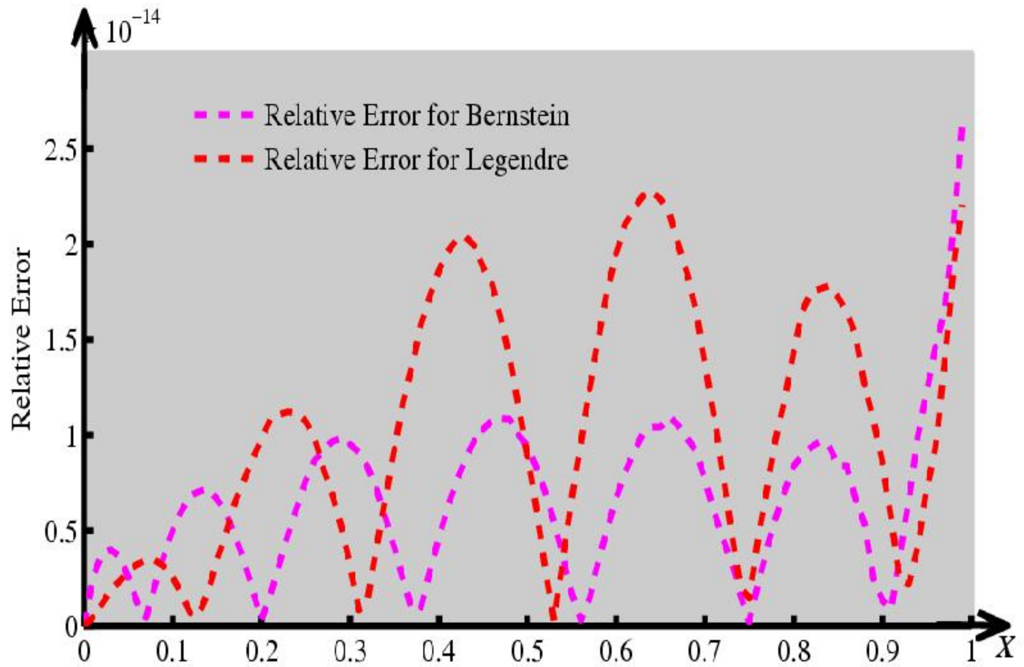


Fig. 1(b): Graphical representation of relative error of example 1 using 11 polynomials.

Here $\theta_0(x)=0$ as specified by the essential boundary conditions of eqn. (5.15b). Now the parameters $\alpha_i (i=1,2,.......,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j=1,2,\ldots,n \tag{5.17a}$$

where

$$D_{i,j}=\int_0^1\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]+N_{i,n}(x)N_{j,n}(x)\right]dx$$
$$-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} \tag{5.17b}$$

$$F_j=\int_0^1\left[-e^x(35+12x+2x^2)N_{j,n}(x)\right]dx-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0}\times(-3)$$
$$-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-4e)+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-e)$$
$$-\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0} \tag{5.17c}$$

Solving the system (5.17a), we obtain the values of the parameters and then substituting these parameters into eqn. (5.16), we get the approximate solution of the BVP (5.15).

The maximum absolute errors for this problem are shown in **Table 2.**

Example 3: Consider the linear differential equation [63]

$$\frac{d^7u}{dx^7}=u-7e^x,\ 0\le x\le 1 \qquad \text{5.18a)}$$

subject to the boundary conditions

$$u(0)=1,\ u(1)=0,\ u'(0)=0, u'(1)=-e, u''(0)=-1, u''(1)=-2e, u'''(0)=-2. \tag{5.18b}$$

Table 2: Maximum absolute errors for the example 2.

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	6.2597190E-026
0.1	0.0994653826	0.0994653826	1.1934898E-015	0.0994653826	5.8911209E-014
0.2	0.1954244413	0.1954244413	1.0824674E-015	0.1954244413	1.0269563E-014
0.3	0.2834703496	0.2834703496	3.8857806E-016	0.2834703496	1.0608181E-013
0.4	0.3580379274	0.3580379274	1.7763568E-015	0.3580379274	6.8500761E-014
0.5	0.4121803177	0.4121803177	9.9920072E-016	0.4121803177	9.6977981E-014
0.6	0.4373085121	0.4373085121	8.8817842E-016	0.4373085121	7.1442852E-014
0.7	0.4228880686	0.4228880686	1.1657342E-015	0.4228880686	6.1228800E-014
0.8	0.3560865486	0.3560865486	1.1102230E-016	0.3560865486	5.0015547E-014
0.9	0.2213642800	0.2213642800	2.2204460E-016	0.2213642800	5.9674488E-015
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand the maximum absolute error has been found by Siddiqi and Muzammal Iftikhar [64] is 7.482×10^{-10}

In Fig. 2(a), the exact and approximate solutions are given and a plot of relative errors are shown in Fig. 2(b) of example 2 for $n=12$. It is observed from Fig. 2(b) that the error is nearly the order 10^{-13}.

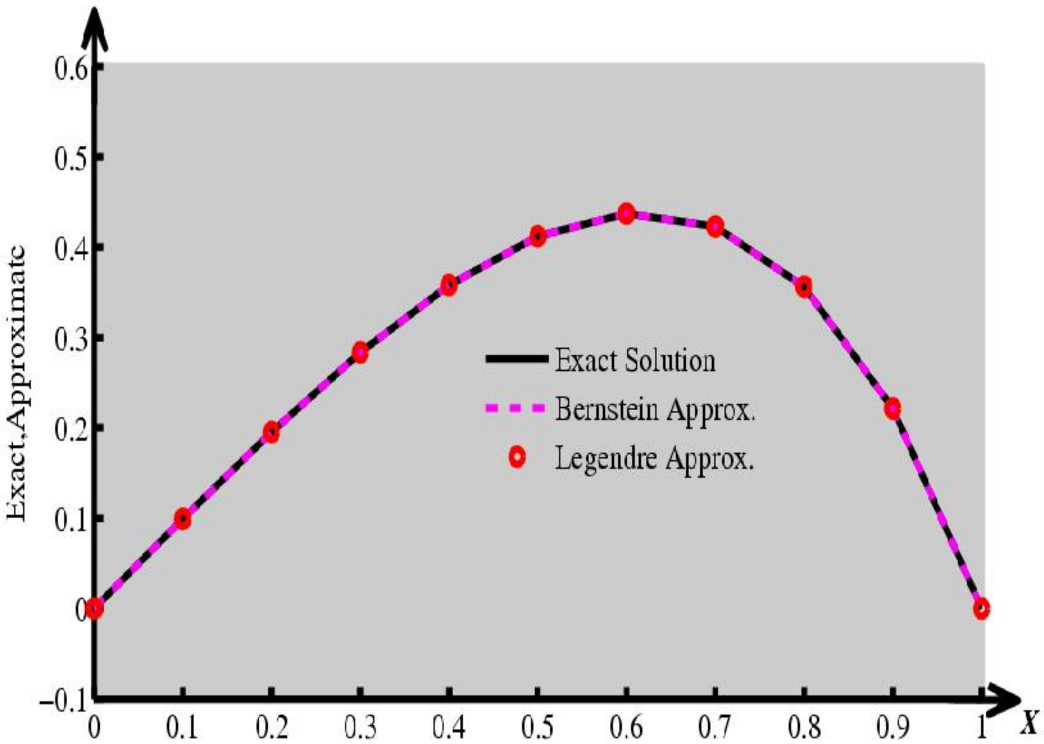


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 12 polynomials.

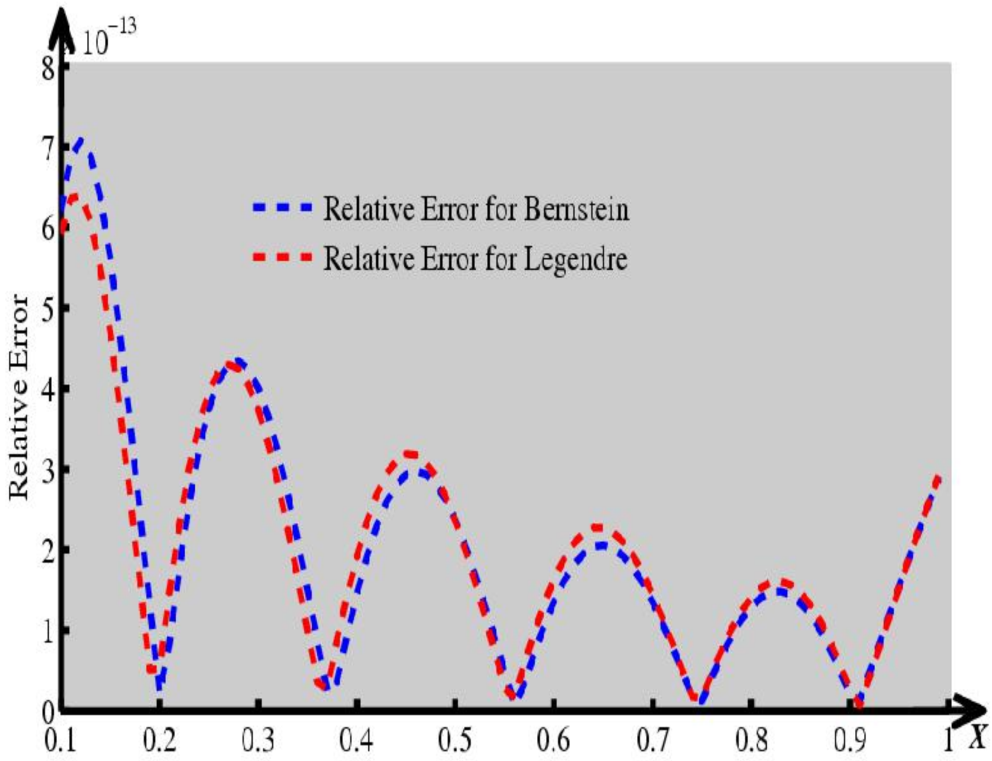


Fig. 2(b): Graphical representation of relative error of example 2 using 12 polynomials.

The analytic solution of the above system is, $u(x) = (1-x)e^x$.

Using the method described in section (5.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \geq 1 \tag{5.19}$$

Here $\theta_0(x) = 1 - x$ as specified by the essential boundary conditions of eqn. (5.18b). Now the parameters $\alpha_i (i = 1,2,......., n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, \; j = 1,2,\ldots,n \tag{5.20a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + N_{i,n}(x)N_{j,n}(x) \right] dx$$

$$- \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} \tag{5.20b}$$

$$F_j=\int_0^1\left\{-7e^x N_{j,n}(x)-\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d\theta}{dx}\right]+\theta_0 N_{j,n}(x)\right\}dx$$

$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0}\times(-2)-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-2e)+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\right]_{x=0}\times(-1)$$

$$+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-e) \tag{5.20c}$$

Solving the system (5.20a), we obtain the values of the parameters and then substituting these parameters into eqn. (5.19), we obtain the approximate solution of the BVP (5.18).

The maximum absolute errors for this problem with Bernstein and Legendre polynomials are shown in **Table 3**.

Table 3: Maximum absolute errors for the example 3.

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	1.1102230E-016	0.9946538263	2.5535130E-016
0.2	0.9771222065	0.9771222065	1.1102230E-016	0.9771222065	9.5479180E-017
0.3	0.9449011653	0.9449011653	3.3306691E-016	0.9449011653	2.9976022E-015
0.4	0.8950948186	0.8950948186	6.6613381E-016	0.8950948186	1.6653345E-014
0.5	0.8243606354	0.8243606354	1.1102230E-016	0.8243606354	7.3274720E-015
0.6	0.7288475202	0.7288475202	1.1102230E-016	0.7288475202	1.4210855E-014
0.7	0.6041258122	0.6041258122	0.0000000E+000	0.6041258122	8.3266727E-015
0.8	0.4451081857	0.4451081857	5.5511151E-017	0.4451081857	6.3837824E-015
0.9	0.2459603111	0.2459603111	1.1102230E-016	0.2459603111	2.1094237E-015
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the contrary the maximum absolute error has been obtained by Siddiqi *et al* [63] is 1.5×10^{-14}

Figs. 3(a) and 3(b) deal with the exact and approximate solutions, and the relative errors of example 3 for $n=12$. It is found from Fig. 3(b) that the error is of the order 10^{-14}

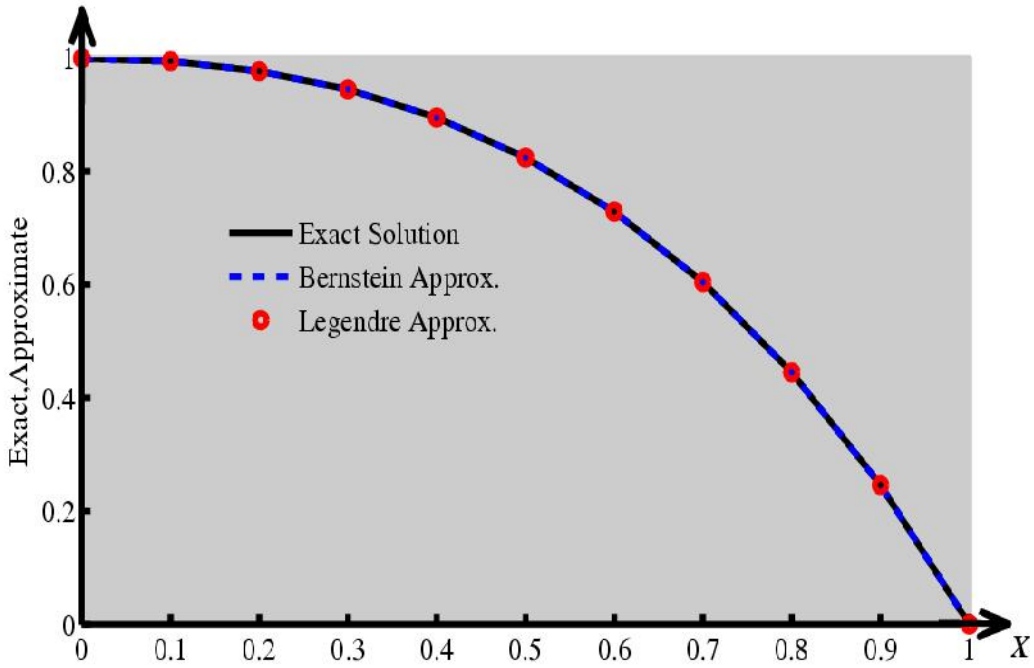


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 12 polynomials.

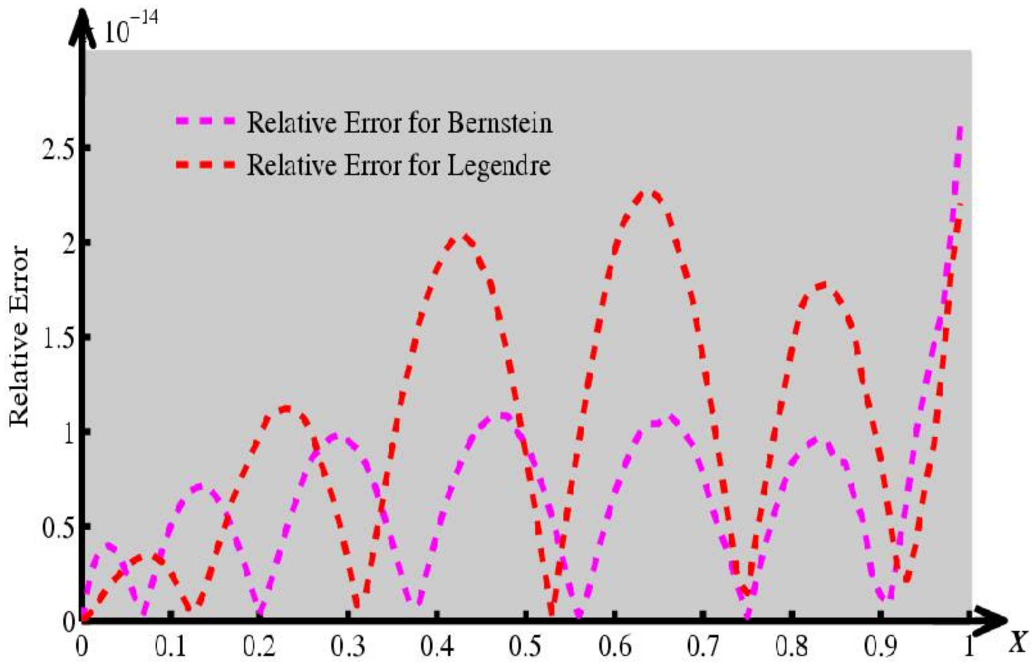


Fig. 3(b): Graphical representation of relative error of example 3 using 12 polynomials

Example 4: Consider the **nonlinear** differential equation [63]

$$\frac{d^7u}{dx^7} = u^2 e^{-x},\ 0 \le x \le 1 \tag{5.21a}$$

subject to the following boundary conditions

$u(0)=1,\, u(1)=e,\, u'(0)=1, u'(1)=e, u''(0)=1, u''(1)=e, u'''(0)=1\,.$ (5.21b)

The exact solution of this BVP is $u(x)=e^{x}$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i\, N_{i,n}(x),\; n\geq 1 \tag{5.22}$$

Here $\theta_0(x)=1-x(1-e)$ is specified by the essential boundary conditions in (5.21b). Also $N_{i,n}(0)=N_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Using eqn.(5.22) into eqn. (5.21a), the Galerkin weighted residual eqns. are

$$\int_0^1\left[\frac{d^7\tilde{u}}{dx^7}-\tilde{u}^2e^{-x}\right]N_{k,n}(x)dx=0, k=1,2,\cdots,n \tag{5.23}$$

Integrating first term of (5.23) by parts, we obtain

$$\int_0^1\frac{d^7\tilde{u}}{dx^7}N_{k,n}(x)dx=\left[N_{k,n}(x)\frac{d^6\tilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx \;\; [\text{Since}\, N_{k,n}(0)=N_{k,n}(1)=0\,]$$

$$=-\left[\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[N_{k,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\left[\frac{d^3}{dx^3}\left[N_{k,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$+\int_0^1\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\left[\frac{d^3}{dx^3}\left[N_{k,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[N_{k,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[N_{k,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[N_{k,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (5.24)$$

Putting eqn. (5.24) into eqn. (5.23) and using approximation for $\tilde{u}(x)$ given in eqn. (5.22) and after applying the conditions given in eqn. (5.21b) and rearranging the terms for the resulting eqns. we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{dN_{i,n}(x)}{dx} - 2\theta_0 e^{-x}\, N_{i,n}(x)N_{k,n}(x) - \sum_{j=1}^{n}\alpha_j (N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\right]dx\right.$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0} + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}$$

$$\left.-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0} - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}\right]\alpha_i = \int_0^1\left[-\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right.$$

$$\left.+\theta_0{}^2 e^{-x}N_{k,n}(x)\right]dx - \left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=0} - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=0} + \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3\theta_0}{dx^3}\right]_{x=1} - \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}\times e$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0} + \left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e - \left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0} \qquad (5.25)$$

The above equation (5.25) is equivalent to matrix form

$$(D+B)A = G \qquad (5.26a)$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k} = \int_0^1\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{dN_{i,n}(x)}{dx} - 2\theta_0 e^{-x}\, N_{i,n}(x)N_{k,n}(x)\right]dx - \left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0} + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1} \tag{5.26b}$$

$$b_{i,k}=-\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\,dx \tag{5.26c}$$

$$g_k=\int_0^1\left[-\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}N_{k,n}(x)\right]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=0}$$
$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}\times e$$
$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0} \tag{5.26d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (5.21a). That is, to find initial coefficients we solve the system

$$DA=G \tag{5.27a}$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1\frac{d^6N_{k,n}(x)}{dx^6}\frac{dN_{i,n}(x)}{dx}dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$
$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}$$
$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1} \tag{5.27b}$$

$$g_k=\int_0^1-\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^5\theta_0}{dx^5}\right]_{x=0}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^4\theta_0}{dx^4}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$
$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}\times e+\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e$$
$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0} \qquad (5.27c)$$

Once the initial values of α_i are obtained from eqn. (5.27a), they are substituted into eqn. (5.26a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (5.22), we obtain an approximate solution of the BVP (5.21).

The maximum absolute errors for 11 pieces of Bernstein and Legendre polynomials are shown in **Table 4** with 6 iterations.

Table 4: Maximum absolute errors of example 4 using 6 iterations.

x	Exact Results	11 Bernstein Polynomials		11 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	1.6084911E-012	1.1051709181	1.6084911E-011
0.2	1.2214027582	1.2214027582	1.3298251E-012	1.2214027580	1.3298251E-011
0.3	1.3498588076	1.3498588076	5.0746074E-012	1.3498588075	5.0746074E-011
0.4	1.4918246976	1.4918246976	7.1547213E-012	1.4918246976	7.1547213E-012
0.5	1.6487212707	1.6487212707	6.2061467E-012	1.6487212707	6.2061467E-011
0.6	1.8221188004	1.8221188004	2.9898306E-012	1.8221188004	2.9898306E-012
0.7	2.0137527075	2.0137527075	1.8429702E-013	2.0137527075	1.8429702E-013
0.8	2.2255409285	2.2255409285	6.5281114E-014	2.2255409282	6.5281114E-011
0.9	2.4596031112	2.4596031112	1.6604496E-012	2.4596031112	1.6604496E-011
1.0	2.7182818285	2.7182818285	0.0000000E+000	2.7182818285	0.0000000E+000

On the contrary the maximum absolute error has been obtained by Siddiqi *et al* [63] is 7.586×10^{-10}

We depict the exact and approximate solutions in Fig. 4(a) and a plot of relative errors in Fig. 4(b) of example 4 for $n=11$. From Fig. 4(b) we observe that the error is nearly the order 10^{-8}

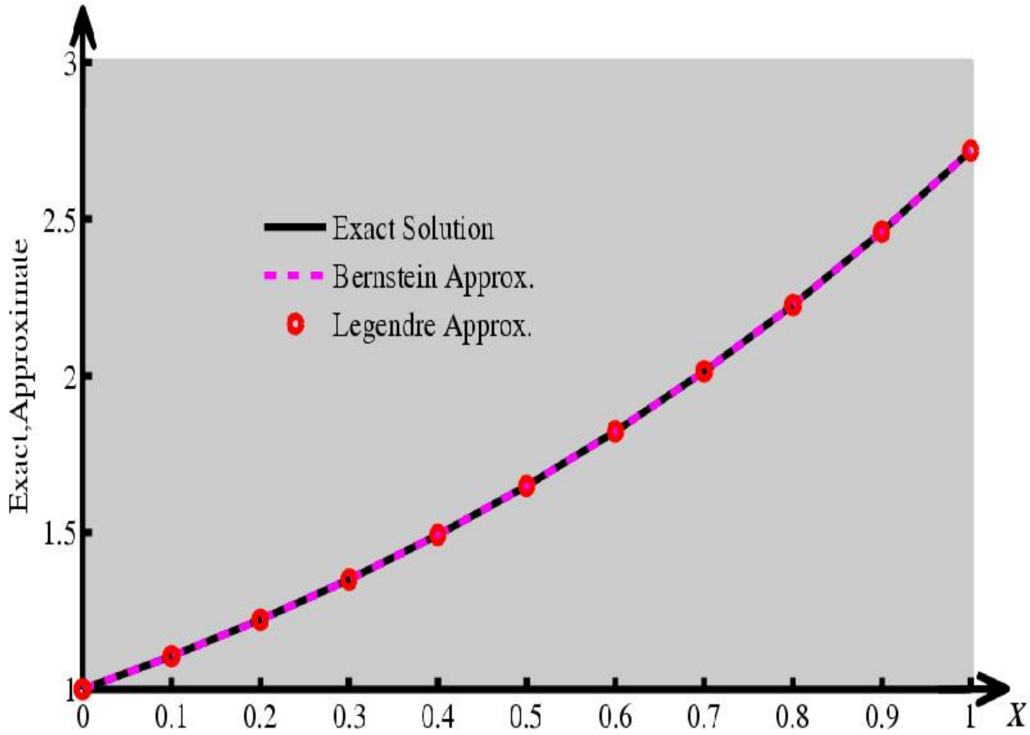


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 11 polynomials.

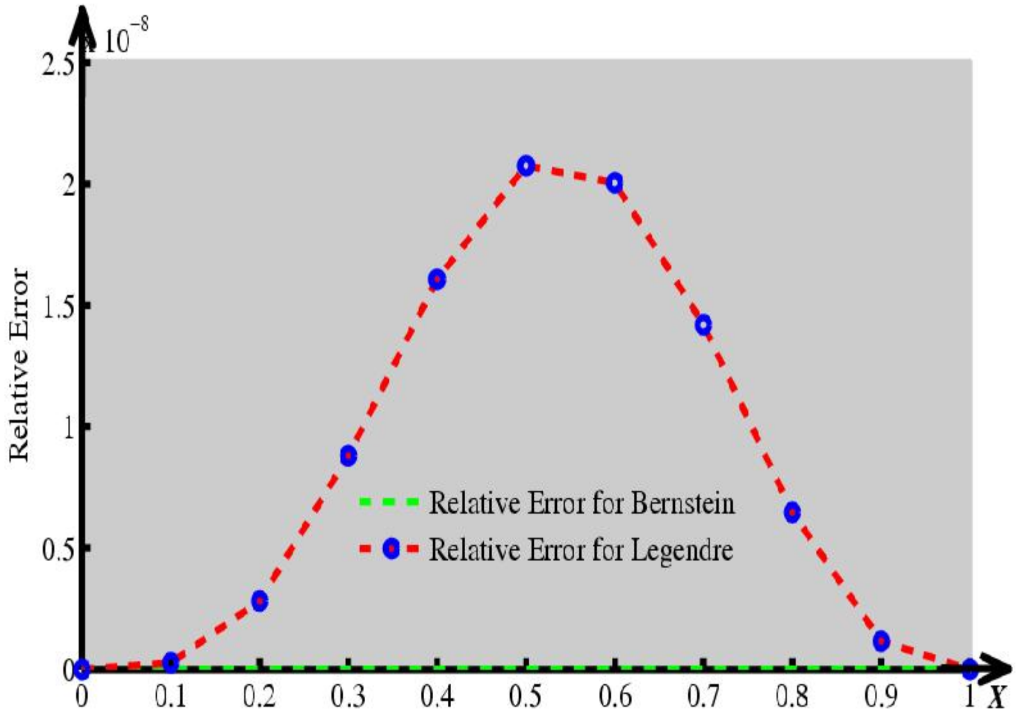


Fig. 4(b): Graphical representation of relative error of example 4 using 11 polynomials.

5.4 Conclusions

In this chapter, Galerkin method has been applied for the numerical solution of general seventh order linear and nonlinear BVPs using Bernstein and Legendre polynomials as trial functions. We see from the tables that the numerical results obtained by our method are better than other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. We observe that the numerical solutions coincide with the exact solution in some cases which are shown in Figs. [1-4].

CHAPTER 6

Eighth Order Boundary Value Problems

.6.1 Introduction

Eighth order differential equations represent physics of some hydrodynamic stability problems. We can find from a book written by Chandrasekhar [9] that when an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order. When the instability sets in as overstability, it is modelled by an eighth order ordinary differential equation. Eighth order differential equations are also modelled while considering the motion of a cylindrical shell. The existence and uniqueness theorem of solution of such type of BVP was presented by Agarwal [8]. The literature on the numerical solutions of eighth order BVPs and associated eigenvalue problems is found to be rare.

Some researchers have developed few methods for computing approximations to the solutions of eighth order BVPs. For this, Shen [65] investigated eighth order ordinary differential equations occurring bending and axial vibrations. An eighth order differential equation governing in torsional vibration of uniform beams was derived by [66]. Paliwal and Pande [67] developed equations for the equilibrium in terms of displacement components for an orthotropic thin circular cylindrical shell under a load that is not symmetric about the axis of the shell, which resulted in eighth order differential equations. Over the years, there are several authors who worked on eighth order BVPs by using different methods. For example, finite difference methods for the solution of eighth order BVPs were presented by Boutayeb and Twizell [69]. Twizell *et al* [70] derived numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [71] solved the eighth order BVP using octic spline. Siddiqi and Ghazala Akram [72, 73] used nonic spline and nonpolynomial respectively for the solution of eighth order linear special case BVPs. Siddiqi *et al* [74] applied

Variational Iteration Technique to solve eighth order BVPs. Kasi and Raju [75] used Quintic B-spline Collocation Method for solving eighth order BVPs. Further Scott and Watts [76] presented a numerical method for the solution of linear BVPs applying a combination of superposition and orthonormalization. Inc and Evans [77] solved eighth order BVPs using Adomain decomposition method. Liu and Wu [78] investigated differential quadrature solutions of eighth order differential equations. Scott and Watson [79] proved that Chow-Yorke algorithm was globally convergent for a class of spline collocation approximations to nonlinear BVPs. Wazwaz [80] applied modified decomposition method for approximating solution of higher order BVPs with two point boundary conditions. Modified adomain decomposition method was developed in [81] to find the analytical solution of the eighth order BVPs.

In this chapter, Galerkin method with Bernstein and Legendre polynomials as basis functions is used for the numerical solutions of general eighth order linear and nonlinear differential equations for two different cases of boundary conditions. In this method, the basis functions are transformed into a new set of basis functions which vanish at the boundary where the essential types of boundary conditions are defined and a matrix formulation is derived for solving the eighth order BVPs. The numerical results of the proposed method are compared with both the exact solution and the results of the other methods. A comparison of the results obtained by the present method with the results obtained by the previous methods reveals that the present method is of high precision, efficient and convenient

However, in this chapter, the formulation for solving linear eighth order BVP by Galerkin weighted residual method with Bernstein and Legendre polynomials is given in portion 6.2. Particularly, two formulations are described by using two types of boundary conditions in sections (6.2.1) and (6.2.2) respectively. Then we deduce similar formulation for nonlinear problems in the next section. Then numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation and the conclusions of the present chapter are given in the last section.

6.2 Matrix Derivation using Galerkin method

Here we first derive the matrix formulation for eighth order linear BVP and then extend our idea for solving nonlinear BVP. Because of this reason, we consider a general eighth order linear BVP of the type:

$$a_8 \frac{d^8u}{dx^8} + a_7 \frac{d^7u}{dx^7} + a_6 \frac{d^6u}{dx^6} + a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2}$$

$$+ a_1 \frac{du}{dx} + a_0 u = r \ , \ a < x < b \tag{6.1a}$$

subject to the following two types of boundary conditions

TypeI: $u(a) = A_0$, $\quad u(b) = B_0$, $\quad u'(a) = A_1$, $\quad u'(b) = B_1$,

$u''(a) = A_2$, $\quad u''(b) = B_2$, $\quad u'''(a) = A_3$, $\quad u'''(b) = B_3$ (6.1b)

TypeII: $u(a) = A_0$, $\quad u(b) = B_0$, $\quad u''(a) = A_2$, $\quad u''(b) = B_2$,

$u^{(iv)}(a)=A_4$, $\quad u^{(iv)}(b)=B_4$, $\quad u^{(vi)}(a)=A_6$, $\quad u^{(vi)}(b)=B_6$ (6.1c)

where $A_i, B_i, i = 0,1,2,3,4,6$ are finite real constants and $a_i, i = 0,1,\cdots 8$ and r are all continuous functions defined on the interval [a,b]. The boundary value problem (6.1) is solved with both cases of the boundary conditions of type TypeI and TypeII.

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (6.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x + a$, and thus we have:

$$c_8 \frac{d^8u}{dx^8} + c_7 \frac{d^7u}{dx^7} + c_6 \frac{d^6u}{dx^6} + c_5 \frac{d^5u}{dx^5} + c_4 \frac{d^4u}{dx^4} + c_3 \frac{d^3u}{dx^3} + c_2 \frac{d^2u}{dx^2}$$

$$+ c_1 \frac{du}{dx} + c_0 u = s, \ 0 < x < 1 \tag{6.2a}$$

$u(0) = A_0$, $\quad \frac{1}{b-a} u'(0) = A_1$, $\quad \frac{1}{(b-a)^2} u''(0) = A_2$,

$u(1) = B_0$, $\quad \frac{1}{b-a} u'(1) = B_1$, $\quad \frac{1}{(b-a)^2} u''(1) = B_2$,

$\frac{1}{(b-a)^3} u'''(0) = A_3$, $\quad \frac{1}{(b-a)^3} u'''(1) = B_3$ (6.2b) and

$$u(0)=A_0,\qquad \frac{1}{(b-a)^2}u''(0)=A_2,\qquad \frac{1}{(b-a)^4}u^{(iv)}(0)=A_4,$$

$$u(1)=B_0,\qquad \frac{1}{(b-a)^2}u''(1)=B_2,\qquad \frac{1}{(b-a)^4}u^{(iv)}(1)=B_4,$$

$$\frac{1}{(b-a)^6}u^{(vi)}(0)=A_6,\quad \frac{1}{(b-a)^6}u^{(vi)}(1)=B_6 \tag{6.2c}$$

where

$$c_8=\frac{1}{(b-a)^8}a_8((b-a)x+a),\qquad c_7=\frac{1}{(b-a)^7}a_7((b-a)x+a),$$

$$c_6=\frac{1}{(b-a)^6}a_6((b-a)x+a),\qquad c_5=\frac{1}{(b-a)^5}a_5((b-a)x+a),$$

$$c_4=\frac{1}{(b-a)^4}a_4((b-a)x+a),\qquad c_3=\frac{1}{(b-a)^3}a_3((b-a)x+a),$$

$$c_2=\frac{1}{(b-a)^2}a_2((b-a)x+a),\qquad c_1=\frac{1}{b-a}a_1((b-a)x+a),$$

$$c_0=a_0((b-a)x+a),\qquad s=r((b-a)x+a)$$

To solve the boundary value problem (6.2) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\ge 1 \tag{6.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0)=N_{i,n}(1)=0$, for each $i=1,2,3,\ldots,n.$

Applying eqn. (6.3) into eqn. (6.2a), the weighted residual equations are

$$\int_0^1\left[c_8\frac{d^8\tilde{u}}{dx^8}+c_7\frac{d^7\tilde{u}}{dx^7}+c_6\frac{d^6\tilde{u}}{dx^6}+c_5\frac{d^5\tilde{u}}{dx^5}+c_4\frac{d^4\tilde{u}}{dx^4}+c_3\frac{d^3\tilde{u}}{dx^3}+c_2\frac{d^2\tilde{u}}{dx^2}\right.$$

$$\left.+c_1\frac{d\tilde{u}}{dx}+c_0\tilde{u}-s\right]N_{j,n}(x)dx \tag{6.4}$$

6.2.1 Formulation I

In this section, we obtain the matrix formulation by using the boundary conditions of type I

Integrating by parts the terms up to second derivative on the left hand side of (6.4), we obtain

$$\int_0^1 c_8 \frac{d^8\tilde{u}}{dx^8} N_{j,n}(x)dx = \left[c_8 N_{j,n}(x) \frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx \ \text{[Since } N_{j,n}(0) = N_{j,n}(1) = 0 \text{]}$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{6.5}$$

$$\int_0^1 c_7 \frac{d^7\tilde{u}}{dx^7} N_{j,n}(x)dx = \left[c_7 N_{j,n}(x)\frac{d^6\tilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{6.6}$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (6.7)$$

$$\int_0^1 c_5 \frac{d^5\widetilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x)\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (6.8)$$

$$\int_0^1 c_4 \frac{d^4\widetilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x)\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (6.9)$$

$$\int_0^1 c_3 \frac{d^3\widetilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x)\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (6.10)$$

$$\int_0^1 c_2 \frac{d^2\widetilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x)\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (6.11)$$

Substituting eqns. (6.5) – (6.11) into eqn. (6.4) and using approximation for $\widetilde{u}(x)$ given in equation (6.3) and after applying the boundary conditions given in type I, eqn. (6.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (6.12a)$$

where

$$D_{i,j} = \int_0^1 \left\{\left[-\frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\right.\right.$$
$$+\frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]$$
$$\left.\left.+c_1 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x)\right\}dx - \left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{6.12b}$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]\right.\right.$$

$$-\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2N_{j,n}(x)\right]$$

$$\left.\left.-c_1N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0N_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3$$

$$+\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3+\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2$$

$$-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$+\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) A_1 + \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 - \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$-\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 - \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 + \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 - \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$+\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$-\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 \tag{6.12c}$$

Solving the system (6.12a), we find the values of the parameters α_i and then substituting these parameters into eqn. (6.3), we get the approximate solution of BVP (6.2). If we replace x by $\dfrac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (6.1).

6.2.2 Formulation II

In this portion, matrix formulation is given by applying the boundary conditions of type II.

Similarly of section (6.2.1), integrating by parts the terms consisting eighth, seventh, sixth, fifth, fourth, third, and second derivatives on the left hand side of (6.4) and applying the boundary conditions prescribed in type II, eqn (6.2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\; j = 1,2,\ldots,n \qquad (6.13a)$$

where

$$\begin{aligned}
D_{i,j} = \int_0^1 \Bigg\{ \Bigg[&-\frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] \\
&- \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \\
&+ c_1 N_{j,n}(x)\Bigg] \frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x) \Bigg\} dx + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&- \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&- \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right] \frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&- \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right] \frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+ \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right] \frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}
\end{aligned}$$

$$-\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} -\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} +\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} -\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} -\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} +\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} +\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} -\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{6.13b}$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]\right.\right.$$
$$-\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2N_{j,n}(x)\right]$$
$$\left.\left.-c_1N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0N_{j,n}(x)\right\}dx-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} -\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1} +\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$
$$-\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} +\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0} +\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} +\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1} -\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} -\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0} -\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0} -\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} +\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1} -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0} +\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0} -\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} +\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6 -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6$$

$$+\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4 -\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$+\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2 -\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4 +\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2 +\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$+\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2 -\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2 +\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$+\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2 -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2 \qquad (6.13c)$$

Solving the system (6.13a), we find the values of the parameters α_i and then substituting these parameters into eqn. (6.3), we get the approximate solution of

BVP (6.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (6.1).

For nonlinear eighth order BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (6.12) and (6.13). Then using the iterative method we find the numerical approximations for desired nonlinear BVP.

6.3 Numerical examples and results

To test the applicability of the proposed method, we consider five linear and two nonlinear problems with both types of boundary conditions. For all examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB* 10**. The convergence of linear BVP is calculated by

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using *n*-th polynomials and δ (depends on the problem) which is less than 10^{-12}. The convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^N\right| < \delta$$

Where $\delta < 10^{-11}$ and N is the Newton's iteration number

Example 1: Consider the linear differential equation [75]

$$\frac{d^8u}{dx^8} + \frac{d^7u}{dx^7} + 2\frac{d^6u}{dx^6} + 2\frac{d^5u}{dx^5} + 2\frac{d^4u}{dx^4} + 2\frac{d^3u}{dx^3} + 2\frac{d^2u}{dx^2} + \frac{du}{dx} + u$$
$$= 14\cos x - 16\sin x - 4x\sin x,\ 0 \le x \le 1 \qquad (6.14a)$$

subject to the boundary conditions of type I in eqn. (2b):

$$u(0) = 0,\ u(1) = 0,\ u'(0) = -1, u'(1) = 2\sin 1, u''(0) = 0, u''(1) = 4\cos 1 + 2\sin 1$$
$$u'''(0) = 7, u'''(1) = 6\cos 1 - 6\sin 1. \qquad (6.14b)$$

The analytic solution of the above problem is, $u(x) = (x^2 - 1)\sin x$.

Using the method illustrated in (6.2.1), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \;\; n \geq 1 \tag{6.15}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (6.14b). Now the parameters α_i $(i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j ,\, j = 1,2,\ldots,n \tag{6.16a}$$

where

$$\begin{aligned}
D_{i,j} = \int_0^1 \Bigg\{ \Bigg[& -\frac{d^7}{dx^7}\big[N_{j,n}(x)\big] + \frac{d^6}{dx^6}\big[N_{j,n}(x)\big] - \frac{d^5}{dx^5}\big[2N_{j,n}(x)\big] + \frac{d^4}{dx^4}\big[2N_{j,n}(x)\big] - \frac{d^3}{dx^3}\big[2N_{j,n}(x)\big] \\
& + \frac{d^2}{dx^2}\big[2N_{j,n}(x)\big] - \frac{d}{dx}\big[2N_{j,n}(x)\big] + N_{j,n}(x) \Bigg] \frac{d}{dx}[N_{i,n}(x)] + N_{i,n}(x) N_{j,n}(x) \Bigg\} dx \\
& - \left[\frac{d}{dx}\big[N_{j,n}(x)\big] \frac{d^6}{dx^6}\big[N_{i,n}(x)\big]\right]_{x=1} + \left[\frac{d}{dx}\big[N_{j,n}(x)\big] \frac{d^6}{dx^6}\big[N_{i,n}(x)\big]\right]_{x=0} \\
& + \left[\frac{d^2}{dx^2}\big[N_{j,n}(x)\big] \frac{d^5}{dx^5}\big[N_{i,n}(x)\big]\right]_{x=1} - \left[\frac{d^2}{dx^2}\big[N_{j,n}(x)\big] \frac{d^5}{dx^5}\big[N_{i,n}(x)\big]\right]_{x=0} \\
& - \left[\frac{d^3}{dx^3}\big[N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=1} + \left[\frac{d^3}{dx^3}\big[N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=0} \\
& - \left[\frac{d}{dx}\big[N_{j,n}(x)\big] \frac{d^5}{dx^5}\big[N_{i,n}(x)\big]\right]_{x=1} + \left[\frac{d}{dx}\big[N_{j,n}(x)\big] \frac{d^5}{dx^5}\big[N_{i,n}(x)\big]\right]_{x=0} \\
& + \left[\frac{d^2}{dx^2}\big[N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=1} - \left[\frac{d^2}{dx^2}\big[N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=0} \\
& - \left[\frac{d}{dx}\big[2N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=1} + \left[\frac{d}{dx}\big[2N_{j,n}(x)\big] \frac{d^4}{dx^4}\big[N_{i,n}(x)\big]\right]_{x=0}
\end{aligned} \tag{6.16b}$$

$$F_j = \int_0^1 (14\cos x - 16\sin x - 4x\sin x) N_{j,n}(x) dx + \Bigg\{ -\frac{d^4}{dx^4}\big[N_{j,n}(x)\big] + \frac{d^3}{dx^3}\big[N_{j,n}(x)\big]$$

$$-\frac{d^2}{dx^2}\left[2N_{j,n}(x)\right]+\frac{d}{dx}\left[2N_{j,n}(x)\right]\Bigg\}_{x=1}\times(6\cos1-6\sin1)+\Bigg\{\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]$$

$$-\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]+\frac{d^2}{dx^2}\left[2N_{j,n}(x)\right]-\frac{d}{dx}\left[2N_{j,n}(x)\right]\Bigg\}_{x=0}\times7+\Bigg\{\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]$$

$$-\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[2N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[2N_{j,n}(x)\right]+\frac{d}{dx}\left[2N_{j,n}(x)\right]\Bigg\}_{x=1}\times(4\cos1+2\sin1)$$

$$+\Bigg\{-\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[2N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[2N_{j,n}(x)\right]$$

$$-\frac{d^2}{dx^2}\left[2N_{j,n}(x)\right]+\frac{d}{dx}\left[2N_{j,n}(x)\right]\Bigg\}_{x=1}\times2\sin1-\Bigg\{\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]-\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]$$

$$+\frac{d^4}{dx^4}\left[2N_{j,n}(x)\right]-\frac{d^3}{dx^3}\left[2N_{j,n}(x)\right]+\frac{d^2}{dx^2}\left[2N_{j,n}(x)\right]-\frac{d}{dx}\left[2N_{j,n}(x)\right]\Bigg\}_{x=0} \quad (6.16c)$$

Solving the system (6.16a) we obtain the values of the parameters and then substituting these parameters into eqn. (6.15), we get the approximate solution of the BVP (6.14) for different values of n.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 1.**

Table 1: Maximum absolute errors for the example 1.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
9	3.149×10^{-9}	3.149×10^{-9}	4.679×10^{-6} (Kasi and Raju [75])
10	2.193×10^{-11}	6.239×10^{-11}	
11	3.593×10^{-13}	3.592×10^{-13}	
12	3.164×10^{-15}	9.660×10^{-14}	

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n=12$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-13}.

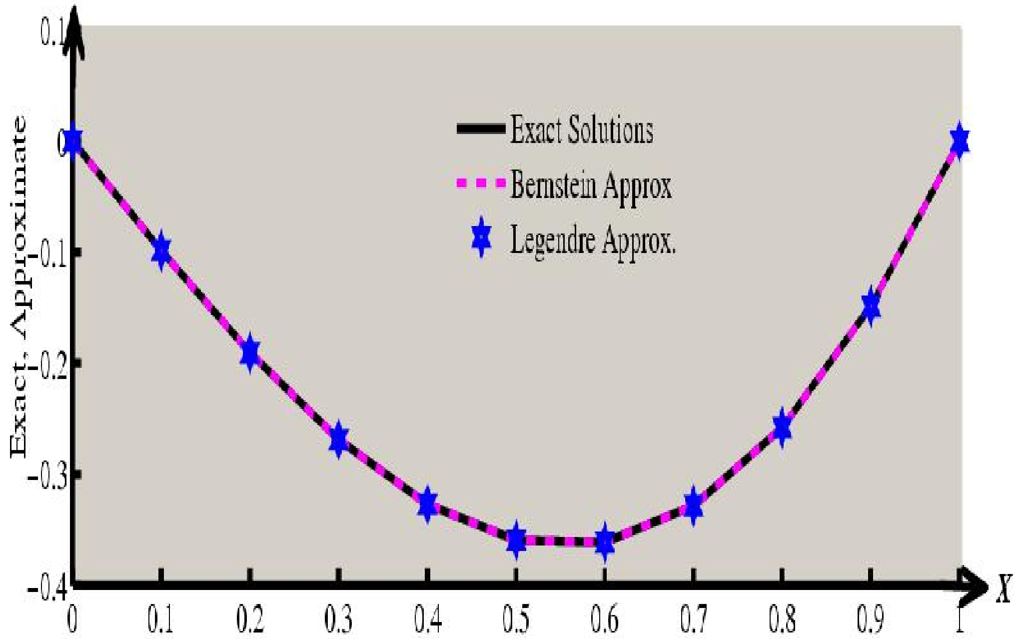


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 12 polynomials.

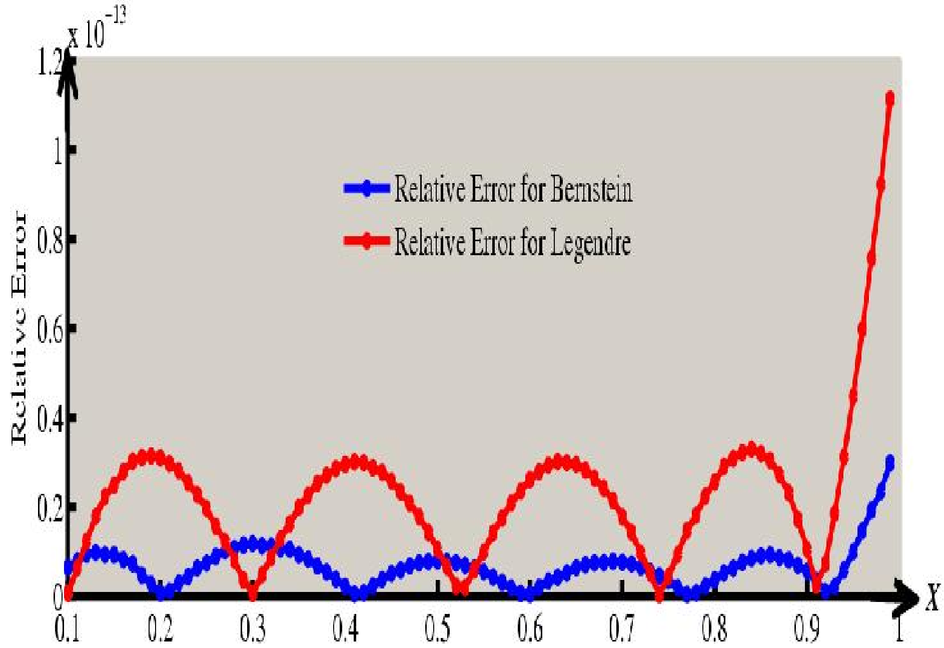


Fig. 1(b): Graphical representation of relative error of example 1 using 12 polynomials.

Example 2: Consider the linear differential equation [75]

$$\frac{d^8u}{dx^8} - u = -8e^x, \quad 0 \le x \le 1 \tag{6.17a}$$

subject to the boundary conditions of type I in eqn. (2b):

$$u(0) = 1,\ u(1) = 0,\ u'(0) = 0, u'(1) = -e, u''(0) = -1, u''(1) = -2e, u'''(0) = -2,$$
$$u'''(1) = -3e\ . \tag{6.17b}$$

The analytic solution of the above system is, $u(x) = (1-x)e^x$.

Employing the method given in (6.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{6.18}$$

Here $\theta_0(x) = 1 - x$ is specified by the essential boundary conditions of equation (6.17b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \tag{6.19a}$$

where

$$\begin{aligned}
D_{i,j} = &\int_0^1 \left[-\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - N_{i,n}(x)N_{j,n}(x) \right] dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}[N_{i,n}(x)] \right]_{x=1} \\
&+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=0}
\end{aligned} \tag{6.20b}$$

$$\begin{aligned}
F_j = &\int_0^1 \left[-8e^x N_{j,n}(x) + \frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d\theta_0}{dx} + \theta_0 N_{j,n}(x) \right] dx - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-3e) \\
&+ \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-2) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-2e) \\
&- \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-1) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-e)
\end{aligned} \tag{6.20c}$$

Solving the system (6.20a) we obtain the values of the parameters and then substituting these parameters into eqn. (6.19), we get the approximate solution of the BVP (6.18).

The numerical results for this problem are summarized in **Table 2**.

Table 2: Maximum absolute errors for the example 2.

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	0.0000000E+000	0.9946538263	4.4408921E-016
0.2	0.9771222065	0.9771222065	7.7715612E-016	0.9771222065	6.2172489E-015
0.3	0.9449011653	0.9449011653	1.5543122E-015	0.9449011653	3.5527137E-015
0.4	0.8950948186	0.8950948186	7.7715612E-016	0.8950948186	8.6597396E-015
0.5	0.8243606354	0.8243606354	5.5511151E-016	0.8243606354	2.9976022E-015
0.6	0.7288475202	0.7288475202	2.2204460E-016	0.7288475202	1.0214052E-014
0.7	0.6041258122	0.6041258122	6.6613381E-016	0.6041258122	2.2204460E-016
0.8	0.4451081857	0.4451081857	1.6653345E-016	0.4451081857	8.6042284E-015
0.9	0.2459603111	0.2459603111	1.1102230E-016	0.2459603111	5.5511151E-016
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand Kasi and Raju [75] obtained the accuracy nearly the order 10^{-6} on using Quintic B-spline collocation method.

In Fig. 2(a), the exact and approximate solutions are given and a plot of relative errors are shown in Fig. 2(b) of example 2 for $n=12$. It is observed from Fig. 2(b) that the error is nearly the order 10^{-14}.

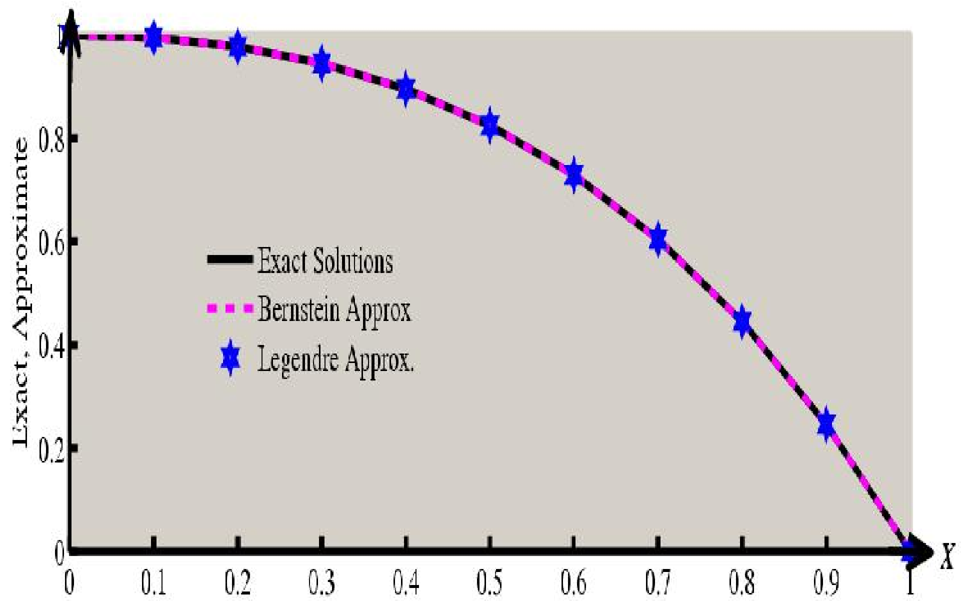


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 12 polynomials.

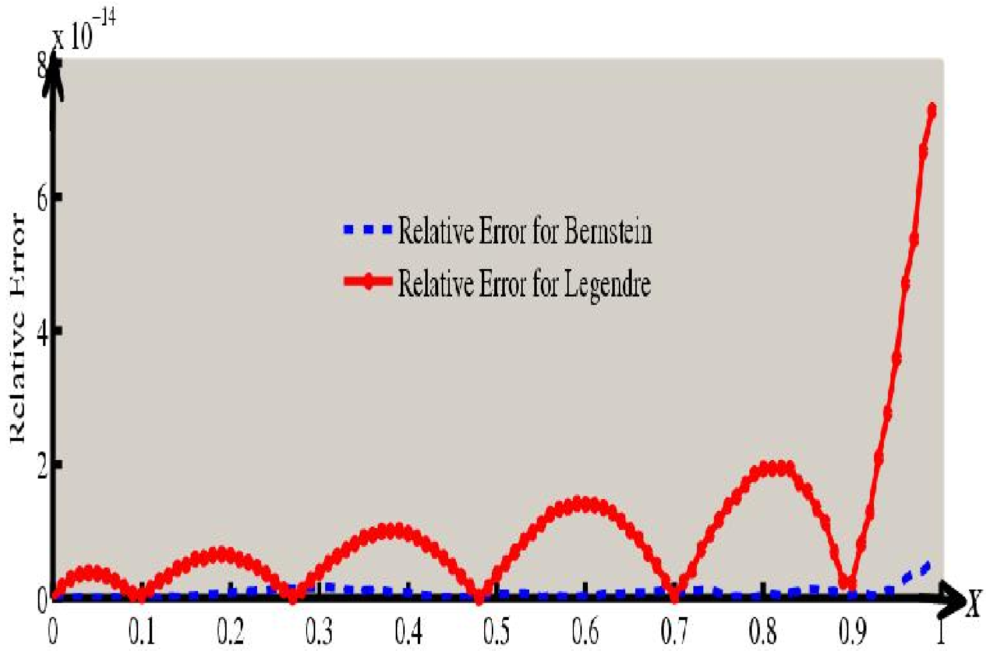


Fig. 2(b): Graphical representation of relative error of example 2 using 12 polynomials.

Example 3: Consider the linear differential equation [74]

$$\frac{d^8u}{dx^8} = u,\ \ 0 \le x \le \frac{\pi}{2} \tag{6.21a}$$

subject to the boundary conditions of type I in eqn. (2b):

$$u(0) = 1, u(\frac{\pi}{2}) = 1,\ u'(0) = 1, u'(\frac{\pi}{2}) = -1, u''(0) = -1, u''(\frac{\pi}{2}) = -1, u'''(0) = -1,$$

$$u'''(\frac{\pi}{2}) = 1\ . \tag{6.21b}$$

The analytic solution of the above system is, $u(x) = \sin x + \cos x$.

To use Bernstein and Legendre polynomials over [0, 1], we convert the BVP (6.21) to an equivalent BVP on [0,1] by letting $x = \frac{\pi}{2}x$. Thus the BVP (6.21) is equivalent to the BVP

$$\frac{d^8u}{dx^8} = \frac{\pi^8}{256}\ u,\ \ 0 \le x \le 1 \tag{6.22a}$$

$$u(0) = 1, u(1) = 1,\ u'(0) = \frac{2}{\pi}, u'(1) = -\frac{2}{\pi}, u''(0) = -\frac{4}{\pi^2}, u''(1) = -\frac{4}{\pi^2}, u'''(0) = -\frac{8}{\pi^3},$$

$$u'''(1) = \frac{8}{\pi^3} \tag{6.22b}$$

Applying the method mentioned in (6.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{6.23}$$

Here $\theta_0(x) = (1-2x)^2$ is specified by the essential boundary conditions of equation (6.22b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \tag{6.24a}$$

where

$$\begin{aligned} D_{i,j} = {} & \int_0^1 \left[-\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - \frac{\pi^8}{256} N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=1} \\ & + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} \\ & - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=1} \\ & + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right] \right]_{x=0} \end{aligned} \tag{6.24b}$$

$$\begin{aligned} F_j = {} & \int_0^1 \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d\theta_0}{dx} + \frac{\pi^8}{256}\theta_0 N_{j,n}(x) \right] dx - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=1} \times \left(\frac{8}{\pi^3} \right) \\ & + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(-\frac{8}{\pi^3} \right) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1} \times \left(-\frac{4}{\pi^2} \right) \\ & - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(-\frac{4}{\pi^2} \right) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=1} \times \left(-\frac{2}{\pi} \right) \\ & + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(\frac{2}{\pi} \right) \end{aligned} \tag{6.24c}$$

Solving the system (6.24a), we have the values of the parameters and then substituting these parameters into eqn. (6.23), we get the approximate solution of the BVP (6.22) for different values of *n*. If we replace x by $\frac{\pi}{2}x$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (6.21).

The maximum absolute errors, shown in **Table 3,** are listed to compare with existing results.

Table 3: Maximum absolute errors for the example 3.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
9	9.297×10^{-11}	9.297×10^{-11}	6.9188×10^{-11} (Siddiqi *et al* [74])
10	2.310×10^{-13}	6.297×10^{-12}	
11	2.456×10^{-15}	2.398×10^{-14}	
12	6.661×10^{-16}	9.104×10^{-15}	

In Figs. 3(a) and 3(b), the exact and approximate solutions, and the relative errors of example 3 for $n=12$ are depicted respectively. We see from Fig. 3(b) that the error is nearly the order 10^{-14}

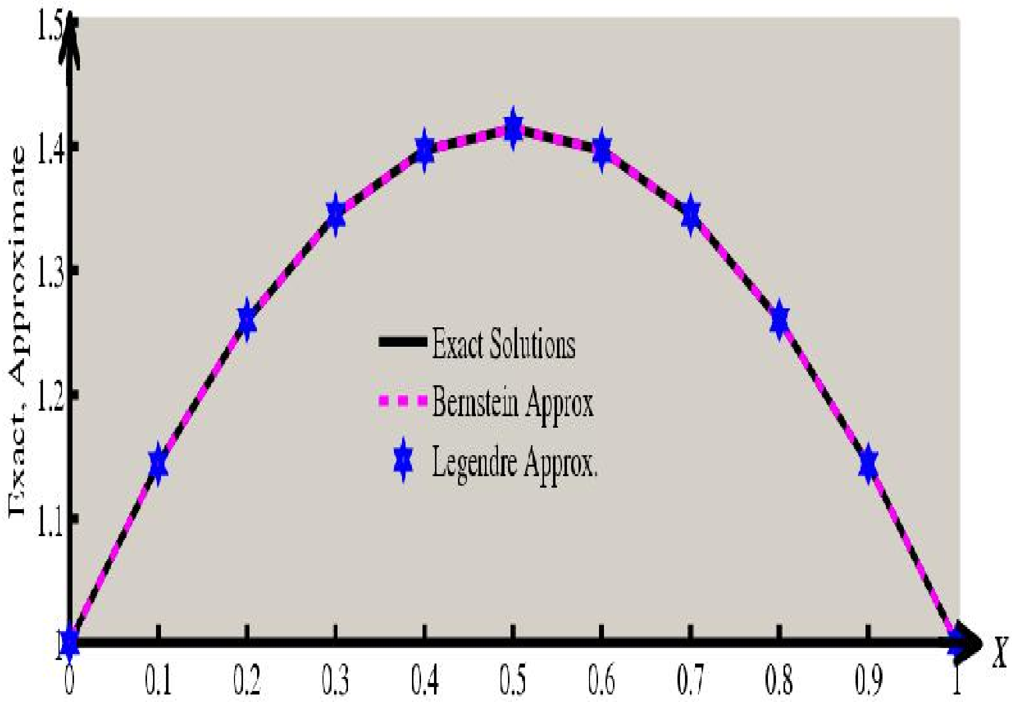
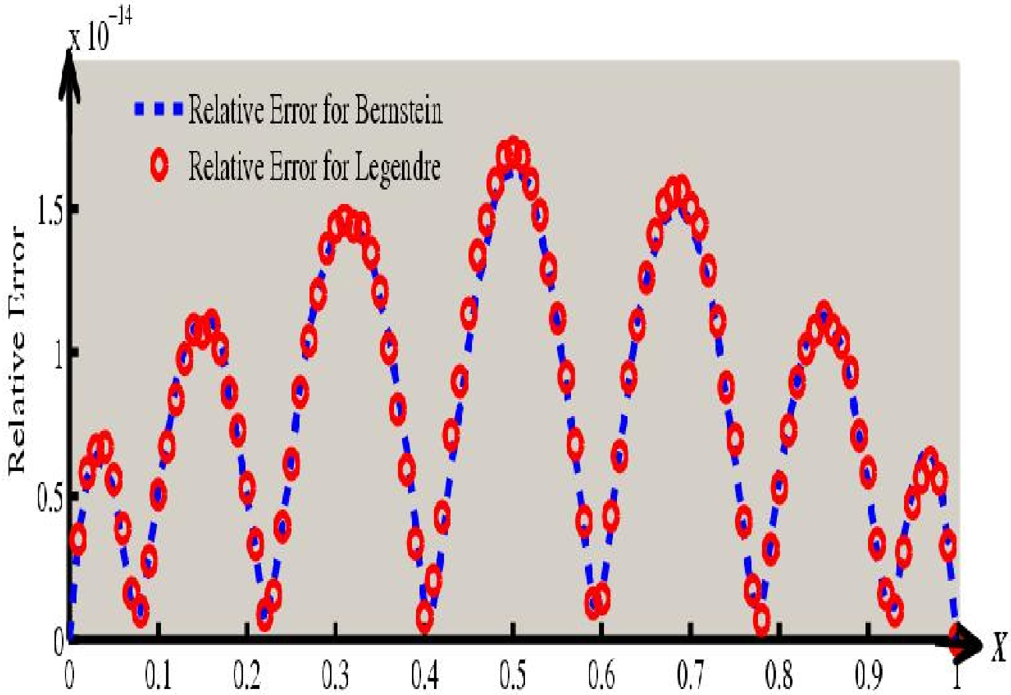


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 12 polynomials.



3(b): Graphical representation of relative error of example 3 using 12 polynomials.

Example 4: Consider the linear boundary value problem [71]

$$\frac{d^8u}{dx^8} - xu = -(55+17x+x^2-x^3)e^x, \quad -1 \le x \le 1 \tag{6.25a}$$

$$u(-1) = u(1) = 0, u''(-1) = \frac{2}{e}, u''(1) = -6e, u^{(iv)}(-1) = -\frac{4}{e}, u^{(iv)}(1) = -20e,$$
$$u^{(vi)}(-1) = -\frac{18}{e}, u^{(vi)}(1) = -42e. \tag{6.25b}$$

The analytic solution of the above system is, $u(x) = (1-x^2)e^x$.

To use Bernstein and Legendre polynomials over [0, 1], we convert the BVP (6.25) to an equivalent BVP on [0, 1] by letting $x = 2x-1$. Thus the BVP (6.25) is equivalent to the BVP

$$\frac{1}{2^8}\frac{d^8u}{dx^8} - (2x-1)u = -(55+17(2x-1)+(2x-1)^2-(2x-1)^3)e^{(2x-1)}, 0 \le x \le 1 \tag{6.26a}$$

$$u(0) = u(1) = 0, \frac{1}{4}u''(0) = \frac{2}{e}, \frac{1}{4}u''(1) = -6e, \frac{1}{16}u^{(iv)}(0) = -\frac{4}{e}, \frac{1}{16}u^{(iv)}(1) = -20e,$$

$$\frac{1}{64}u^{(vi)}(0) = -\frac{18}{e}, \frac{1}{64}u^{(vi)}(1) = -42e \tag{6.26b}$$

Applying the method illustrated in (6.2.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \geq 1 \tag{6.27}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (6.26b). Now the parameters $\alpha_i \; (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \tag{6.28a}$$

where

$$\begin{aligned} D_{i,j} = &\int_0^1 \left[-\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 256(2x-1)N_{i,n}(x)N_{j,n}(x) \right] dx \\ &+ \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}[N_{i,n}(x)] \right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}[N_{i,n}(x)] \right]_{x=0} \\ &+ \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)] \right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}[N_{i,n}(x)] \right]_{x=0} \\ &+ \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}[N_{i,n}(x)] \right]_{x=1} - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}[N_{i,n}(x)] \right]_{x=0} \end{aligned} \tag{6.28b}$$

$$\begin{aligned} F_j = &\int_0^1 -(55 + 17(2x-1) + (2x-1)^2 - (2x-1)^3)e^{(2x-1)} N_{j,n}(x)dx \\ &+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-42 \times 64e) - 64\left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(\frac{18}{e} \right) \\ &+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-320\; e) - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(-\frac{64}{e} \right) \\ &+ \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-24e) - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} \times \left(\frac{8}{e} \right) \end{aligned} \tag{6.28c}$$

Solving the system (6.28a), we have the values of the parameters and then substituting these parameters into eqn. (6.27), we get the approximate solution of the BVP (6.26) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (6.25).

In **Table 4**, we tabulate the maximum absolute errors to compare with the previous results.

Table 4: Maximum absolute errors for the example 4.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Bernstein	Reference Results
10	4.294×10^{-10}	9.975×10^{-9}	9.443×10^{-3} (Siddiqi and Twizell [71])
11	9.805×10^{-12}	4.294×10^{-10}	
12	4.590×10^{-13}	9.859×10^{-12}	
13	3.078×10^{-14}	8.218×10^{-13}	

In figs. 4(a) and 4(b) we have given the exact and approximate solutions, and the relative error of example 4 for $n = 13$. From fig. 4(b) we observed that the error is nearly the order 10^{-12}.

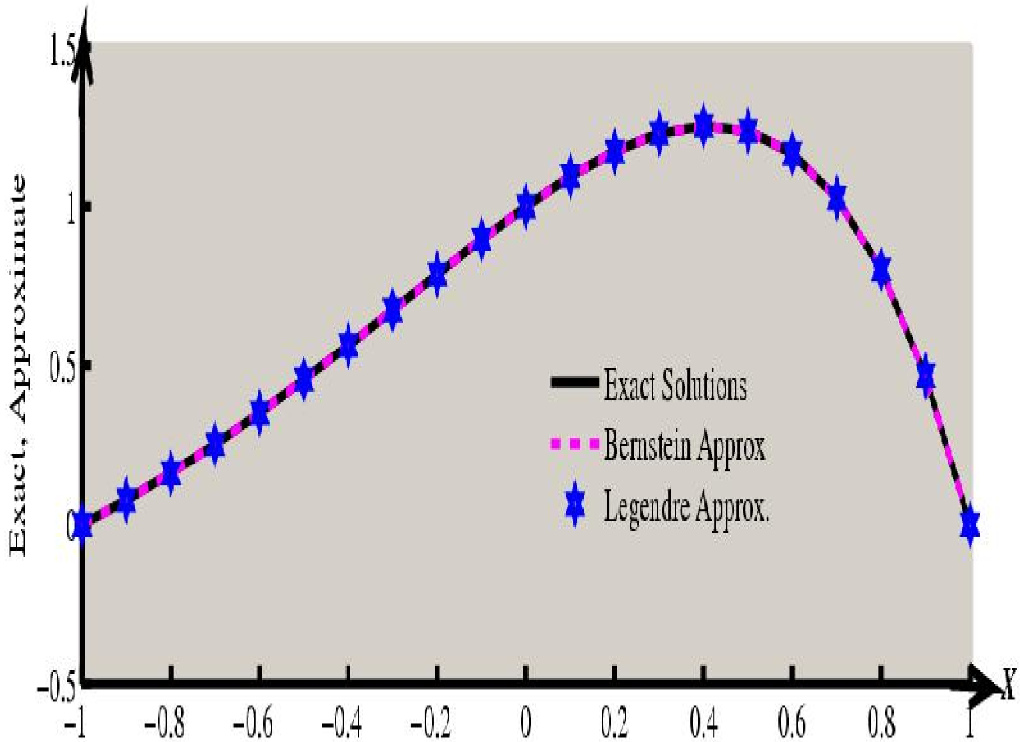


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 13 polynomials.

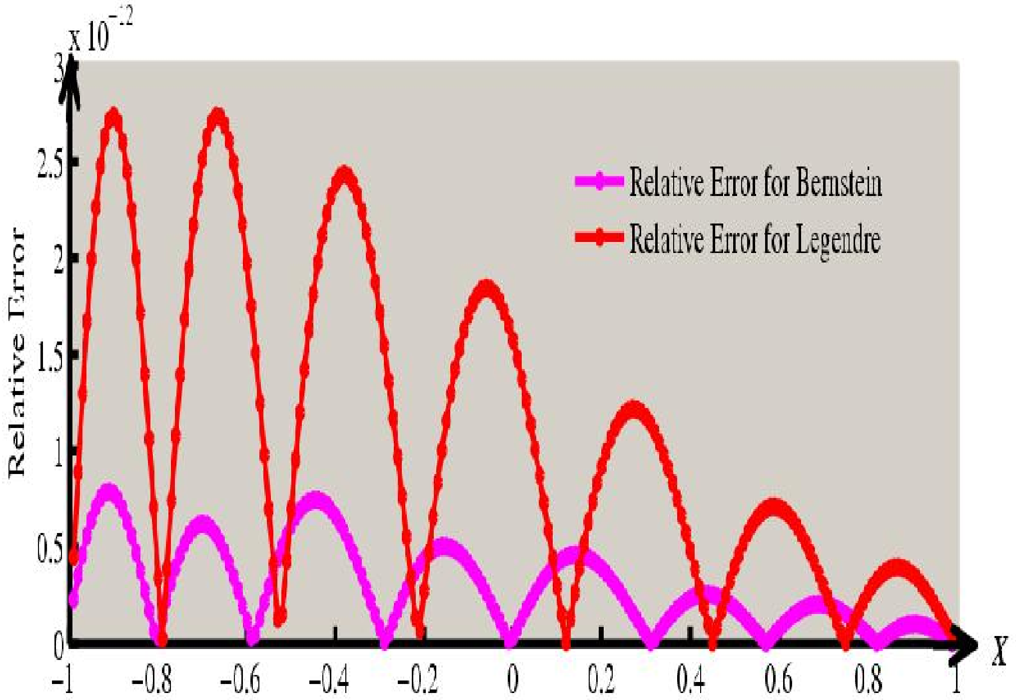


Fig. 4(b): Graphical representation of relative error of example 9 using 13 polynomials.

Example 5: Consider the linear BVP [71]

$$\frac{d^8u}{dx^8} - u = 8(2x\sin x - 7\cos x),\ -1 \le x \le 1 \tag{6.29a}$$

$$u(-1) = u(1) = 0, u''(-1) = 4\sin(-1) + 2\cos(-1), u''(1) = -4\sin(1) + 2\cos(1),$$

$$u^{(iv)}(-1) = -8\sin(-1) - 12\cos(-1), u^{(iv)}(1) = 8\sin(1) - 12\cos(1),$$

$$u^{(vi)}(-1) = 12\sin(-1) + 30\cos(-1), u^{(vi)}(1) = -12\sin(1) + 30\cos(1). \tag{6.29b}$$

The analytic solution of the above system is, $u(x) = (x^2 - 1)\cos x$.

To use Bernstein and Legendre polynomials over [0, 1], we convert the BVP (6.29) to an equivalent BVP on [0, 1] by letting $x = 2x - 1$. Thus the BVP (6.29) is equivalent to the BVP

$$\frac{1}{2^8}\frac{d^8u}{dx^8} - u = 8(2(2x-1)\sin(2x-1) - 7\cos(2x-1)),\ 0 \le x \le 1 \tag{6.30a}$$

$$u(0) = u(1) = 0, \frac{1}{4}u''(0) = 4\sin(-1) + 2\cos(-1), \frac{1}{4}u''(1) = -4\sin(1) + 2\cos(1),$$

$$\frac{1}{16}u^{(iv)}(0) = -8\sin(-1) - 12\cos(-1), \frac{1}{16}u^{(iv)}(1) = 8\sin(1) - 12\cos(1),$$

$$\frac{1}{64}u^{(vi)}(0) = 12\sin(-1) + 30\cos(-1), \frac{1}{64}u^{(vi)}(1) = -12\sin(1) + 30\cos(1) \qquad (6.30b)$$

Using the method mentioned in (6.2.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \geq 1 \qquad (6.31)$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (6.30b). Now the parameters α_i $(i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, \; j = 1,2,\ldots,n \qquad (6.32a)$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 256 N_{i,n}(x) N_{j,n}(x) \right] dx + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1}$$
$$- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=1}$$
$$- \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=1}$$
$$- \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=0} \qquad (6.32b)$$

$$F_j = \int_0^1 2048\; (2(2x-1)\sin(2x-1) - 7\cos(2x-1))\, N_{j,n}(x) dx$$
$$+ 64\left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-12\sin 1 + 30\cos 1) - 64\left[\frac{d}{dx}\left[N_{j,n}(x)\right] \right]_{x=0}$$
$$\times (-12\sin 1 + 30 s\cos 1) + 16\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=1} \times (8\sin 1 - 12\cos 1)$$
$$-16\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \right]_{x=0} \times (8\sin 1 - 12\cos 1) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1}$$
$$\times (-16\sin 1 + 8\cos 1) - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-16\sin 1 + 8\cos 1) \qquad (6.32c)$$

Solving the system (6.32a), we have the values of the parameters and then substituting these parameters into eqn. (6.31), we get the approximate solution of the BVP (6.30) for different values of n. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (6.29).

The maximum absolute errors, shown in **Table 5,** are tabulated to compare with existing results

Table 5: Maximum absolute errors for the example 5.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
10	9.286×10^{-9}	9.286×10^{-9}	7.937×10^{-2} (Siddiqi and Twizell [71])
11	3.524×10^{-11}	8.998×10^{-12}	
12	8.950×10^{-12}	9.000×10^{-12}	
13	4.718×10^{-14}	2.676×10^{-13}	

We illustrated graphically the exact and approximate solutions in Fig. 5(a) and the relative errors in Fig. 5(b) of example 5 for $n=13$. It is clear from Fig. 5(b) that the error is of order 10^{-13}.

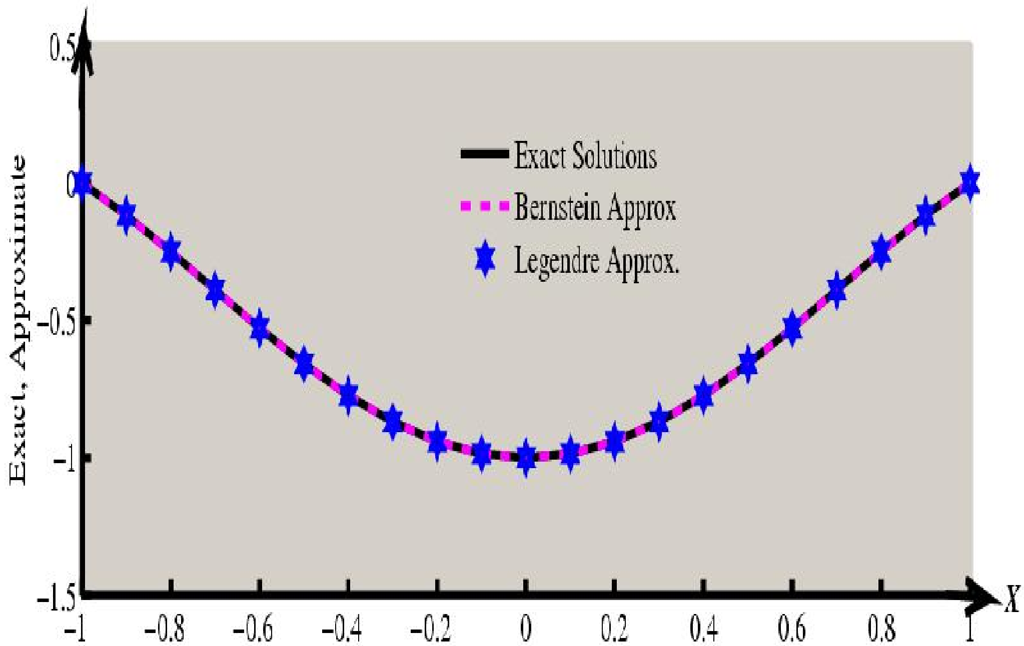
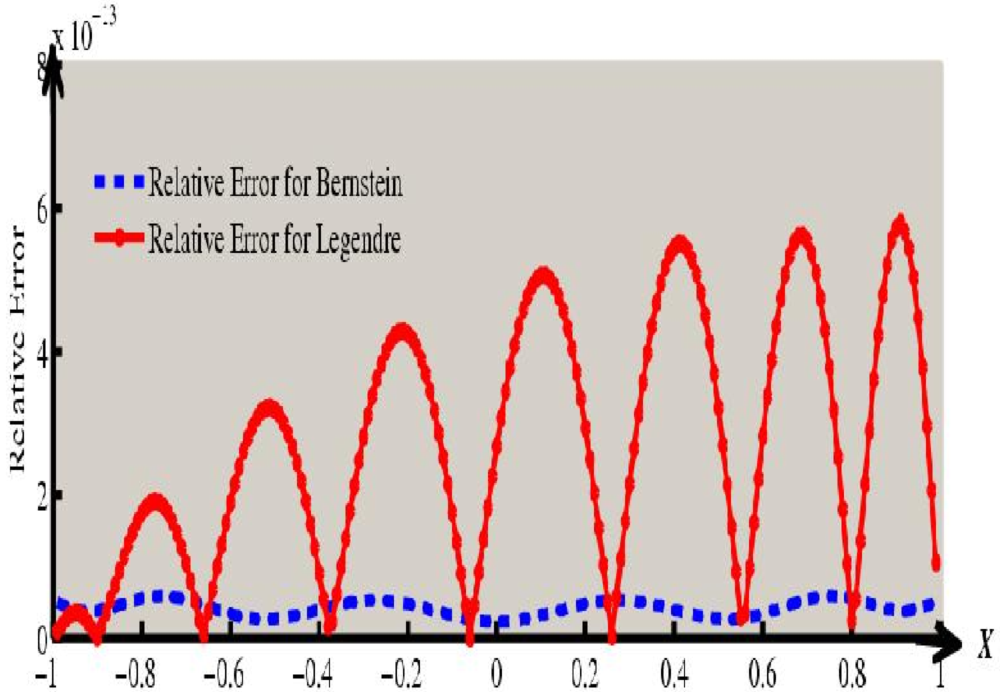


Fig. 5(a): Graphical representation of exact and approximate solutions of example 5 using 13 polynomials.



5(b): Graphical representation of relative error of example 5 using 13 polynomials.

Example 6: Consider the **nonlinear** differential equation [75]

$$\frac{d^8u}{dx^8} = u^2 e^{-x},\ 0 \le x \le 1 \qquad (6.33a)$$

subject to the boundary conditions of type I defined in eqn. (2b)

$$u(0)=1,\ u(1)=e,\ u'(0)=1, u'(1)=e, u''(0)=1, u''(1)=e, u'''(0)=1, u'''(1)=e. \qquad (6.33b)$$

The exact solution of this BVP is $u(x)=e^x$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ \ n \ge 1 \qquad (6.34)$$

Here $\theta_0(x) = 1 - x(1-e)$ is specified by the essential boundary conditions in (6.33b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Using eqn. (6.34) into eqn. (6.33a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^8\tilde{u}}{dx^8} - \tilde{u}^2 e^{-x} \right] N_{k,n}(x)dx = 0, k = 1,2,\cdots,n \qquad (6.35)$$

Integrating first term of (6.35) by parts, we obtain

$$\int_0^1 \frac{d^8\tilde{u}}{dx^8} N_{k,n}(x)\,dx = \left[N_{k,n}(x)\frac{d^7\tilde{u}}{dx^7}\right]_0^1 - \int_0^1 \frac{dN_{k,n}(x)}{dx}\frac{d^7\tilde{u}}{dx^7}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \int_0^1 \frac{d^2N_{k,n}(x)}{dx^2}\frac{d^6\tilde{u}}{dx^6}dx \text{ [Since } N_{k,n}(1) = N_{k,n}(0) = 0\text{]}$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d^3N_{k,n}(x)}{dx^3}\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^4N_{k,n}(x)}{dx^4}\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$- \int_0^1 \frac{d^5N_{k,n}(x)}{dx^5}\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$- \left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^6N_{k,n}(x)}{dx^6}\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$- \left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^7N_{k,n}(x)}{dx^7}\frac{d\tilde{u}}{dx}dx \qquad (6.36)$$

Putting eqn. (6.36) into eqn. (6.35) and using approximation for $\tilde{u}(x)$ given in eqn. (6.34) and after applying the conditions given in eqn. (6.33b) and rearranging the terms for the resulting eqns. we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^7N_{k,n}(x)}{dx^7}\frac{dN_{i,n}(x)}{dx}-2\theta_0e^{-x}N_{i,n}(x)N_{k,n}(x)-\sum_{j=1}^{n}\alpha_j(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\right]dx\right.$$
$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$
$$\left.-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0}\right]\alpha_i$$
$$=\int_0^1\left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}N_{k,n}(x)\right]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$
$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\theta_0}{dx^4}\right]_{x=0}$$
$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\theta_0}{dx^4}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=1}\times e+\left[\frac{d^4N_{k,n}(x)}{dx^4}\right]_{x=0}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e$$
$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}\times e-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0} \tag{6.37}$$

The above equation (6.37) is equivalent to matrix form

$$(D+B)A=G \tag{6.38a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{d^7N_{k,n}(x)}{dx^7}\frac{dN_{i,n}(x)}{dx}-2\theta_0e^{-x}N_{i,n}(x)N_{k,n}(x)\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}$$
$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$
$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=1}$$
$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4N_{i,n}(x)}{dx^4}\right]_{x=0} \tag{6.38b}$$

$$b_{i,k} = -\sum_{j=1}^{n} \alpha_j \int_0^1 (N_{i,n}(x) N_{j,n}(x) N_{k,n}(x)) e^{-x} \, dx \tag{6.38c}$$

$$\begin{aligned} g_k = & \int_0^1 \left[\frac{d^7 N_{k,n}(x)}{dx^7} \frac{d\theta_0}{dx} + {\theta_0}^2 e^{-x} N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^6\theta_0}{dx^6} \right]_{x=1} \\ & - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^6\theta_0}{dx^6} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^5\theta_0}{dx^5} \right]_{x=0} \\ & + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^4\theta_0}{dx^4} \right]_{x=1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^4\theta_0}{dx^4} \right]_{x=0} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=1} \times e \\ & + \left[\frac{d^4 N_{k,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=1} \times e - \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=0} \\ & - \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=1} \times e - \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=0} \end{aligned} \tag{6.38d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (6.33a). That is, to find initial coefficients we solve the system

$$DA = G \tag{6.39a}$$

whose matrices are constructed from

$$\begin{aligned} d_{i,k} = & \int_0^1 - \frac{d^7 N_{k,n}(x)}{dx^7} \frac{dN_{i,n}(x)}{dx} dx - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^6 N_{i,n}(x)}{dx^6} \right]_{x=1} \\ & + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^6 N_{i,n}(x)}{dx^6} \right]_{x=0} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^5 N_{i,n}(x)}{dx^5} \right]_{x=1} \\ & - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^5 N_{i,n}(x)}{dx^5} \right]_{x=0} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=1} \\ & + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^4 N_{i,n}(x)}{dx^4} \right]_{x=0} \end{aligned} \tag{6.39b}$$

$$g_k = \int_0^1 \frac{d^7 N_{k,n}(x)}{dx^7}\frac{d\theta_0}{dx}dx + \left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$-\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=0} + \left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^4\theta_0}{dx^4}\right]_{x=0}$$

$$-\left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^4\theta_0}{dx^4}\right]_{x=0} - \left[\frac{d^4 N_{k,n}(x)}{dx^4}\right]_{x=1} \times e + \left[\frac{d^4 N_{k,n}(x)}{dx^4}\right]_{x=0}$$

$$+\left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=1} \times e - \left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=0} - \left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=1} \times e$$

$$-\left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=0} \tag{6.39c}$$

Once the initial values of α_i are obtained from eqn. (6.39a), they are substituted into eqn. (6.38a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (6.34), we obtain an approximate solution of the BVP (6.33).

The maximum absolute errors are shown in **Table 6** with 7 iterations.

Table 6: Maximum absolute errors of example 6 using 7 iterations.

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	2.5979219E-014	1.1051709181	1.6084911E-011
0.2	1.2214027582	1.2214027580	7.0188300E-013	1.2214027582	1.3298251E-011
0.3	1.3498588076	1.3498588075	2.6412206E-012	1.3498588076	5.0746074E-011
0.4	1.4918246976	1.4918246976	5.0834892E-012	1.4918246976	7.1547213E-012
0.5	1.6487212707	1.6487212707	6.2061467E-012	1.6487212707	6.2061467E-011
0.6	1.8221188004	1.8221188004	5.0610627E-012	1.8221188004	2.9898306E-012
0.7	2.0137527075	2.0137527075	2.6179059E-012	2.0137527075	1.8429702E-013
0.8	2.2255409285	2.2255409282	6.9233508E-013	2.2255409285	6.5281114E-011
0.9	2.4596031112	2.4596031112	2.5757174E-014	2.4596031112	1.6604496E-011
1.0	2.7182818285	2.7182818285	0.0000000E+000	2.7182818285	0.0000000E+000

On the other hand the maximum absolute error has been obtained by Kasi and Raju [75] is 9.799×10^{-5}

We have shown the exact and approximate solutions in Fig. 6(a) and the relative errors in Fig. 6(b) of example 6 for $n = 12$. It is clear from Fig. 6(b) that the error is of order 10^{-12}.

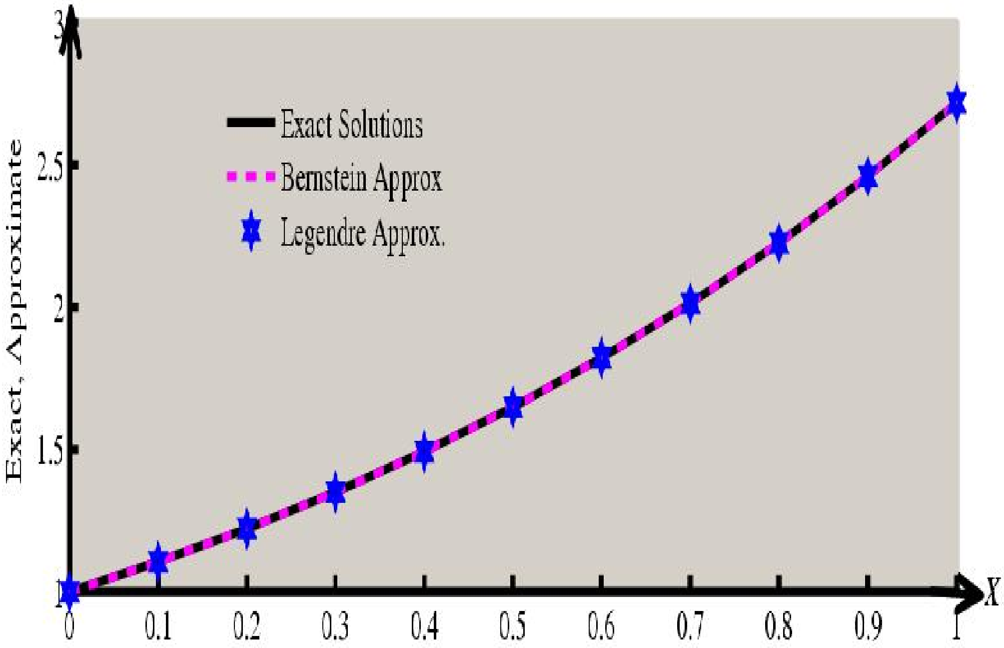


Fig. 6(a): Graphical representation of exact and approximate solutions of example 6 using 12 polynomials.

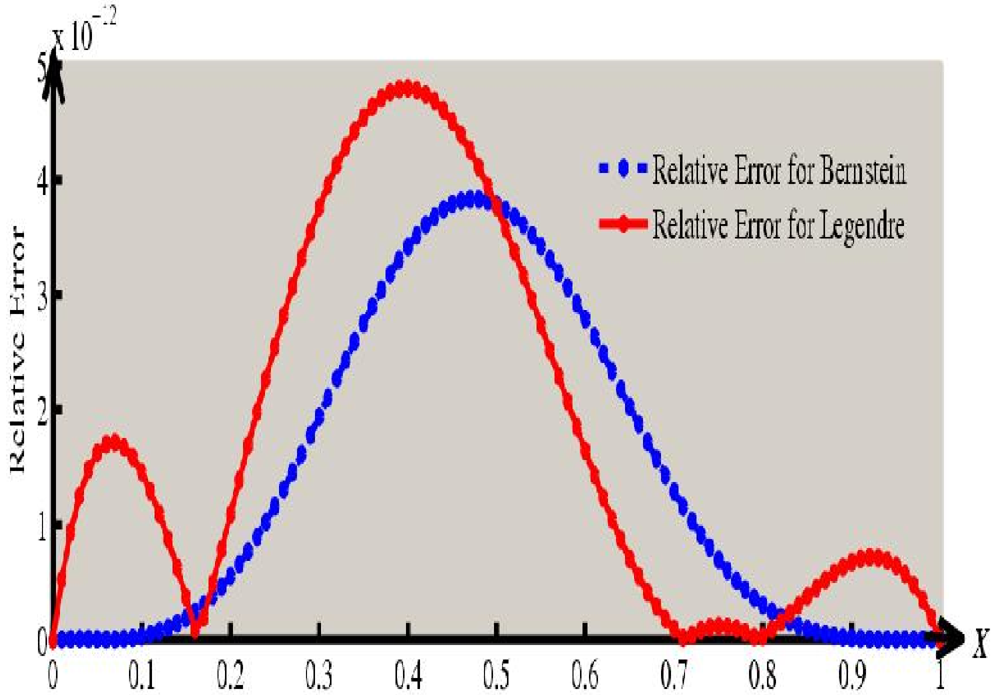


Fig. 6(b): Graphical representation of relative error of example 6 using 12 polynomials.

Example 7: Consider the nonlinear differential equation [82]

$$\frac{d^8u}{dx^8} = 7!\times e^{-8u} - 2\times 7!(1+x)^{-8},\ 0 \le x \le e^{\frac{1}{2}} - 1 \tag{6.40a}$$

with boundary conditions type II, defined in eqn. (1c)

$$u(0) = 0,\ u(e^{\frac{1}{2}} - 1) = \frac{1}{2},\ u''(0) = -1, u''(e^{\frac{1}{2}} - 1) = -e^{-1}, u^{(iv)}(0) = -6,$$

$$u^{(iv)}(e^{\frac{1}{2}} - 1) = -6e^{-2}, u^{(vi)}(0) = -120, u^{(vi)}(e^{\frac{1}{2}} - 1) = -120e^{-3}. \tag{6.40b}$$

The exact solution of this BVP is, $u(x) = \ln(1+x)$.

To use Bernstein and Legendre polynomials over [0, 1], we convert the BVP (6.40) to an equivalent BVP on [0, 1] by letting $x = (e^{\frac{1}{2}} - 1)x$. Thus the BVP (6.40) is equivalent to the BVP

$$\frac{d^8u}{dx^8} = 7!(e^{\frac{1}{2}} - 1)^8 \times e^{-8u} - 2\times 7!(e^{\frac{1}{2}} - 1)^8 (1+x)^{-8},\ 0 \le x \le 1 \tag{6.41a}$$

$$u(0) = 0,\ u(1) = \frac{1}{2},\ u''(0) = -(e^{\frac{1}{2}} - 1)^2, u''(1) = -e^{-1}(e^{\frac{1}{2}} - 1)^2, u^{(iv)}(0) = -6(e^{\frac{1}{2}} - 1)^4,$$

$$u^{(iv)}(1) = -6e^{-2}(e^{\frac{1}{2}} - 1)^4, u^{(vi)}(0) = -120(e^{\frac{1}{2}} - 1)^6, u^{(vi)}(1) = -120e^{-3}(e^{\frac{1}{2}} - 1)^6 \tag{6.41b}$$

Consider the approximate solution of $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{6.42}$$

Here $\theta_0(x) = \frac{x}{2}$ is specified by the essential boundary conditions in (6.41b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Applying eqn. (6.42) into eqn. (6.41a), the Galerkin weighted residual eqns. are

$$\int_0^1 \left[\frac{d^8\widetilde{u}}{dx^8} - 7!(e^{\frac{1}{2}} - 1)^8 \left[e^{-8\widetilde{u}} - 2\left[1 + (e^{\frac{1}{2}} - 1) \right]^{-8} \right] \right] N_{k,n}(x)dx = 0, k = 1,2,\ldots,n \tag{6.43}$$

In the same way of example 6, integrating first term of (6.43) by parts we obtain

$$\int_0^1 \frac{d^8\tilde{u}}{dx^8} N_{k,n}(x)\,dx = -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^6\tilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$

$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^2\tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^7N_{k,n}(x)}{dx^7}\frac{d\tilde{u}}{dx}dx \qquad (6.44)$$

Putting eqn. (6.44) into eqn. (6.43) and using approximation for $\tilde{u}(x)$ given in eqn. (6.42) and after applying the boundary conditions given in eqn. (6.41b) and rearranging the terms for the resulting eqns. we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^7N_{k,n}(x)}{dx^7}\frac{dN_{i,n}(x)}{dx}\right]dx + \left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}\right.$$

$$-\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$\left.-\left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}\right]\alpha_i = (-2)\times(7!)(e^{\frac{1}{2}}-1)^8\int_0^1\left[1+(e^{\frac{1}{2}}-1)x\right]^{-8}N_{k,n}(x)dx$$

$$+\int_0^1 \frac{d^7N_{k,n}(x)}{dx^7}\frac{d\theta_0}{dx}dx + (7!)(e^{\frac{1}{2}}-1)^8\int_0^1\left[e^{-8\left[\theta_0+\sum\limits_{j=1}^{n}\alpha_j N_{j,n}(x)\right]}\right]N_{k,n}(x)dx$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=0} - \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-120e^{-3}(e^{\frac{1}{2}}-1)^6) - \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-120(e^{\frac{1}{2}}-1)^6)$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(-6e^{-2}(e^{\frac{1}{2}}-1)^4)-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times(-6(e^{\frac{1}{2}}-1)^4)$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}(-e^{-1}(e^{\frac{1}{2}}-1)^2)-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-(e^{\frac{1}{2}}-1)^2) \qquad (6.45)$$

The above equation (6.45) is equivalent to matrix form

$$DA = B + G \qquad (6.46a)$$

where the elements of the square matrix D and the column matrices B and G are given by

$$d_{i,k} = \int_0^1 \left[-\frac{d^7N_{k,n}(x)}{dx^7}\frac{dN_{i,n}(x)}{dx}\right]dx + \left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (6.46b)$$

$$b_k = (7!)(e^{\frac{1}{2}}-1)^8\int_0^1\left[e^{-8\left[\theta_0+\sum_{j=1}^{n}\alpha_j N_{j,n}(x)\right]}\right]N_{k,n}(x)dx \qquad (6.46c)$$

$$g_k = \int_0^1\left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d\theta_0}{dx} - 2\times(7!)(e^{\frac{1}{2}}-1)^8\left[1+(e^{\frac{1}{2}}-1)x\right]^{-8}N_{k,n}(x)\right]dx$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-120e^{-3}(e^{\frac{1}{2}}-1)^6)-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-120(e^{\frac{1}{2}}-1)^6)$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(-6e^{-2}(e^{\frac{1}{2}}-1)^4)-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times(-6(e^{\frac{1}{2}}-1)^4)$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}(-e^{-1}(e^{\frac{1}{2}}-1)^2)-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-(e^{\frac{1}{2}}-1)^2) \qquad (6.46d)$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (6.41a). That is, to find initial coefficients we solve the system

$$DA = G \qquad (6.47a)$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1\left[-\frac{d^7N_{k,n}(x)}{dx^7}\frac{dN_{i,n}(x)}{dx}\right]dx+\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[N_{k,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{k,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^6}{dx^6}\left[N_{k,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (6.47b)$$

$$g_k=\int_0^1\left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d\theta_0}{dx}-2\times(7!)(e^{\frac{1}{2}}-1)^8\left[1+(e^{\frac{1}{2}}-1)x\right]^{-8}N_{k,n}(x)\right]dx$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times(-120e^{-3}(e^{\frac{1}{2}}-1)^6)-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-120(e^{\frac{1}{2}}-1)^6)$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times(-6e^{-2}(e^{\frac{1}{2}}-1)^4)-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times(-6(e^{\frac{1}{2}}-1)^4)$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}(-e^{-1}(e^{\frac{1}{2}}-1)^2)-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-(e^{\frac{1}{2}}-1)^2) \qquad (6.47c)$$

Once the initial values of the coefficients α_i are obtained from eqn. (6.47a), they are substituted into eqn. (6.46a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (6.42), we obtain an approximate solution of the BVP (6.41). If we replace x by $(e^{\frac{1}{2}}-1)x$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (6.40).

The maximum absolute errors, using different number of polynomials by the present method with 5 iterations and the previous results obtained so far, are summarized in **Table 7.**

Table 7: Maximum absolute errors of example 7 using 5 iterations.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
8	6.910×10^{-7}	5.915×10^{-7}	4.44×10^{-8} (Djidjeli *et al* [82])
9	8.918×10^{-8}	7.905×10^{-8}	
10	9.875×10^{-9}	9.860×10^{-9}	
11	5.795×10^{-10}	9.980×10^{-10}	

The exact and approximate solutions are depicted in Fig. 7(a) and a plot of the relative errors are shown in Fig. 7(b) of example 7 for $n=11$. We observed from Fig. 7(b) that the error is of the order 10^{-6}.

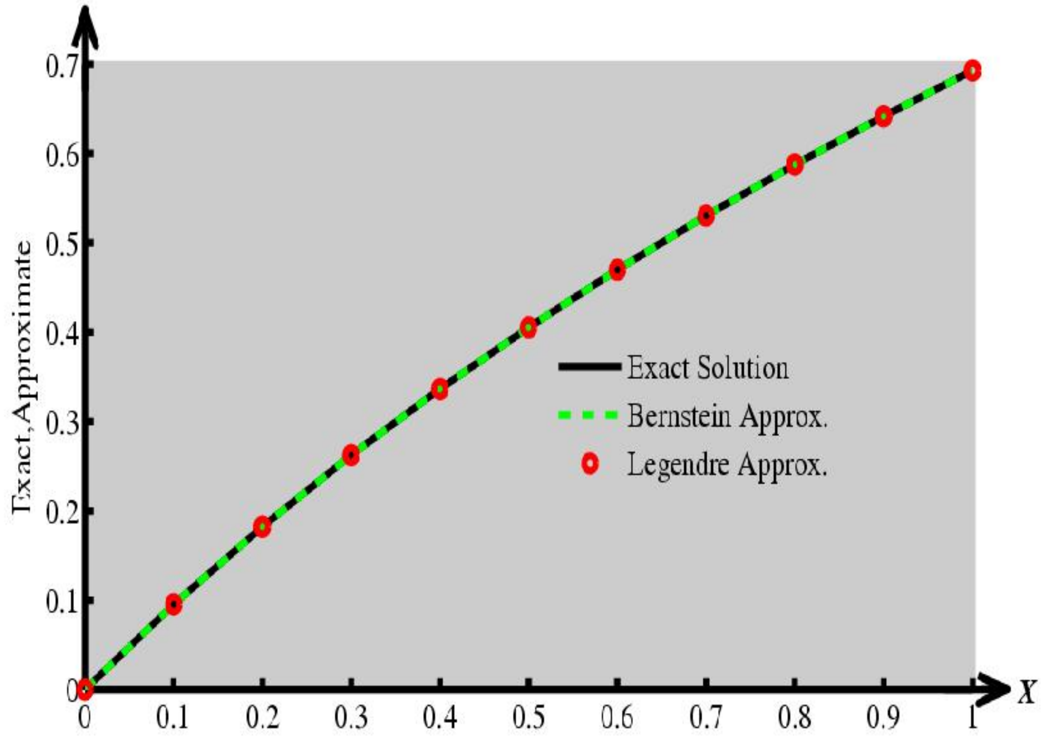


Fig. 7(a): Graphical representation of exact and approximate solutions of example 7 using 11 polynomials.

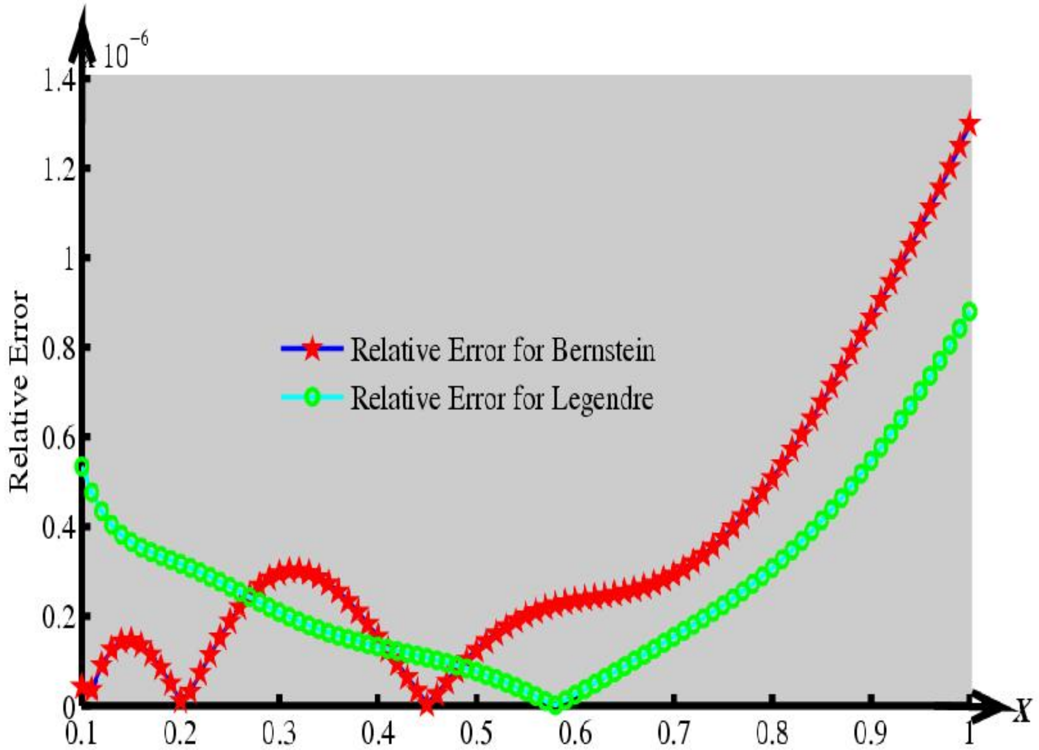


Fig.7 (b): Graphical representation of relative error of example 7 using 11 polynomials.

6.4 Conclusions

This chapter has been considered to find the numerical solutions of the general eighth order linear and nonlinear differential equations using Bernstein and Legendre polynomials as basis functions in the Galerkin method with two different types of boundary conditions. From the tables we see that the numerical results obtained by our method are better than the other existing methods. In addition, Bernstein polynomials yield the better results than the Legendre polynomials. We may also notice that the numerical solutions coexist with the exact solution even lower order Bernstein and Legendre polynomials are used in the approximation which are shown in Figs. [1-7].

CHAPTER 7

Ninth Order Boundary Value Problems

7.1 Introduction

It is observed that the ninth order BVPs are known to arise in hydrodynamic, hydro magnetic stability and applied sciences. The literature of numerical analysis contains little on the solution of the ninth order BVPs. The existence and uniqueness theorem of solutions of BVP was presented in a book written by Agarwal [8] which does not contain any numerical examples. Recently, the BVPs of ninth order have been developed due to their mathematical importance and the potential for applications in hydrodynamic, hydro magnetic stability. Few techniques including finite-difference, polynomial and nonpolynomial spline, homotopy perturbation and decomposition have been used to solve such problems [82 – 86]. Most of these techniques used so far are well known that they provide the solution only at grid points. The modified variational iteration and homotopy perturbation methods have been applied for solving tenth and ninth order BVPs and twelfth order BVPs by Mohy-ud-Din and Yildirim [87] and Mohamed Othman *et al* [88] respectively. Nadjafi and Zahmatkesh [89] also investigated the homotopy perturbation method for solving higher order BVPs.

The aim of this chapter is to apply Galerkin weighted residual method for solving ninth order BVPs. In this method, we exploit Bernstein and Legendre polynomials as basis functions which are modified into to a new set of basis functions to satisfy the corresponding homogeneous boundary conditions where the essential types of boundary conditions are mentioned. The method is formulated as a rigorous matrix form.

Moreover, the formulation for solving linear ninth order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials is presented in the portion 7.2. Then one numerical example of linear BVP is considered to verify the proposed formulation and the solution is compared with the existing methods. We have given the conclusions of this chapter in the last portion 9.4.

7.2 Formulation using Galerkin method

In this section, we employed the Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions for the numerical solution of a general ninth order linear BVP of the following form:

$$a_9 \frac{d^9u}{dx^9} + a_8 \frac{d^8u}{dx^8} + a_7 \frac{d^7u}{dx^7} + a_6 \frac{d^6u}{dx^6} + a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2}$$

$$+ a_1 \frac{du}{dx} + a_0 u = r,\ a < x < b \tag{7.1a}$$

subject to the boundary conditions

$$u(a) = A_0, \quad u(b) = B_0, \quad u'(a) = A_1, \quad u'(b) = B_1,$$

$$u''(a) = A_2, \quad u''(b) = B_2, \quad u'''(a) = A_3, \quad u'''(b) = B_3,$$

$$u^{(iv)}(a) = A_4 \tag{7.1b}$$

Where $A_i, i = 0,1,2,3,4$ and $B_j, j = 0,1,2,3$ are finite real constants and $a_i, i = 0,1,\cdots 9$ and r are all continuous functions defined on the interval [a, b]. The BVP (7.1) is solved with the boundary conditions of eqn (7.1b).

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (7.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x + a$, and thus we have:

$$c_9 \frac{d^9u}{dx^9} + c_8 \frac{d^8u}{dx^8} + c_7 \frac{d^7u}{dx^7} + c_6 \frac{d^6u}{dx^6} + c_5 \frac{d^5u}{dx^5} + c_4 \frac{d^4u}{dx^4} + c_3 \frac{d^3u}{dx^3} + c_2 \frac{d^2u}{dx^2}$$

$$+ c_1 \frac{du}{dx} + c_0 u = s,\ 0 < x < 1 \tag{7.2a}$$

$$u(0) = A_0, \quad \frac{1}{b-a} u'(0) = A_1, \quad \frac{1}{(b-a)^2} u''(0) = A_2,$$

$$u(1) = B_0, \quad \frac{1}{b-a} u'(1) = B_1, \quad \frac{1}{(b-a)^2} u''(1) = B_2,$$

$$\frac{1}{(b-a)^3} u'''(0) = A_3, \quad \frac{1}{(b-a)^3} u'''(1) = B_3, \quad \frac{1}{(b-a)^4} u^{(iv)}(0) = A_4 \tag{7.2b}$$

where

$$c_9 = \frac{1}{(b-a)^9} a_9((b-a)x+a), \qquad c_8 = \frac{1}{(b-a)^8} a_8((b-a)x+a),$$

$$c_7 = \frac{1}{(b-a)^7} a_7((b-a)x+a), \qquad c_6 = \frac{1}{(b-a)^6} a_6((b-a)x+a),$$

$$c_5 = \frac{1}{(b-a)^5} a_5((b-a)x+a), \qquad c_4 = \frac{1}{(b-a)^4} a_4((b-a)x+a),$$

$$c_3 = \frac{1}{(b-a)^3} a_3((b-a)x+a), \qquad c_2 = \frac{1}{(b-a)^2} a_2((b-a)x+a),$$

$$c_1 = \frac{1}{b-a} a_1((b-a)x+a), \qquad c_0 = a_0((b-a)x+a),$$

$$s = r((b-a)x+a)$$

For solving the boundary value problem (7.2) by the Galerkin weighted residual method we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \ge 1 \tag{7.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0,$ for each $i = 1,2,3,\ldots,n.$

Putting eqn. (7.3) into eqn. (7.2a), the weighted residual equations are

$$\int_0^1 \left[c_9 \frac{d^7\widetilde{u}}{dx^7} + c_8 \frac{d^7\widetilde{u}}{dx^7} + c_7 \frac{d^7\widetilde{u}}{dx^7} + c_6 \frac{d^6\widetilde{u}}{dx^6} + c_5 \frac{d^5\widetilde{u}}{dx^5} + c_4 \frac{d^4\widetilde{u}}{dx^4} + c_3 \frac{d^3\widetilde{u}}{dx^3} + c_2 \frac{d^2\widetilde{u}}{dx^2} \right.$$

$$\left. + c_1 \frac{d\widetilde{u}}{dx} + c_0\widetilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{7.4}$$

Integrating by parts the terms up to second derivative on the left hand side of (7.4), we have

$$\int_0^1 c_9 \frac{d^9\widetilde{u}}{dx^9} N_{j,n}(x)dx = \left[c_9 N_{j,n}(x) \frac{d^8\widetilde{u}}{dx^8} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_9 N_{j,n}(x)\right] \frac{d^8\widetilde{u}}{dx^8} dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right] \frac{d^7\widetilde{u}}{dx^7} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right] \frac{d^7\widetilde{u}}{dx^7} dx \; [\text{Since } N_{j,n}(0) = N_{j,n}(1) = 0]$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$-\int_0^1\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_9N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{7.5}$$

$$\int_0^1 c_8 \frac{d^8\widetilde{u}}{dx^8} N_{j,n}(x)dx = \left[c_8 N_{j,n}(x) \frac{d^7\widetilde{u}}{dx^7} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^7\widetilde{u}}{dx^7} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right] \frac{d^6\widetilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right] \frac{d^5\widetilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right] \frac{d^4\widetilde{u}}{dx^4} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right] \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right] \frac{d^2\widetilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right] \frac{d\widetilde{u}}{dx} \right]_0^1$$

$$- \int_0^1 \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] \frac{d\widetilde{u}}{dx} dx \qquad (7.6)$$

$$\int_0^1 c_7 \frac{d^7\tilde{u}}{dx^7} N_{j,n}(x)dx = \left[c_7 N_{j,n}(x) \frac{d^6\tilde{u}}{dx^6} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^6\tilde{u}}{dx^6} dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (7.7)$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x) \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^5\tilde{u}}{dx^5} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\,dx \tag{7.8}$$

$$\int_0^1 c_5 \frac{d^5\widetilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x)\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\,dx \tag{7.9}$$

$$\int_0^1 c_4 \frac{d^4\widetilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x)\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\,dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\,dx \tag{7.10}$$

$$\int_0^1 c_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x) \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx} dx \tag{7.11}$$

$$\int_0^1 c_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x) \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx} dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx} dx \tag{7.12}$$

Substituting eqns. (7.5) – (7.12) into eqn. (7.4) and using approximation for $\tilde{u}(x)$ given in eqn. (7.3) and after applying the boundary conditions given in eqn. (7.2b) and rearranging the terms for the resulting equations we get a system of eqns. in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j,\ j = 1,2,\ldots,n \tag{7.13a}$$

where

$$D_{i,j} = \int_0^1 \left\{ \left[\frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] - \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] \right.\right.$$

$$\left. + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] + c_1 N_{j,n}(x) \right]$$

$$\left. \times \frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x) \right\} dx - \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$+ \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$- \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1} \qquad (7.13b)$$

$$F_j=\int_0^1\Bigg\{sN_{j,n}(x)+\Bigg[-\frac{d^8}{dx^8}\left[c_9N_{j,n}(x)\right]+\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right]$$

$$+\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]$$

$$+\frac{d}{dx}\left[c_2N_{j,n}(x)\right]-c_1N_{j,n}(x)\Bigg]\frac{d\theta_0}{dx}-c_0\theta_0N_{j,n}(x)\Bigg\}dx+\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} -\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1} +\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\theta_0}{dx^4}\right]_{x=1}$$

$$+\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$-\left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1+\left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1-\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3$$

$$+\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1+\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1$$

$$+\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1$$

$$-\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3$$

$$+\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2$$

$$-\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1 + \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1$$

$$+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3 - \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3$$

$$-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2$$

$$+\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1$$

$$+\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2 - \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1$$

$$+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1 - \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 \qquad (7.13c)$$

Solving the system (7.13a), we find the values of the parameters α_i and then substituting these parameters into eqn. (7.3), we get the approximate solution of the BVP (7.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (7.1).

7.3 Numerical examples and results

To test the applicability of the proposed method, we consider one linear problem which is available in the literature only so far. For the example, the solution obtained by the proposed method is compared with the exact solution. All the calculations are performed by ***MATLAB*** **10**. The convergence of linear BVP is calculated by

$$E = \left|\widetilde{u}_{n+1}(x) - \widetilde{u}_n(x)\right| < \delta$$

where $\widetilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-12}.

Example 1: Consider the following ninth order linear differential equation [86, 87, 88, 89]

$$\frac{d^9u}{dx^9} = u - 9e^x, \ \ 0 \le x \le 1 \tag{7.14a}$$

subject to the boundary conditions

$$u(0) = 1,\ u(1) = 0,\ u'(0) = 0, u'(1) = -e, u''(0) = -1, u''(1) = -2e, u'''(0) = -2,$$
$$u'''(1) = -3e, u^{(iv)}(0) = -3. \tag{7.14b}$$

The analytic solution of the above system is, $u(x) = (1-x)e^x$.

Employing the method given in (7.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ \ n \ge 1 \tag{7.15}$$

Here $\theta_0(x) = 1 - x$ is specified by the essential boundary conditions of equation (7.14b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \tag{7.16a}$$

where

$$\begin{aligned} D_{i,j} = &\int_0^1 \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - N_{i,n}(x)N_{j,n}(x) \right]dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\ &+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} \\ &- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\ &+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^4}{dx^4}\left[N_{i,n}(x)\right]\right]_{x=1} \end{aligned} \tag{7.16b}$$

$$F_j = \int_0^1 \left[-9e^x N_{j,n}(x) - \frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d\theta_0}{dx} + \theta_0 N_{j,n}(x) \right] dx$$

$$+ \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-3) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-3e)$$

$$- \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-2) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-2e)$$

$$+ \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=0} \times (-1) + \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right] \right]_{x=1} \times (-e) \qquad (7.16c)$$

Solving the system (7.16a) we obtain the values of the parameters and then substituting these parameters into eqn. (7.15), we get the approximate solution of the BVP (7.14).

The numerical results for this problem are summarized in **Table 1**.

Table 1: Maximum absolute errors for the example 1.

x	Exact Results	12 Bernstein Polunomials		12 Legendre Polunomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	1.5543122E-015	0.9946538263	1.4432899E-015
0.2	0.9771222065	0.9771222065	3.9968029E-015	0.9771222065	3.2196468E-015
0.3	0.9449011653	0.9449011653	3.6637360E-015	0.9449011653	4.5519144E-015
0.4	0.8950948186	0.8950948186	4.7739590E-015	0.8950948186	4.8849813E-015
0.5	0.8243606354	0.8243606354	2.2204460E-015	0.8243606354	3.1086245E-015
0.6	0.7288475202	0.7288475202	4.6629367E-015	0.7288475202	5.6621374E-015
0.7	0.6041258122	0.6041258122	0.0000000E+000	0.6041258122	4.4408921E-016
0.8	0.4451081857	0.4451081857	3.1641356E-015	0.4451081857	3.1641356E-015
0.9	0.2459603111	0.2459603111	3.3306691E-016	0.2459603111	4.4408921E-016
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand, it is observed that the accuracy is found nearly the order 10^{-10} in [86], [87] by Wazwaz; Mohy-ud-Din and Ahmed Yildirim and nearly the order 10^{-9} in [88], [89] by Mohamed Othman *et al*; Nadjafi and Zahmatkesh respectively.

In Fig. 1(a), the exact and approximate solutions are given and a plot of relative errors are shown in Fig. 1(b) of example 1 for $n = 12$. We see from the Fig. 1(b) that the error is nearly the order 10^{-13}

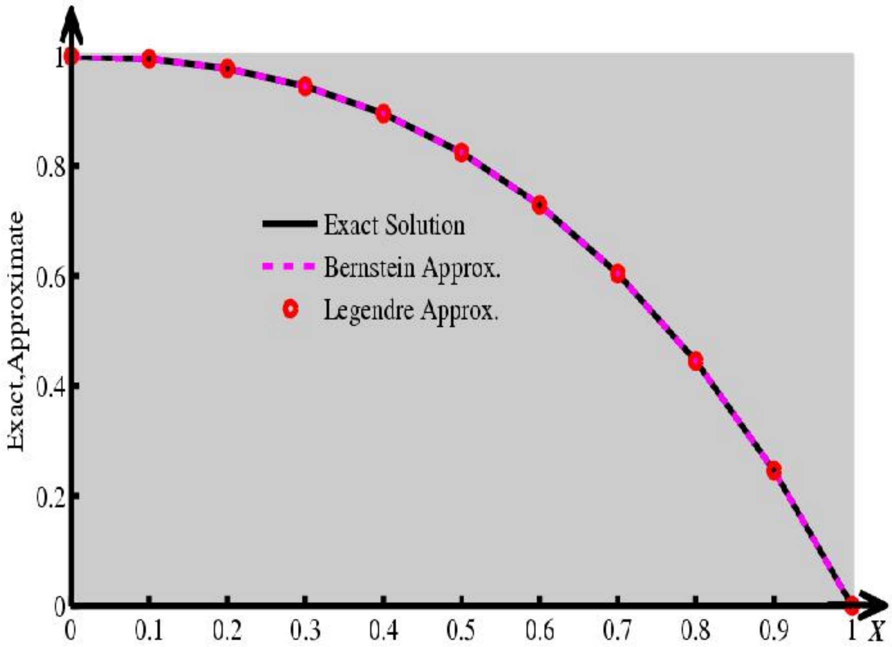


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 12 polynomials.

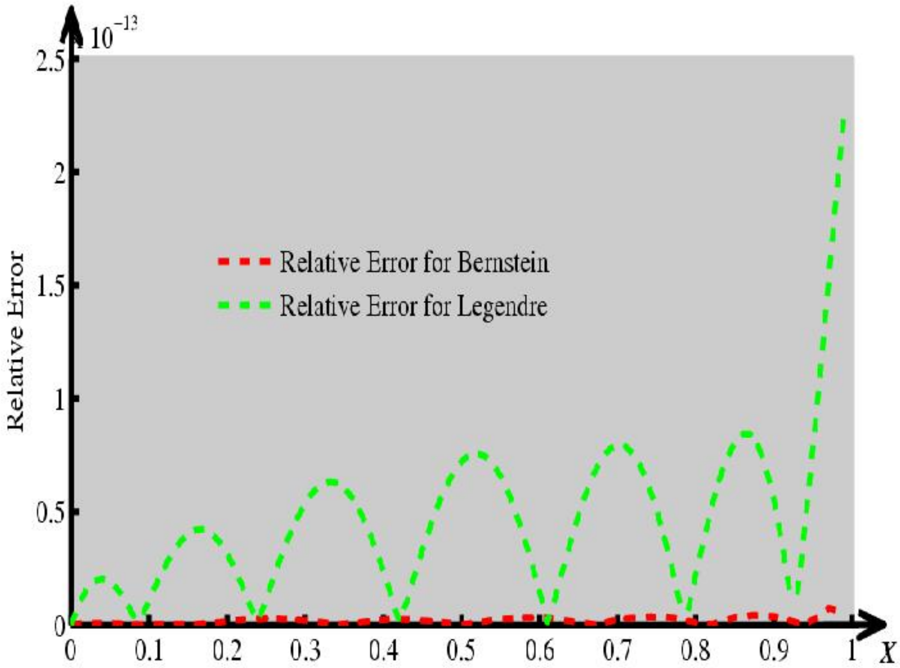


Fig. 1(b): Graphical representation of relative error of example 1 using 12 polynomials.

7.4 Conclusions

In this chapter, we have used Bernstein and Legendre, the piecewise continuous and differentiable polynomials as basis functions for the numerical solution of ninth order linear BVP in the Galerkin method. We find from the table that the numerical results obtained by our method are superior to other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. We claim that any ninth order BVP can be solved with high accuracy using the method discussed in this chapter.

CHAPTER 8

Tenth Order Boundary Value Problems

8.1 Introduction

In the literature of BVPs we observe that the higher order differential equations arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Particularly eighth, tenth and more even higher order BVPs arise in hydro magnetic stability analysis. We can also find from a book written by Chandrasekhar [9] that when an infinite horizontal layer of fluid is heated from below with the assumption that a uniform magnetic field is also used across in the same direction as gravity and the fluid is under the action of rotation, instability sets in. When instability sets in as ordinary convection, the ordinary differential equation is tenth order. The existence and uniqueness theorem of solution of BVP was presented in a book by Agarwal [8] without any numerical examples. Finite difference methods for the solution of such problems were developed by Boutayeb and Twizell [69], Twizell et *al* [70] and Djidjeli *et al* [82]. Siddiqi and Twizell developed spline solutions for the linear sixth order BVP in [90] and for the eighth order problem in [71]. Usmani [13] used quartic splines for the numerical solution of fourth order BVP. Ghazala and Siddiqi [72] applied nonic spline solution technique for eighth order BVPs. From the literature we observe that the tenth order BVP has been attempted to solve numerically by a few researchers, namely, Siddiqi and Twizell [91] solved tenth order BVPs using tenth degree spline where some unexpected results for the solution and higher order derivatives were obtained near the boundaries of the interval and Siddiqi and Ghazala [60] presented the solutions of tenth order BVPs by eleventh degree spline. Muhammad Aslam Noor *et al* [94] derived reliable algorithm for solving tenth order BVPs using variational iterative method. Fazhan and Xiuying [95] used variational iteration method for the numerical solution of tenth order BVPs. The modified decomposition method has been used extensively by Wazwaz [80] for the approximate solutions to BVPs of higher order. Inayat Ullah *et al* [96] presented the numerical solutions of higher order nonlinear BVPs by new iterative method.

In this chapter, Galerkin method with Bernstein and Legendre polynomials as basis functions is applied for the numerical solution of tenth order linear and nonlinear BVPs for two different types of boundary conditions. In this method, the Bernstein and Legendre polynomial basis functions are modified into a new set of basis functions where the *Dirichlet* type of boundary conditions are presented and a matrix formulation is derived for solving the tenth order BVPs. Numerical results of the method are tabulated to compare the errors with those developed previous.

The formulation for solving linear tenth order BVP by Galerkin weighted residual method with Bernstein and Legendre polynomials is described section 8.2. Two formulations are described considering two types of boundary conditions in sections 8.2.1 and 8.2.2 respectively. Then several numerical examples are given to verify the proposed formulations in section 8.3 and the conclusions of this chapter are mentioned in the last portion 8.4.

8.2 Description of the Method

In the present chapter, we extend Galerkin method with Bernstein and Legendre polynomials as basis functions for the numerical solution of a general tenth order linear BVP given by

$$a_{10}\frac{d^{10}u}{dx^{10}}+a_9\frac{d^9u}{dx^9}+a_8\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}$$

$$+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r\ ,\ a<x<b \qquad (8.1a)$$

subject to the following two types of boundary conditions

TypeI: $u(a)=A_0,\quad u(b)=B_0,\quad u'(a)=A_1,\quad u'(b)=B_1,$

$u''(a)=A_2,\quad u''(b)=B_2,\quad u'''(a)=A_3,\quad u'''(b)=B_3,$

$u^{(iv)}(a)=A_4,\quad u^{(iv)}(b)=B_4 \qquad (8.1b)$

TypeII: $u(a)=A_0,\quad u(b)=B_0,\quad u''(a)=A_2,\quad u''(b)=B_2,$

$$u^{(iv)}(a) = A_4, \qquad u^{(iv)}(b) = B_4, \qquad u^{(vi)}(a) = A_6, \qquad u^{(vi)}(b) = B_6,$$

$$u^{(viii)}(a) = A_8, \qquad u^{(viii)}(b) = B_8 \tag{8.1c}$$

where $A_i, B_i, i = 0,1,2,3,4,6,8$ are finite real constants and $a_i, i = 0,1,\cdots 10$ and r are all continuous functions defined on the interval [*a,b*]. The BVP (8.1) is solved with both the boundary conditions of type I and type II.

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (8.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$c_{10}\frac{d^{10}u}{dx^{10}} + c_9\frac{d^9u}{dx^9} + c_8\frac{d^8u}{dx^8} + c_7\frac{d^7u}{dx^7} + c_6\frac{d^6u}{dx^6} + c_5\frac{d^5u}{dx^5} + c_4\frac{d^4u}{dx^4} + c_3\frac{d^3u}{dx^3}$$

$$+ c_2\frac{d^2u}{dx^2} + c_1\frac{du}{dx} + c_0 u = s, \; 0 < x < 1 \tag{8.2a}$$

$$u(0) = A_0, \qquad \frac{1}{b-a}u'(0) = A_1, \qquad \frac{1}{(b-a)^2}u''(0) = A_2,$$

$$u(1) = B_0, \qquad \frac{1}{b-a}u'(1) = B_1, \qquad \frac{1}{(b-a)^2}u''(1) = B_2,$$

$$\frac{1}{(b-a)^3}u'''(0) = A_3, \qquad \frac{1}{(b-a)^3}u'''(1) = B_3, \qquad \frac{1}{(b-a)^4}u^{(iv)}(0) = A_4,$$

$$\frac{1}{(b-a)^4}u^{(iv)}(1) = B_4 \tag{8.2b}$$

and

$$u(0) = A_0, \qquad \frac{1}{(b-a)^2}u''(0) = A_2, \qquad \frac{1}{(b-a)^4}u^{(iv)}(0) = A_4,$$

$$u(1) = B_0, \qquad \frac{1}{(b-a)^2}u''(1) = B_2, \qquad \frac{1}{(b-a)^4}u^{(iv)}(1) = B_4,$$

$$\frac{1}{(b-a)^6}u^{(vi)}(0) = A_6, \qquad \frac{1}{(b-a)^6}u^{(vi)}(1) = B_6, \qquad \frac{1}{(b-a)^8}u^{(viii)}(0) = A_8,$$

$$\frac{1}{(b-a)^8}u^{(viii)}(1) = B_8 \tag{8.2c}$$

where

$$c_{10} = \frac{1}{(b-a)^{10}} a_{10}((b-a)x+a), \qquad c_9 = \frac{1}{(b-a)^9} a_9((b-a)x+a),$$

$$c_8 = \frac{1}{(b-a)^8} a_8((b-a)x+a), \qquad c_7 = \frac{1}{(b-a)^7} a_7((b-a)x+a),$$

$$c_6 = \frac{1}{(b-a)^6} a_6((b-a)x+a), \qquad c_5 = \frac{1}{(b-a)^5} a_5((b-a)x+a),$$

$$c_4 = \frac{1}{(b-a)^4} a_4((b-a)x+a), \qquad c_3 = \frac{1}{(b-a)^3} a_3((b-a)x+a),$$

$$c_2 = \frac{1}{(b-a)^2} a_2((b-a)x+a), \qquad c_1 = \frac{1}{b-a} a_1((b-a)x+a),$$

$$c_0 = a_0((b-a)x+a), \qquad s = r((b-a)x+a)$$

For the numerical solution of boundary value problem (8.2) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \ge 1 \tag{8.3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,3,\ldots,n.$

Putting eqn. (8.3) into eqn. (8.2a), the weighted residual equations are

$$\int_0^1 \left[c_{10} \frac{d^{10}\tilde{u}}{dx^{10}} + c_9 \frac{d^9\tilde{u}}{dx^9} + c_8 \frac{d^8\tilde{u}}{dx^8} + c_7 \frac{d^7\tilde{u}}{dx^7} + c_6 \frac{d^6\tilde{u}}{dx^6} + c_5 \frac{d^5\tilde{u}}{dx^5} + c_4 \frac{d^4\tilde{u}}{dx^4} + c_3 \frac{d^3\tilde{u}}{dx^3} \right.$$

$$\left. + c_2 \frac{d^2\tilde{u}}{dx^2} + c_1 \frac{d\tilde{u}}{dx} + c_0\tilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{8.4}$$

8.2.1 Formulation I

In this portion, we have derived the matrix formulation by applying the boundary conditions of type I.

Integrating by parts the terms up to second derivative on the left hand side of (8.4), we get

$$\int_0^1 c_{10} \frac{d^{10}\tilde{u}}{dx^{10}} N_{j,n}(x)dx = \left[c_{10} N_{j,n}(x) \frac{d^9\tilde{u}}{dx^9} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^9\tilde{u}}{dx^9}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}dx \text{ [Since } N_{j,n}(0) = N_{j,n}(1) = 0 \text{]}$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1$$
$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3} \right]_0^1$$
$$- \int_0^1 \frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6} \right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.5)$$

$$\int_0^1 c_9\frac{d^9\widetilde{u}}{dx^9}N_{j,n}(x)dx=\left[c_9N_{j,n}(x)\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$-\int_0^1 \frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.6)$$

$$\int_0^1 c_8 \frac{d^8\widetilde{u}}{dx^8} N_{j,n}(x)dx = \left[c_8 N_{j,n}(x)\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{8.7}$$

$$\int_0^1 c_7 \frac{d^7\widetilde{u}}{dx^7} N_{j,n}(x)dx = \left[c_7 N_{j,n}(x)\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (8.8)$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (8.9)$$

$$\int_0^1 c_5 \frac{d^5\widetilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x) \frac{d^4\widetilde{u}}{dx^4} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx} \right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.10)$$

$$\int_0^1 c_4 \frac{d^4\widetilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x) \frac{d^3\widetilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.11)$$

$$\int_0^1 c_3 \frac{d^3\widetilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x) \frac{d^2\widetilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.12)$$

$$\int_0^1 c_2 \frac{d^2\widetilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x) \frac{d\widetilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (8.13)$$

Substituting eqns. (8.5) – (8.13) into eqn (8.4) and using approximation for $\tilde{u}(x)$ given in equation (8.3) and after applying the boundary conditions given in eqn. (8.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j\,,\; j = 1,2,\ldots,n \tag{8.14a}$$

where

$$\begin{aligned}
D_{i,j} = \int_0^1 \Bigg\{ \Bigg[& -\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] + \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] - \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] \\
& -\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \\
& + c_1 N_{j,n}(x)\Bigg]\frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x) \Bigg\} dx - \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& + \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& - \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& + \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& - \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& + \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& - \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
& + \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}
\end{aligned}$$

$$+\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{8.14b}$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right]-\frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right]+\frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]\right.\right.$$

$$+\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2 N_{j,n}(x)\right]$$

$$\left.\left.-c_1 N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0 N_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=1}-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=0}$$

$$-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}-\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d}{dx}\left[c_7(x) N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}$$

$$+\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$+\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)A_1$$

$$-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$+\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$-\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$+\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3+\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$+\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2-\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)A_1$$

$$-\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$+\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$-\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2+\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$+\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1-\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$+\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$-\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$+\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2-\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$+\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$-\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$+\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$+\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)A_1$$

$$+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1 \qquad (8.14c)$$

Solving the system (8.14a), we find the values of the parameters α_i and then substituting these parameters into eqn. (8.3), we get the approximate solution of the BVP (8.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (8.1).

8.2.2 Formulation II

In this section, we have used the boundary conditions of type II for obtaining the matrix formulation.

In the same way of section (8.2.1), integrating by parts the terms up to second derivative on the left hand side of (8.4), and after applying the boundary conditions prescribed in type II, eqn (2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\; j = 1,2,\ldots,n \tag{8.15a}$$

where

$$\begin{aligned}
D_{i,j} = \int_0^1 \Bigg\{ \Bigg[&-\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] + \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] - \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] \\
&-\frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \\
&+ c_1 N_{j,n}(x)\Bigg]\frac{d}{dx}\left[N_{i,n}(x)\right] + c_0 N_{i,n}(x) N_{j,n}(x)\Bigg\} dx + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&-\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+\left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}
\end{aligned}$$

$$+\left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (8.15b)$$

$$F_j = \int_0^1 \left\{ sN_{j,n}(x) + \left[\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] - \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] + \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] - \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] \right.\right.$$

$$\left.\left. + \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] - \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] \right] + \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] - \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] + \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \right.$$

$$\left. - c_1 N_{j,n}(x)\frac{d\theta_0}{dx} - c_0\theta_0 N_{j,n}(x) \right\} dx + - \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=0}$$

$$- \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=0} - \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1}$$

$$+ \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=1} + \left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=0}$$

$$+ \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1} - \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1}$$

$$+ \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=0} + \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0}$$

$$+ \left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=1} - \left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=0} - \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1} + \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0}$$

$$- \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=1} + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=0} + \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0}$$

$$+ \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=1} - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=1}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=1} + \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\theta_0}{dx} \right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1} -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0} +\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0} -\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} +\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1} -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0} +\left[\frac{d}{dx}\left[c_{10} N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^8 B_8$$

$$-\left[\frac{d}{dx}\left[c_{10} N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^8 A_8+\left[\frac{d^3}{dx^3}\left[c_{10} N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6$$

$$-\left[\frac{d^3}{dx^3}\left[c_{10} N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6+\left[\frac{d^5}{dx^5}\left[c_{10} N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d^5}{dx^5}\left[c_{10} N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^7}{dx^7}\left[c_{10} N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10} N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6$$

$$+\left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6-\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$+\left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4-\left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6$$

$$-\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6+\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$+\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4 - \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2 + \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4 + \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2 + \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2 \qquad (8.15c)$$

Solving the system (8.15a), we find the values of the parameters α_i and then substituting these parameters into eqn. (8.3), we get the approximate solution of the BVP (8.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (8.1).

For nonlinear tenth-order BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (8.14) and (8.15). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

8.3 Numerical examples and results

In this section, we consider five linear and two nonlinear problems consisting of both types of boundary conditions. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB* 10**. The convergence of linear BVP is calculated by

$$E = \left|\widetilde{u}_{n+1}(x) - \widetilde{u}_n(x)\right| < \delta$$

where $\widetilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which varies from 10^{-11} to 10^{-13}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\widetilde{u}_n^{N+1} - \widetilde{u}_n^{N}\right| < \delta$$

where δ is less than 10^{-12} and N is the Newton's iteration number.

Example 1: Consider the linear differential equation [60, 93]

$$\frac{d^{10}u}{dx^{10}} - (x^2 - 2x)u = 10\cos x - (x-1)^3 \sin x, \quad -1 \le x \le 1 \tag{8.16a}$$

subject to the boundary conditions of type I in eqn. (1b):

$$u(-1) = 2\sin 1,\ u(1) = 0,\ u'(-1) = -2\cos 1 - \sin 1, u'(1) = \sin 1, u''(-1) = 2\cos 1 - 2\sin 1,$$

$$u''(1) = 2\cos 1 u'''(-1) = 2\cos 1 + 3\sin 1, u'''(1) = -3\sin 1, u^{(iv)}(-1) = -4\cos 1 + 2\sin 1,$$

$$u^{(iv)}(1) = -4\cos 1\,. \tag{8.16b}$$

The analytic solution of the above problem is, $u(x) = (x-1)\sin x$.

The equivalent BVP over [0, 1] to the BVP (8.16) is,

$$\frac{1}{2^{10}}\frac{d^{10}u}{dx^{10}} - ((2x-1)^2 - 2(2x-1))u = 10\cos(2x-1) - ((2x-1)-1)^3 \sin(2x-1), 0 < x < 1 \tag{8.17a}$$

$$u(0) = 2\sin 1, u(1) = 0,\ \frac{1}{2}u'(0) = -2\cos 1 - \sin 1, \frac{1}{2}u'(1) = \sin 1, \frac{1}{4}u''(0) = 2\cos 1 - 2\sin 1,$$

$$\frac{1}{4}u''(1) = 2\cos 1, \frac{1}{8}u'''(0) = 2\cos 1 + 3\sin 1, \frac{1}{8}u'''(1) = -3\sin 1, \frac{1}{16}u^{(iv)}(0) = -4\cos 1 + 2\sin 1,$$

$$\frac{1}{16}u^{(iv)}(1) = -4\cos 1 \tag{8.17b}$$

Using the method illustrated in (8.2.1), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{8.18}$$

Here $\theta_0(x)=(1-x)\times 2\sin 1$ is specified by the essential boundary conditions of equation (8.17b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j=1,2,\ldots,n \tag{8.19a}$$

where

$$\begin{aligned}
D_{i,j} &= \int_0^1 \left[-\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 2^{10}((2x-1)^2-2(2x-1))\,N_{i,n}(x)N_{j,n}(x) \right] dx \\
&\quad - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&\quad + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&\quad - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&\quad + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}
\end{aligned} \tag{8.19b}$$

$$\begin{aligned}
F_j &= \int_0^1 \Big\{ 2^{10}\left[10\cos(2x-1) - ((2x-1)-1)^3\sin(2x-1)\right] N_{j,n}(x) + \frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\frac{d\theta_0}{dx} \\
&\quad + 2^{10}((2x-1)^2-2(2x-1))\theta_0 N_{j,n}(x) \Big\} dx + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}(-64\cos 1) \\
&\quad - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0}(-64\cos 1 + 32\sin 1) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1}(-24\sin 1) \\
&\quad + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0}(16\cos 1 + 24\sin 1) + \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}(8\cos 1) \\
&\quad - \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}(8\cos 1 - 8\sin 1) - \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1}(2\sin 1) \\
&\quad + \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0}(-4\cos 1 - 2\sin 1)
\end{aligned} \tag{8.19c}$$

Solving the system (8.19a) we obtain the values of the parameters and then substituting these parameters into eqn. (8.18), we get the approximate solution of the BVP (8.17) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (8.16).

The maximum absolute errors, using different number of polynomials by the present method are summarized in **Table 1.**

Table 1: Maximum absolute errors for the example 1.

x	Exact Results	14 Bernstein Polynomials		14 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
-1.0	1.6829419696	1.6829419696	0.0000000E+000	1.6829419696	0.0000000E+000
-0.8	1.2912409636	1.2912409636	5.9729999E-014	1.2912409636	5.6177285E-014
-0.6	0.9034279574	0.9034279574	4.1522341E-014	0.9034279574	4.4741988E-014
-0.4	0.5451856792	0.5451856792	1.4754864E-013	0.5451856792	1.3855583E-013
-0.2	0.2384031970	0.2384031970	6.9666495E-015	0.2384031970	5.3013149E-015
0.0	-0.0000000000	0.0000000000	1.7186252E-013	0.0000000000	1.5987212E-013
0.2	-0.1589354646	-0.1589354646	6.5225603E-015	-0.1589354646	1.8735014E-014
0.4	-0.2336510054	-0.2336510054	1.4749313E-013	-0.2336510054	1.4238610E-013
0.6	-0.2258569894	-0.2258569894	4.1439074E-014	-0.2258569894	5.6316063E-014
0.8	-0.1434712182	-0.1434712182	5.9979799E-014	-0.1434712182	5.7537308E-014
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand, it is observed that the accuracy is found nearly the order 10^{-8} in [60] by Siddiqi and Akram and nearly the order 10^{-6} in [93] by Kasi and Raju respectively.

Example 2: Consider the linear BVP [93]

$$\frac{d^{10}u}{dx^{10}} + 5u = 10\cos x + 4(x-1)\sin x, \quad 0 \le x \le 1 \qquad 8.20a)$$

$$u(0) = u(1) = 0,\ u'(0) = -1, u'(1) = \sin 1, u''(0) = 2, u''(1) = 2\cos 1, u'''(0) = 1, u'''(1) = -3\sin 1,$$
$$u^{(iv)}(0) = -4, u^{(iv)}(1) = -4\cos 1. \qquad (8.20b)$$

The analytic solution of the above problem is $u(x) = (x-1)\sin x$.

Using the method illustrated in (8.2.1), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ n \ge 1 \qquad (8.21)$$

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 14$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-12}.

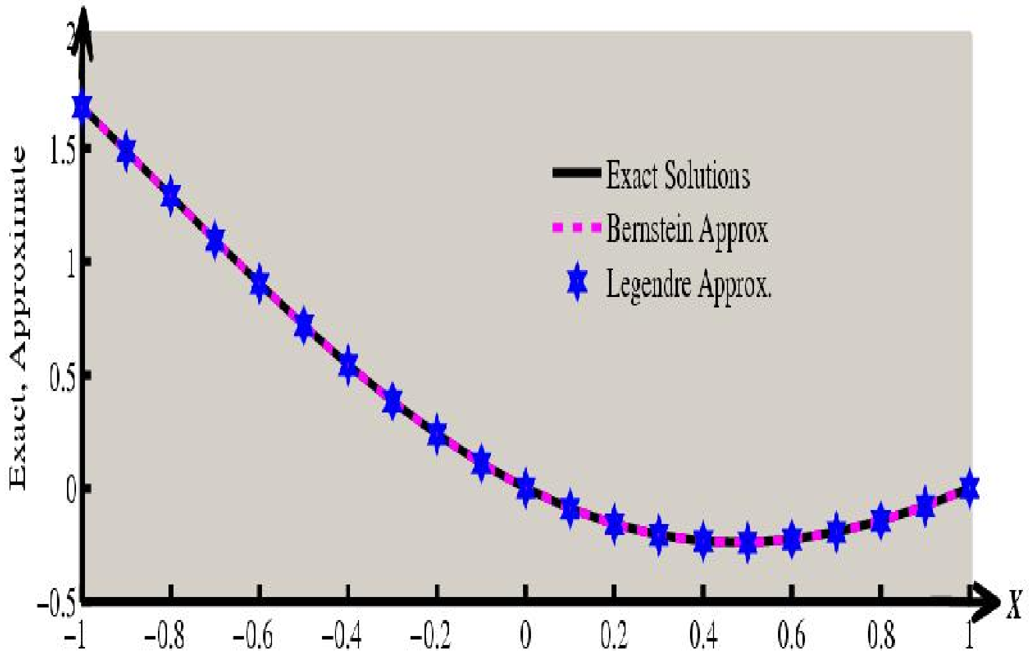


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 14 polynomials.

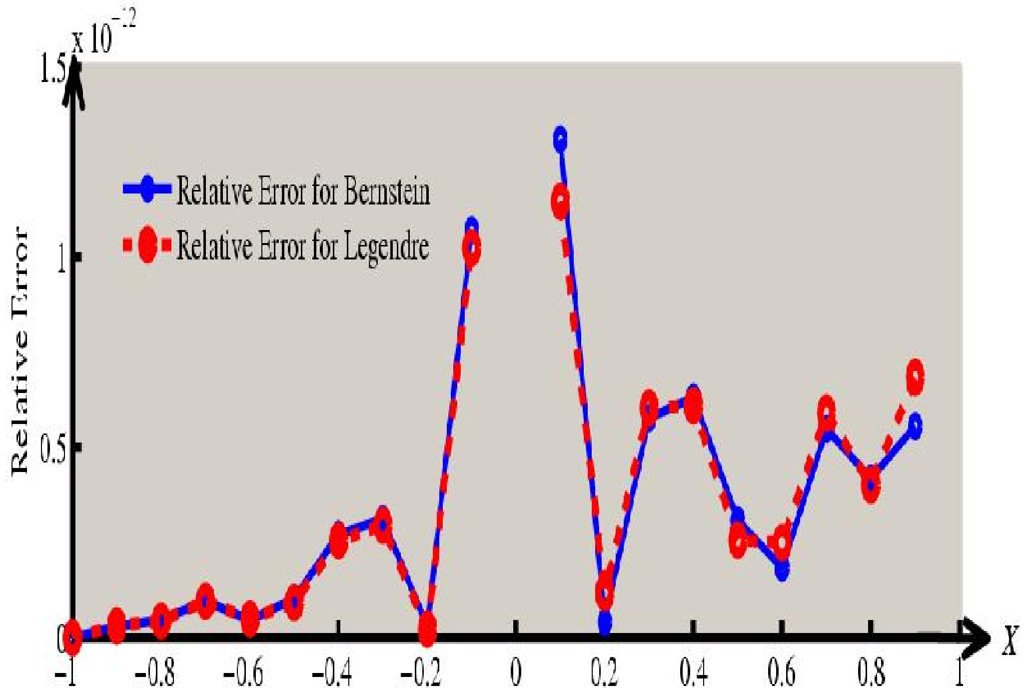


Fig. 1(b): Graphical representation of relative error of example 1 using 14 polynomials.

Here $\theta_0(x)=0$ is specified by the essential boundary conditions of equation (8.20b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j=1,2,\ldots,n \tag{8.22a}$$

where

$$\begin{aligned}
D_{i,j} = \int_0^1 &\left[-\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + 5\,N_{i,n}(x)N_{j,n}(x) \right]dx \\
&- \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&+ \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&- \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} \\
&+ \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}
\end{aligned} \tag{8.22b}$$

$$\begin{aligned}
F_j = \int_0^1 &\left[10\cos x - 4(x-1)\sin x\right]N_{j,n}(x)dx + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1} \times(-4\cos 1) \\
&- \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0} \times(-4) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1} \times(-3\sin 1) \\
&+ \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0} + \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1} \times 2\cos 1 \\
&- \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0} \times 2 - \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1} \times \sin 1 \\
&+ \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0} \times(-1)
\end{aligned} \tag{8.22c}$$

Solving the system (8.22a) we obtain the values of the parameters and then substituting these parameters into eqn. (8.21), we get the approximate solution of the BVP (8.20) for different values of *n*.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 2.**

Table 2: Maximum absolute errors for the example 2.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Bernstein	Reference Results
11	9.791×10^{-12}	9.791×10^{-12}	7.942×10^{-6} (Kasi and Raju [93])
12	6.520×10^{-14}	6.509×10^{-14}	
13	3.053×10^{-16}	3.969×10^{-15}	
14	8.326×10^{-17}	4.996×10^{-15}	

Figs. 2(a) and 2(b) deal with the exact and approximate solutions, and the relative errors of example 2 for $n = 14$. From Fig. 2(b) we observe that the error is nearly the order 10^{-13}

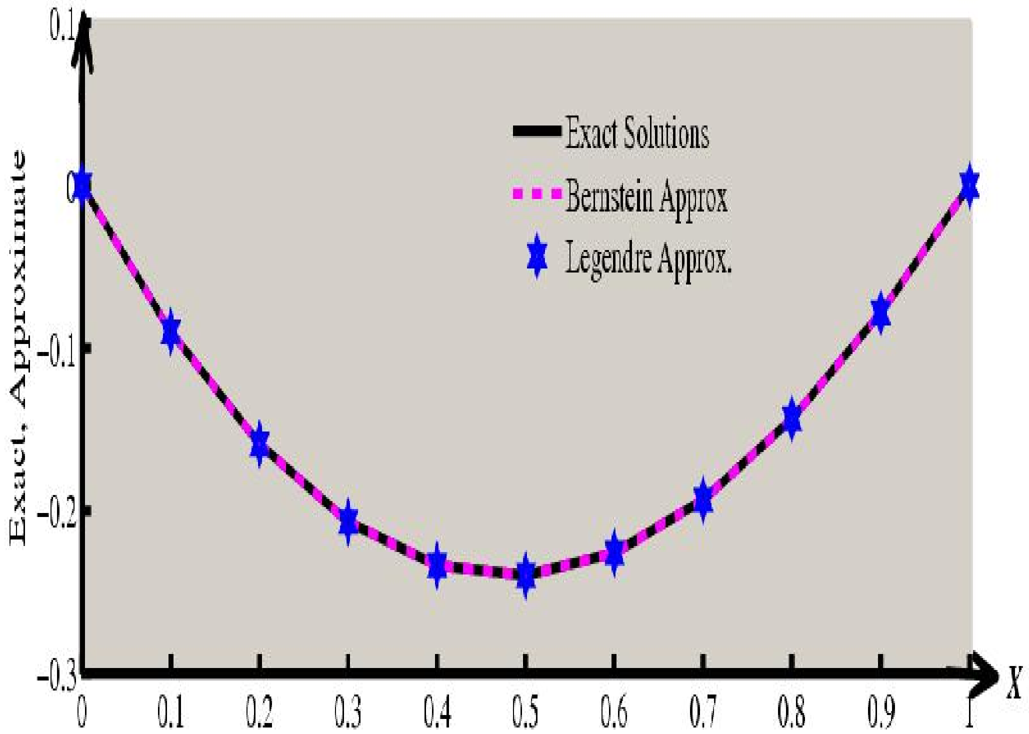


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 14 polynomials.

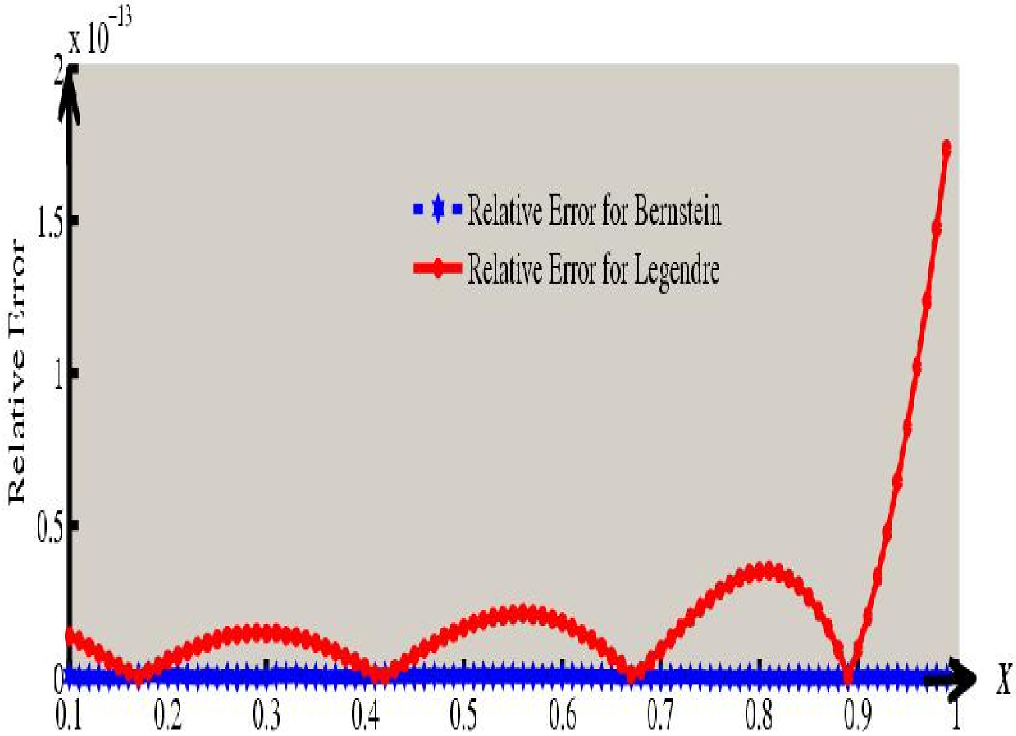


Fig. 2(b): Graphical representation of relative error of example 2 using 14 polynomials.

Example 3: Consider the linear differential equation [87, 93]

$$\frac{d^{10}u}{dx^{10}} = -8e^{x} + \frac{d^{2}u}{dx^{2}}, \quad 0 \le x \le 1 \tag{8.23a}$$

subject to the boundary conditions of type I in eqn. (2b):

$$u(0) = 1,\ u(1) = 0,\ u'(0) = 0, u'(1) = -e, u''(0) = -1, u''(1) = -2e, u'''(0) = -2,$$

$$u'''(1) = -3e, u^{(iv)}(0) = -3, u^{(iv)}(1) = -4e\ . \tag{8.23b}$$

The analytic solution of the above problem is, $u(x) = (1-x)e^{x}$.

Applying the method given in (8.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ n \ge 1 \tag{8.24}$$

Here $\theta_0(x) = 1 - x$ is specified by the essential boundary conditions of equation (8.23b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \tag{8.25a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^9}{dx^9}\left[N_{j,n}(x)\right] + \frac{d}{dx}\left[N_{j,n}(x)\right] \right] \frac{d}{dx}\left[N_{i,n}(x)\right] dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^8}{dx^8}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^8}{dx^8}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right] \frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right] \frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \frac{d^6}{dx^6}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$- \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right] \frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0} \quad (8.25b)$$

$$F_j = \int_0^1 \left\{ -8e^x N_{j,n}(x) + \left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right] - \frac{d}{dx}\left[N_{j,n}(x)\right] \right] \frac{d\theta_0}{dx} \right\} dx + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=1} (-4e)$$

$$- \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right] \right]_{x=0} (-3) - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=1} (-3e) + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right] \right]_{x=0} (-2)$$

$$+ \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right] \right]_{x=1} (-2e) - \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right] \right]_{x=0} (-1) - \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right] \right]_{x=1} (-e)$$

(8.25c)

Solving the system (8.25a) we obtain the values of the parameters and then substituting these parameters into eqn. (8.24), we get the approximate solution of the BVP (8.23) for different values of n.

Example 4: Consider the linear boundary value problem [91]

$$\frac{d^{10}u}{dx^{10}} - xu = -(89 + 21x + x^2 - x^3)e^x, \quad -1 \le x \le 1 \quad (8.26a)$$

subject to the boundary conditions of type II in eqn. (1c):

$$u(-1) = u(1) = 0,\ u''(-1) = \frac{2}{e}, u''(1) = -6e, u^{(iv)}(-1) = -\frac{4}{e}, u^{(iv)}(1) = -20e, u^{(vi)}(-1) = -\frac{18}{e},$$

$$u^{(vi)}(1) = -42e, u^{(viii)}(-1) = -\frac{40}{e}, u^{(viii)}(1) = -72e\,. \quad (8.26b)$$

The maximum absolute errors using different number of polynomials and to compare with existing methods are shown in **Table 3.**

Table 3: Maximum absolute errors for the example 3.

x	Exact Results	13 Bernstein Polynomials		13 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	1.1102230E-016	0.9946538263	2.8865799E-015
0.2	0.9771222065	0.9771222065	1.1102230E-016	0.9771222065	9.4368957E-015
0.3	0.9449011653	0.9449011653	5.5511151E-016	0.9449011653	6.7723605E-015
0.4	0.8950948186	0.8950948186	4.4408921E-016	0.8950948186	6.5503158E-015
0.5	0.8243606354	0.8243606354	2.2204460E-016	0.8243606354	1.2767565E-014
0.6	0.7288475202	0.7288475202	7.7715612E-016	0.7288475202	4.3298698E-015
0.7	0.6041258122	0.6041258122	2.2204460E-016	0.6041258122	8.3266727E-015
0.8	0.4451081857	0.4451081857	3.8857806E-016	0.4451081857	9.1593400E-015
0.9	0.2459603111	0.2459603111	3.3306691E-016	0.2459603111	2.9976022E-015
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand the accuracy is found nearly the order 10^{-6} in [87] by Mohy-ud-Din and Yildirim and in [93] by Kasi and Raju respectively.

We depict the exact and approximate solutions in Fig. 3(a) and a plot of relative errors in Fig. 3(b) of example 3 for $n = 13$. From Fig. 3(b) we observe that the error is nearly the order 10^{-13}

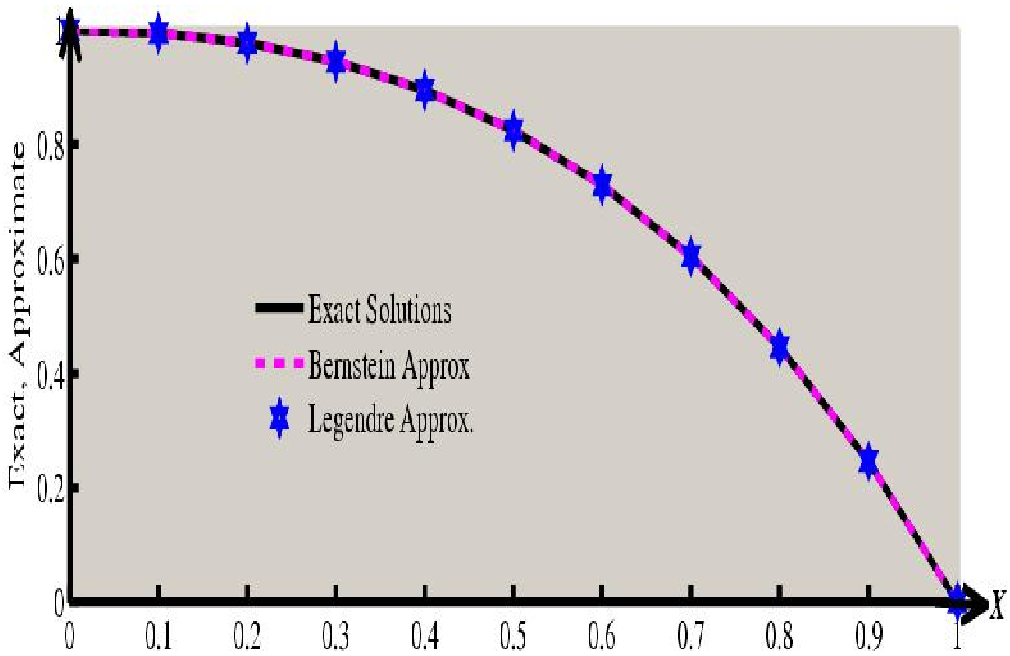


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 13 polynomials.

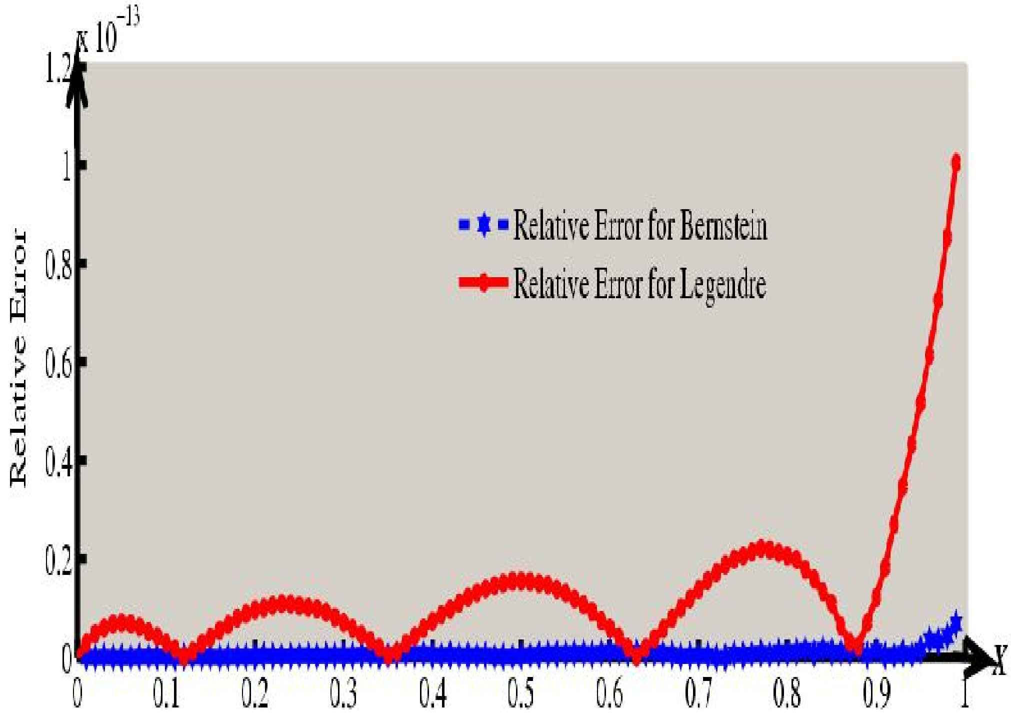


Fig. 3(b): Graphical representation of relative error of example 3 using 13 polynomials.

The analytic solution of the above problem is $u(x) = (1-x^2)e^x$.

The equivalent BVP over [0, 1] to the BVP (8.26) is

$$\frac{1}{2^{10}}\frac{d^{10}u}{dx^{10}} - (2x-1)u = -(89+21(2x-1)+(2x-1)^2-(2x-1)^3)e^{(2x-1)}, 0<x<1 \quad (8.27a)$$

$$u(0)=u(1)=0,\ \frac{1}{4}u''(0)=\frac{2}{e}, \frac{1}{4}u''(1)=-6e, \frac{1}{16}u^{(iv)}(0)=-\frac{4}{e}, \frac{1}{16}u^{(iv)}(1)=-20e,$$

$$\frac{1}{64}u^{(vi)}(0)=-\frac{18}{e}, \frac{1}{64}u^{(vi)}(1)=-42e, \frac{1}{256}u^{(viii)}(0)=-\frac{40}{e}, \frac{1}{256}u^{(viii)}(1)=-72e \quad (8.27b)$$

Applying the method given in (8.2.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \geq 1 \quad (8.28)$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (8.27b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j, j = 1,2,\ldots,n \quad (8.29a)$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 2^{10}(2x-1)\,N_{i,n}(x)N_{j,n}(x) \right] dx$$
$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \quad (8.29b)$$

$$F_j = \int_0^1 2^{10}\left[-(89 + 21(2x-1) + (2x-1)^2 - (2x-1)^3)e^{(2x-1)}\right]N_{j,n}(x)dx$$
$$+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=1}(-256\times 72e) - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=0}\left(-\frac{256\times 40}{e}\right)$$
$$+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1}(-64\times 42e) - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0}\left(-\frac{64\times 18}{e}\right)$$
$$+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}(-16\times 20e) - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0}\times\left(-\frac{64}{e}\right)$$
$$+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-24e) + \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}\times\left(\frac{8}{e}\right) \quad (8.29c)$$

Solving the system (8.29a) we obtain the values of the parameters and then substituting these parameters into eqn. (8.28), we get the approximate solution of the BVP (8.27) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (8.26).

In **Table 4**, we tabulate the maximum absolute errors to compare with the previous results.

Table 4: Maximum absolute errors for the example 4.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
11	7.684×10^{-9}	7.684×10^{-9}	0.5163×10^{-2} (Siddiqi and Twizell [91])
12	5.316×10^{-11}	5.295×10^{-11}	
13	7.196×10^{-12}	4.714×10^{-12}	
14	9.615×10^{-13}	3.106×10^{-13}	

Example 5: Consider the linear BVP [91]

$$\frac{d^{10}u}{dx^{10}}+u=-10(2x\sin x-9\cos x),\quad -1\le x\le 1 \tag{8.30a}$$

$$u(-1)=u(1)=0,\ u''(-1)=-4\sin1+2\cos1=u''(1),\ u^{(iv)}(-1)=8\sin1-12\cos1=u^{(iv)}(1)$$

$$u^{(vi)}(-1)=-12\sin1+30\cos1=u^{(vi)}(1),\ u^{(viii)}(-1)=16\sin1-56\cos1=u^{(viii)}(1)\ . \tag{8.30b}$$

The analytic solution of the above problem is $u(x)=(x^2-1)\cos x$.

The equivalent BVP over [0, 1] to the BVP (8.30) is,

$$\frac{1}{2^{10}}\frac{d^{10}u}{dx^{10}}+u=-10(2(2x-1)\sin(2x-1)-9\cos(2x-1)),0<x<1 \tag{8.31a}$$

$$u(0)=u(1)=0,\ \frac{1}{4}u''(0)=\frac{1}{4}u''(1)=-4\sin1+2\cos1,\frac{1}{16}u^{(iv)}(0)=\frac{1}{16}u^{(iv)}(1)=8\sin1-12\cos1$$

$$\frac{1}{64}u^{(vi)}(0)=\frac{1}{64}u^{(vi)}(1)=-12\sin1+30\cos1,\frac{1}{256}u^{(viii)}(0)=\frac{1}{256}u^{(viii)}(1)=16\sin1-56\cos1 \tag{8.31b}$$

Employing the method mentioned in (8.2.2), we approximate $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\ge1 \tag{8. 32}$$

Here $\theta_0(x)=0$ is specified by the essential boundary conditions of equation (8.31b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n}D_{i,j}\,\alpha_i=F_j,\ j=1,2,\ldots,n \tag{8.33a}$$

We illustrated graphically the exact and approximate solutions in Fig. 4(a) and the relative errors in Fig. 4(b) of example 4 for $n=14$. It is clear from Fig. 4(b) that the error is of order 10^{-11}.

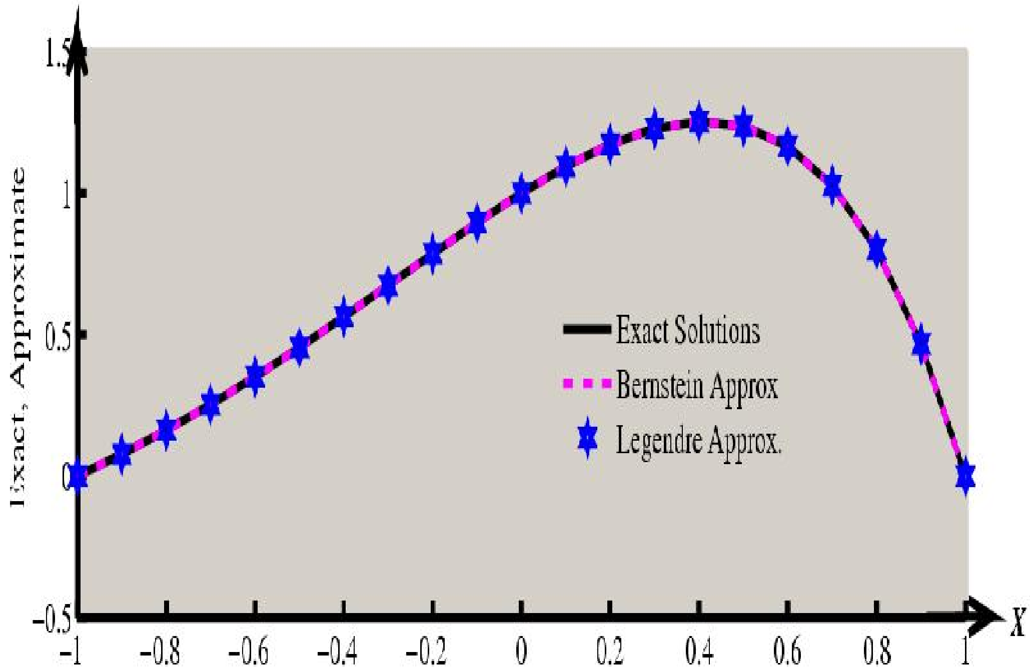


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 14 polynomials.

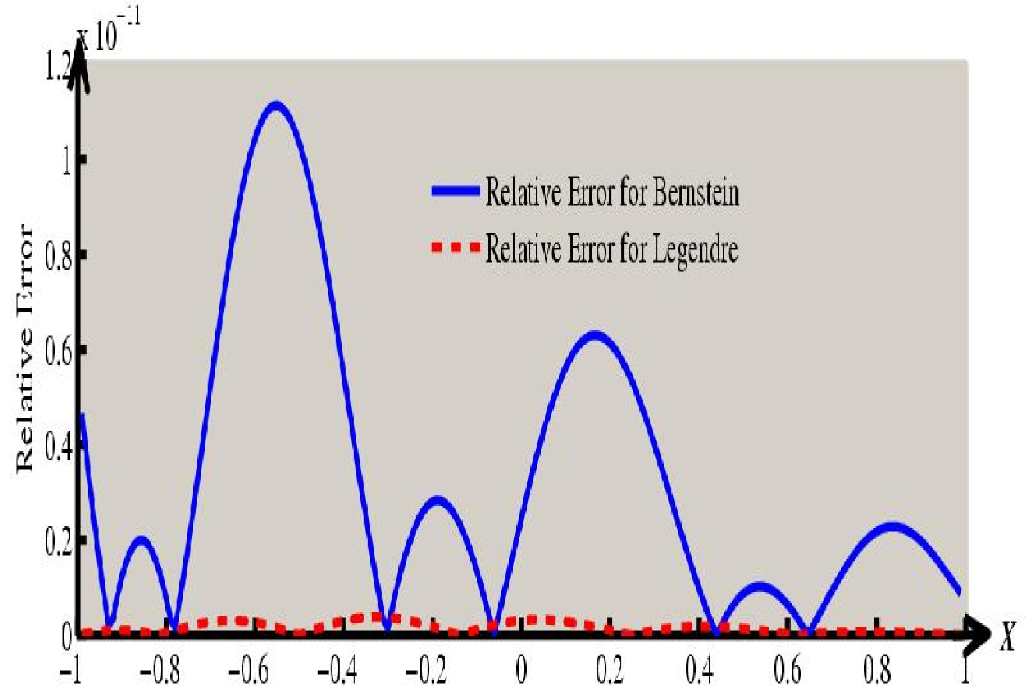


Fig. 4(b): Graphical representation of relative error of example 4 using 14 polynomials.

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + 2^{10}\, N_{i,n}(x)N_{j,n}(x) \right]dx$$
$$+\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (8.33b)$$

$$F_j = \int_0^1 2^{10}\left[-10(2(2x-1)\sin(2x-1) - 9\cos(2x-1))\right]N_{j,n}(x)dx$$
$$+\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=1} \times (256 \times 16 \sin 1 - 256 \times 56 \cos 1) - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=0}$$
$$\times (256 \times 16 \sin 1 - 256 \times 56 \cos 1) + \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1} \times (-64 \times 12 \sin 1 + 64 \times 30 \cos 1)$$
$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0} (-64 \times 12 \sin 1 + 64 \times 30 \cos 1) + \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}$$
$$\times (16 \times 8 \sin 1 - 16 \times 12 \cos 1) - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1} \times (16 \times 8 \sin 1 - 16 \times 12 \cos 1)$$
$$+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1} \times (-16 \sin 1 + 8 \cos 1) - \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0} \times (-16 \sin 1 + 8 \cos 1)$$

(8.33c)

Solving the system (8.33a) we obtain the values of the parameters and then substituting these parameters into eqn. (8.32), we get the approximate solution of the BVP (8.31) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (8.30).

The maximum absolute errors, shown in **Table 5,** are tabulated to compare with existing results

Table 5: Maximum absolute errors for the example 5.

x	Exact Results	13 Bernstein Polynomials		13 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
-1.0	0.0000000000	0.0000000000	3.3865378E-023	0.0000000000	0.0000000E+000
-0.8	-0.2508144154	-0.2508144154	1.3156143E-014	-0.2508144154	2.6040281E-013
-0.6	-0.5282147935	-0.5282147935	5.5178084E-014	-0.5282147935	5.7065463E-013
-0.4	-0.7736912350	-0.7736912350	6.0951244E-014	-0.7736912350	6.4326322E-013
-0.2	-0.9408639147	-0.9408639147	3.5638159E-014	-0.9408639147	7.9802831E-013
0.0	-1.0000000000	-1.0000000000	1.1568524E-013	-1.0000000000	9.9997788E-013
0.2	-0.9408639147	-0.9408639147	4.0856207E-014	-0.9408639147	7.9791729E-013
0.4	-0.7736912350	-0.7736912350	6.4614980E-014	-0.7736912350	6.4348527E-013
0.6	-0.5282147935	-0.5282147935	5.6288307E-014	-0.5282147935	5.7054361E-013
0.8	-0.2508144154	-0.2508144154	1.1768364E-014	-0.2508144154	2.6040281E-013
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the contrary the maximum absolute error has been found by Siddiqi and Twizell [91] is 0.9090×10^{-3}

In Figs. 5(a) and 5(b), the exact and approximate solutions, and the relative errors of example 5 for $n = 13$ are depicted respectively. We see from Fig. 5(b) that the error is nearly the order 10^{-11}

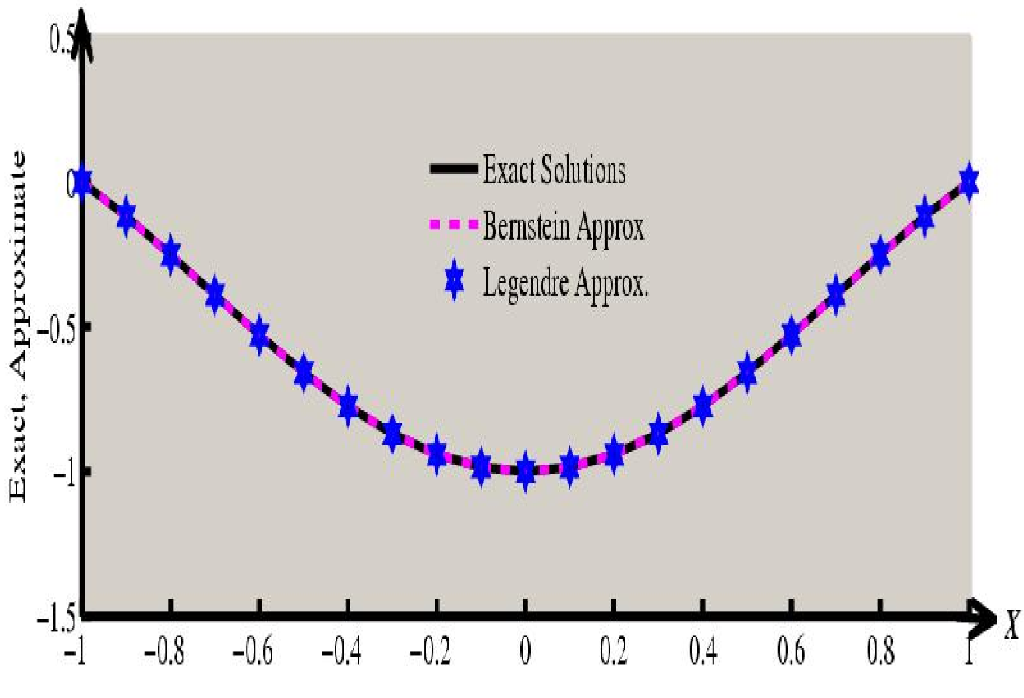


Fig. 5(a): Graphical representation of exact and approximate solutions of example 5 using 13 polynomials.

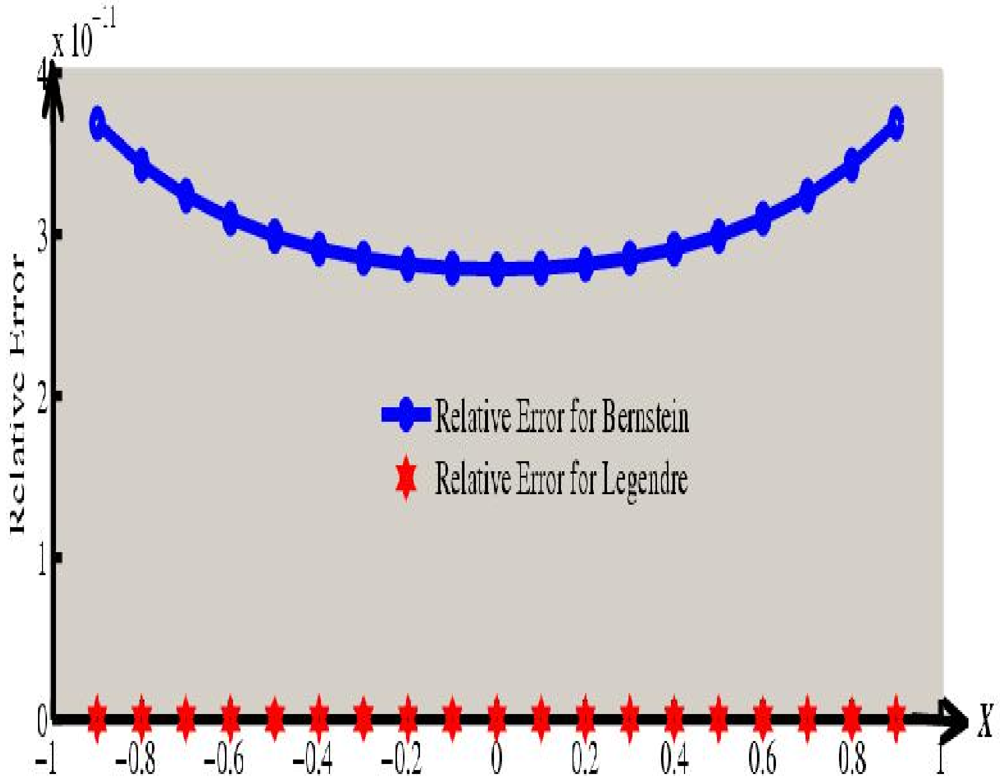


Fig. 5(b): Graphical representation of relative error of example 5 using 13 polynomials.

Example 6: Consider the tenth-order **nonlinear** differential equation [93]

$$\frac{d^{10}u}{dx^{10}} - \frac{d^3u}{dx^3} = 2e^x u^2,\ 0 \le x \le 1 \tag{8.34a}$$

subject to the boundary conditions of type I defined in eqn. (2b)

$$u(0) = 1,\ u(1) = e^{-1}, u'(0) = -1, u'(1) = -e^{-1}, u''(0) = 1, u''(1) = e^{-1}, u'''(0) = -1,$$

$$u'''(1) = -e^{-1}, u^{(iv)}(0) = 1, u^{(iv)}(1) = e^{-1}. \tag{8.34b}$$

The exact solution of this BVP is $u(x) = e^{-x}$.

Consider the approximate solution of $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{8.35}$$

Here $\theta_0(x) = 1 - x(1 - e^{-1})$ is specified by the essential boundary conditions in (8.34b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1, 2, \ldots, n$.

Using eqn. (8.35) into eqn. (8.34a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^{10}\tilde{u}}{dx^{10}} - \frac{d^3\tilde{u}}{dx^3} - 2e^x\tilde{u}^2 \right] N_{k,n}(x)dx = 0, k = 1,2,\cdots,n \tag{8.36}$$

Integrating **1st** and **2nd** terms of (8.36) by parts, we obtain

$$\int_0^1 \frac{d^{10}\tilde{u}}{dx^{10}} N_{k,n}(x) = -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^8\tilde{u}}{dx^8} \right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^6\tilde{u}}{dx^6} \right]_0^1$$

$$+ \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^5N_{k,n}(x)}{dx^5} \frac{d^4\tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^6N_{k,n}(x)}{dx^6} \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^7N_{k,n}(x)}{dx^7} \frac{d^2\tilde{u}}{dx^2} \right]_0^1$$

$$+ \left[\frac{d^8N_{k,n}(x)}{dx^8} \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^9 N_{k,n}(x)}{dx^9} \frac{d\tilde{u}}{dx} dx \tag{8.37}$$

$$\int_0^1 \frac{d^3\tilde{u}}{dx^3} N_{k,n}(x) = -\left[\frac{dN_{k,n}(x)}{dx} \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2 N_{k,n}(x)}{dx^2} \frac{d\tilde{u}}{dx} dx \tag{8.38}$$

Putting eqn. (8.37) and (8.38) into eqn. (8.36) and using approximation for $\tilde{u}(x)$ given in eqn. (8.35) and after applying the conditions given in eqn. (8.34b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n} \left[\int_0^1 \left[\left\{ -\frac{d^9N_{k,n}(x)}{dx^9} - \frac{d^2N_{k,n}(x)}{dx^2} \right\} \frac{dN_{i,n}(x)}{dx} - 4\theta_0 e^x N_{i,n}(x)N_{k,n}(x) \right.\right.$$

$$\left. -2\sum_{j=1}^{n} \alpha_j (N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^x \right] dx - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8N_{i,n}(x)}{dx^8} \right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8N_{i,n}(x)}{dx^8} \right]_{x=0}$$

$$+ \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^7N_{i,n}(x)}{dx^7} \right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^7N_{i,n}(x)}{dx^7} \right]_{x=0} - \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^6N_{i,n}(x)}{dx^6} \right]_{x=1}$$

$$\left. + \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^6N_{i,n}(x)}{dx^6} \right]_{x=0} + \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^5N_{i,n}(x)}{dx^5} \right]_{x=1} - \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^5N_{i,n}(x)}{dx^5} \right]_{x=0} \right] \alpha_i$$

$$= \int_0^1 \left[\left\{ \frac{d^9N_{k,n}(x)}{dx^9} + \frac{d^2N_{k,n}(x)}{dx^2} \right\} \frac{d\theta_0}{dx} + 2\theta_0^2 e^x N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8\theta_0}{dx^8} \right]_{x=1}$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{8}\theta_0}{dx^{8}}\right]_{x=0}-\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{7}\theta_0}{dx^{7}}\right]_{x=1}+\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{7}\theta_0}{dx^{7}}\right]_{x=0}$$

$$+\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{6}\theta_0}{dx^{6}}\right]_{x=1}-\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{6}\theta_0}{dx^{6}}\right]_{x=0}-\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{5}\theta_0}{dx^{5}}\right]_{x=1}$$

$$+\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{5}\theta_0}{dx^{5}}\right]_{x=0}+\left[\frac{d^{5}N_{k,n}(x)}{dx^{5}}\right]_{x=1}\times e^{-1}-\left[\frac{d^{5}N_{k,n}(x)}{dx^{5}}\right]_{x=0}$$

$$-\left[\frac{d^{6}N_{k,n}(x)}{dx^{6}}\right]_{x=1}\times(-e^{-1})+\left[\frac{d^{6}N_{k,n}(x)}{dx^{6}}\right]_{x=0}(-1)+\left[\frac{d^{7}N_{k,n}(x)}{dx^{7}}\right]_{x=1}\times e^{-1}$$

$$-\left[\frac{d^{7}N_{k,n}(x)}{dx^{7}}\right]_{x=0}-\left[\frac{d^{8}N_{k,n}(x)}{dx^{8}}\right]_{x=1}\times(-e^{-1})-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}(-e^{-1})$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-1)+\left[\frac{d^{8}N_{k,n}(x)}{dx^{8}}\right]_{x=0}(-1) \tag{8.39}$$

The above equation (8.39) is equivalent to matrix form

$$(D+B)A=G \tag{8.40a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[\left\{-\frac{d^{9}N_{k,n}(x)}{dx^{9}}-\frac{d^{2}N_{k,n}(x)}{dx^{2}}\right\}\frac{dN_{i,n}(x)}{dx}-4\theta_0 e^{x}N_{i,n}(x)N_{k,n}(x)\right]dx$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{8}N_{i,n}(x)}{dx^{8}}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{8}N_{i,n}(x)}{dx^{8}}\right]_{x=0}+\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{7}N_{i,n}(x)}{dx^{7}}\right]_{x=1}$$

$$-\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{6}N_{i,n}(x)}{dx^{6}}\right]_{x=1}+\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{6}N_{i,n}(x)}{dx^{6}}\right]_{x=0}-\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{7}N_{i,n}(x)}{dx^{7}}\right]_{x=0}$$

$$+\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{5}N_{i,n}(x)}{dx^{5}}\right]_{x=1}-\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{5}N_{i,n}(x)}{dx^{5}}\right]_{x=0} \tag{8.40b}$$

$$b_{i,k}=-2\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{x}\,dx \tag{8.40c}$$

$$g_k = \int_0^1 \left[\left\{ \frac{d^9 N_{k,n}(x)}{dx^9} + \frac{d^2 N_{k,n}(x)}{dx^2} \right\} \frac{d\theta_0}{dx} + 2\theta_0{}^2 e^x N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8\theta_0}{dx^8} \right]_{x=1}$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8\theta_0}{dx^8} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7\theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7\theta_0}{dx^7} \right]_{x=0}$$

$$+ \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^6\theta_0}{dx^6} \right]_{x=1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^6\theta_0}{dx^6} \right]_{x=0} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5\theta_0}{dx^5} \right]_{x=1}$$

$$+ \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5\theta_0}{dx^5} \right]_{x=0} + \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=1} \times e^{-1} - \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=0}$$

$$- \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=1} \times (-e^{-1}) + \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=0} (-1) + \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=1} \times e^{-1}$$

$$- \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=0} - \left[\frac{d^8 N_{k,n}(x)}{dx^8} \right]_{x=1} \times (-e^{-1}) + \left[\frac{d^8 N_{k,n}(x)}{dx^8} \right]_{x=0} (-1)$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} (-e^{-1}) + \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0} (-1) \qquad (8.40d)$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (8.34a). That is, to find initial coefficients we solve the system

$$DA = G \qquad (8.41a)$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left[\left\{ -\frac{d^9 N_{k,n}(x)}{dx^9} - \frac{d^2 N_{k,n}(x)}{dx^2} \right\} \frac{dN_{i,n}(x)}{dx} \right] dx - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8 N_{i,n}(x)}{dx^8} \right]_{x=1}$$

$$+ \left[\frac{dN_{k,n}(x)}{dx} \frac{d^8 N_{i,n}(x)}{dx^8} \right]_{x=0} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 N_{i,n}(x)}{dx^7} \right]_{x=1}$$

$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 N_{i,n}(x)}{dx^7} \right]_{x=0} + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^6 N_{i,n}(x)}{dx^6} \right]_{x=0}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0} \qquad (8.41b)$$

$$g_k=\int_0^1\left\{\frac{d^9N_{k,n}(x)}{dx^9}+\frac{d^2N_{k,n}(x)}{dx^2}\right\}\frac{d\theta_0}{dx}dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=0}$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e^{-1}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}$$

$$-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}\times(-e^{-1})+\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0}(-1)+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times e^{-1}$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}-\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=1}\times(-e^{-1})+\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=0}(-1)$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}(-e^{-1})+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}(-1) \qquad (8.41c)$$

Once the initial values of α_i are obtained from eqn. (8.41a), they are substituted into eqn. (8.40a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (8.35), we obtain an approximate solution of the BVP (8.34).

Numerical results for example 6 are shown in the following **Table 6**.

Table 6: Numerical results for example 6 using 6 iterations

Nunber of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
10	4.380×10^{-8}	4.163×10^{-8}	5.722×10^{-6} (Kasi and Raju [93])
11	7.680×10^{-9}	7.564×10^{-9}	
12	5.315×10^{-10}	3.625×10^{-10}	
13	7.160×10^{-12}	4.600×10^{-11}	

Example 7: Consider the tenth-order **nonlinear** differential equation [80, 87, 88, 89]

$$\frac{d^{10}u}{dx^{10}} = u^2 e^{-x},\ 0 \le x \le 1 \tag{8.42a}$$

subject to the boundary conditions of type II in eqn. (2c):

$$u(0) = 1,\ u(1) = e, u''(0) = 1, u''(1) = e, u^{(iv)}(0) = 1, u^{(iv)}(1) = e, u^{(vi)}(0) = 1,$$

$$u^{(vi)}(1) = e, u^{(viii)}(0) = 1, u^{(viii)}(1) = e. \tag{8.42b}$$

The exact solution of this BVP is $u(x) = e^x$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ \ n \ge 1 \tag{8.43}$$

Here $\theta_0(x) = 1 - x(1-e)$ is specified by the essential boundary conditions in (8.42b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Using eqn. (8.43) into eqn. (8.42a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^{10}\tilde{u}}{dx^{10}} - \tilde{u}^2 e^{-x}\right] N_{k,n}(x)dx = 0, k = 1,2,\cdots,n \tag{8.44}$$

Fig. 6(a) depicts the exact and approximate solutions and the relative errors are shown in Fig. 6(b) of example 6 for $n = 13$. We see from Fig. 6(b) that the error is nearly the order 10^{-11}.

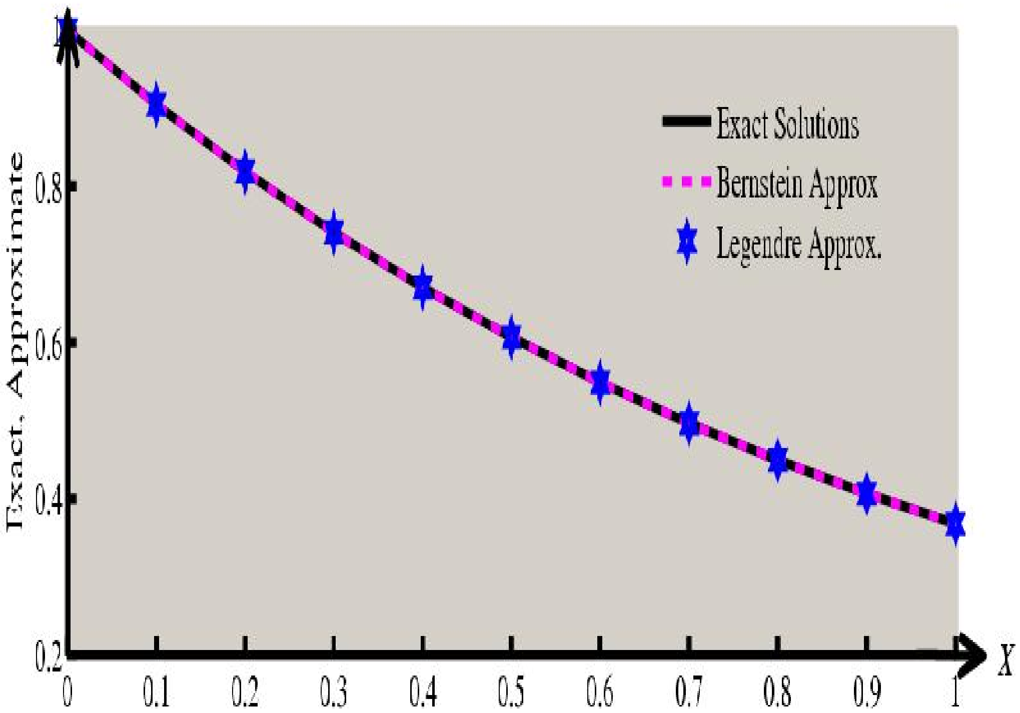


Fig. 6(a): Graphical representation of exact and approximate solutions of example 6 using 13 polynomials.

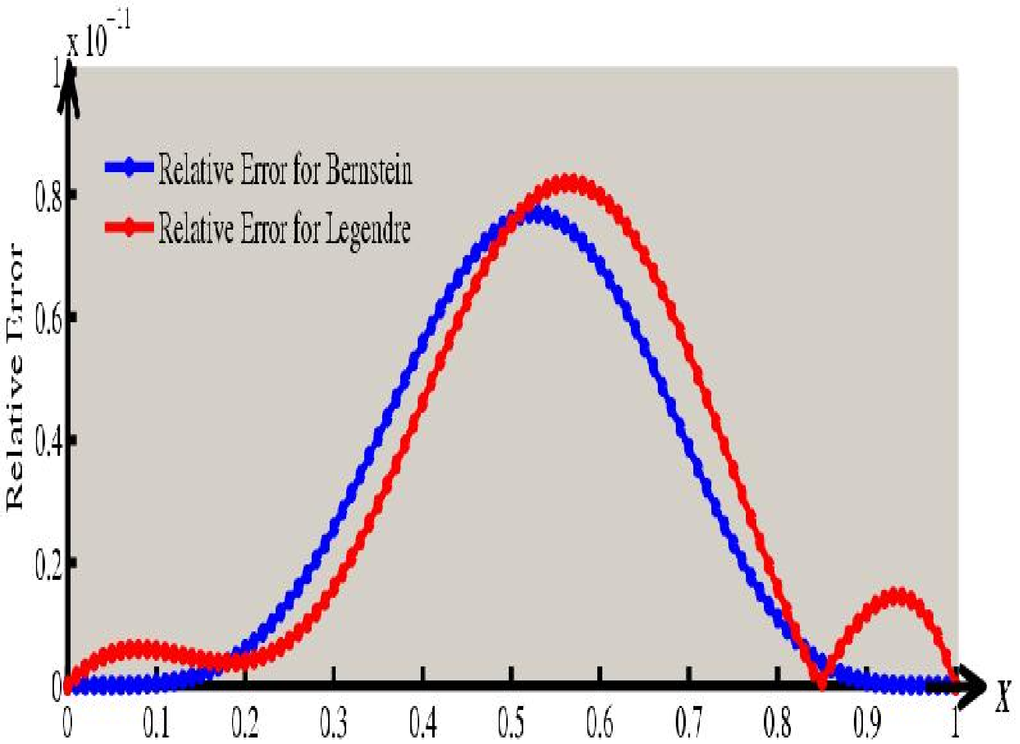


Fig. 6(b): Graphical representation of relative error of example 6 using 13 polynomials.

Integrating first term of (8.44) by parts, we obtain

$$\int_0^1 \frac{d^{10}\tilde{u}}{dx^{10}} N_{k,n}(x) = -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^8\tilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\tilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^9N_{k,n}(x)}{dx^9}\frac{d\tilde{u}}{dx}dx \tag{8.45}$$

Putting eqn. (8.45) into eqn. (8.44) and using approximation for $\tilde{u}(x)$ given in eqn. (8.43) and after applying the boundary conditions given in eqn. (8.42b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^9N_{k,n}(x)}{dx^9}\frac{dN_{i,n}(x)}{dx} - 2\theta_0 e^{-x} N_{i,n}(x)N_{k,n}(x) - \sum_{j=1}^{n}\alpha_j (N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\right]dx\right.$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0} + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0} + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1} - \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}$$

$$\left.+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{dN_{i,n}(x)}{dx}\right]_{x=1} - \left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{dN_{i,n}(x)}{dx}\right]_{x=0}\right]\alpha_i = \int_0^1\left[\frac{d^9N_{k,n}(x)}{dx^9}\frac{d\theta_0}{dx}\right.$$

$$\left.+\theta_0^2 e^{-x}N_{k,n}(x)\right]dx - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1} + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0} - \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3\theta_0}{dx^3}\right]_{x=1} + \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times e - \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times e + \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} + \left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e - \left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}$$

$$+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times e-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0} \tag{8.46}$$

The above equation (8.46) is equivalent to matrix form

$$(D+B)A=G \tag{8.47a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{d^9N_{k,n}(x)}{dx^9}\frac{dN_{i,n}(x)}{dx}-2\theta_0e^{-x}N_{i,n}(x)N_{k,n}(x)\right]dx+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}$$
$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}$$
$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{dN_{i,n}(x)}{dx}\right]_{x=1}$$
$$-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{dN_{i,n}(x)}{dx}\right]_{x=0} \tag{8.47b}$$

$$b_{i,k}=-\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\,dx \tag{8.47c}$$

$$g_k=\int_0^1\left[\frac{d^9N_{k,n}(x)}{dx^9}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}N_{k,n}(x)\right]dx-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$
$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$
$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d\theta_0}{dx}\right]_{x=1}$$
$$+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times e-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times e+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}$$
$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times e-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times e-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}$$

(8.47d)

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (8.42a). That is, to find initial coefficients we solve the system

$$DA = G \tag{8.48a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left[-\frac{d^9 N_{k,n}(x)}{dx^9} \frac{dN_{i,n}(x)}{dx} \right] dx + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 N_{i,n}(x)}{dx^7} \right]_{x=1}$$
$$- \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 N_{i,n}(x)}{dx^7} \right]_{x=0} + \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5 N_{i,n}(x)}{dx^5} \right]_{x=1} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5 N_{i,n}(x)}{dx^5} \right]_{x=0}$$
$$+ \left[\frac{d^6 N_{k,n}(x)}{dx^6} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=1} - \left[\frac{d^6 N_{k,n}(x)}{dx^6} \frac{d^3 N_{i,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^8 N_{k,n}(x)}{dx^8} \frac{dN_{i,n}(x)}{dx} \right]_{x=1}$$
$$- \left[\frac{d^8 N_{k,n}(x)}{dx^8} \frac{dN_{i,n}(x)}{dx} \right]_{x=0} \tag{8.48b}$$

$$g_k = \int_0^1 \frac{d^9 N_{k,n}(x)}{dx^9} \frac{d\theta_0}{dx} dx - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 \theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^7 \theta_0}{dx^7} \right]_{x=0}$$
$$- \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5 \theta_0}{dx^5} \right]_{x=1} + \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^5 \theta_0}{dx^5} \right]_{x=0} - \left[\frac{d^6 N_{k,n}(x)}{dx^6} \frac{d^3 \theta_0}{dx^3} \right]_{x=1}$$
$$+ \left[\frac{d^6 N_{k,n}(x)}{dx^6} \frac{d^3 \theta_0}{dx^3} \right]_{x=0} - \left[\frac{d^8 N_{k,n}(x)}{dx^8} \frac{d\theta_0}{dx} \right]_{x=1} + \left[\frac{d^8 N_{k,n}(x)}{dx^8} \frac{d\theta_0}{dx} \right]_{x=1}$$
$$+ \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} \times e - \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0} + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=1} \times e + \left[\frac{d^3 N_{k,n}(x)}{dx^3} \right]_{x=0}$$
$$+ \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=1} \times e - \left[\frac{d^5 N_{k,n}(x)}{dx^5} \right]_{x=0} + \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=1} \times e - \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=0}$$

(8.48c)

Once the initial values of α_i are obtained from eqn. (8.48a), they are substituted into eqn. (8.47a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (8.43), we obtain an approximate solution of the BVP (8.42).

Numerical results for example 7 are shown in the following **Table 7**.

Table 7: Numerical results for example 7 using 6 iterations

x	Exact Results	12 Bernstein Polynomials		12 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	2.5979219E-014	1.1051709181	1.6084911E-012
0.2	1.2214027582	1.2214027582	7.0188300E-013	1.2214027582	1.3298251E-011
0.3	1.3498588076	1.3498588076	2.6412206E-012	1.3498588076	5.0746074E-012
0.4	1.4918246976	1.4918246976	5.0834892E-012	1.4918246976	7.1547213E-012
0.5	1.6487212707	1.6487212707	6.2061467E-012	1.6487212707	6.2061467E-012
0.6	1.8221188004	1.8221188004	5.0610627E-012	1.8221188004	2.9898306E-012
0.7	2.0137527075	2.0137527075	2.6179059E-012	2.0137527075	1.8429702E-013
0.8	2.2255409285	2.2255409285	6.9233508E-013	2.2255409285	6.5281114E-014
0.9	2.4596031112	2.4596031112	2.5757174E-014	2.4596031112	1.6604496E-012
1.0	2.7182818285	2.7182818285	0.0000000E+000	2.7182818285	0.0000000E+000

In Figs. 7(a) and 7(b) we have given the exact and approximate solutions, and the relative errors of example 7 for $n = 12$. From Fig. 7(b) we observed that the error is nearly the order 10^{-12}.

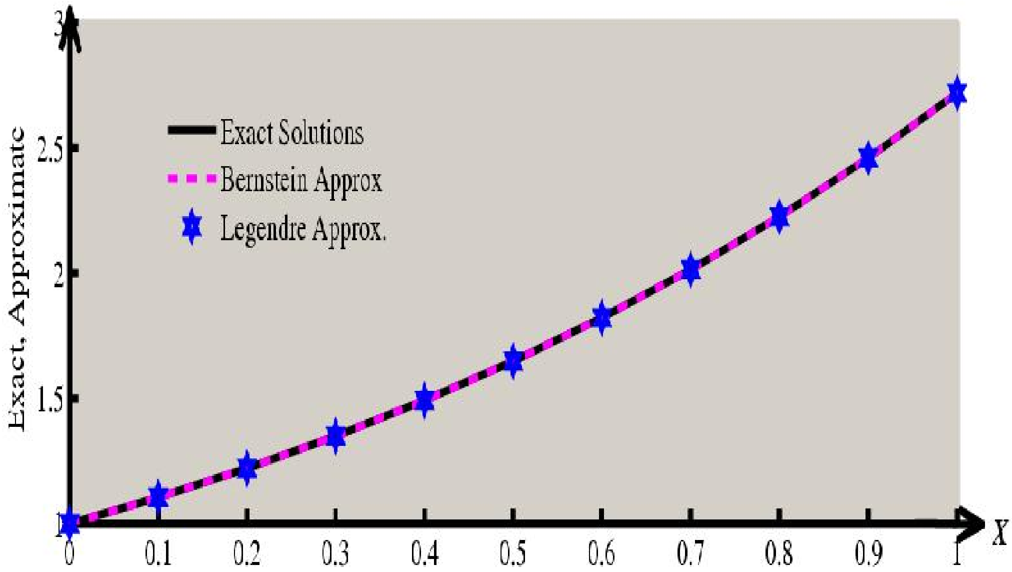


Fig. 7(a): Graphical representation of exact and approximate solutions of example 7 using 12 polynomials.

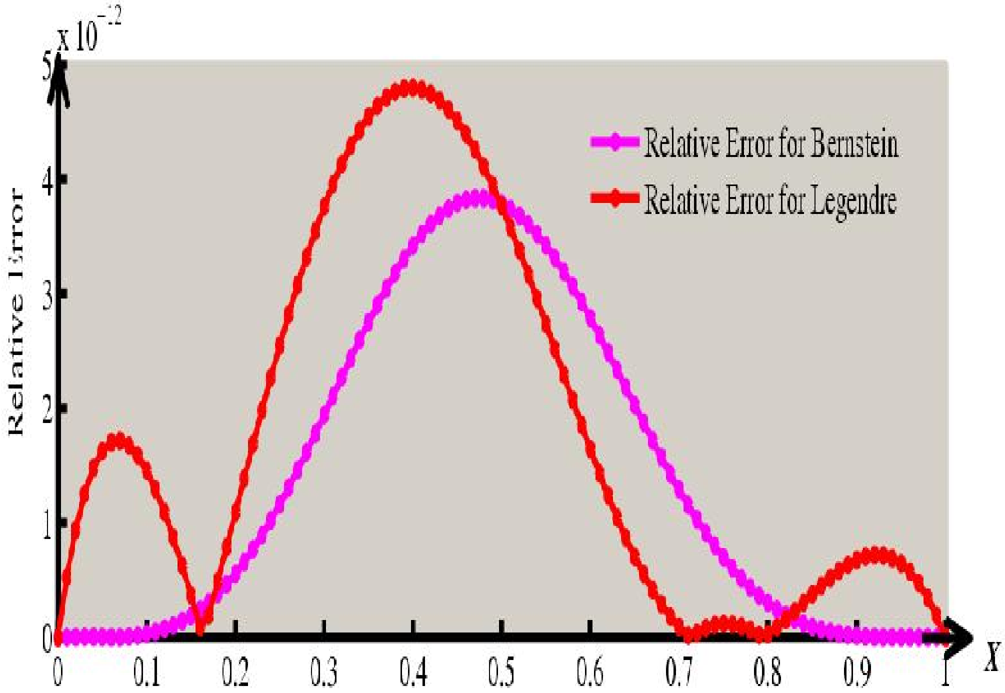


Fig. 7(b): Graphical representation of relative error of example 7 using 12 polynomials.

On the other hand, it is observed that the accuracy is found nearly the order 10^{-6} in [80], [87] and [89] by Wazwaz; Mohy-ud-Din and Yildirim; Nadjafi and Zahmatkesh respectively and nearly the order 10^{-8} in [88] by Mohamed Othman *et al*.

8.4 Conclusions

In this chapter, we have solved tenth order linear and nonlinear BVPs numerically by the Galerkin method. We have used both the Bernstein and Legendre polynomials as basis functions for two different types of boundary conditions. We see from the tables that the numerical results obtained by our method are superior to other existing methods. Also, we get better results for Bernstein polynomials than the Legendre polynomials. It may also note that the numerical solutions co-exist with the exact solution even lower order Bernstein and Legendre polynomials are used in the approximation and are shown in Figs. [1-7].

CHAPTER 9

Eleventh Order Boundary Value Problems

9.1 Introduction

The eleventh order BVPs is presented due to their mathematical importance and the potential for applications in numerous fields of science and engineering. Agarwal [8] derived the theorems stating the conditions for the existence and uniqueness of solutions of such BVPs while no numerical methods are included there in. In the literature of numerical analysis the eleventh order BVPs have not taken much attention. He [97, 98] applied Variational iteration technique for solving nonlinear initial and BVPs. Siddiqi *et al* [99] used Variational iteration technique to obtain numerical approximations for eleventh order BVPs there by converting the original problem into a system of integral equations. Very recently Amjad Hussain *et al* [100] derived the numerical solutions of eleventh order BVPs using differential transformation method.

This chapter is considered for an application of Galerkin method for the numerical solution of linear and nonlinear eleventh order BVPs with Bernstein and Legendre polynomials as basis functions. In this method, the basis functions are modified into a new set of basis functions which vanish at the boundary where the essential type of boundary conditions is mentioned and a matrix formulation is derived for solving the eleventh order BVPs. Results of some numerical examples are tabulated to compare the errors with those developed before.

Moreover, the formulation for solving linear eleventh order BVP by Galerkin weighted residual method with Bernstein and Legendre polynomials is presented in section 9.2. Then several numerical examples are given to verify the proposed formulation in section 9.3 and the conclusions are illustrated in section 9.4.

9.2 Matrix Formulation

In the present chapter, the numerical solution of eleventh order BVP is derived by the Galerkin method with standard (Bernstein and Legendre) polynomials as basis functions. The problem has the following form:

$$a_{11}\frac{d^{11}u}{dx^{11}}a_{10}\frac{d^{10}u}{dx^{10}}+a_9\frac{d^9u}{dx^9}+a_8\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}$$

$$+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r,\; a<x<b \qquad (9.1a)$$

subject to the boundary conditions

$$u(a)=A_0,\quad u(b)=B_0,\quad u'(a)=A_1,\quad u'(b)=B_1,$$

$$u''(a)=A_2,\quad u''(b)=B_2,\quad u'''(a)=A_3,\quad u'''(b)=B_3,$$

$$u^{(iv)}(a)=A_4,\quad u^{(iv)}(b)=B_4,\quad u^{(v)}(a)=A_5 \qquad (9.1b)$$

where A_i, $i=0,1,2,3,4,5$ and B_j, $j=0,1,2,3,4$ are finite real constants and $a_i, i=0,1,\cdots 11$ and r are all continuous functions defined on the interval [a,b]. The BVP (9.1) is solved with the boundary conditions of eqn. (9.1b).

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (9.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$c_{11}\frac{d^{11}u}{dx^{11}}+c_{10}\frac{d^{10}u}{dx^{10}}+c_9\frac{d^9u}{dx^9}+c_8\frac{d^8u}{dx^8}+c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}$$

$$+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=s,\; 0<x<1 \qquad (9.2a)$$

$$u(0)=A_0,\quad \frac{1}{b-a}u'(0)=A_1,\quad \frac{1}{(b-a)^2}u''(0)=A_2,$$

$$u(1)=B_0,\quad \frac{1}{b-a}u'(1)=B_1,\quad \frac{1}{(b-a)^2}u''(1)=B_2,$$

$$\frac{1}{(b-a)^3}u'''(0)=A_3,\quad \frac{1}{(b-a)^3}u'''(1)=B_3,\quad \frac{1}{(b-a)^4}u^{(iv)}(0)=A_4,$$

$$\frac{1}{(b-a)^4}u^{(iv)}(1)=B_4,\quad \frac{1}{(b-a)^5}u^{(iv)}(0)=A_5 \qquad (9.2b)$$

where

$$c_{11}=\frac{1}{(b-a)^{11}}a_{11}((b-a)x+a),\qquad c_{10}=\frac{1}{(b-a)^{10}}a_{10}((b-a)x+a),$$

$c_9 = \dfrac{1}{(b-a)^9} a_9((b-a)x+a),$ $\qquad c_8 = \dfrac{1}{(b-a)^8} a_8((b-a)x+a),$

$c_7 = \dfrac{1}{(b-a)^7} a_7((b-a)x+a),$ $\qquad c_6 = \dfrac{1}{(b-a)^6} a_6((b-a)x+a),$

$c_5 = \dfrac{1}{(b-a)^5} a_5((b-a)x+a),$ $\qquad c_4 = \dfrac{1}{(b-a)^4} a_4((b-a)x+a),$

$c_3 = \dfrac{1}{(b-a)^3} a_3((b-a)x+a),$ $\qquad c_2 = \dfrac{1}{(b-a)^2} a_2((b-a)x+a),$

$c_1 = \dfrac{1}{b-a} a_1((b-a)x+a),$ $\qquad c_0 = a_0((b-a)x+a),$

$s = r((b-a)x+a)$

To solve the boundary value problem (9.2) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \geq 1 \tag{9.3}$$

where $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0,$ for each $i = 1,2,3,\ldots,n.$

Putting eqn. (9.3) into eqn. (9.2a), the weighted residual equations are

$$\int_0^1 \left[c_{11}\frac{d^{11}\tilde{u}}{dx^{11}} + c_{10}\frac{d^{10}\tilde{u}}{dx^{10}} + c_9\frac{d^9\tilde{u}}{dx^9} + c_8\frac{d^8\tilde{u}}{dx^8} + c_7\frac{d^7\tilde{u}}{dx^7} + c_6\frac{d^6\tilde{u}}{dx^6} + c_5\frac{d^5\tilde{u}}{dx^5} + c_4\frac{d^4\tilde{u}}{dx^4} \right.$$

$$\left. + c_3\frac{d^3\tilde{u}}{dx^3} + c_2\frac{d^2\tilde{u}}{dx^2} + c_1\frac{d\tilde{u}}{dx} + c_0\tilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{9.4}$$

Integrating by parts the terms up to second derivative on the left hand side of (9.4), we get

$$\int_0^1 c_{11}\frac{d^{11}\tilde{u}}{dx^{11}} N_{j,n}(x)dx = \left[c_{11}N_{j,n}(x)\frac{d^{10}\tilde{u}}{dx^{10}} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}dx \text{ [Since } N_{j,n}(0) = N_{j,n}(1) = 0\text{]}$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$
$$+ \int_0^1 \frac{d^4}{dx^4}[c_{11}N_{j,n}(x)]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$
$$- \int_0^1 \frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$
$$- \left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$- \left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$- \left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$+ \int_0^1 \frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{9.5}$$

$$\int_0^1 c_{10}\frac{d^{10}\widetilde{u}}{dx^{10}}N_{j,n}(x)dx = \left[c_{10}N_{j,n}(x)\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\int_0^1 \frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (9.6)$$

$$\int_0^1 c_9 \frac{d^9\widetilde{u}}{dx^9} N_{j,n}(x)dx = \left[c_9 N_{j,n}(x)\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$- \int_0^1 \frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (9.7)$$

$$\int_0^1 c_8 \frac{d^8\widetilde{u}}{dx^8} N_{j,n}(x)dx = \left[c_8 N_{j,n}(x)\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$-\int_0^1\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{9.8}$$

$$\int_0^1 c_7\frac{d^7\widetilde{u}}{dx^7}N_{j,n}(x)dx=\left[c_7N_{j,n}(x)\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \tag{9.9}$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (9.10)$$

$$\int_0^1 c_5 \frac{d^5\tilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x)\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (9.11)$$

$$\int_0^1 c_4 \frac{d^4\tilde{u}}{dx^4} N_{j,n}(x)dx = \left[c_4 N_{j,n}(x) \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (9.12)$$

$$\int_0^1 c_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx = \left[c_3 N_{j,n}(x) \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_3 N_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (9.13)$$

$$\int_0^1 c_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = \left[c_2 N_{j,n}(x) \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx$$

$$= -\int_0^1 \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \qquad (9.14)$$

Putting eqn. (9.5) – (9.14) into eqn. (9.4) and using approximation for $\tilde{u}(x)$ given in eqn. (9.3) and after applying the boundary conditions given in eqn. (9.2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (9.15a)$$

where

$$D_{i,j} = \int_0^1 \Bigg\{ \Bigg[\frac{d^{10}}{dx^{10}}\left[c_{11} N_{j,n}(x)\right] - \frac{d^9}{dx^9}\left[c_{10} N_{j,n}(x)\right] + \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] - \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right]$$

$$+ \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]$$

$$+\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]-\frac{d}{dx}\left[c_2N_{j,n}(x)\right]+c_1N_{j,n}(x)\Bigg]\frac{d}{dx}\left[N_{i,n}(x)\right]+c_0N_{i,n}(x)N_{j,n}(x)\Bigg\}dx$$

$$-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \qquad (9.15b)$$

$$F_j = \int_0^1 \left\{ sN_{j,n}(x) + \left[-\frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right] + \frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] - \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right] + \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right] \right.\right.$$

$$- \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right] + \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right] - \frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right] + \frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right] + \frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]$$

$$\left.\left. + \frac{d}{dx}\left[c_2 N_{j,n}(x)\right] - c_1 N_{j,n}(x) \right]\frac{d\theta_0}{dx} - c_0\theta_0 N_{j,n}(x) \right\} dx + \left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9} \right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9} \right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8} \right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8} \right]_{x=0}$$

$$- \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=1}$$

$$+ \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8} \right]_{x=1} - \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=0}$$

$$- \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=0} - \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8} \right]_{x=0}$$

$$- \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=0} - \left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=1} + \left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7} \right]_{x=1}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6} \right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1} + \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5} \right]_{x=1}$$

$$- \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right] \right]_{x=0} \times (b-a)^5 A_5 - \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right] \right]_{x=1} \times (b-a)^4 B_4$$

$$+ \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right] \right]_{x=0} \times (b-a)^4 A_4 + \left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right] \right]_{x=1} \times (b-a)^3 B_3$$

$$-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3-\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2$$

$$+\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2+\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5A_5$$

$$+\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3$$

$$+\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2$$

$$-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1$$

$$-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5A_5-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4+\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3$$

$$-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3-\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2$$

$$+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2+\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5A_5$$

$$+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3$$

$$+\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1+\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) A_1$$

$$-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5 A_5-\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$+\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1$$

$$-\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1+\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4-\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3+\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a) B_1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a) A_1+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^2}{dx^2}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2+\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2$$

$$-\left[\frac{d}{dx}\left[c_4N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2-\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\right]_{x=1}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1 \qquad (9.15c)$$

Solving the system (9.15a), we find the values of the parameters α_i and then substituting these parameters into eqn. (9.3), we get the approximate solution of the BVP (9.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (9.1).

For nonlinear eleventh-order BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (9.15). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

9.3 Numerical examples and results

To test the applicability of the proposed method, we consider two linear and one nonlinear problems. For all examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB 10***. The convergence of linear BVP is calculated by

$$E=\left|\tilde{u}_{n+1}(x)-\tilde{u}_n(x)\right|<\delta$$

where $\widetilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-13}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\widetilde{u}_n^{N+1} - \widetilde{u}_n^{N}\right| < \delta$$

where $\delta < 10^{-12}$ and N is the Newton's iteration number

Example 1: Consider the linear differential equation of eleventh-order [99, 100]

$$\frac{d^{11}u}{dx^{11}} - u = -22(5+x)e^x, \quad 0 \le x \le 1 \tag{9.16a}$$

subject to the boundary conditions

$$u(0)=1,\ u(1)=0,\ u'(0)=1, u'(1)=-2e, u''(0)=-1, u''(1)=-6e, u'''(0)=-5,$$
$$u'''(1)=-12e, u^{(iv)}(0)=-11, u^{(iv)}(1)=-20e, u^{(v)}(0)=-19\,. \tag{9.16b}$$

The analytic solution of the above problem is, $u(x)=(1-x^2)e^x$.

Using the method given in section (9.2), we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ n \ge 1 \tag{9.17}$$

Here $\theta_0(x) = 1 - x$ is specified by the essential boundary conditions of equation (9.16b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j=1,2,\ldots,n \tag{9.18a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right] \frac{d}{dx}\left[N_{i,n}(x)\right] - N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^9}{dx^9}\left[N_{i,n}(x)\right] \right]_{x=1}$$
$$+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right] \frac{d^9}{dx^9}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right] \frac{d^8}{dx^8}\left[N_{i,n}(x)\right] \right]_{x=1}$$
$$- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right] \frac{d^8}{dx^8}\left[N_{i,n}(x)\right] \right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right] \frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \quad (9.\ 18b)$$

$$F_j=\int_0^1\left\{-22(5+x)e^xN_{j,n}(x)-\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]\frac{d\theta_0}{dx}+\theta_0N_{j,n}(x)\right\}dx$$

$$-\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0}(-19)-\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1}(-20e)$$

$$+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0}(-11)-\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1}(-6e)$$

$$+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}(-12e)-\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}(-5)$$

$$+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0}(-1)+\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1}(-2e)$$

$$-\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=0} \quad (9.18c)$$

Solving the system (9.18a) we obtain the values of the parameters and then substituting these parameters into eqn. (9.17), we get the approximate solution of the BVP (9.16) for different values of n.

Example 2: Consider the linear BVP [99, 100]

$$\frac{d^{11}u}{dx^{11}}+u=22(5\sin x+x\cos x)+(1-x^2)(\sin x+\cos x),\quad 0\le x\le 1 \quad (9.19a)$$

$$u(0)=1, u(1)=0,\ u'(0)=0, u'(1)=-2\cos 1, u''(0)=-3, u''(1)=4\sin 1-2\cos 1, u'''(0)=0,$$

$$u'''(1)=6\sin 1+6\cos 1, u^{(iv)}(0)=13, u^{(iv)}(1)=-8\sin 1+12\cos 1, u^{(v)}(0)=0\ . \quad (9.19b)$$

The analytic solution of the above problem is, $u(x)=(1-x^2)\cos x$.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 1.**

Table 1: Maximum absolute errors for the example 1.

x	Exact Results	14 Bernstein Polynomials		14 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.0000000E+000
0.1	1.0941192089	1.0941192089	1.3322676E-015	1.0941192089	2.4424907E-015
0.2	1.1725466478	1.1725466478	4.4408921E-016	1.1725466478	1.9984014E-015
0.3	1.2283715149	1.2283715149	1.1102230E-015	1.2283715149	3.3306691E-015
0.4	1.2531327460	1.2531327460	6.6613381E-016	1.2531327460	5.1070259E-015
0.5	1.2365409530	1.2365409530	4.4408921E-016	1.2365409530	8.8817842E-016
0.6	1.1661560322	1.1661560322	8.8817842E-016	1.1661560322	4.8849813E-015
0.7	1.0270138808	1.0270138808	2.2204460E-016	1.0270138808	6.2172489E-015
0.8	0.8011947343	0.8011947343	9.9920072E-016	0.8011947343	2.1094237E-015
0.9	0.4673245911	0.4673245911	5.5511151E-017	0.4673245911	3.1641356E-015
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand, it is observed that the accuracy is found nearly the order 10^{-12} in [99] by Siddiqi *et al* and nearly the order 10^{-13} in [100] by Amjad Hussain *et al*

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 14$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-14}.

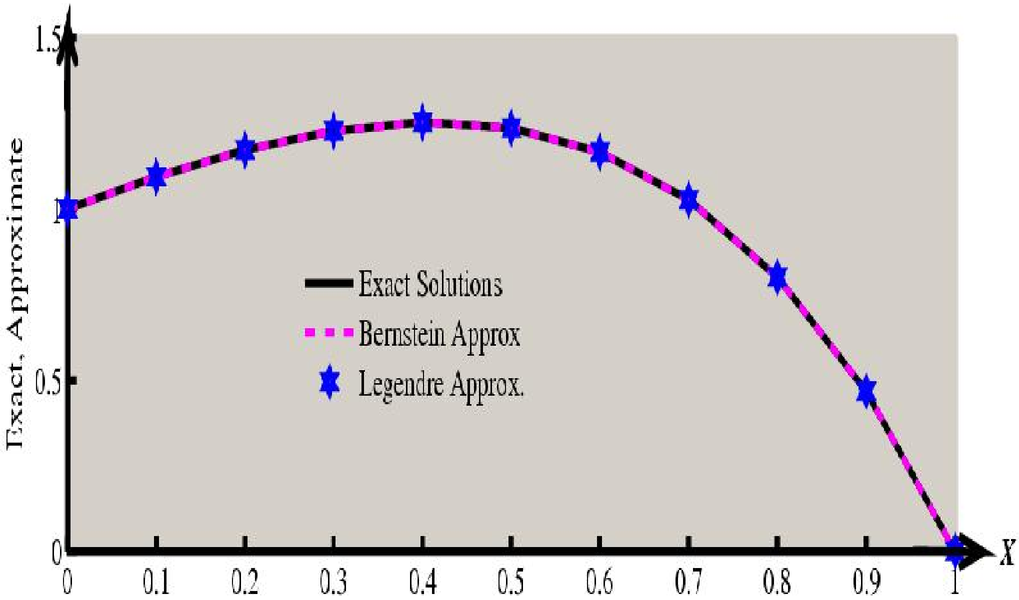


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 14 polynomials.

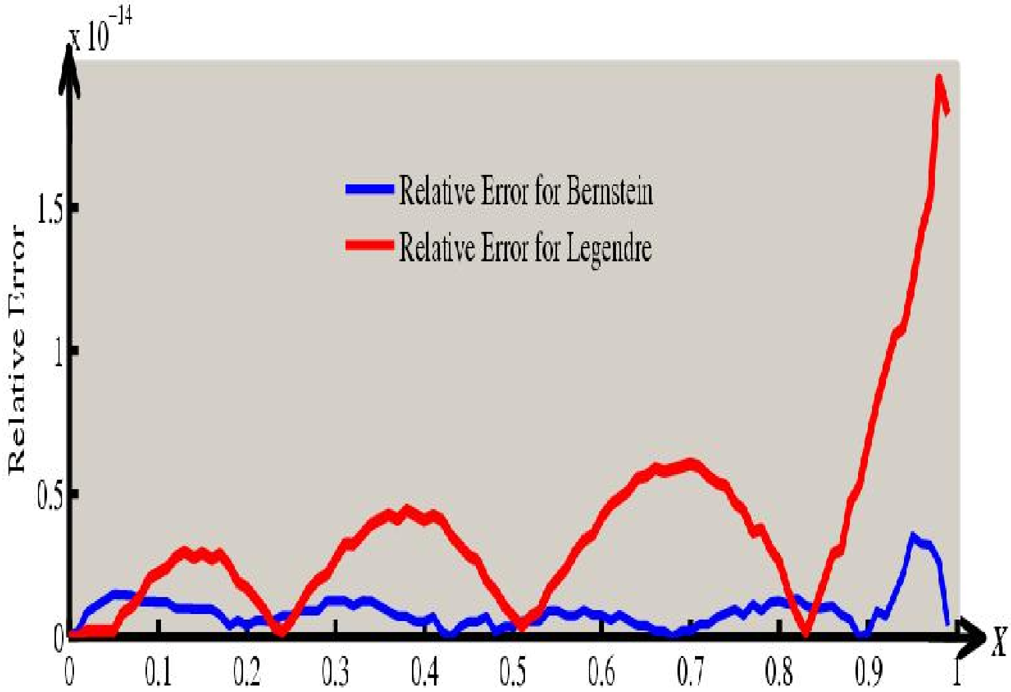


Fig. 1(b): Graphical representation of relative error of example 1 using 14 polynomials.

Employing the method mentioned in (9.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{9.20}$$

Here $\theta_0(x) = 1 - x$ is specified by the essential boundary conditions of equation (9.19b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j, j = 1,2,\ldots,n \tag{9.21a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^{10}}{dx^{10}}[N_{j,n}(x)] \frac{d}{dx}[N_{i,n}(x)] + N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d}{dx}[N_{j,n}(x)] \frac{d^9}{dx^9}[N_{i,n}(x)] \right]_{x=1}$$

$$+ \left[\frac{d}{dx}[N_{j,n}(x)] \frac{d^9}{dx^9}[N_{i,n}(x)] \right]_{x=0} + \left[\frac{d^2}{dx^2}[N_{j,n}(x)] \frac{d^8}{dx^8}[N_{i,n}(x)] \right]_{x=1}$$

$$- \left[\frac{d^2}{dx^2}[N_{j,n}(x)] \frac{d^8}{dx^8}[N_{i,n}(x)] \right]_{x=0} - \left[\frac{d^3}{dx^3}[N_{j,n}(x)] \frac{d^7}{dx^7}[N_{i,n}(x)] \right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}-\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1} \quad (9.21b)$$

$$F_j=\int_0^1\left\{\left[22(5\sin x+x\cos x)+(1-x^2)(\sin x+\cos x)\right]N_{j,n}(x)-\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]\frac{d\theta_0}{dx}\right.$$

$$\left.-\theta_0 N_{j,n}(x)dx\right\}-\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1}\times(-8\sin 1+12\cos 1)+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0}\times 13$$

$$+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}\times(6\sin 1+6\cos 1)-\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1}\times(4\sin 1-2\cos 1)$$

$$+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0}\times(-3)+\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1}(-2\cos 1) \quad (9.21c)$$

Solving the system (9.21a) we obtain the values of the parameters and then substituting these parameters into eqn. (9.20), we get the approximate solution of the BVP (9.19) for different values of *n*.

The maximum absolute errors, shown in **Table 2,** are listed to compare with existing results.

Table 2: Maximum absolute errors for the example 2.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
12	7.669×10^{-11}	7.669×10^{-11}	
13	8.826×10^{-14}	8.848×10^{-14}	5.148×10^{-12} (Siddiqi *et al* [99])
14	5.551×10^{-16}	2.442×10^{-15}	9.560×10^{-13} (Amjad Hussain *et al* [100])
15	8.882×10^{-16}	2.220×10^{-16}	

We have shown the exact and approximate solutions in Fig. 2(a) and the relative errors in Fig. 2(b) of example 2 for $n=15$. It is found from Fig. 2(b) that the error is of the order 10^{-14}.

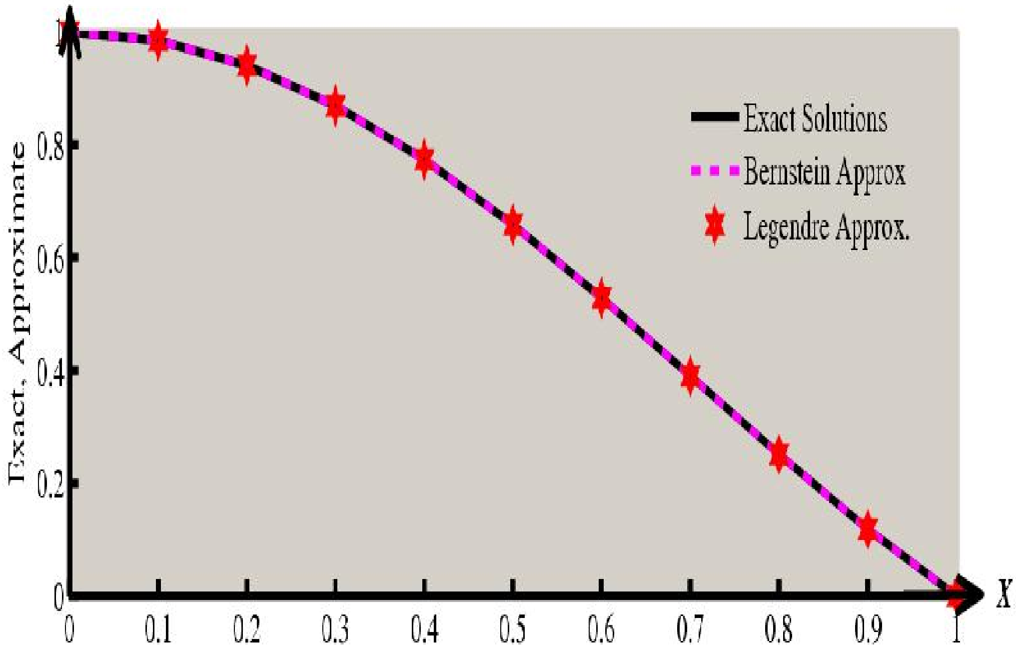


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 15 polynomials.

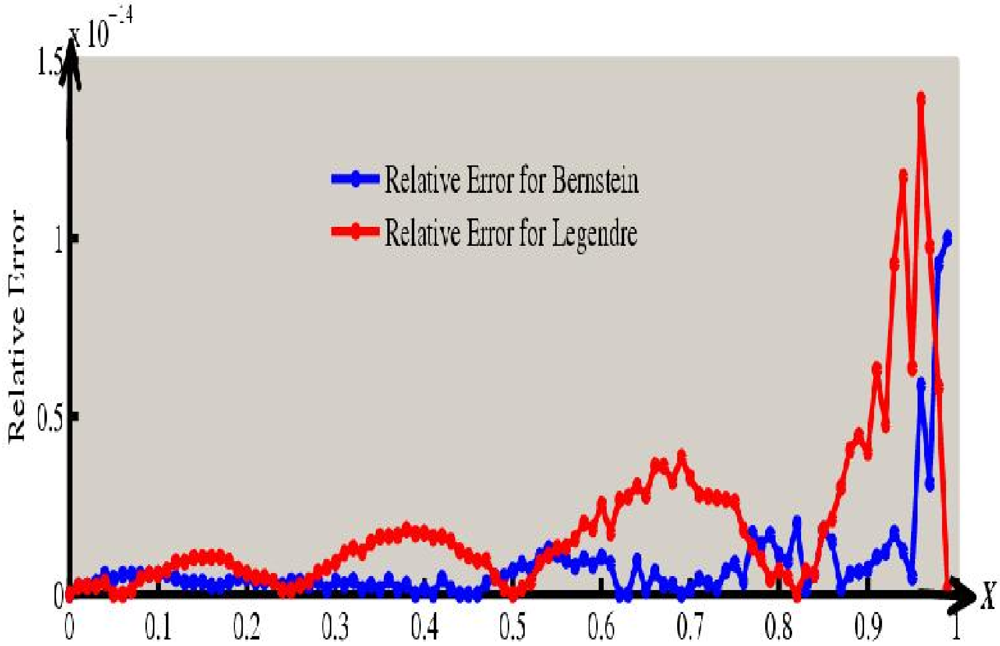


Fig. 2(b): Graphical representation of relative error of example 2 using 15 polynomials.

Example 3: Consider the following **nonlinear** differential equation [99]

$$\frac{d^{11}u}{dx^{11}} = 11(\cos x - \sin x) - x(\cos x + \sin x) - u^2 + x^2(1 - 2\sin x \cos x),\ 0 \le x \le 1 \qquad (9.22a)$$

subject to the boundary conditions

$u(0)=0, u(1)=\sin 1-\cos 1,\ u'(0)=-1, u'(1)=2\sin 1, u''(0)=2, u''(1)=\sin 1+3\cos 1,$

$u'''(0)=3, u'''(1)=-4\sin 1+2\cos 1, u^{(iv)}(0)=-4, u^{(iv)}(1)=-3\sin 1-5\cos 1, u^{(v)}(0)=-5.$

(9.22b)

The exact solution of this BVP is $u(x)=x(\sin x-\cos x)$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\geq 1 \tag{9.23}$$

Here $\theta_0(x)=1+x(\sin 1-\cos 1-1)$ is specified by the essential boundary conditions in (9.22b). Also $N_{i,n}(0)=N_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Putting eqn. (9.23) into eqn. (9.22a), the Galerkin weighted residual equations are

$$\int_0^1\left[\frac{d^{11}\tilde{u}}{dx^{11}}+\tilde{u}^2-11(\cos x-\sin x)+x(\cos x+\sin x)-x^2(1-2\sin x\cos x)\right]N_{k,n}(x)dx=0$$

(9.24)

Integrating 1[st] term of (9.24) by parts, we obtain

$$\int_0^1\frac{d^{11}\tilde{u}}{dx^{11}}N_{k,n}(x)=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\tilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\tilde{u}}{dx^8}\right]_0^1-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\tilde{u}}{dx^7}\right]_0^1$$
$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\tilde{u}}{dx^6}\right]_0^1-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d^9N_{k,n}(x)}{dx^9}\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\tilde{u}}{dx}dx \tag{9.25}$$

Using eqn. (9.25) into eqn. (9.24) and using approximation for $\tilde{u}(x)$ given in eqn. (9.23) and after applying the conditions given in eqn. (9.22b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}+2\theta_0 N_{i,n}(x)N_{k,n}(x)+\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))\right]\right]dx$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{9}N_{i,n}(x)}{dx^{9}}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{9}N_{i,n}(x)}{dx^{9}}\right]_{x=0}+\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{8}N_{i,n}(x)}{dx^{8}}\right]_{x=1}$$

$$-\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{8}N_{i,n}(x)}{dx^{8}}\right]_{x=0}-\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{7}N_{i,n}(x)}{dx^{7}}\right]_{x=1}+\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{7}N_{i,n}(x)}{dx^{7}}\right]_{x=0}$$

$$+\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{6}N_{i,n}(x)}{dx^{6}}\right]_{x=1}-\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{6}N_{i,n}(x)}{dx^{6}}\right]_{x=0}-\left[\frac{d^{5}N_{k,n}(x)}{dx^{5}}\frac{d^{5}N_{i,n}(x)}{dx^{5}}\right]_{x=1}\Bigg]\alpha_i$$

$$=\int_0^1\Bigg[-\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}-\theta_0^{\,2}N_{k,n}(x)+\{x(\cos x+\sin x)-11(\cos x-\sin x)$$

$$-x^2(1-2\sin x\cos x)\}\Big]N_{k,n}(x)+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{9}\theta_0}{dx^{9}}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{9}\theta_0}{dx^{9}}\right]_{x=0}$$

$$-\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{8}\theta_0}{dx^{8}}\right]_{x=1}+\left[\frac{d^{2}N_{k,n}(x)}{dx^{2}}\frac{d^{8}\theta_0}{dx^{8}}\right]_{x=0}+\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{7}\theta_0}{dx^{7}}\right]_{x=1}$$

$$-\left[\frac{d^{3}N_{k,n}(x)}{dx^{3}}\frac{d^{7}\theta_0}{dx^{7}}\right]_{x=0}-\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{6}\theta_0}{dx^{6}}\right]_{x=1}+\left[\frac{d^{4}N_{k,n}(x)}{dx^{4}}\frac{d^{6}\theta_0}{dx^{6}}\right]_{x=0}$$

$$+\left[\frac{d^{5}N_{k,n}(x)}{dx^{5}}\frac{d^{5}\theta_0}{dx^{5}}\right]_{x=1}-\left[\frac{d^{5}N_{k,n}(x)}{dx^{5}}\right]_{x=0}\times(-5)-\left[\frac{d^{6}N_{k,n}(x)}{dx^{6}}\right]_{x=1}\times(-3\sin1-5\cos1)$$

$$+\left[\frac{d^{6}N_{k,n}(x)}{dx^{6}}\right]_{x=0}\times(-4)+\left[\frac{d^{7}N_{k,n}(x)}{dx^{7}}\right]_{x=1}\times(-4\sin1+2\cos1)-\left[\frac{d^{7}N_{k,n}(x)}{dx^{7}}\right]_{x=0}\times3$$

$$-\left[\frac{d^{8}N_{k,n}(x)}{dx^{8}}\right]_{x=1}\times(\sin1+3\cos1)+\left[\frac{d^{8}N_{k,n}(x)}{dx^{8}}\right]_{x=0}\times2+\left[\frac{d^{9}N_{k,n}(x)}{dx^{9}}\right]_{x=1}\times(2\sin1)$$

$$-\left[\frac{d^{9}N_{k,n}(x)}{dx^{9}}\right]_{x=0}\times(-1) \tag{9.26}$$

The above equation (9.26) is equivalent to matrix form

$$(D+B)A=G \tag{9.27a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}+2\theta_0N_{i,n}(x)N_{k,n}(x)\ \right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=1}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1} \qquad (9.27b)$$

$$b_{i,k}=\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))\,dx \qquad (9.27c)$$

$$g_k=\int_0^1\left[-\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}-\theta_0{}^2N_{k,n}(x)+\{x(\cos x+\sin x)-11(\cos x-\sin x)\right.$$

$$\left.-x^2(1-2\sin x\cos x)\}\right]N_{k,n}(x)dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-5)-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}$$

$$\times(-3\sin 1-5\cos 1)+\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0}\times(-4)+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times(-4\sin 1+2\cos 1)$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}\times 3-\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=1}\times(\sin 1+3\cos 1)+\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=0}\times 2$$

$$+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times(2\sin 1)-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}\times(-1) \qquad (9.27\text{d})$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (9.22a). That is, to find initial coefficients we solve the system

$$DA = G \qquad (9.28\text{a})$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1 \frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=1}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=0}$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}$$

$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1} \qquad (9.28\text{b})$$

$$g_k=\int_0^1\Big[-\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}+\{x(\cos x+\sin x)-11(\cos x-\sin x)$$

$$-x^2(1-2\sin x\cos x)\}\Big]N_{k,n}(x)dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=0}+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-5)-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}$$

$$\times(-3\sin 1-5\cos 1)+\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0}\times(-4)+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times(-4\sin 1+2\cos 1)$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}\times 3-\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=1}\times(\sin 1+3\cos 1)+\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=0}\times 2$$

$$+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times(2\sin 1)-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}\times(-1) \qquad (9.28c)$$

Once the initial values of α_i are obtained from eqn. (9.28a), they are substituted into eqn. (9.27a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (9.23), we obtain an approximate solution of the BVP (9.22).

Numerical results for example 3 are shown in the following **Table 3**.

Table 3: Numerical results for example 3 using 6 iterations

x	Exact Results	13 Bernstein Polynomials		13 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	5.3209018E-027
0.1	-0.0895170749	-0.0895170749	9.7560848E-015	-0.0895170749	9.7977182E-015
0.2	-0.1562794494	-0.1562794494	1.0380585E-014	-0.1562794494	1.0602630E-014
0.3	-0.1979448847	-0.1979448847	2.9445890E-013	-0.1979448847	2.9437563E-013
0.4	-0.2126570607	-0.2126570607	8.5739749E-013	-0.2126570607	8.5667584E-013
0.5	-0.1990785116	-0.1990785116	1.3202772E-012	-0.1990785116	1.3194446E-012
0.6	-0.1564158849	-0.1564158849	1.2643497E-012	-0.1564158849	1.2642110E-012
0.7	-0.0844371500	-0.0844371500	7.2382378E-013	-0.0844371500	7.2435113E-013
0.8	0.0165195052	0.0165195052	1.9494822E-013	0.0165195052	1.9536109E-013
0.9	0.1455452472	0.1455452472	2.6478819E-014	0.1455452472	2.6062486E-014
1.0	0.3011686789	0.3011686789	0.0000000E+000	0.3011686789	0.0000000E+000

On the contrary the maximum absolute error has been found by Siddiqi *et al* [99] is 4.415×10^{-10}

We have shown the exact and approximate solutions in Fig. 3(a) and the relative errors in Fig. 3(b) of example 3 for $n = 13$. It is found from Fig. 3(b) that the error is of the order 10^{-11}

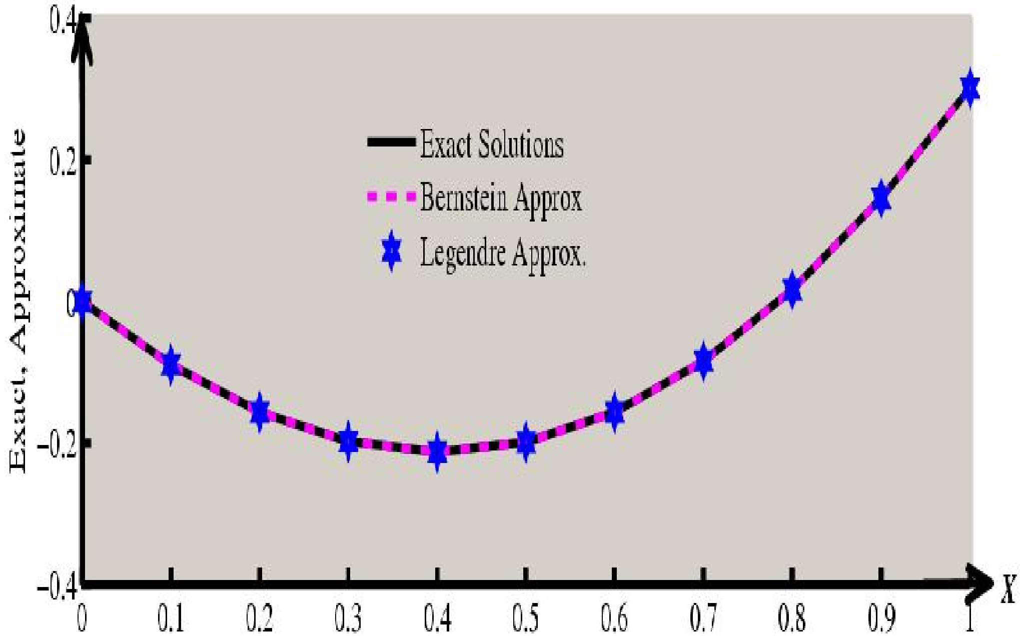


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 13 polynomials.

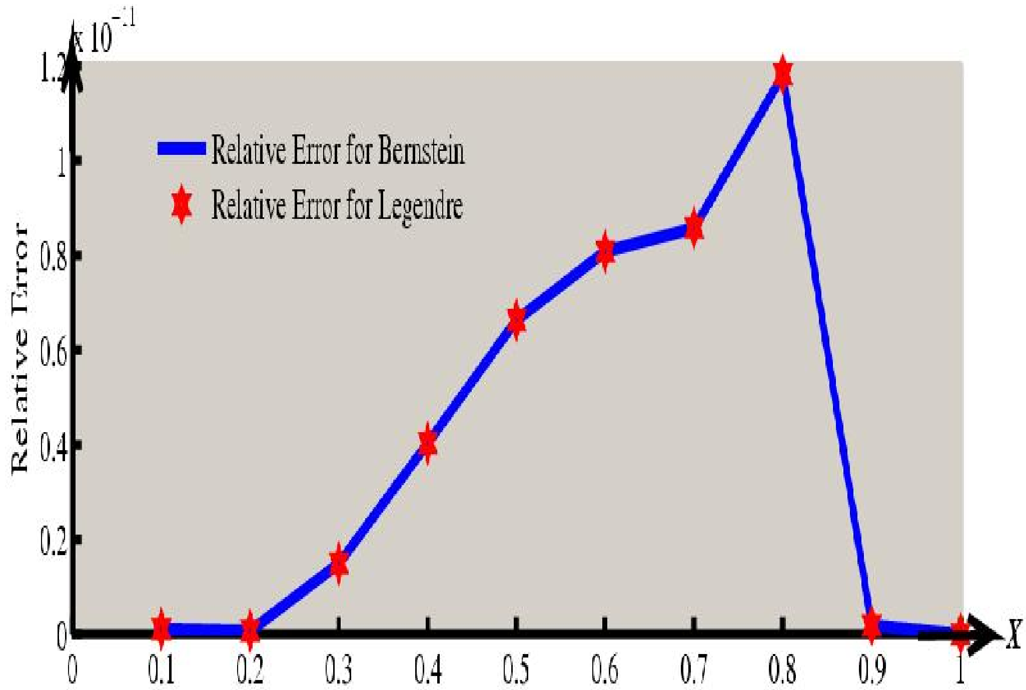


Fig. 3(b): Graphical representation of relative error of example 3 using 13 polynomials.

9.4 Conclusions

In this chapter, we have used Bernstein and Legendre polynomials as basis functions for the numerical solution of eleventh order linear and nonlinear BVPs in the Galerkin method. It is clear from the tables that the numerical results obtained by our method are superior to other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. It may also notice that the numerical solutions are identical with the exact solution even lower order Bernstein and Legendre polynomials are used in the approximation.

CHAPTER 10

Twelfth Order Boundary Value Problems

10.1 Introduction

The higher order BVPs are known to arise in hydrodynamic, hydro magnetic stability and applied sciences. Recently the twelfth order BVPs have been investigated due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability. Chandrasekhar [9] determined that if an infinite horizontal layer of fluid is heated from below with the supposition that a uniform magnetic field is also used across in the same direction as gravity and the fluid is under the action of rotation, instability sets in. When the instability sets in as overstability, it is modelled by twelfth order BVP. The literature on the numerical solutions of twelfth order BVPs and associated eigenvalue problems is found not to be too much. Finite difference methods for the solution of such problems were developed in [40, 41, 68, 69, 82]. Twizell *et al* [70] presented numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [29, 30] solved the tenth and twelfth order BVPs using tenth and twelfth degree splines respectively. Siddiqi and Ghazala Akram [57, 60] developed the solutions of tenth and twelfth order BVPs applying eleventh and thirteen degree spline respectively. Usmani [13] presented the solution of fourth order BVP using quartic splines. Kudri and mulhem [102] derived the numerical solutions of twelfth order BVPs using adomain decomposition method. Approximate solutions of twelfth order BVPs were presented by Mohy-ud-Din *et al* [103]. Mirmoradi *et al* [104] solved twelfth order BVPs by the homotopy perturbation method. Noor and Mohy-ud-Din [105] used variational iteration method for solving twelfth order BVPs applying He's polynomials.

These are few numerical techniques are available to solve twelfth order BVPs. For this, we have used Galerkin method for the numerical solution of the twelfth order BVPs with Bernstein and Legendre polynomials as basis functions for two different cases of boundary conditions. In this method, the basis functions are modified into a new set of basis functions which vanish at the boundary where the essential type of boundary conditions is mentioned and a matrix formulation is derived for solving the twelfth order BVPs. Results of some numerical examples are tabulated to compare the errors with those developed before.

However, the formulation for solving linear twelfth order BVP by Galerkin weighted residual method with Bernstein and Legendre polynomials is presented in section 10.2. Two formulations are given by applying two types of boundary conditions in sections 10.2.1 and 10.2.2 respectively. Then several numerical examples are given to verify the proposed formulation. Finally, the conclusions of this chapter are presented in section 10.4.

10.2 Formulation by the Galerkin method

In this present chapter, the solution of twelfth order BVP is derived by the Galerkin method with standard (Bernstein and Legendre) polynomials as basis functions for two different types of boundary conditions. The problem has the form

$$a_{12}\frac{d^{12}u}{dx^{12}}+a_{11}\frac{d^{11}u}{dx^{11}}a_{10}\frac{d^{10}u}{dx^{10}}+a_9\frac{d^9u}{dx^9}+a_8\frac{d^8u}{dx^8}+a_7\frac{d^7u}{dx^7}+a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}$$

$$+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r,\ a<x<b \tag{10.1a}$$

subject to the following two types of boundary conditions

TypeI: $u(a)=A_0,\quad u(b)=B_0,\quad u'(a)=A_1,\quad u'(b)=B_1,$

$u''(a)=A_2,\quad u''(b)=B_2,\quad u'''(a)=A_3,\quad u'''(b)=B_3,$

$u^{(iv)}(a)=A_4,\quad u^{(iv)}(b)=B_4,\quad u^{(v)}(a)=A_5,\quad u^{(v)}(b)=B_5$

(10.1b)

TypeII: $u(a)=A_0,\quad u(b)=B_0,\quad u''(a)=A_2,\quad u''(b)=B_2,$

$u^{(iv)}(a)=A_4,\quad u^{(iv)}(b)=B_4,\quad u^{(vi)}(a)=A_6,\quad u^{(vi)}(b)=B_6,$

$u^{(viii)}(a)=A_8,\quad u^{(viii)}(b)=B_8,\quad u^{(x)}(a)=A_{10},\quad u^{(x)}(b)=B_{10}$

(10.1b)

where $A_i, B_i, i = 0,1,2,3,4,6,8,10$ are finite real constants and $a_i, i = 0,1,\cdots 12$ and r are all continuous functions defined on the interval [*a,b*]. The BVP (10.1) is solved with both cases of the boundary conditions of Type I and Type II.

Since our aim is to use the Bernstein and Legendre polynomials as trial functions which are derived over the interval [0, 1], so the BVP (10.1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x+a$, and thus we have:

$$c_{12}\frac{d^{12}u}{dx^{12}}+c_{11}\frac{d^{11}u}{dx^{11}}+c_{10}\frac{d^{10}u}{dx^{10}}+c_9\frac{d^9u}{dx^9}+c_8\frac{d^8u}{dx^8}+c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}$$

$$+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=s,\ 0<x<1 \qquad (10.2a)$$

$$u(0)=A_0, \qquad \frac{1}{b-a}u'(0)=A_1, \qquad \frac{1}{(b-a)^2}u''(0)=A_2,$$

$$u(1)=B_0, \qquad \frac{1}{b-a}u'(1)=B_1, \qquad \frac{1}{(b-a)^2}u''(1)=B_2,$$

$$\frac{1}{(b-a)^3}u'''(0)=A_3, \qquad \frac{1}{(b-a)^3}u'''(1)=B_3, \qquad \frac{1}{(b-a)^4}u^{(iv)}(0)=A_4,$$

$$\frac{1}{(b-a)^4}u^{(iv)}(1)=B_4, \qquad \frac{1}{(b-a)^5}u^{(v)}(1)=B_5, \qquad \frac{1}{(b-a)^5}u^{(iv)}(0)=A_5$$

$$(10.2b)$$

and

$$u(0)=A_0, \qquad \frac{1}{(b-a)^2}u''(0)=A_2, \qquad \frac{1}{(b-a)^4}u^{(iv)}(0)=A_4,$$

$$u(1)=B_0, \qquad \frac{1}{(b-a)^2}u''(1)=B_2, \qquad \frac{1}{(b-a)^4}u^{(iv)}(1)=B_4,$$

$$\frac{1}{(b-a)^6}u^{(vi)}(0)=A_6, \qquad \frac{1}{(b-a)^6}u^{(vi)}(1)=B_6, \qquad \frac{1}{(b-a)^8}u^{(viii)}(0)=A_8,$$

$$\frac{1}{(b-a)^8}u^{(viii)}(1)=B_8, \qquad \frac{1}{(b-a)^{10}}u^{(x)}(0)=A_{10}, \qquad \frac{1}{(b-a)^{10}}u^{(x)}(1)=B_{10}$$

$$(10.2c)$$

where

$c_{12} = \frac{1}{(b-a)^{12}} a_{12}((b-a)x+a),$ $\qquad$ $c_{11} = \frac{1}{(b-a)^{11}} a_{11}((b-a)x+a),$

$c_{10} = \frac{1}{(b-a)^{10}} a_{10}((b-a)x+a),$ $\qquad$ $c_{9} = \frac{1}{(b-a)^{9}} a_{9}((b-a)x+a),$

$c_{8} = \frac{1}{(b-a)^{8}} a_{8}((b-a)x+a),$ $\qquad$ $c_{7} = \frac{1}{(b-a)^{7}} a_{7}((b-a)x+a),$

$c_{6} = \frac{1}{(b-a)^{6}} a_{6}((b-a)x+a),$ $\qquad$ $c_{5} = \frac{1}{(b-a)^{5}} a_{5}((b-a)x+a),$

$c_{4} = \frac{1}{(b-a)^{4}} a_{4}((b-a)x+a),$ $\qquad$ $c_{3} = \frac{1}{(b-a)^{3}} a_{3}((b-a)x+a),$

$c_{2} = \frac{1}{(b-a)^{2}} a_{2}((b-a)x+a),$ $\qquad$ $c_{1} = \frac{1}{b-a} a_{1}((b-a)x+a),$

$c_{0} = a_{0}((b-a)x+a),$ $\qquad$ $s = r((b-a)x+a)$

To solve the boundary value problem (10.2) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \; n \geq 1 \tag{10.3}$$

where $\theta_0(x)$ is specified by the essential boundary conditions, $N_{i,n}(x)$ are the Bernstein or Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $N_{i,n}(0) = N_{i,n}(1) = 0,$ for each $i = 1,2,3,\ldots,n.$

Putting eqn. (10.3) into eqn. (10.2a), the weighted residual equations are

$$\int_0^1 \left[c_{12}\frac{d^{12}\tilde{u}}{dx^{12}} + c_{11}\frac{d^{11}\tilde{u}}{dx^{11}} + c_{10}\frac{d^{10}\tilde{u}}{dx^{10}} + c_{9}\frac{d^{9}\tilde{u}}{dx^{9}} + c_{8}\frac{d^{8}\tilde{u}}{dx^{8}} + c_{7}\frac{d^{7}\tilde{u}}{dx^{7}} + c_{6}\frac{d^{6}\tilde{u}}{dx^{6}} + c_{5}\frac{d^{5}\tilde{u}}{dx^{5}} \right.$$

$$\left. + c_{4}\frac{d^{4}\tilde{u}}{dx^{4}} + c_{3}\frac{d^{3}\tilde{u}}{dx^{3}} + c_{2}\frac{d^{2}\tilde{u}}{dx^{2}} + c_{1}\frac{d\tilde{u}}{dx} + c_0\tilde{u} - s \right] N_{j,n}(x)dx = 0 \tag{10.4}$$

10.2.1 Formulation I

In this section, we obtain the matrix formulation by applying the boundary conditions of type I.

Integrating by parts the terms up to second derivative on the left hand side of (10.4), we get

$$\int_0^1 c_{12}\frac{d^{12}\tilde{u}}{dx^{12}}N_{j,n}(x)dx = \left[c_{12}N_{j,n}(x)\frac{d^{11}\tilde{u}}{dx^{11}}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{11}\tilde{u}}{dx^{11}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}dx \text{ [Since } N_{j,n}(0) = N_{j,n}(1) = 0\text{]}$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^{8}\tilde{u}}{dx^{8}}\right]_0^1$$
$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^{8}\tilde{u}}{dx^{8}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^{8}\tilde{u}}{dx^{8}}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}\tilde{u}}{dx^{7}}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}\tilde{u}}{dx^{7}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^{8}\tilde{u}}{dx^{8}}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}\tilde{u}}{dx^{7}}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^{6}\tilde{u}}{dx^{6}}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^{6}\tilde{u}}{dx^{6}}dx$$

$$= -\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}\tilde{u}}{dx^{9}}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^{8}\tilde{u}}{dx^{8}}\right]_0^1$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}\tilde{u}}{dx^{7}}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^{6}\tilde{u}}{dx^{6}}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^{5}\tilde{u}}{dx^{5}}\right]_0^1$$
$$- \int_0^1 \frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\frac{d^{5}\tilde{u}}{dx^{5}}dx$$

$$=-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\widetilde{u}}{dx^{10}}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\widetilde{u}}{dx^{10}}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\widetilde{u}}{dx^{10}}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+\int_0^1\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\widetilde{u}}{dx^{10}}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1-\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$+\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^{11}}{dx^{11}}\left[c_{12}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (10.5)$$

$$\int_0^1 c_{11}\frac{d^{11}\widetilde{u}}{dx^{11}}N_{j,n}(x)dx=\left[c_{11}N_{j,n}(x)\frac{d^{10}\widetilde{u}}{dx^{10}}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^{10}\widetilde{u}}{dx^{10}}dx$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}dx\ [\text{Since } N_{j,n}(0)=N_{j,n}(1)=0\,]$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}[c_{11}N_{j,n}(x)]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$-\int_0^1\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$+\int_0^1 \frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (10.6)$$

$$\int_0^1 c_{10}\frac{d^{10}\widetilde{u}}{dx^{10}}N_{j,n}(x)dx = \left[c_{10}N_{j,n}(x)\frac{d^9\widetilde{u}}{dx^9}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^9\widetilde{u}}{dx^9}dx$$

$$= -\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\int_0^1\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\int_0^1\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (10.7)$$

$$\int_0^1 c_9\frac{d^9\widetilde{u}}{dx^9}N_{j,n}(x)dx=\left[c_9N_{j,n}(x)\frac{d^8\widetilde{u}}{dx^8}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^8\widetilde{u}}{dx^8}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$- \int_0^1 \frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_9 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_9 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_9 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1$$

$$+ \left[\frac{d^4}{dx^4}\left[c_9 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_9 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_9 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1$$

$$- \left[\frac{d^7}{dx^7}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^8}{dx^8}\left[c_9 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (10.8)$$

$$\int_0^1 c_8 \frac{d^8\widetilde{u}}{dx^8} N_{j,n}(x)dx = \left[c_8 N_{j,n}(x)\frac{d^7\widetilde{u}}{dx^7}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^7\widetilde{u}}{dx^7}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+ \int_0^1 \frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_8 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\frac{d^2\widetilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}\right]_0^1$$

$$-\int_0^1 \frac{d^7}{dx^7}\left[c_8 N_{j,n}(x)\right]\frac{d\widetilde{u}}{dx}dx \qquad (10.9)$$

$$\int_0^1 c_7 \frac{d^7\widetilde{u}}{dx^7} N_{j,n}(x)dx = \left[c_7 N_{j,n}(x)\frac{d^6\widetilde{u}}{dx^6}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^6\widetilde{u}}{dx^6}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\widetilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\widetilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\widetilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1 - \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 + \int_0^1 \frac{d^6}{dx^6}\left[c_7 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.10)$$

$$\int_0^1 c_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = \left[c_6 N_{j,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \int_0^1 \frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\int_0^1 \frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1 + \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 - \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d^5}{dx^5}\left[c_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.11)$$

$$\int_0^1 c_5 \frac{d^5\tilde{u}}{dx^5} N_{j,n}(x)dx = \left[c_5 N_{j,n}(x)\frac{d^4\tilde{u}}{dx^4}\right]_0^1 - \int_0^1 \frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$

$$+\int_0^1\frac{d^4}{dx^4}\left[c_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.12)$$

$$\int_0^1 c_4\frac{d^4\tilde{u}}{dx^4}N_{j,n}(x)dx=\left[c_4 N_{j,n}(x)\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.13)$$

$$\int_0^1 c_3\frac{d^3\tilde{u}}{dx^3}N_{j,n}(x)dx=\left[c_3 N_{j,n}(x)\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.14)$$

$$\int_0^1 c_2\frac{d^2\tilde{u}}{dx^2}N_{j,n}(x)dx=\left[c_2 N_{j,n}(x)\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx$$

$$=-\int_0^1\frac{d}{dx}\left[c_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \qquad (10.15)$$

Substituting eqns. (10.5) – (10.15) into eqn. (10.4) and using approximation for $\tilde{u}(x)$ given in eqn. (10.3) and after applying the boundary conditions given in eqn. (10.2b) and rearranging the terms for the resulting eqns. we get a system of eqns. in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (10.16a)$$

where

$$D_{i,j} = \int_0^1 \left\{\left[-\frac{d^{11}}{dx^{11}}\left[c_{12}N_{j,n}(x)\right] + \frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right] - \frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] + \frac{d^8}{dx^8}\left[c_9N_{j,n}(x)\right]\right.\right.$$

$$-\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]$$

$$\left.\left.+\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right] - \frac{d}{dx}\left[c_2N_{j,n}(x)\right] + c_1N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + c_0N_{i,n}(x)N_{j,n}(x)\right\}dx$$

$$-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$- \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$- \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+ \left[\frac{d}{dx}\left[c_{9}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_{9}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+ \left[\frac{d^2}{dx^2}\left[c_{9}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{9}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$- \left[\frac{d}{dx}\left[c_{8}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_{8}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (10.16b)$$

$$F_j = \int_0^1 \Bigg\{ sN_{j,n}(x) + \left[\frac{d^{11}}{dx^{11}}\left[c_{12}N_{j,n}(x)\right] - \frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right] + \frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right] - \frac{d^8}{dx^8}\left[c_{9}N_{j,n}(x)\right]\right.$$

$$- \frac{d^7}{dx^7}\left[c_{8}N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[c_{7}N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[c_{6}N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[c_{5}N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[c_{4}N_{j,n}(x)\right]$$

$$\left. - \frac{d^2}{dx^2}\left[c_{3}N_{j,n}(x)\right] + \frac{d}{dx}\left[c_{2}N_{j,n}(x)\right] - c_1N_{j,n}(x)\right]\frac{d\theta_0}{dx} - c_0\theta_0 N_{j,n}(x)\Bigg\}dx + \left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\theta_0}{dx^{10}}\right]_{x=1}$$

$$- \left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\frac{d^{10}\theta_0}{dx^{10}}\right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1}$$

$$+ \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=1} - \left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=0} - \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$+ \left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0} + \left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+ \left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1} - \left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}+\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}+\left[\frac{d}{dx}\left[c_{9}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_{9}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_{9}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_{9}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_{8}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d}{dx}\left[c_{8}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}-\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^5B_5$$

$$+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5A_5+\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4A_4-\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3$$

$$+\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3A_3+\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2B_2$$

$$-\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2A_2-\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$+\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^5B_5$$

$$-\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5A_5-\left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4B_4$$

$$+\left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4A_4+\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3B_3$$

$$-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1-\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^5 B_5$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5 A_5+\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3+\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^5 B_5$$

$$-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^5 A_5-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$

$$-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^3 A_3-\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$

$$-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^5 B_5$$

$$-\left[\frac{d^2}{dx^2}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^5 A_5 + \left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^4 B_4$$

$$-\left[\frac{d^3}{dx^3}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^4 A_4 - \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$+\left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 + \left[\frac{d^4}{dx^4}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3$$

$$-\left[\frac{d^5}{dx^5}\left[c_8 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$+\left[\frac{d^6}{dx^6}\left[c_8 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^5 B_5$$

$$-\left[\frac{d}{dx}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^5 A_5 - \left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^4 B_4$$

$$+\left[\frac{d^2}{dx^2}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^4 A_4 + \left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$-\left[\frac{d^3}{dx^3}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 - \left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$+\left[\frac{d^4}{dx^4}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$-\left[\frac{d^5}{dx^5}\left[c_7 N_{j,n}(x)\right]\right]_{x=0} \times (b-a) A_1 + \left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^4 B_4$$

$$-\left[\frac{d}{dx}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^4 A_4 - \left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^3 B_3$$

$$+\left[\frac{d^2}{dx^2}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^3 A_3 + \left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2$$

$$-\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=1} \times (b-a) B_1$$

$$+\left[\frac{d^4}{dx^4}\left[c_6 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 B_3$$
$$+\left[\frac{d^3}{dx^3}\left[c_6 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^3 A_3-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$
$$+\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2+\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$
$$-\left[\frac{d^3}{dx^3}\left[c_5 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$
$$-\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$
$$+\left[\frac{d^2}{dx^2}\left[c_4 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)A_1+\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=1}\times(b-a)B_1$$
$$-\left[\frac{d}{dx}\left[c_3 N_{j,n}(x)\right]\right]_{x=0}\times(b-a)A_1 \qquad (10.16c)$$

Solving the system (10.16a), we find the values of the parameters α_i and then substituting these parameters into eqn. (10.3), we get the approximate solution of the BVP (10.2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (10.1).

10.2.2 Formulation II

In this portion, we formulate the matrix form by using the boundary conditions of type II.

In the same way of section (10.2.1), integrating by parts the terms up to second derivative on the left hand side of (10.4), and after applying the conditions prescribed in type II, eqn. (2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j = 1,2,\ldots,n \qquad (10.17a)$$

where

$$D_{i,j}=\int_0^1\left\{\left[-\frac{d^{11}}{dx^{11}}\left[c_{12}N_{j,n}(x)\right]+\frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right]-\frac{d^{9}}{dx^{9}}\left[c_{10}N_{j,n}(x)\right]+\frac{d^{8}}{dx^{8}}\left[c_{9}N_{j,n}(x)\right]\right.\right.$$

$$-\frac{d^{7}}{dx^{7}}\left[c_{8}N_{j,n}(x)\right]+\frac{d^{6}}{dx^{6}}\left[c_{7}N_{j,n}(x)\right]-\frac{d^{5}}{dx^{5}}\left[c_{6}N_{j,n}(x)\right]+\frac{d^{4}}{dx^{4}}\left[c_{5}N_{j,n}(x)\right]-\frac{d^{3}}{dx^{3}}\left[c_{4}N_{j,n}(x)\right]$$

$$\left.\left.+\frac{d^{2}}{dx^{2}}\left[c_{3}N_{j,n}(x)\right]-\frac{d}{dx}\left[c_{2}N_{j,n}(x)\right]+c_{1}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]+c_{0}N_{i,n}(x)N_{j,n}(x)\right\}dx$$

$$+\left[\frac{d^{2}}{dx^{2}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}}{dx^{9}}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{2}}{dx^{2}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{9}}{dx^{9}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^{4}}{dx^{4}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{4}}{dx^{4}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^{6}}{dx^{6}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{5}}{dx^{5}}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{6}}{dx^{6}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{5}}{dx^{5}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^{8}}{dx^{8}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{3}}{dx^{3}}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{8}}{dx^{8}}\left[c_{12}N_{j,n}(x)\right]\frac{d^{3}}{dx^{3}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^{9}}{dx^{9}}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^{9}}{dx^{9}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^{3}}{dx^{3}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^{3}}{dx^{3}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^{5}}{dx^{5}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{5}}{dx^{5}}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^{5}}{dx^{5}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{5}}{dx^{5}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^{7}}{dx^{7}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^{7}}{dx^{7}}\left[c_{11}N_{j,n}(x)\right]\frac{d^{7}}{dx^{7}}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^{9}}{dx^{9}}\left[c_{11}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^{9}}{dx^{9}}\left[c_{11}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0}$$

$$-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \qquad (10.17b)$$

$$F_j=\int_0^1\left\{sN_{j,n}(x)+\left[\frac{d^{11}}{dx^{11}}\left[c_{12}N_{j,n}(x)\right]-\frac{d^{10}}{dx^{10}}\left[c_{11}N_{j,n}(x)\right]+\frac{d^9}{dx^9}\left[c_{10}N_{j,n}(x)\right]-\frac{d^8}{dx^8}\left[c_9N_{j,n}(x)\right]\right.\right.$$

$$+\frac{d^7}{dx^7}\left[c_8N_{j,n}(x)\right]-\frac{d^6}{dx^6}\left[c_7N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[c_6N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[c_5N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4N_{j,n}(x)\right]$$

$$\left.\left.-\frac{d^2}{dx^2}\left[c_3N_{j,n}(x)\right]+\frac{d}{dx}\left[c_2N_{j,n}(x)\right]-c_1N_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0N_{j,n}(x)\right\}dx-\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_{12}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_{12}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$

$$-\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^6}{dx^6}\left[c_{12}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^8}{dx^8}\left[c_{12}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^{10}}{dx^{10}}\left[c_{12}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=1}-\left[\frac{d}{dx}\left[c_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^5}{dx^5}\left[c_{11}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}$$

$$+\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^7}{dx^7}\left[c_{11}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}$$

$$-\left[\frac{d^9}{dx^9}\left[c_{11}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$

$$-\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^6}{dx^6}\left[c_{10}N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^8}{dx^8}\left[c_{10}N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}-\left[\frac{d}{dx}\left[c_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[c_9N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^5}{dx^5}\left[c_9N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{d^7}{dx^7}\left[c_9N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_8N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^6}{dx^6}\left[c_8N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d}{dx}\left[c_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_7N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$+\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{d^5}{dx^5}\left[c_7N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}$$

$$+\left[\frac{d^2}{dx^2}\left[c_6N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_6N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d}{dx}\left[c_5N_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}$$

$$-\left[\frac{d^3}{dx^3}\left[c_5N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_4N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}$$

$$+\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{d}{dx}\left[c_3N_{j,n}(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{10}B_{10}$$

$$-\left[\frac{d}{dx}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{10}A_{10}+\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{8}B_{8}$$

$$-\left[\frac{d^3}{dx^3}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{8}A_{8}+\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{6}B_{6}$$

$$-\left[\frac{d^5}{dx^5}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{6}A_{6}+\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{4}B_{4}$$

$$-\left[\frac{d^7}{dx^7}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{4}A_{4}+\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{2}B_{2}$$

$$-\left[\frac{d^9}{dx^9}\left[c_{12}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{2}A_{2}-\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{8}B_{8}$$

$$+\left[\frac{d^2}{dx^2}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{8}A_{8}-\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{6}B_{6}$$

$$+\left[\frac{d^4}{dx^4}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{6}A_{6}-\left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{4}B_{4}$$

$$+\left[\frac{d^6}{dx^6}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{4}A_{4}-\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{2}B_{2}$$

$$+\left[\frac{d^8}{dx^8}\left[c_{11}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^{2}A_{2}+\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^{8}B_{8}$$

$$-\left[\frac{d}{dx}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^8 A_8+\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6$$

$$-\left[\frac{d^3}{dx^3}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6+\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d^5}{dx^5}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4+\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2$$

$$-\left[\frac{d^7}{dx^7}\left[c_{10}N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2-\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6$$

$$+\left[\frac{d^2}{dx^2}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4+\left[\frac{d^4}{dx^4}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2+\left[\frac{d^6}{dx^6}\left[c_9N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$+\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^6 B_6-\left[\frac{d}{dx}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^6 A_6$$

$$+\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4-\left[\frac{d^3}{dx^3}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$+\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2-\left[\frac{d^5}{dx^5}\left[c_8N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4+\left[\frac{d^2}{dx^2}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$-\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2+\left[\frac{d^4}{dx^4}\left[c_7N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$+\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^4 B_4-\left[\frac{d}{dx}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^4 A_4$$

$$+\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\right]_{x=1}\times(b-a)^2 B_2-\left[\frac{d^3}{dx^3}\left[c_6N_{j,n}(x)\right]\right]_{x=0}\times(b-a)^2 A_2$$

$$-\left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2 + \left[\frac{d^2}{dx^2}\left[c_5 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2$$

$$+\left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=1} \times (b-a)^2 B_2 - \left[\frac{d}{dx}\left[c_4 N_{j,n}(x)\right]\right]_{x=0} \times (b-a)^2 A_2 \qquad (10.17c)$$

Solving the system (10.17a), we find the values of the parameters α_i and then substituting these parameters into eqn. (10.3), we get the approximate solution of the BVP (10.2). If we replace x by $\frac{x-a}{b-a}$ in $\widetilde{u}(x)$, then we get the desired approximate solution of the BVP (10.1).

For nonlinear twelfth-order BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (10.16) and (10.17). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

10.3 Numerical examples and results

To test the applicability of the proposed method, we consider four linear and two nonlinear problems consisting of both types of boundary conditions. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB* 10**. The convergence of linear BVP is calculated by

$$E = \left|\widetilde{u}_{n+1}(x) - \widetilde{u}_n(x)\right| < \delta$$

where $\widetilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ (depends on the problem) which is less than 10^{-13}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\widetilde{u}_n^{N+1} - \widetilde{u}_n^{N}\right| < \delta$$

where $\delta < 10^{-12}$ and N is the Newton's iteration number.

Example 1: Consider the linear differential equation [57, 92, 102, 104]

$$\frac{d^{12}u}{dx^{12}} + xu = -(120 + 23x + x^3)e^x, \ \ 0 \le x \le 1 \tag{10.18a}$$

subject to boundary conditions of type I in eqn. (2b):

$$u(0) = u(1) = 0,\ u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u'''(0) = -3, u'''(1) = -9e,$$

$$u^{(iv)}(0) = -8, u^{(iv)}(1) = -16e, u^{(v)}(0) = -15, u^{(v)}(1) = -25e. \tag{10.18b}$$

The analytic solution of the above problem is, $u(x) = x(1-x)e^x$.

Using the method illustrated in section (10.2.1), we approximate $u(x)$ in a form similar to (10.3) as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{10.19}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (10.18b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j,\ j = 1,2,\ldots,n \tag{10.20a}$$

where

$$\begin{aligned}
D_{i,j} = &\int_0^1 \left[-\frac{d^{11}}{dx^{11}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + x N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&- \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1} \\
&+ \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0}
\end{aligned} \tag{10.20b}$$

$$F_j = \int_0^1 -(120+23x+x^3)e^x N_{j,n}(x)\,dx - \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1}(-25e)$$

$$+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0}(-15)+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}(-16e)-\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}(-8)$$

$$+\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1}(-4e)-\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1}(-9e)+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0}(-3)$$

$$-\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]_{x=1}(-e)+\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]_{x=0} \qquad (10.20c)$$

Solving the system (10.20a) we obtain the values of the parameters and then substituting these parameters into eqn. (10.19), we get the approximate solution of the BVP (10.18) for different values of *n*.

The maximum absolute errors, using different number of polynomials by the present method and the previous results obtained so far, are summarized in **Table 1.**

Table 1: Maximum absolute errors for the example 1

x	Exact Results	15 Bernstein Polynomials		15 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	6.2580283E-026
0.1	0.0994653826	0.0994653826	2.7755576E-017	0.0994653826	1.3877788E-016
0.2	0.1954244413	0.1954244413	0.0000000E+000	0.1954244413	2.4980018E-015
0.3	0.2834703496	0.2834703496	3.8857806E-016	0.2834703496	1.3877788E-015
0.4	0.3580379274	0.3580379274	7.7715612E-016	0.3580379274	1.8873791E-015
0.5	0.4121803177	0.4121803177	1.2212453E-015	0.4121803177	3.5527137E-015
0.6	0.4373085121	0.4373085121	1.2767565E-015	0.4373085121	1.9984014E-015
0.7	0.4228880686	0.4228880686	8.3266727E-016	0.4228880686	6.6613381E-016
0.8	0.3560865486	0.3560865486	5.5511151E-017	0.3560865486	2.0539126E-015
0.9	0.2213642800	0.2213642800	1.1102230E-016	0.2213642800	6.6613381E-016
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the other hand, it is observed that the accuracy is found nearly the order 10^{-9} in [57], [102] by Siddiqi and Akram; Kudri and Mulhem and nearly the order 10^{-14} and 10^{-8} in [92], [104] by Lamnii *et al* and Mirmoradi *et al* respectively.

Now the exact and approximate solutions are depicted in Fig. 1(a) and the relative errors are shown in Fig. 1(b) of example 1 for $n = 15$. It is observed from Fig. 1(b) that the error is nearly the order 10^{-14}.

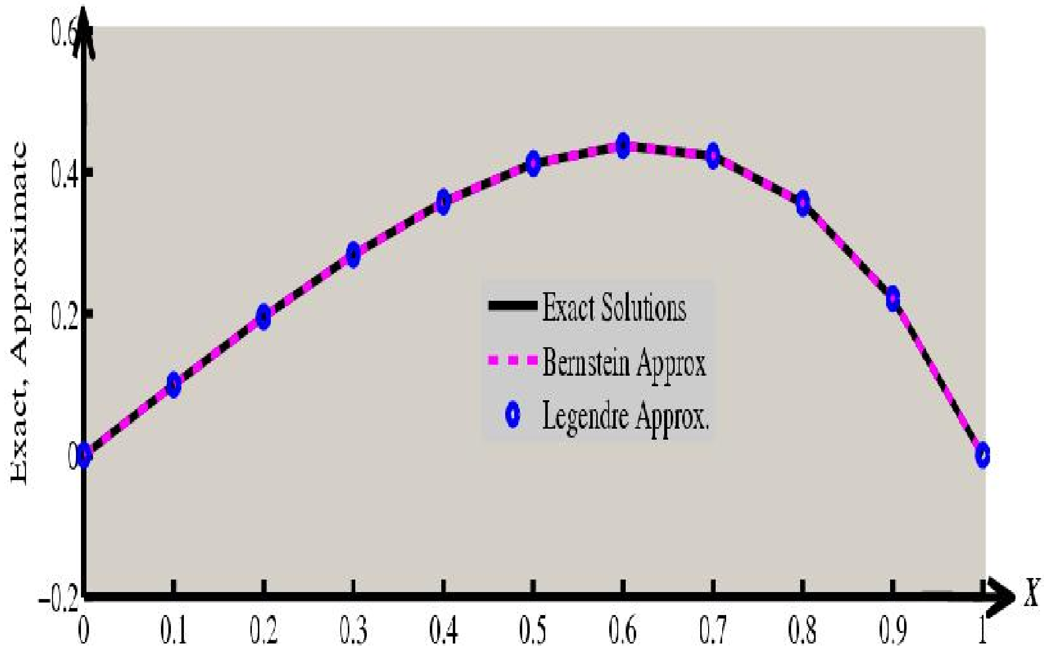


Fig. 1(a): Graphical representation of exact and approximate solutions of example 1 using 15 polynomials.

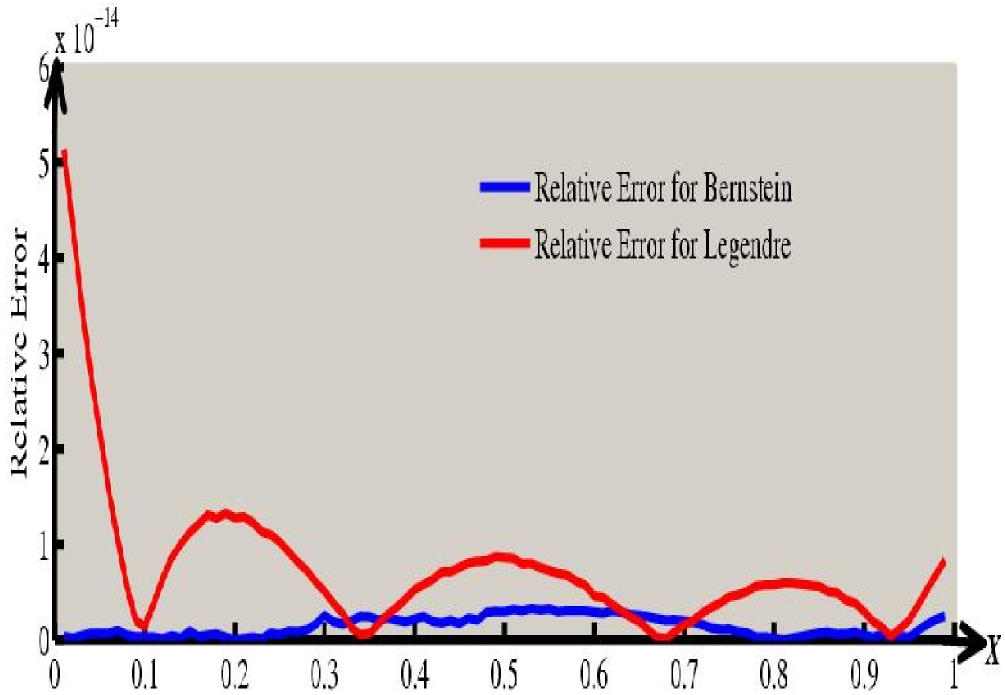


Fig. 1(b): Graphical representation of relative error of example 1 using 15 polynomials.

Example 2: Consider the linear differential equation [57, 102, 104]

$$\frac{d^{12}u}{dx^{12}} - u = -12(2x\cos x + 11\sin x), \quad -1 \le x \le 1 \tag{10.21a}$$

subject to the boundary conditions of type I in eqn. (1b):

$$u(-1) = u(1) = 0,\ u'(-1) = u'(1) = 2\sin 1, u''(-1) = -u''(1) = -4\cos 1 - 2\sin 1,$$

$$u'''(-1) = u'''(1) = 6\cos 1 - 6\sin 1, u^{(iv)}(-1) = -u^{(iv)}(1) = 8\cos 1 + 12\sin 1,$$

$$u^{(v)}(-1) = u^{(v)}(1) = -20\cos 1 + 10\sin 1\,. \tag{10.21b}$$

The analytic solution of the above problem is, $u(x) = (x^2 - 1)\sin x$.

The equivalent BVP over [0, 1] to the BVP (10.21) is,

$$\frac{1}{2^{12}}\frac{d^{10}u}{dx^{10}} - u = -12(2(2x-1)\cos(2x-1) + 11\sin(2x-1)), 0 < x < 1 \tag{10.22a}$$

$$u(0) = u(1) = 0,\ \frac{1}{2}u'(0) = \frac{1}{2}u'(1) = 2\sin 1, \frac{1}{4}u''(0) = -\frac{1}{4}u''(1) = -4\cos 1 - 2\sin 1,$$

$$\frac{1}{8}u'''(0) = \frac{1}{8}u'''(1) = 6\cos 1 - 6\sin 1, \frac{1}{16}u^{(iv)}(0) = -\frac{1}{16}u^{(iv)}(1) = 8\cos 1 + 12\sin 1,$$

$$\frac{1}{32}u^{(v)}(0) = \frac{1}{32}u^{(v)}(1) = -20\cos 1 + 10\sin 1 \tag{10.22b}$$

Employing the method mentioned in (8.2.1), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{10.23}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (10.22b). Now the parameters $\alpha_i\ (i = 1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j, j = 1,2,\ldots,n \tag{10.24a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^{11}}{dx^{11}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 2^{12} N_{i,n}(x)N_{j,n}(x) \right]dx - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\frac{d^{10}}{dx^{10}}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0} -\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=0} +\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} -\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{10.24b}$$

$$F_j = \int_0^1 2^{12}\left[-12(2(2x-1)\cos(2x-1)+11\sin(2x-1))\right]N_{j,n}(x)dx$$

$$-\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=1}\times 32(-20\cos 1+10\sin 1)+\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\right]_{x=0}\times 32(-20\cos 1+10\sin 1)$$

$$+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}\times 16(-8\cos 1-12\sin 1)-\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}\times 16(8\cos 1+12\sin 1)$$

$$+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=0}\times 8(6\cos 1-6\sin 1)-\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\right]_{x=1}\times 8(6\cos 1-6\sin 1)$$

$$+\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1}\times 4(4\cos 1+2\sin 1)-\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=0}\times 4(-4\cos 1-2\sin 1)$$

$$-\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]_{x=1}\times(4\sin 1)+\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\right]_{x=0}\times(4\sin 1) \tag{10.24c}$$

Solving the system (10.24a) we obtain the values of the parameters and then substituting these parameters into eqn. (10.23), we get the approximate solution of the BVP (10.22) for different values of *n*. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (10.21).

The maximum absolute errors, shown in **Table 2,** are listed to compare with existing results.

Table 2: Maximum absolute errors for the example 2.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Legendre	Reference Results
13	1.325×10^{-11}	1.325×10^{-11}	4.69×10^{-5} (Siddiqi and Akram [57])
14	5.277×10^{-12}	5.277×10^{-12}	3.777×10^{-9} (Kudri and Mulhem [102])
15	8.049×10^{-16}	3.592×10^{-14}	3.900×10^{-9} (Mirmoradi *et al* [104])
16	6.661×10^{-16}	6.839×10^{-14}	

Example 3: Consider the linear differential equation [29]

$$\frac{d^{12}u}{dx^{12}} + xu = -(120 + 23x + x^3)e^x, \quad 0 \le x \le 1 \tag{10.25a}$$

subject to the boundary conditions of type II in eqn. (2c):

$$u(0) = u(1) = 0,\ u''(0) = 0, u''(1) = -4e, u^{(iv)}(0) = -8, u^{(iv)}(1) = -16e, u^{(vi)}(0) = -24,$$
$$u^{(vi)}(1) = -36e, u^{(viii)}(0) = -48, u^{(viii)}(1) = -64e,\ u^{(x)}(0) = -80, u^{(x)}(1) = -100e \tag{10.25b}$$

The analytic solution of the above problem is, $u(x) = x(1-x)e^x$.

Applying the method illustrated in section (10.2.2), we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \tag{10.26}$$

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (10.25b). Now the parameters $\alpha_i\ (i = 1,2,\dots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\, \alpha_i = F_j,\ j = 1,2,\dots,n \tag{10.27a}$$

where

$$D_{i,j} = \int_0^1 \left[-\frac{d^{11}}{dx^{11}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + x N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$-\left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=1}$$
$$-\left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=0} + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=1}$$

We have shown the exact and approximate solutions in Fig. 2(a) and the relative errors in Fig. 2(b) of example 2 for $n = 16$. It is found from Fig. 2(b) that the error is of the order 10^{-13}

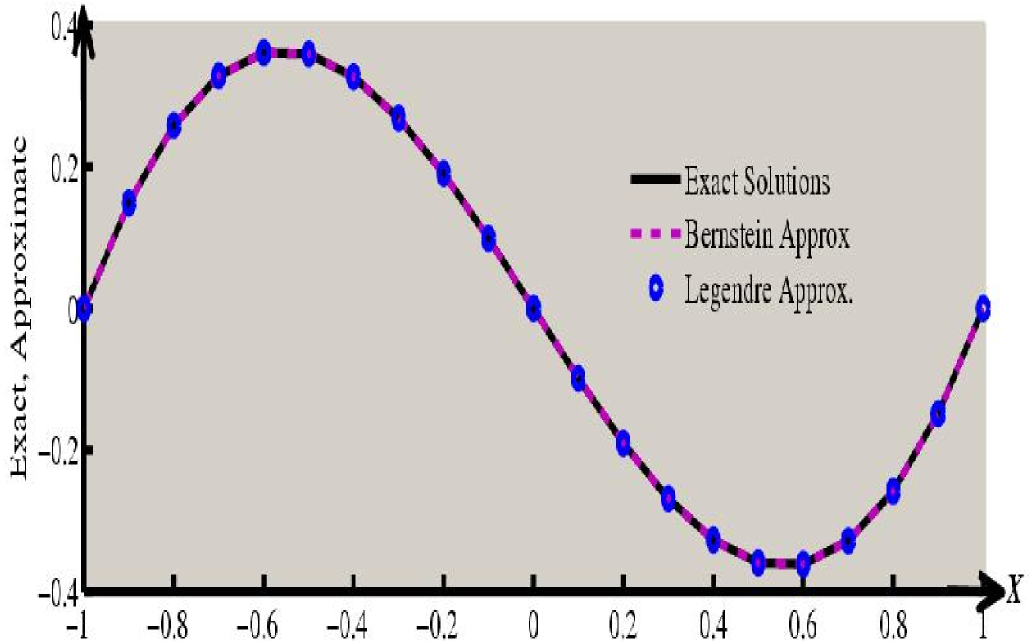


Fig. 2(a): Graphical representation of exact and approximate solutions of example 2 using 16 polynomials.

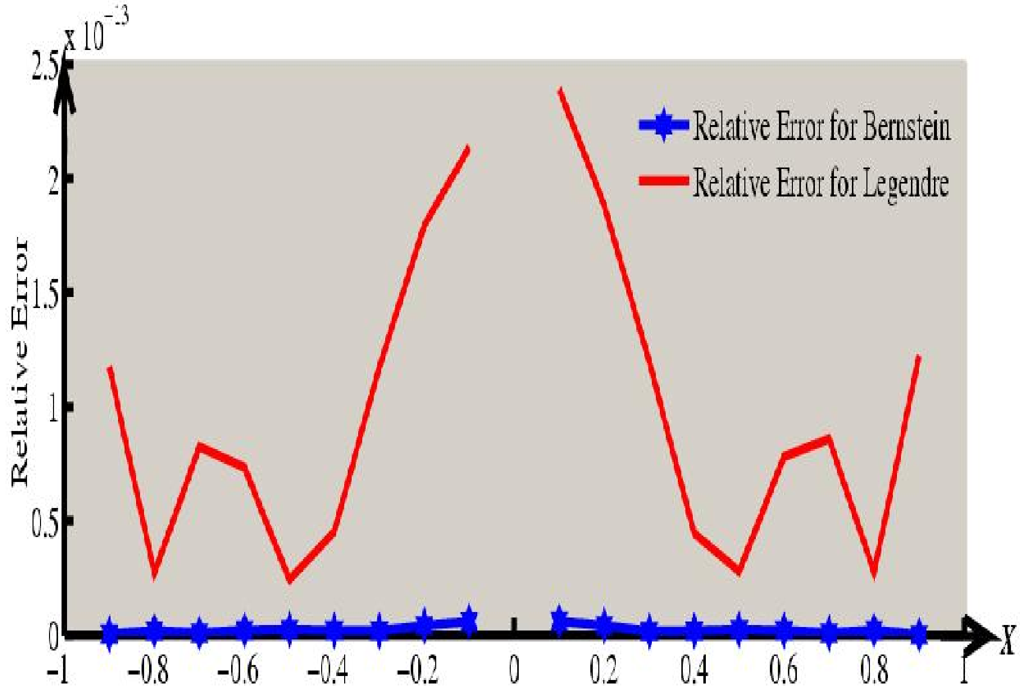


Fig. 2(b): Graphical representation of relative error of example 2 using 16 polynomials.

$$-\left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right]\right]_{x=0}+\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=1}$$

$$-\left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right]\right]_{x=0} \tag{10.27b}$$

$$F_j = \int_0^1 -(120+23x+x^3)e^x N_{j,n}(x)\,dx + \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=1}(-100e)$$

$$-\left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=0}(-80)+\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1}(-64e)$$

$$-\left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0}(-48)-\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1}(-36e)$$

$$+\left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0}(-24)+\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1}(-16e)$$

$$-\left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0}(-8)+\left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1}(-4e) \tag{10.27c}$$

Solving the system (10.27a) we obtain the values of the parameters and then substituting these parameters into eqn. (10.26), we get the approximate solution of the BVP (10.25) for different values of n.

In **Table 3**, we tabulate the maximum absolute errors to compare with the previous results.

Example 4: Consider the linear boundary value problem [29]

$$\frac{d^{12}u}{dx^{12}} - u = -12(2x\cos x + 11\sin x), \ -1 \le x \le 1 \tag{10.28a}$$

$$u(-1)=u(1)=0,\ u''(-1)=-4\cos 1-2\sin 1=-u''(1),\ u^{(iv)}(-1)=8\cos 1+12\sin 1=-u^{(iv)}(1),$$

$$u^{(vi)}(-1)=-12\cos 1-30\sin 1=-u^{(vi)}(1), u^{(viii)}(-1)=16\cos 1+56\sin 1=-u^{(viii)}(1),$$

$$u^{(x)}(-1)=-20\cos 1-90\sin 1=-u^{(x)}(1). \tag{10.28b}$$

Table 3: Maximum absolute errors for the example 3

x	Exact Results	14 Bernstein Polynomials		14 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	6.2580241E-026	0.0000000000	0.0000000E+000
0.1	0.0994653826	0.0994653826	2.3592239E-016	0.0994653826	1.6524004E-012
0.2	0.1954244413	0.1954244413	7.7715612E-016	0.1954244413	3.1429859E-012
0.3	0.2834703496	0.2834703496	8.3266727E-016	0.2834703496	4.3262061E-012
0.4	0.3580379274	0.3580379274	5.5511151E-016	0.3580379274	5.0854876E-012
0.5	0.4121803177	0.4121803177	2.1649349E-015	0.4121803177	5.3470006E-012
0.6	0.4373085121	0.4373085121	1.9984014E-015	0.4373085121	5.0849880E-012
0.7	0.4228880686	0.4228880686	1.6653345E-016	0.4228880686	4.3254844E-012
0.8	0.3560865486	0.3560865486	1.3877788E-015	0.3560865486	3.1423752E-012
0.9	0.2213642800	0.2213642800	8.3266727E-016	0.2213642800	1.6519563E-012
1.0	0.0000000000	0.0000000000	0.0000000E+000	0.0000000000	0.0000000E+000

On the contrary the maximum absolute error has been found by Siddiqi and Twizell [29] is 0.5582×10^{-2}

We have shown the exact and approximate solutions in fig. 3(a) and the relative error in fig. 3(b) of example 3 for $n=14$. It is found from fig. 3(b) that the error is of the order 10^{-11}

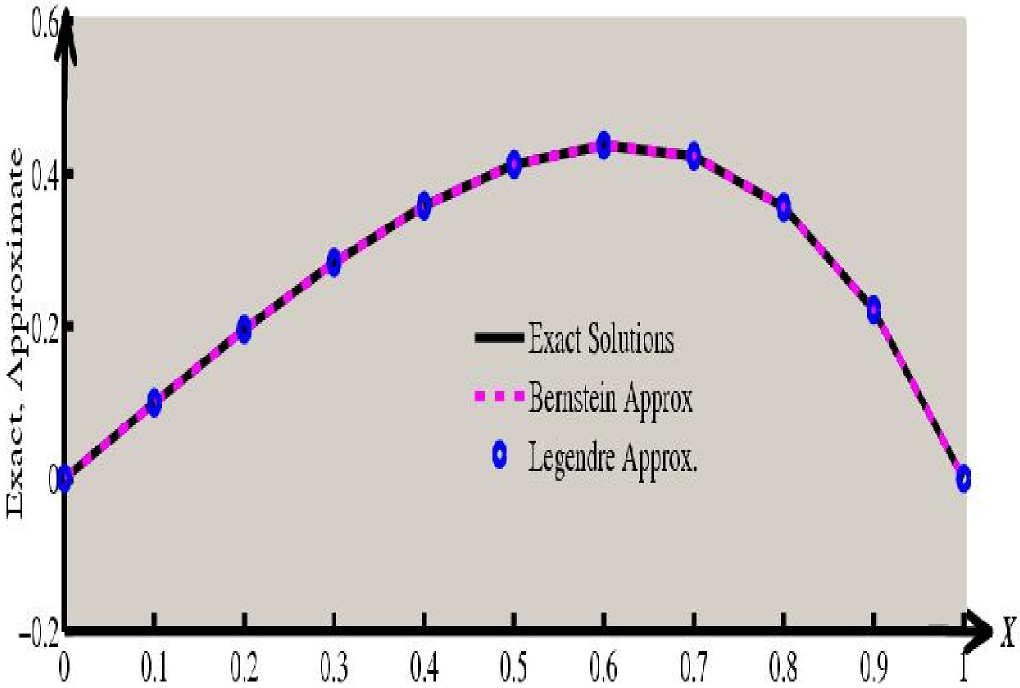


Fig. 3(a): Graphical representation of exact and approximate solutions of example 3 using 14 polynomials.

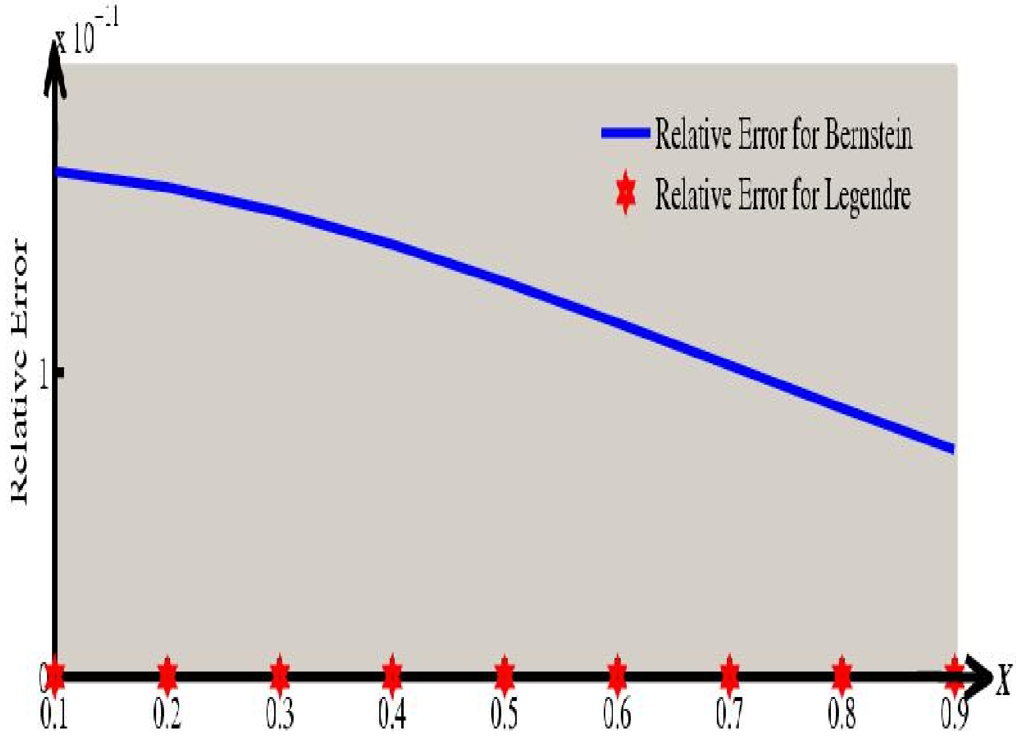


Fig. 3(b): Graphical representation of relative error of example 3 using 14 polynomials.

The analytic solution of the above problem is $u(x)=(x^2-1)\sin x$.

The equivalent BVP over [0, 1] to the BVP (10.28) is,

$$\frac{1}{2^{12}}\frac{d^{10}u}{dx^{10}}-u=-12(2(2x-1)\cos(2x-1)+11\sin(2x-1)),0<x<1 \tag{10.29a}$$

$$u(0)=u(1)=0,\frac{1}{4}u''(0)=-\frac{1}{4}u''(1)=-4\cos 1-2\sin 1,\frac{1}{16}u^{(iv)}(0)=-\frac{1}{16}u^{(iv)}(1)=8\cos 1+12\sin 1,$$

$$\frac{1}{64}u^{(vi)}(0)=-\frac{1}{64}u^{(vi)}(1)=-12\cos 1-30\sin 1,\frac{1}{256}u^{(viii)}(0)=-\frac{1}{256}u^{(viii)}(1)=16\cos 1+56\sin 1,$$

$$\frac{1}{2^{10}}u^{(x)}(0)=-\frac{1}{2^{10}}u^{(x)}(1)=-20\cos 1-90\sin 1 \tag{10.29b}$$

Employing the method given in (10.2.2), we approximate $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\geq 1 \tag{10.30}$$

Here $\theta_0(x)=0$ is specified by the essential boundary conditions of equation (10.29b). Now the parameters $\alpha_i\ (i=1,2,\ldots,n)$ satisfy the linear system

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j \,,\, j = 1,2,\ldots,n \tag{10.31a}$$

where

$$\begin{aligned}
D_{i,j} =& \int_0^1 \left[-\frac{d^{11}}{dx^{11}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] - 2^{12} N_{i,n}(x) N_{j,n}(x) \right] dx - \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^2}{dx^2}\left[N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^4}{dx^4}\left[N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^6}{dx^6}\left[N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^8}{dx^8}\left[N_{j,n}(x)\right]\frac{d^3}{dx^3}\left[N_{i,n}(x)\right] \right]_{x=0} + \left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=1} \\
&- \left[\frac{d^{10}}{dx^{10}}\left[N_{j,n}(x)\right]\frac{d}{dx}\left[N_{i,n}(x)\right] \right]_{x=0}
\end{aligned} \tag{10.31b}$$

$$\begin{aligned}
F_j =& \int_0^1 2^{12}\left[-12(2(2x-1)\cos(2x-1) + 11\sin(2x-1))\right] N_{j,n}(x)\,dx \\
&+ \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=1} \times 2^{10}(20\cos 1 + 90\sin 1) - \left[\frac{d}{dx}\left[N_{j,n}(x)\right]\right]_{x=0} \times 2^{10}(-20\cos 1 - 90\sin 1) \\
&+ \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=1} \times 256(-16\cos 1 - 56\sin 1) - \left[\frac{d^3}{dx^3}\left[N_{j,n}(x)\right]\right]_{x=0} \times 256(16\cos 1 + 56\sin 1) \\
&+ \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=1} \times 64(12\cos 1 + 30\sin 1) - \left[\frac{d^5}{dx^5}\left[N_{j,n}(x)\right]\right]_{x=0} \times 64(-12\cos 1 - 30\sin 1) \\
&+ \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=1} \times 16(-8\cos 1 - 12\sin 1) - \left[\frac{d^7}{dx^7}\left[N_{j,n}(x)\right]\right]_{x=0} \times 16(8\cos 1 + 12\sin 1) \\
&- \left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=0} \times (-4\cos 1 - 2\sin 1) + \left[\frac{d^9}{dx^9}\left[N_{j,n}(x)\right]\right]_{x=1} \times (4\cos 1 + 2\sin 1)
\end{aligned} \tag{10.31c}$$

Solving the system (10.31a) we obtain the values of the parameters and then substituting these parameters into eqn. (10.30), we get the approximate solution of the BVP (10.29) for different values of n. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (10.28).

The maximum absolute errors using different number of polynomials and to compare with existing methods are shown in **Table 4.**

Table4: Maximum absolute errors for the example 4

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Bernstein	Reference Results
13	9.202×10^{-11}	9.201×10^{-11}	0.6901×10^{-3} (Siddiqi and Twizell [29])
14	4.474×10^{-14}	4.827×10^{-14}	
15	4.169×10^{-14}	4.843×10^{-14}	
16	9.873×10^{-14}	3.675×10^{-14}	

We depict the exact and approximate solutions in Fig. 4(a) and a plot of relative errors in Fig. 4(b) of example 4 for $n=16$. From Fig. 4(b) we observe that the error is nearly the order 10^{-13}

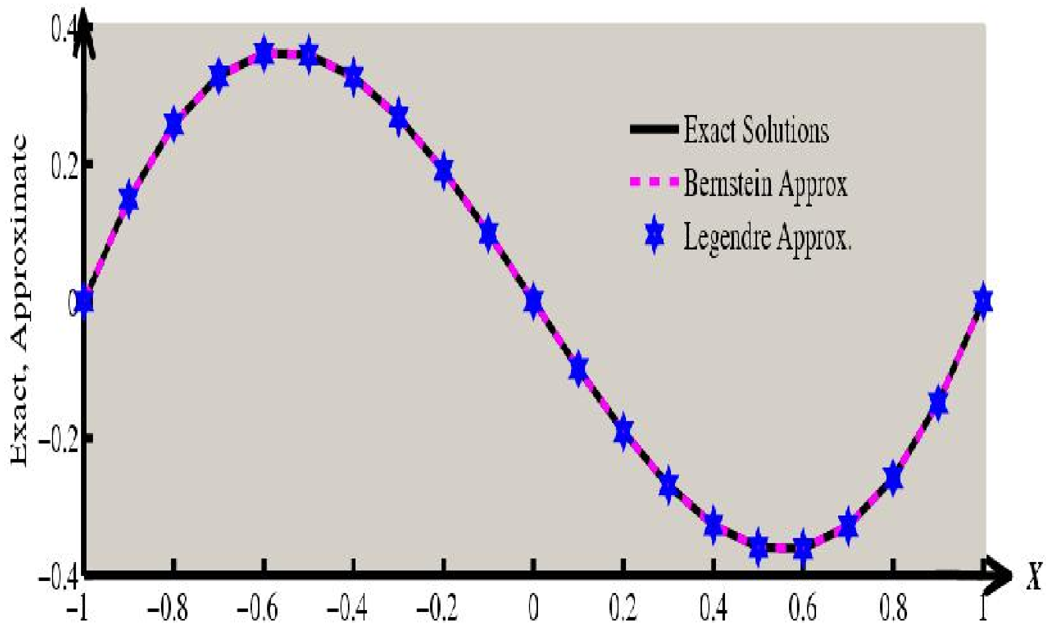


Fig. 4(a): Graphical representation of exact and approximate solutions of example 4 using 16 polynomials.

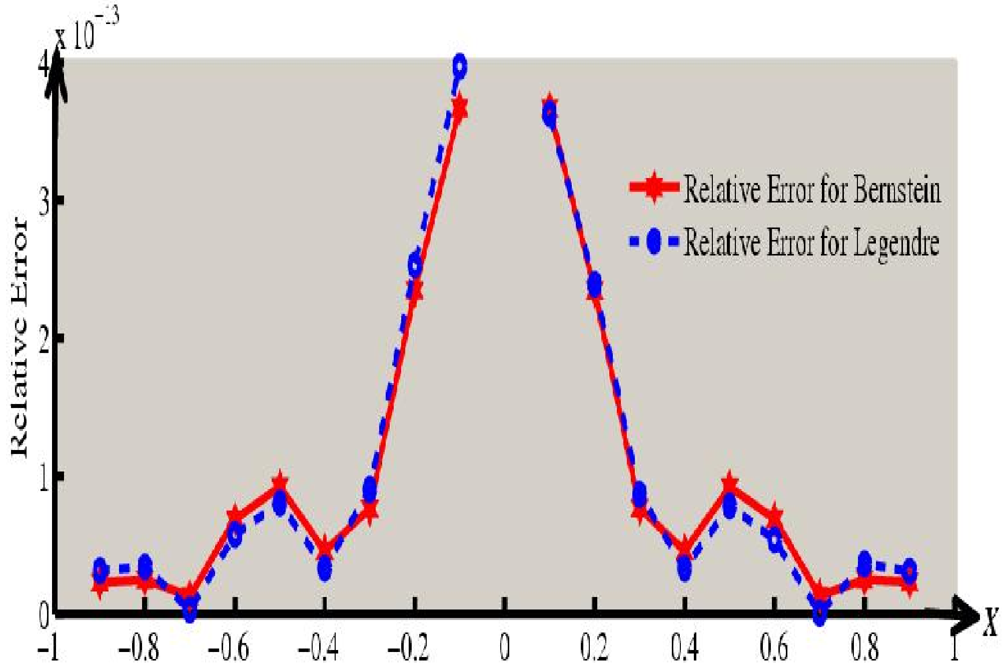


Fig. 4(b): Graphical representation of relative error of example 4 using 16 polynomials.

Example 5: Consider the twelfth order **nonlinear** differential equation [80, 89, 103, 105]

$$\frac{d^{12}u}{dx^{12}} = 2e^{x}u^{2} + \frac{d^{3}u}{dx^{3}},\ 0 \le x \le 1 \qquad (10.32a)$$

subject to the boundary conditions of type I defined in eqn. (2b)

$$u(0)=1, u(1)=e^{-1}, u'(0)=1, u'(1)=e^{-1}, u''(0)=1, u''(1)=e^{-1}, u^{(iii)}(0)=1,$$
$$u^{(iii)}(1)=e^{-1}, u^{(iv)}(0)=1, u^{(iv)}(1)=e^{-1}, u^{(v)}(0)=1, u^{(v)}(1)=e^{-1}. \qquad (10.32b)$$

The exact solution of this BVP is $u(x)=e^{-x}$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x),\ n \ge 1 \qquad (10.33)$$

Here $\theta_0(x) = 1 - x(1-e^{-1})$ is specified by the essential boundary conditions in (10.32b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$.

Using eqn. (10.33) into eqn. (10.32a), the Galerkin weighted residual eqns. are

$$\int_0^1 \left[\frac{d^{12}\tilde{u}}{dx^{12}} - 2e^x \tilde{u}^2 - \frac{d^3\tilde{u}}{dx^3} \right] N_{k,n}(x)dx = 0, k = 1,2,\cdots,n \tag{10.34}$$

Integrating 1[st] and 3[rd] terms of (10.34) by parts, we obtain

$$\int_0^1 \frac{d^{12}\tilde{u}}{dx^{12}} N_{k,n}(x) = -\left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}\tilde{u}}{dx^{10}} \right]_0^1 + \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^9\tilde{u}}{dx^9} \right]_0^1 - \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^8\tilde{u}}{dx^8} \right]_0^1$$
$$+ \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^7\tilde{u}}{dx^7} \right]_0^1 - \left[\frac{d^5N_{k,n}(x)}{dx^5} \frac{d^6\tilde{u}}{dx^6} \right]_0^1 + \left[\frac{d^6N_{k,n}(x)}{dx^6} \frac{d^5\tilde{u}}{dx^5} \right]_0^1 - \left[\frac{d^7N_{k,n}(x)}{dx^7} \frac{d^4\tilde{u}}{dx^4} \right]_0^1$$
$$+ \left[\frac{d^8N_{k,n}(x)}{dx^8} \frac{d^3\tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^9N_{k,n}(x)}{dx^9} \frac{d^2\tilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^{10}N_{k,n}(x)}{dx^{10}} \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^{11}N_{k,n}(x)}{dx^{11}} \frac{d\tilde{u}}{dx} dx$$

(10.35)

$$\int_0^1 \frac{d^3\tilde{u}}{dx^3} N_{k,n}(x) = -\left[\frac{dN_{k,n}(x)}{dx} \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^2N_{k,n}(x)}{dx^2} \frac{d\tilde{u}}{dx} dx \tag{10.36}$$

Using eqns. (10.35) and (10.36) into eqn. (10.34) and using approximation for $\tilde{u}(x)$ given in eqn. (10.33) and after applying the conditions given in eqn. (10.32b) and rearranging the terms for the resulting eqns. we obtain

$$\sum_{i=1}^{n} \left[\int_0^1 \left[\left\{ -\frac{d^{11}N_{k,n}(x)}{dx^{11}} - \frac{d^2N_{k,n}(x)}{dx^2} \right\} \frac{dN_{i,n}(x)}{dx} - 4\theta_0 e^x N_{i,n}(x)N_{k,n}(x) \right.\right.$$
$$\left. - 2\sum_{j=1}^{n} \alpha_j (N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^x \right] dx - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=1}$$
$$+ \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=0} + \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^9N_{i,n}(x)}{dx^9} \right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2} \frac{d^9N_{i,n}(x)}{dx^9} \right]_{x=0}$$
$$- \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^8N_{i,n}(x)}{dx^8} \right]_{x=1} + \left[\frac{d^3N_{k,n}(x)}{dx^3} \frac{d^8N_{i,n}(x)}{dx^8} \right]_{x=0} + \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^7N_{i,n}(x)}{dx^7} \right]_{x=1}$$
$$\left. - \left[\frac{d^4N_{k,n}(x)}{dx^4} \frac{d^7N_{i,n}(x)}{dx^7} \right]_{x=0} - \left[\frac{d^5N_{k,n}(x)}{dx^5} \frac{d^6N_{i,n}(x)}{dx^6} \right]_{x=1} + \left[\frac{d^5N_{k,n}(x)}{dx^5} \frac{d^6N_{i,n}(x)}{dx^6} \right]_{x=0} \right] \alpha_i$$

$$= \int_0^1 \left[\left\{ \frac{d^{11}N_{k,n}(x)}{dx^{11}} + \frac{d^2 N_{k,n}(x)}{dx^2} \right\} \frac{d\theta_0}{dx} + 2\theta_0{}^2 e^x N_{k,n}(x) \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}\theta_0}{dx^{10}} \right]_{x=1}$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}\theta_0}{dx^{10}} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^9\theta_0}{dx^9} \right]_{x=1} + \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^9\theta_0}{dx^9} \right]_{x=0}$$

$$+ \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^8\theta_0}{dx^8} \right]_{x=1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^8\theta_0}{dx^8} \right]_{x=0} - \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^7\theta_0}{dx^7} \right]_{x=1}$$

$$+ \left[\frac{d^4 N_{k,n}(x)}{dx^4} \frac{d^7\theta_0}{dx^7} \right]_{x=0} + \left[\frac{d^5 N_{k,n}(x)}{dx^5} \frac{d^6\theta_0}{dx^6} \right]_{x=1} - \left[\frac{d^5 N_{k,n}(x)}{dx^5} \frac{d^6\theta_0}{dx^6} \right]_{x=0}$$

$$- \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=1} \times e^{-1} + \left[\frac{d^6 N_{k,n}(x)}{dx^6} \right]_{x=0} + \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=1} \times e^{-1} - \left[\frac{d^7 N_{k,n}(x)}{dx^7} \right]_{x=0}$$

$$- \left[\frac{d^8 N_{k,n}(x)}{dx^8} \right]_{x=1} \times e^{-1} + \left[\frac{d^8 N_{k,n}(x)}{dx^8} \right]_{x=0} + \left[\frac{d^9 N_{k,n}(x)}{dx^9} \right]_{x=1} \times e^{-1} - \left[\frac{d^9 N_{k,n}(x)}{dx^9} \right]_{x=0}$$

$$- \left[\frac{d^{10} N_{k,n}(x)}{dx^{10}} \right]_{x=1} \times e^{-1} + \left[\frac{d^{10} N_{k,n}(x)}{dx^{10}} \right]_{x=0} - \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=1} \times e^{-1} + \left[\frac{dN_{k,n}(x)}{dx} \right]_{x=0}$$

(10.37)

The above equation (10.37) is equivalent to matrix form

$$(D + B)A = G \qquad (10.38a)$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k} = \int_0^1 \left[\left\{ -\frac{d^{11}N_{k,n}(x)}{dx^{11}} - \frac{d^2 N_{k,n}(x)}{dx^2} \right\} \frac{dN_{i,n}(x)}{dx} - 4\theta_0 e^x N_{i,n}(x) N_{k,n}(x) \right] dx$$

$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=1} + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=0}$$

$$+ \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^9 N_{i,n}(x)}{dx^9} \right]_{x=1} - \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^8 N_{i,n}(x)}{dx^8} \right]_{x=1}$$

$$+ \left[\frac{d^3 N_{k,n}(x)}{dx^3} \frac{d^8 N_{i,n}(x)}{dx^8} \right]_{x=0} - \left[\frac{d^2 N_{k,n}(x)}{dx^2} \frac{d^9 N_{i,n}(x)}{dx^9} \right]_{x=0}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}$$

$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0} \qquad (10.38b)$$

$$b_{i,k}=-2\sum_{j=1}^{n}\alpha_j\int_0^1 (N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^x\,dx \qquad (10.38c)$$

$$g_k=\int_0^1\left[\left\{\frac{d^{11}N_{k,n}(x)}{dx^{11}}+\frac{d^2N_{k,n}(x)}{dx^2}\right\}\frac{d\theta_0}{dx}+2{\theta_0}^2e^xN_{k,n}(x)\right]dx+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{10}\theta_0}{dx^{10}}\right]_{x=1}$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{10}\theta_0}{dx^{10}}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^8\theta_0}{dx^8}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^8\theta_0}{dx^8}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=1}$$

$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^6\theta_0}{dx^6}\right]_{x=1}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}\times e^{-1}+\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0}+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times e^{-1}$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}-\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=1}\times e^{-1}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=0}$$

$$+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times e^{-1}-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\right]_{x=1}\times e^{-1}$$

$$+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\right]_{x=0}-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times e^{-1}+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0} \qquad (10.38d)$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (10.32a). That is, to find initial coefficients we solve the system

$$DA = G \tag{10.39a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \left\{ -\frac{d^{11}N_{k,n}(x)}{dx^{11}} - \frac{d^{2}N_{k,n}(x)}{dx^{2}} \right\} \frac{dN_{i,n}(x)}{dx} dx - \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=1}$$
$$+ \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}N_{i,n}(x)}{dx^{10}} \right]_{x=0} + \left[\frac{d^{2}N_{k,n}(x)}{dx^{2}} \frac{d^{9}N_{i,n}(x)}{dx^{9}} \right]_{x=1}$$
$$- \left[\frac{d^{2}N_{k,n}(x)}{dx^{2}} \frac{d^{9}N_{i,n}(x)}{dx^{9}} \right]_{x=0} - \left[\frac{d^{3}N_{k,n}(x)}{dx^{3}} \frac{d^{8}N_{i,n}(x)}{dx^{8}} \right]_{x=1}$$
$$+ \left[\frac{d^{3}N_{k,n}(x)}{dx^{3}} \frac{d^{8}N_{i,n}(x)}{dx^{8}} \right]_{x=0} + \left[\frac{d^{4}N_{k,n}(x)}{dx^{4}} \frac{d^{7}N_{i,n}(x)}{dx^{7}} \right]_{x=1}$$
$$- \left[\frac{d^{4}N_{k,n}(x)}{dx^{4}} \frac{d^{7}N_{i,n}(x)}{dx^{7}} \right]_{x=0} - \left[\frac{d^{5}N_{k,n}(x)}{dx^{5}} \frac{d^{6}N_{i,n}(x)}{dx^{6}} \right]_{x=1}$$
$$+ \left[\frac{d^{5}N_{k,n}(x)}{dx^{5}} \frac{d^{6}N_{i,n}(x)}{dx^{6}} \right]_{x=0} \tag{10.39b}$$

$$g_k = \int_0^1 \left[\frac{d^{11}N_{k,n}(x)}{dx^{11}} + \frac{d^{2}N_{k,n}(x)}{dx^{2}} \frac{d\theta_0}{dx} \right] dx + \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}\theta_0}{dx^{10}} \right]_{x=1}$$
$$- \left[\frac{dN_{k,n}(x)}{dx} \frac{d^{10}\theta_0}{dx^{10}} \right]_{x=0} - \left[\frac{d^{2}N_{k,n}(x)}{dx^{2}} \frac{d^{9}\theta_0}{dx^{9}} \right]_{x=1} + \left[\frac{d^{2}N_{k,n}(x)}{dx^{2}} \frac{d^{9}\theta_0}{dx^{9}} \right]_{x=0}$$
$$+ \left[\frac{d^{3}N_{k,n}(x)}{dx^{3}} \frac{d^{8}\theta_0}{dx^{8}} \right]_{x=1} - \left[\frac{d^{3}N_{k,n}(x)}{dx^{3}} \frac{d^{8}\theta_0}{dx^{8}} \right]_{x=0} - \left[\frac{d^{4}N_{k,n}(x)}{dx^{4}} \frac{d^{7}\theta_0}{dx^{7}} \right]_{x=1}$$
$$+ \left[\frac{d^{4}N_{k,n}(x)}{dx^{4}} \frac{d^{7}\theta_0}{dx^{7}} \right]_{x=0} + \left[\frac{d^{5}N_{k,n}(x)}{dx^{5}} \frac{d^{6}\theta_0}{dx^{6}} \right]_{x=1} - \left[\frac{d^{5}N_{k,n}(x)}{dx^{5}} \frac{d^{6}\theta_0}{dx^{6}} \right]_{x=0}$$
$$- \left[\frac{d^{6}N_{k,n}(x)}{dx^{6}} \right]_{x=1} \times e^{-1} + \left[\frac{d^{6}N_{k,n}(x)}{dx^{6}} \right]_{x=0} + \left[\frac{d^{7}N_{k,n}(x)}{dx^{7}} \right]_{x=1} \times e^{-1}$$
$$- \left[\frac{d^{7}N_{k,n}(x)}{dx^{7}} \right]_{x=0} - \left[\frac{d^{8}N_{k,n}(x)}{dx^{8}} \right]_{x=1} \times e^{-1} + \left[\frac{d^{8}N_{k,n}(x)}{dx^{8}} \right]_{x=0}$$

$$+\left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=1} \times e^{-1} - \left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=0} - \left[\frac{d^{10} N_{k,n}(x)}{dx^{10}}\right]_{x=1} \times e^{-1}$$

$$+\left[\frac{d^{10} N_{k,n}(x)}{dx^{10}}\right]_{x=0} - \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1} \times e^{-1} + \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0} \qquad (10.39c)$$

Once the initial values of α_i are obtained from eqn. (10.39a), they are substituted into eqn.(10.38a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (10.33), we obtain an approximate solution of the BVP (10.32).

Numerical results for example 5 are shown in the following **Table 5**.

Table 5: Numerical results for example 5 using 6 iterations

x	Exact Results	14 Bernstein Polynomials		14 Legendre Polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000	1.0000000000	0.000000E-000
0.1	1.1051709181	1.1051709181	2.5979219E-014	1.1051709181	5.436494E-012
0.2	1.2214027582	1.2214027582	7.0188300E-013	1.2214027582	7.342582E-013
0.3	1.3498588076	1.3498588076	2.6412206E-012	1.3498588076	9.542484E-012
0.4	1.4918246976	1.4918246976	5.0834892E-012	1.4918246976	1.738262E-012
0.5	1.6487212707	1.6487212707	6.2061467E-012	1.6487212707	4.990510E-012
0.6	1.8221188004	1.8221188004	5.0610627E-012	1.8221188004	2.407930E-012
0.7	2.0137527075	2.0137527075	2.6179059E-012	2.0137527075	4.307570E-013
0.8	2.2255409285	2.2255409285	6.9233508E-013	2.2255409285	7.753470E-012
0.9	2.4596031112	2.4596031112	2.5757174E-014	2.4596031112	3.203970E-012
1.0	2.7182818285	2.7182818285	0.0000000E+000	2.7182818285	0.000000E-000

On the contrary the maximum absolute errors have been found by Wazwaz [80], Nadjafi and Zahmatkesh [89], Mohy-ud-Din *et al* [103] and Noor and Mohy-ud-Din [105] is 5.22×10^{-7}.

We depict the exact and approximate solutions in Fig. 5(a) and a plot of relative errors in Fig. 5(b) of example 5 for $n=14$. From Fig. 5(b) we observe that the error is nearly the order 10^{-8}.

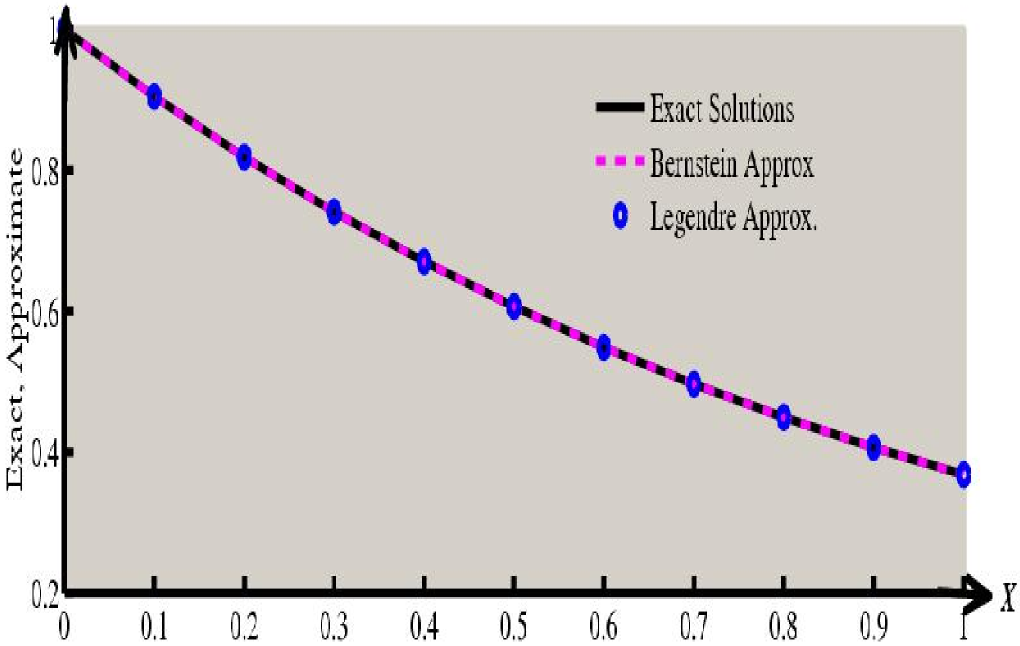


Fig. 5(a): Graphical representation of exact and approximate solutions of example 5 using 14 polynomials.

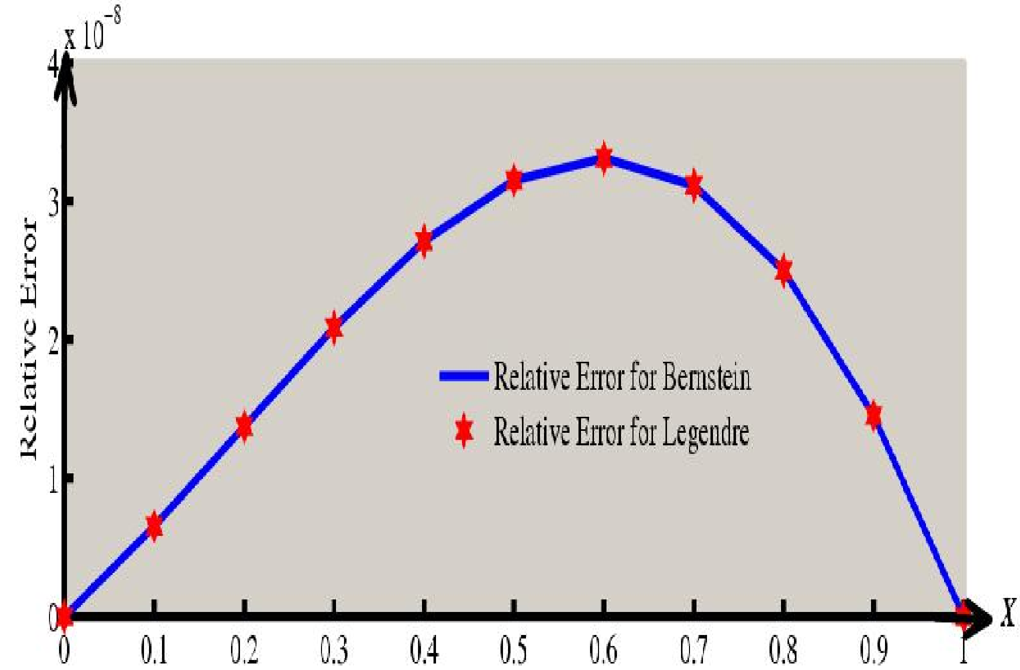


Fig. 5(b): Graphical representation of relative error of example 5 using 14 polynomials.

Example 6: Consider the **nonlinear** differential equation [82, 103, 105]

$$\frac{d^{12}u}{dx^{12}} = \frac{1}{2}e^{-x}u^2, 0 \le x \le 1 \tag{10.40a}$$

with boundary conditions type II, defined in eqn. (2c)

$$u(0)=2,\ u(1)=2e,\ u''(0)=2, u''(1)=2e, u^{(iv)}(0)=2, u^{(iv)}(1)=2e, u^{(vi)}(0)=2,$$

$$u^{(vi)}(1)=2e, u^{(viii)}(0)=2, u^{(viii)}(1)=2e, u^{(x)}(0)=2, u^{(x)}(1)=2e. \qquad (10.40b)$$

The exact solution of this BVP is, $u(x)=2e^{x}$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}\alpha_i N_{i,n}(x),\ n\geq 1 \qquad (10.41)$$

Here $\theta_0(x)=2+2x(e-1)$ is specified by the essential boundary conditions in (10.40b). Also $N_{i,n}(0)=N_{i,n}(1)=0$ for each $i=1,2,\ldots,n$.

Using eqn. (10.41) into eqn. (10.40a), the Galerkin weighted residual eqns. are

$$\int_0^1\left[\frac{d^{12}\tilde{u}}{dx^{12}}-\frac{1}{2}e^{-x}\tilde{u}^2\right]N_{k,n}(x)dx=0, k=1,2,\cdots,n \qquad (10.42)$$

Integrating first term of (10.42) by parts, we obtain

$$\int_0^1\frac{d^{12}\tilde{u}}{dx^{12}}N_{k,n}(x)=-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^{10}\tilde{u}}{dx^{10}}\right]_0^1+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\tilde{u}}{dx^9}\right]_0^1-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^8\tilde{u}}{dx^8}\right]_0^1$$
$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^6\tilde{u}}{dx^6}\right]_0^1+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^7N_{k,n}(x)}{dx^7}\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$
$$+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^9N_{k,n}(x)}{dx^9}\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{d\tilde{u}}{dx}dx$$

(10.43)

Using eqn. (10.43) into eqn. (10.42) and using approximation for $\tilde{u}(x)$ given in eqn. (10.41) and after applying the boundary conditions given in eqn. (10.40b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{dN_{i,n}(x)}{dx}-\theta_0\,e^{-x}\,N_{i,n}(x)N_{k,n}(x)\right.\right.$$
$$\left.-\frac{1}{2}\sum_{j=1}^{n}\alpha_j(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\right]dx+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}$$

$$\left.-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=1}-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=0}\right]\alpha_i$$

$$=\int_0^1\left[\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{d\theta_0}{dx}+\frac{1}{2}\theta_0{}^2e^{-x}N_{k,n}(x)\right]dx-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times 2e$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}\times 2+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times 2e-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times 2$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times 2e-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times 2+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times 2e$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}\times 2+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times 2e-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}\times 2 \qquad (10.44)$$

The above equation (10.44) is equivalent to matrix form

$$(D+B)A=G \qquad (10.45a)$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{dN_{i,n}(x)}{dx}-\theta_0e^{-x}N_{i,n}(x)N_{k,n}(x)\right]dx+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}$$

$$-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1}$$

$$-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=1}$$

$$-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=0} \qquad (10.45b)$$

$$b_{i,k}=-\frac{1}{2}\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))e^{-x}\,dx \qquad (10.45c)$$

$$g_k=\int_0^1\left[\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{d\theta_0}{dx}+\frac{1}{2}\theta_0{}^2e^{-x}N_{k,n}(x)\right]dx-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$

$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$

$$-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1}\times 2e$$

$$-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0}\times 2+\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1}\times 2e-\left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0}\times 2$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=1}\times 2e-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times 2+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times 2e$$

$$-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}\times 2+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times 2e-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}\times 2 \qquad (10.45d)$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (10.40a). That is, to find initial coefficients we solve the system

$$DA = G \tag{10.46a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 -\frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{dN_{i,n}(x)}{dx}dx + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1} - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}$$
$$+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1} - \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}$$
$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1} - \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=0}$$
$$+\left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=1} - \left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3N_{i,n}(x)}{dx^3}\right]_{x=0}$$
$$+\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=1} - \left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}\right]_{x=0} \tag{10.46b}$$

$$g_k = \int_0^1 \frac{d^{11}N_{k,n}(x)}{dx^{11}}\frac{d\theta_0}{dx}dx - \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=1} + \left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^9\theta_0}{dx^9}\right]_{x=0}$$
$$-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=1} + \left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^7\theta_0}{dx^7}\right]_{x=0} - \left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$
$$+\left[\frac{d^6N_{k,n}(x)}{dx^6}\frac{d^5\theta_0}{dx^5}\right]_{x=0} - \left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=1} + \left[\frac{d^8N_{k,n}(x)}{dx^8}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$
$$-\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=1} + \left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}\right]_{x=0} + \left[\frac{dN_{k,n}(x)}{dx}\right]_{x=1} \times 2e$$
$$-\left[\frac{dN_{k,n}(x)}{dx}\right]_{x=0} \times 2 + \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=1} \times 2e - \left[\frac{d^3N_{k,n}(x)}{dx^3}\right]_{x=0} \times 2$$

$$+\left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=1} \times 2e - \left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=0} \times 2 + \left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=1} \times 2e$$

$$-\left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=0} \times 2 + \left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=1} \times 2e - \left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=0} \times 2 \qquad (10.46c)$$

Once the initial values of α_i are obtained from eqn. (10.46a), they are substituted into eqn.(10.45a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (10.41), we obtain an approximate solution of the BVP (10.40).

Numerical results for example 6 are shown in the following **Table 6**.

We illustrated graphically the exact and approximate solutions in Fig. 6(a) and the relative errors in Fig. 6(b) of example 6 for $n = 15$. It is clear from Fig. 6(b) that the error is of order 10^{-8}.

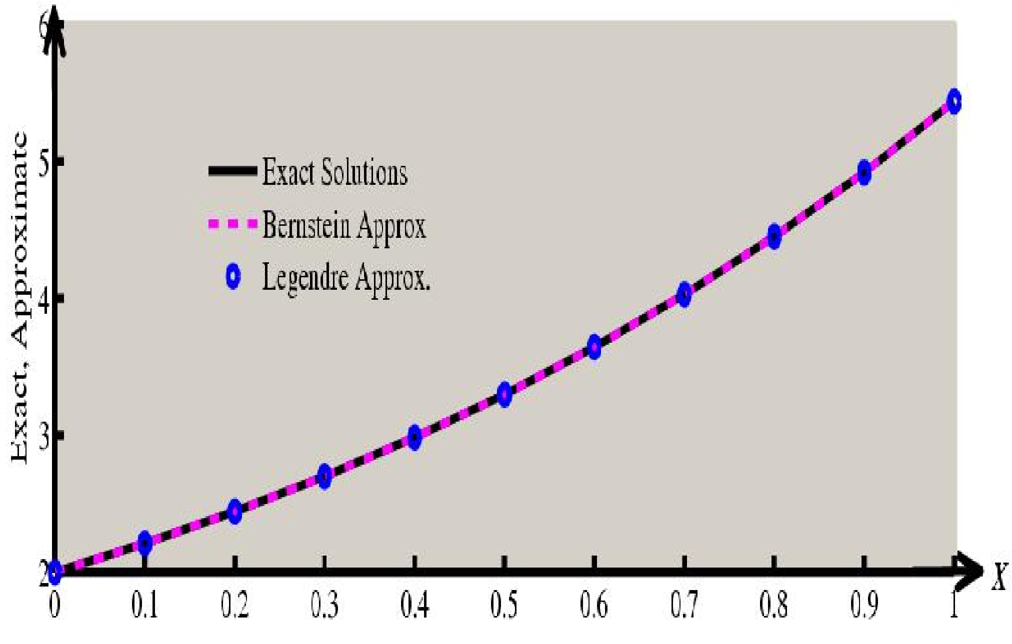


Fig. 6(a): Graphical representation of exact and approximate solutions of example 6 using 15 polynomials.

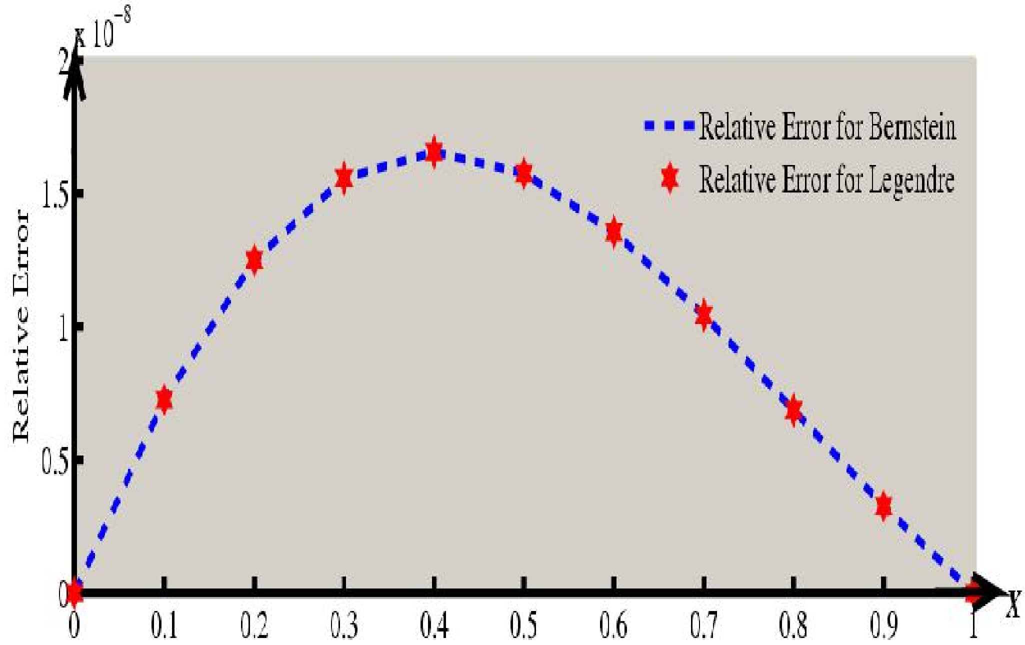


Fig. 6(b): Graphical representation of relative error of example 6 using 15 polynomials.

Table 6: Maximum absolute errors of example 6 using 6 iterations.

Number of Polynomial used	Max. Abs. Error for Bernstein	Max. Abs. Error for Bernstein	Reference Results
12	6.910×10^{-9}	5.915×10^{-9}	2.621×10^{-5} (Kasi and Showri Raju [75]) 6.614×10^{-4} (Mohy-ud-Din *et al* [103]) 6.614×10^{-4} (Noor and Mohy-ud-Din [105])
13	8.918×10^{-10}	7.905×10^{-10}	
14	9.875×10^{-11}	9.860×10^{-11}	
15	9.980×10^{-12}	5.795×10^{-12}	

10.4 Conclusions

In this chapter, we have solved numerically twelfth order linear and nonlinear BVPs by the Galerkin method with Bernstein and Legendre polynomials as basis functions for two different types of boundary conditions. It is observed from the tables that the numerical results obtained by our method are superior to other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. It may also mention that the numerical solutions coincide with the exact solution if we use lower order Bernstein and Legendre polynomials in the approximation which are shown in Figs. [1-6].

Present work and conclusions

The higher order boundary value problems (BVPs) have many applications in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Therefore, in our thesis we have solved numerically higher order (from order four up to order twelve) linear and nonlinear boundary value problems using Galerkin method. The well known Bernstein and Legendre polynomials have been exploited as basis functions in the method.

For this reason, in chapter 1 we have discussed some definitions, theorems, corollaries, etc. which are related to our thesis, the properties of Bernstein and Legendre polynomials with some mathematical well known theorems/formulae that are essential to solve the problems presented in this thesis. The scope and objectives of the thesis are also included in this chapter.

In chapter 2, the numerical solutions of the fourth order linear and nonlinear differential equations using Bernstein and Legendre polynomials as basis functions have been investigated. Formulations I, II and III are derived in matrix form by Galerkin method for two different types of boundary conditions. For the numerical verification of the proposed Formulations I, II and III, we consider four linear and two nonlinear BVPs. The approximate solutions converge to the exact solutions even with desired large significant digits.

In chapter 3, at first the matrix form is formulated for the numerical solution of linear fifth order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials as trial functions and then this idea is extended for solving nonlinear differential equations. To verify the reliability and efficiency of the proposed method two linear and two nonlinear BVPs are considered. The computed results have been presented in tabular forms and graphically. These show that the present method is better than the existing methods.

Chapter 4 has been dealt with sixth order BVPs to find the numerical solutions of linear and nonlinear differential equations using Formulation I and Formulation II for two types of boundary conditions by the Galerkin method. In this method the basis functions are modified into a new set of basis functions to satisfy the corresponding homogeneous form of Dirichlet boundary conditions. Numerical verification of the method has been performed and the results have been obtained are superior to other existing methods considering four linear and two nonlinear BVPs.

Chapter 5 has been provided for the numerical evaluations of seventh order linear and nonlinear BVPs by the Galerkin method where the basis functions Bernstein and Legendre polynomials are modified to satisfy the corresponding homogeneous form of *Dirichlet* boundary conditions. The numerical results of three linear and one nonlinear BVPs have been shown in tables and graphs, and the results are also better than the other existing methods.

In chapter 6, the numerical solutions of eighth order BVPs have been illustrated with five linear and two nonlinear differential equations by Galerkin method with Bernstein and Legendre polynomials as basis functions using Formulation I and Formulation II for two different types of boundary conditions. The numerical results of the proposed method are compared with both the exact solution and the results of the other methods which show that the present method is efficient and convenient.

The aim of the chapter 7 was to apply Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions to find the numerical solutions of ninth order BVPs. The method is formulated as a rigorous matrix form to verify the proposed formulation considering one numerical example of linear BVP. The solution obtained by the present method is better than the existing methods.

In chapter 8, we have considered Formulation I and Formulation II for two different kinds of boundary conditions for the numerical solutions of tenth order BVPs by the Galerkin method. Then we solve five linear and two nonlinear BVPs using these two formulations taking Bernstein and Legendre polynomials as basis functions and we obtain better results than previous results.

Chapter 9 has been included for the numerical solutions of eleventh order linear and nonlinear BVPs with Bernstein and Legendre polynomials as basis functions. The proposed method has been tested on two linear and one nonlinear BVPs to compare the errors with those developed before and the results obtained are superior to other existing methods.

The last chapter 10 is devoted to find the numerical solutions of Twelfth order linear and nonlinear differential equations by the Galerkin method using Formulation I and Formulation II for two types of boundary conditions. In this chapter we have solved four linear and two nonlinear BVPs applying Bernstein and Legendre polynomials as basis functions and we have got better results than other existing methods.

The approximate solutions for BVPs that we have discussed in this thesis have good results but they depend on the basis functions, types of boundary conditions and order of the problem. From the table, we see that the numerical results for the boundary conditions of type I are superior to the boundary conditions of type II in each even order BVPs discussed in chapters 2, 4, 6, 8 and 10. In addition, Bernstein polynomials yield the better results than the Legendre polynomials with same degree. For nonlinear problems, when the order is higher the results will be better. The nonlinear BVPs take long time in testing and calculating to get more accurate results. The Galerkin method needs hard work while deriving approximations especially for these higher order problems. But the algorithm can be coded easily and may be used to solve any higher order BVP.

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