EXISTENCE AND STABILITY OF ISOTROPIC MODULI SPACES

A Thesis Submitted

In partial fulfillment of the requirements for the Degree of

Master of Philosophy

Submitted by

Name: Sharmin Akter

Registration No:236

Session:2011-2012

April - 2017

To

UNIVERSITY OF DHAKA

Department of Mathematics

Faculty of Science

UNIVERSITY OF DHAKA

DHAKA-1000

Dedicated to

My loving parents

Declaration

I hereby declare that, this paper has been done by me under the supervision of **Dr Md. Showkat Ali, Professor**, Department of Applied Mathematics, University of Dhaka, Dhaka-1000.I am hereby declaring that no portion of the work considered in this thesis has been submitted in support of an application for another degree or qualification of this or any other University or Institute of learning either in home or abroad.

Sharmin Akter
April 2017

Certificate

This is to certify that, the thesis entitled **"Existence and Stability of Isotropic Moduli Spaces"** submitted by Sharmin Akter, Registration No: 236, Session: 2011-2012 of University of Dhaka Bangladesh in partial fulfillment of the requirements for the Degree ofMaster of Philosophy in Mathematics of this University is absolutely based upon her work and that the thesis has not been submitted of any degree/diploma or any other academic award anywhere before. This thesis is carried out by the author under the supervision of Dr. Md. Sawkat Ali, Professor, Department of Applied Mathematics, University of Dhaka, Bangladesh.

Dr. Md. Showkat Ali
Professor
Department of Applied Mathematics
University of Dhaka,
Dhaka-1000, Bangladesh

Acknowledgement

I would like to thank the noteworthy recognizance of Almighty's continual mercy and help without which no work would have been possible to accomplish the goal. I am pleased to acknowledge with gratefulness to my supervisor Dr. Md. Sawkat Ali, Professor, Department of Applied Mathematics, University of Dhaka, for his guidance, constant support, intuitive suggestions and relentless encouragement which have been found very benevolent for the outcome of the research.

I would also like to thank Prof. Dr. Md. Shahidul Islam and Prof. Dr. Md. Shafiqul Islam Department of Applied Mathematics, University of Dhaka,member of the Board of Examiners for their outstanding recommendation and counsel in the course of being successful to reach the objectives of this dissertation. I am grateful to the external member of the Board of Examiners, Dr. Md. Kamrul Islam, Professor, Research Centre for Mathematics & Physical Sciences (RCMPS), University of Chittagong, Chittagong, for his valuable suggestions in improving the quality of the work.

I am grateful to all my colleagues for their encouragement and helping mentality in all affairs specially in my research work.

I owe a debt of gratitude to my friends who have assisted me by providing relevant books and valuable suggestions.

I am also indebted to Professor and Chairman of Department of Mathematics, Dr. Ommulo Chandra Mondol Sir, for his support in allowing me to use the departmental facilities in various stages of my work. I wish to thank to the staff of the Department of Mathematics, University of Dhaka, for their cooperation in this work.

Finally, I express my devoted affection to my husband Md. Mijanur Rahaman and all my family members and relatives specially my mother for creating a delightful atmosphere as well as excusing me from family duties in order to complete the courses, research studies and final production of the thesis work.

Abstract

Manifolds are simplifications of our accustomednotions about curves and exteriors to arbitrary dimensional objects. Generally, A Manifolds is a topological space which is homeomorphic to $\mathbb{R}^m$. Connections of manifolds are of central importance in modern geometry in large part because they allow a comparison between the local geometry at one point and the local geometry at another point. It is a well-known fact that, a Riemannian metric on a differentiable manifold induces a Riemannian metric on its submanifold and, hence, a Riemannian connection on the manifold induces a Riemannian connection on its submanifold.

We haveconcerned aboutLie groups and Lie Algebra which deals with the applications of classical mechanics, In the mathematical fields of differential geometry and tensor calculus, differential forms provide a unified approach to defining integrands over curves, surfaces, volumes, and higher-dimensional manifolds

.

In this paper, we established the theorem of Stability of isotropic submanifolds where*X* be a compact complex submanifold of a complex manifold *Y*. The main object of interest in this paper is the set *M* of all holomorphic deformations of *X*inside *Y*, i.e. a point t in *M* can be thought of as a" nearby" compact complex submanifold X_t in *Y*.Instead of analyzing some particularKodaira moduli spaces (as is normally done in twister theory, where moduli spaces of rational curves and quadrics with specific normal bundles have been only considered), This generalizes the result of Merkulov on Isotropic Submanifolds, which is not necessarily on Legendre and Kodaira.

Preface

Many different mathematical methods and concepts are used in classicalmechanics: differential equations and phase flows, smooth mappings and manifolds, Lie groups and Lie algebras, symplectic geometry and ergodic theory.

In Chapter I develops the theory of manifolds of different functions also describe the features of manifolds on vector fields. Connections of various aspects describes possibly with the Levi civita connection, Torsion free connections.

In Chapter II Lie algebras are an essential tool in studying both algebraic groups and Lie groups. we apply the theory of Lie algebras to the study of algebraic groups in characteristic zero and one, the relation between Lie algebras and algebraic groups in characteristic zero is best understood through their categories of representations. Commutator (Lie Bracket) of vector fields have been discussed.

In Chapter III we have discussed about the fibre bundles together with the tangent spaces of a manifold M to the new manifold TM that could be regarded as a bundle of vector spaces. we shall see this can be used to give a very geometrical way of thinking about general tensor structures on manifold.

In Chapter IVwe have discussed about the differential form of Manifolds in the field of modern Calculus, differential Topology and tensors. The idea of differential form being the wedge product of the exterior derivatives forming an exterior Algebra.

In Chapter V, the existence ofKodaria moduli spaces with the family of compact complex isotropic submanifolds in a complex manifold has been discussed. Legendre and isotopic submanifolds with the normal bundle,$N_X|_Y$Probably the conditions for existence of an induced geometric structure on M have the form of vanishing of some cohomology groups associated with normal bundles.

In Chapter VI, we have established the interconnections of isotropic and legendre submanifolds with the Kodaira Moduli Spaces. Different notations and theorem have been studied and finding the stability of compact complex isotropic submanifolds related to Kodaira, therefore that generalizes Merkulov's result to the isotropic submanifold.

Table of Contents

LIST OF FIGURES

Chapter One

Manifolds and Connection

1. Introduction

Riemannian geometry is a special geometry associated with differentiable manifolds, Riemannian manifolds, smooth manifolds, with a Riemannian metric that is an inner product on the tangent space at each point which varies smoothly from point to point. Moreover, it has more applications in physics; geometry forms the natural basis of a physical theory. It is originated with the vision of Bernhard Riemann expressed in his inaugural lecture (On the Hypotheses on which geometry is based). This gives, in particular, local notions of angle, length of curves, surface area, and volume. From those some other global quantities, be derived by integrating local contributions. It enabled Einstein's general relativity theory, made profound impact on theory and representation theory, as well as analysis, and spurred the development of algebraic and topology.

The important notions of Riemannian geometry are based on Manifolds. The calculus of manifold just as topology is based on continuity, so the theory of manifolds is based on smoothness. According to the theory, the universe is smooth Manifold equipped with Pseudo–Riemannian geometry which described the curvature of space time understanding this curvature is essential for the positioning of satellites into orbit around the earth.

1.1 Some Fundamental Definition

1.1.1 The Metric

The metric on a manifold is the basic object of Riemannian geometry. Given a differentiable manifold M, the metric is simply an assignment of a bilinear map $T_pM \otimes T_pM \to \mathbb{R}^n$at each point p ∈ M with the following properties

(i) $g(v, w) = g(w, v)$ (symmetry)

(ii) $g(v, v) > 0$ when $v \neq 0$ (positive definiteness), and

(iii) g varies differentially.

1.1.2 Topology

Let a set X be a non-empty set, A System $\mu = \{\mu_i;\ i \in I, \mu_i \subset X\}$ is called a topology on *X*, if it contains the following three conditions.

(i) The empty set φ and the set *X* belong to μ.

(ii) The union of any number of a finite number of sets μ are in the system μ.

(iii) The Intersection of any finite number of sets μ belongs to μ .

The set of μ are called open. The pair (μ , *X*) consisting of a set *X*, and a Topology μ on x is called a topological space and we define it just as a *X*.

1.1.3 Riemannian metric

A Riemannian metric which is determined by an inner product on each tangent space $T_p M$, can be determined by $< X, Y > := g(X, Y)$ for $X, Y \in T_p M$

The concept of a Riemannian metric on a smooth manifold, which is simply a tensor field determining an inner product at each tangent space.

1.1.4 Housdorff Space

A topological space *X* is called a Housdorff space, if any two distinct Pair's possesses disjoint neighborhoods. Let an example, N_x and N_y possesses disjoint neighborhoods then,

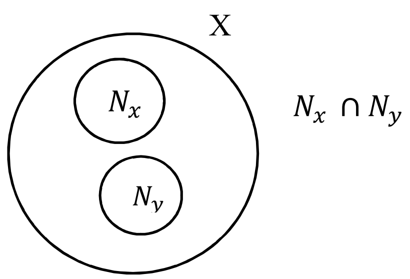

Figure 1.1(a): Housedorff Spaces

1.1.5 Compact Space

A topological space X is called a compact if every covering neighborhood of has a finite sub covering.

1.1.6 Homomorphism

A function f: $X \rightarrow Y$ between two topological spaces X and Y is called a homomorphism if it has the following properties:

(i) f is bijective.

(ii) both f and f^{-1} are contiuous.

Example: Consider X = $\mathbb{R}^n = \{ x^1, x^2, \ldots, x^n \}$ is an open ball considered at $x_0 \in \mathbb{R}^n$ with radius r is $B_r(x_0) = x_0 \in \mathbb{R}^n : \{\sqrt{(x^1 - x_0^1)^2 + (x^2 - x_0^2)^2 + \cdots + (x^1 - x_0^n)^2} < r\}$

1.1.7 Continuous Mapping

A map f: $X \rightarrow Y$ is called continuous at a point $x \in X$, if given any open neighborhood $N(Y) \subset Y$, there exists an open neighborhood $N(x) \subset X$ such that $f(N(x) \subseteq N(y)$.Given an example

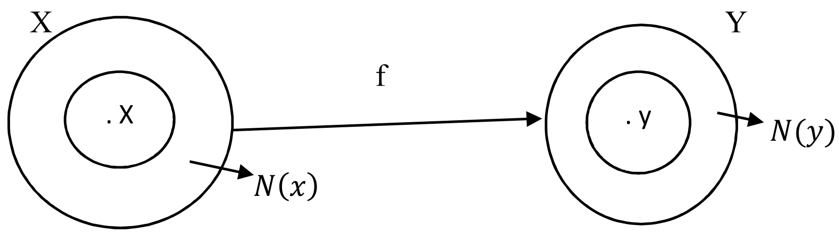

Figure 1.1(b): continuous map

1.1.8 C^∞ Diffeomorphism

Let U and V be two open subsets of $\mathbb{R}$, A function f: $U \rightarrow V$ is called C^∞diffeomorphism, if it satisfies the following.

(i) f is homomorphism.

(ii) If f and f^{-1} are of class C^∞.

1.1.9 Diffeomorphism

A function $U \rightarrow \mathbb{R}^m$ is an isomorphism of C^∞- class and invertible, then f is called a diffeomorphism.

1.1.10 Local Diffeomorphism

A map *f:* $U \rightarrow V$ is a local diffeomorphism if and only if f is smooth and det $(\frac{\partial f}{\partial x})/_{x_0} \neq 0$ at each

Point of $x_0 \epsilon U$, where U and V be open subset in $\mathbb{R}^n$.

1.1.11 Smooth Function

A function *f:* $M \rightarrow \mathbb{R}$ on smooth manifold then *M* is called smooth Function at a point x ϵM,if in a coordinate chart (U, $\emptyset$) with x ϵU, the function $f \circ \emptyset^{-1}$ is smooth on a function of n-variables that is smooth on a function of n-variables $(X^1, X^2$,……, $x^n)$, *n* = dim *M*.

1.1.12 Chart

A chart is a pair (U, $\emptyset$) consisting of a topological manifold M is an open subset *U* of *M* called the domain of the chart together with a homomorphism $\emptyset : U \rightarrow V$ onto an open set *V* in $\mathbb{R}^n$.

1.1.13 Atlas

An atlas of class C^k , on a topological manifold M is a set $\{(U_\alpha, \emptyset_\alpha)$, $\alpha \in I\}$ of charts such that,

(i) The domain $U_\alpha$cover M.

(ii) The homomorphism $\emptyset_\alpha$ satisfy the following compatibility

conditions

$$\emptyset_\alpha \circ {\emptyset_\beta}^{-1} : \emptyset_\beta(U_\alpha \cap U_\beta) \rightarrow \emptyset_\alpha(U_\alpha \cap U_\beta)$$

open subset in $\mathbb{R}^n$ must be a class of C^k .

1.1.14 Pull back Function

Let $f\ :\ X^n \to Y^p$ be a smooth map where X^n be an n-dimensional smooth manifold and g be a function on Y also an P-dimensional smooth manifold, Then the **pull –back** (or reciprocal image) of the function g under the map f is a function on x is,

$$f^*(g)(x) = g\big(f(x)\big) \qquad\qquad where\ x\ \in X.$$

Example: Let M and N be a smooth manifold and

$$f: M \to N$$
$$\uparrow$$
$$x_0 \leftrightarrow f(x_0) = y_0 \in N.$$

Let O_{x_0} be a set of smooth function at x, O_{y_0} be a set of smooth function at the image point. Then the map f: $M \to N$ implies by

$$f^*:\ O_{y_0}\ \to O_{x_0}$$
$$\uparrow$$
$$g \leftrightarrow f^*(g) =\ g \circ f.$$

1.1.15 Push Forward Function

Let M and N be a smooth manifold and f: $M \to N$ induces a push forward map i

$$f_y: T_{x_0} M \leftrightarrow T_{f(x_0)}N\ ; x_0 \in M.$$
$$\uparrow$$
$$\overrightarrow{V_{x_0}} \leftrightarrow f_y\ (\overrightarrow{V_{x_0}}\ )$$

defined by

$f_y\ (\overrightarrow{V_{x_0}}\ )\ .\ g = \overrightarrow{V_{x_0}}\ f^*\ g$, Where V_{x_0} be a set of smooth function at x

1.2 Manifolds

Manifolds are generalizations of our familiar ideas about curves and surfaces to arbitrary dimensional objects. It is to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. It will turn out to be the most general space in which we can use differential and integral calculus with roughly the same facility as in Euclidean space. It should be recalled, though, that calculus in demands special care when curvilinear coordinates are required. In general, it is a topological space which is homeomorphism. Throughout this, all our manifolds are assumed to be smooth, means C^{∞}, or infinitely differentiable, in the differential geometry, we take local co-ordinates on any subset $U \in M$ *as* $(X^1, X^2, \ldots, x^n)$ *or*, X coordinate constitute a map from U to $\mathbb{R}^n$, it is more general ,to use a co-ordinate chart to identify U with its image in $\mathbb{R}^n$ also to identify U with its coordinate representation (X^i) in $\mathbb{R}^n$. Manifolds are roughly speaking; abstract surfaces that are globally look like linear spaces. We shall assume at first that the linear spaces are $\mathbb{R}^n$ for a fixed integer n, which will be the dimension of manifold.

On a finite dimensional vector space V with its standard smooth manifold structure there is a natural identification of each tangent space $T_p V$,Obtained by a vector X $\in V$ with directional derivative of the co-ordinates (X^i)induced on V by any basis this is as,$(X^1, X^2, \ldots, x^n) \leftrightarrow X^i \partial_i$.

$$Xf = \frac{d}{dt} /_{t=0} f(p + tX)$$

A curve in three-dimensional Euclidean space is parameterized locally by a single number *t* as (x(t), y(t), z(t)) while two numbers *u* and *v* parameterize a surface as (x (u, v), y(u, v), z(u, v)). A curve and a surface are considered locally homeomorphism.

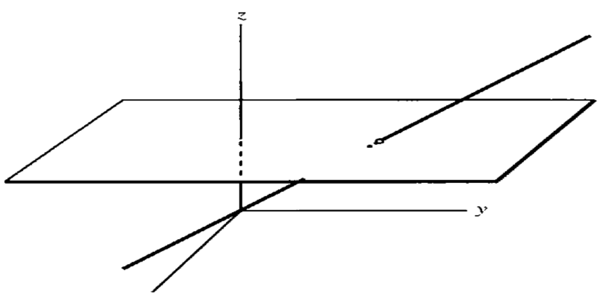

Fig 1.2: The real projective n space $\mathbb{R}^3$ determined by the point on the Line.

There are different types of Manifolds given by the following, on which we will describe the vast descriptions of various deepens that becomes more sophisticated.

1.2.1 Topological Manifold

A topological manifold *M* of dimension n is a topological space with

(i) *M* is Hausdorff, that is, for each pair P_1, P_2 of *M*, there exist V_1 , V_2 such that

$$V_1 \cap V_2 = \emptyset.$$

(ii) Each point $p \in M$ possesses and V homeomorphism to an open subset *U* of $\mathbb{R}^n$.

(iii) *M* satisfies the second count ability axiom.

We could also have defined C^K manifolds by requiring the Coordinate Changes to be C^K maps. c^0-manifold would then denote a Topological manifold).

In particular, the Hausdorff axiom ensures that the limit of a convergent Sequence is unique. This, along with the second countability axiom. If the dimension of *M* is zero, then *M* is a countable set equipped with the discrete topology (every subset of M is an open set). If dim *M* = 1, then *M* is locally homeomorphism to an open interval. If dim *M* = 2, then it is locally homeomorphism to an open disk, Torus of revolution.

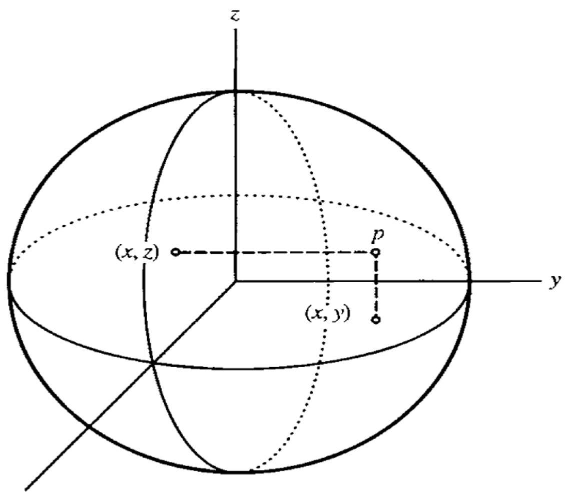

Fig 1.3**:** The surface of a circle is a topological manifold (homeomorphism to s^2).

Example:

- Every $U \in M$ is an open set if and only if $U = M \cap V$ with V (an open set of $\mathbb{R}^n$) is a topological manifold.
- The sphere $S^2 = \{(X, Y, Z) \in \mathbb{R}^3: x^2 + y^2 + Z^2 = 11\}$ with the subspace topology is a topological manifold.
- (The circle S^1) The circle $S^1 = \{(x, y) \in R^2: x^2 + y^2 = 1\}$ with the subspace topology is a topological manifold of dimension1.Conditions (i) and (iii) are inherited from the ambient space.

1.2.2 Riemannian Manifold

A Riemannian Manifold (M, g) is a real differentiable manifold M in which each tangent space is equipped with an inner product g, a Riemannian metric which varies smoothly from point to point. It is a pair (M; g) with M a manifold and g a metric g_m on M.

Example 1:

Let M be a smooth manifold Riemannian metric g on M is a tensor field,

$g : c_2^\infty(TM) \to c_0^\infty(TM)$ for each p $\in M$ the restriction.

$$g_p = g / T_p^m \circ T_p^m : T_p^m (X)$$

with $g_p : (X_p, Y_p) \leftrightarrow g(X, Y)(p)$

is an inner product on the tangent space T_p^m, where the pair (M, g) is called a Riemannian manifold

Example 2:

Let M be a smooth manifold. A **Riemannian metric** g on M is a tensor field

$$g : c_2^\infty(TM) \to c_0^\infty(TM)$$

such that for each $p \in M$ the restriction $g_p$of g to $T_pM \otimes T_pM$ with

$$g_p: (X_p, Y_p) \rightarrow g(X, Y)(p)$$

is an inner product on the tangent space T_pM. The pair (M, g) is called a **Riemannian manifold**. The study of Riemannian manifolds is called Riemannian geometry. Geometric properties of $(M; g)$ which only depend on the metric g are said to be **intrinsic** or metric properties.

Theorem (Cartan–Hadamard). Suppose M is a complete, connected Riemannian n-manifold with all sectional curvatures less than or equal to zero. Then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.

Theorem (Bonnet). Suppose M is a complete, connected Riemannian manifold with all sectional curvatures bounded below by a positive constant, Then M is compact and has a finite fundamental group.

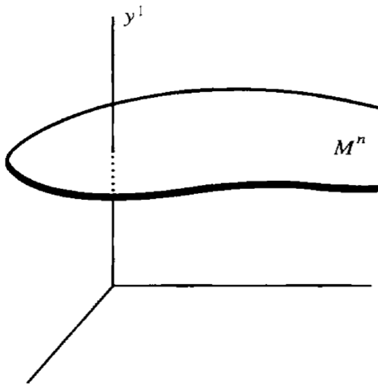

Fig 1.4: $Y^1 = f(X^1 \cdots X^n)$ described an n-dimensional manifold of $\mathbb{R}^{n+1}$.

In Figure 1.4 we have drawn a portion of the manifold M. This M is the graph of a function $f: \mathbb{R}_n \rightarrow \mathbb{R}$ that is, $M = \{((X, Y) \in \mathbb{R}_{n+1} \mid y = f(x, y)\}$. When n = 1, M is a curve; while if n = 2, then it is a surface.

Theorem 1: Every manifold can be given a Riemannian metric.

Proof

If p is a point in a Riemannian manifold $(M; g)$, we define the length or norm of any tangent vector $X \in T_pM$ to be $|X| := \langle X, X \rangle^{1/2}$.

If two non- zero vectors X, $Y \in T_p M$ to be unique if there exists a value $\theta \in [0, \pi]$, satisfying $\text{Cos}\theta = < X, Y > (|X||Y|)$. Here X and Y are orthogonal if their angle is $\pi/2$.

If (M, g) and $(\widetilde{M}, \tilde{g})$are Riemannian manifolds, a diffeomorphism $\emptyset$ from M to $\breve{M}$ is called an isometric if $\emptyset * \tilde{g} = g$. An isometric $\emptyset : (M, g) \rightarrow (M, g)$ is an isometry of M. A composition of isometrics and the inverse of an isometry are again isometrics, so it is a group.

If $(E_1, E_2, \ldots\ldots, E_n)$ is any local frame for TM *and* $(\varphi^1, \ldots \ldots \ldots . \varphi^n)$ is its dual coframe , a Riemannian metric can be written locally as

$$g = g_{ij}\varphi^i \otimes \varphi^j \qquad (1)$$

where $g_{ij} = < E_i, E_j >$ is symmetric in I , j *depends on* $P \in M$, in particular in a coordinate frame, g has the form $g = g_{ij} dx^i \otimes dx^j$

(2)

By introducing the two terms of the symmetry of equation (1) and (2) on g_{ij} , we get

$$g = g_{ij} dx^i dx^j$$

is a Riemannian metric.

This completes the proof. □

Example 3:

- Let $\tilde{g} : \lambda g_0 + (1- \lambda) g_1$, $\lambda \in [0,1]$ is a metric on M is a Manifold.
- Let (,) is an inner product on $\mathbb{R}^n$. An open set $U \subset \mathbb{R}^n$ gets a Riemannian metric. $U_m \cong \mathbb{R}^n$.

1.2.3 Pseudo-Riemannian manifold

A pseudo-Riemannian manifold is a differentiable manifold equipped with a non-degenerate, smooth, symmetric metric tensor Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p, q), where both p and q are non-negative.

In differential geometry, a pseudo-Riemannian manifold (also called a semi-Riemannian manifold) is a generalization of a Riemannian manifold in which the metric tensor need not be positive-definite, but is instead only required to be non-degenerate.

Every tangent space of a pseudo-Riemannian manifold is a pseudo-Euclidean space described by a quadratic form, which may be isotropic. Some basic theorems of Riemannian geometry can be generalized to the pseudo-Riemannian case. In particular, the fundamental theorem of Riemannian geometry is true of pseudo-Riemannian manifolds as well.

1.2.4 Differentiable Manifold

If M is an m-dimensional differentiable manifold, it satisfies the following:

(i) M is a topological space.
(ii) M is provided with a family of pairs $\{(U_i, \emptyset_i)\}$.
(iii) $\{U_i\}$ is a family of open sets which covers M.

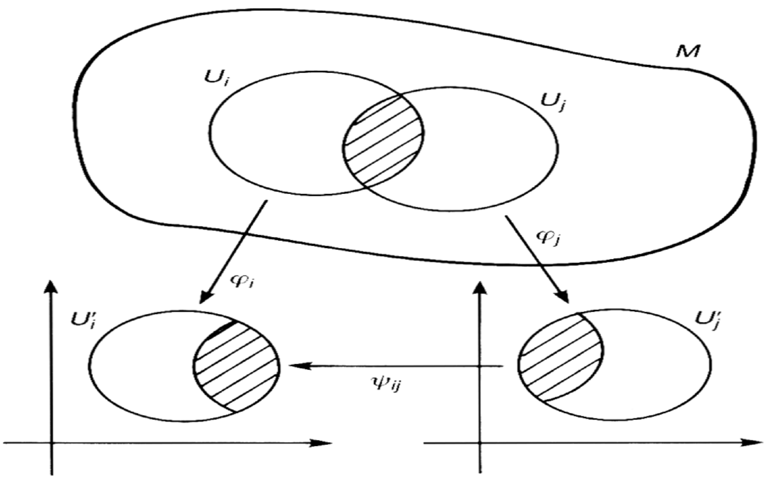

Fig 1.5: A homeomorphism $\emptyset_i$ maps U_i onto an open subset $U_i' \in R^m$,

The pair *(U_i,$\emptyset_i$)* is called a chart while the whole family {*(U_i,$\emptyset_i$)*} is Called an atlas. The subset U_i is called the coordinate, neighborhood while $\emptyset_i$ is the coordinate function or, simply, the coordinate. The homeomorphism $\emptyset_i$ is represented by m functions $\{(X^1(p), X^2\ (p),\ldots,x^m(p))\}$.

From (ii) and (iii), *M* is locally Euclidean. In each coordinate neighborhood,U_i *M* looks like an open subset of R^mwhose element is $\{(X^1, X^2\ ,\ldots, x^m\}$. We do not require that *M* be R^mglobally.

Example 4:

We are living on the earth whose surface is S^2, which does not look $(X^1, X^2\ ,\ldots\ldots, x^m)$like R^2 Globally. However, it looks like an open subset of R^2 locally.

1.2.5 Smooth Manifold

A topological manifold *M*, together with an equivalent class of c^k atlases is called a c^kStructure on *M* and *M* is called $c^k -$ manifold. If *k* = ∞ then M is said to be Smooth manifold.

Example 5:

Let X^n be an n-dimensional smooth manifold and Y^p be an p-dimensional smooth manifold, then *f*: $X^n \rightarrow Y^p$ be a map. The map f is called smooth at a point $x \in X^n$ if $\Psi \circ f \circ \emptyset^{-1}$ is smooth at $\emptyset(x) \in \mathbb{R}^n$ where *(U , ∅)* be the coordinate chart at $x \in U$ and *(V, ∅)* be the coordinate chart at $y \in V$.

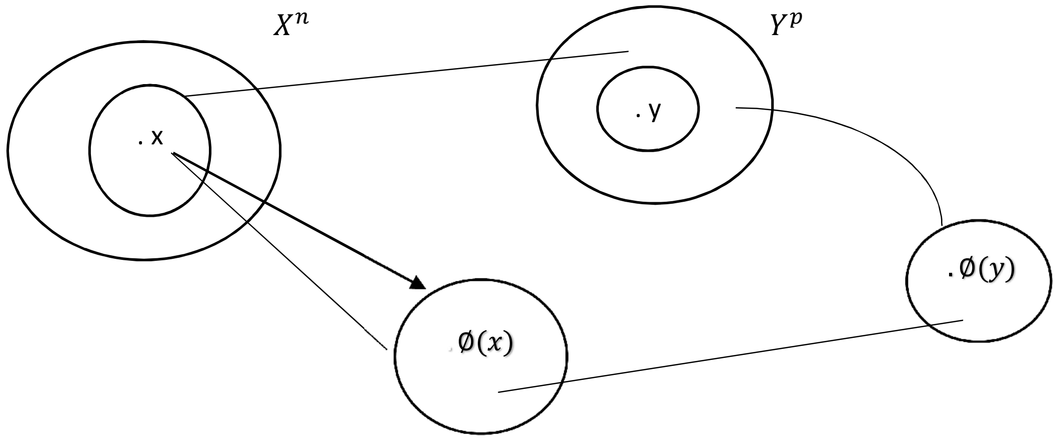

Fig 1.6**:** Smooth mapping on coordinate chart

Such that

$\emptyset : U \to \mathbb{R}^n$ and

$\emptyset^{-1}: \mathbb{R}^n \to U \subset X^n \xrightarrow{f} Y^p \xrightarrow{\Psi} \mathbb{R}^P$

The mapping is $\Psi \circ f \circ \emptyset^{-1}: \mathbb{R}^n \to \mathbb{R}^P$

1.3 Submanifold

Let *M* be a subset of a *k*-dimensional smooth manifold *N*. We say that *M* is a smooth embedded submanifold of *N* of dimension *n* if, given any point m of *M*, there exists a smooth coordinate system $(U^1, U^2, \ldots, u^k)$ defined over Some open set *U in N,* where *m* ∈ U,
with the property that,

$$M \cap U = \{p \in U: U^i(p) = 0 \text{ for } i = n+1, \ldots, k\}$$

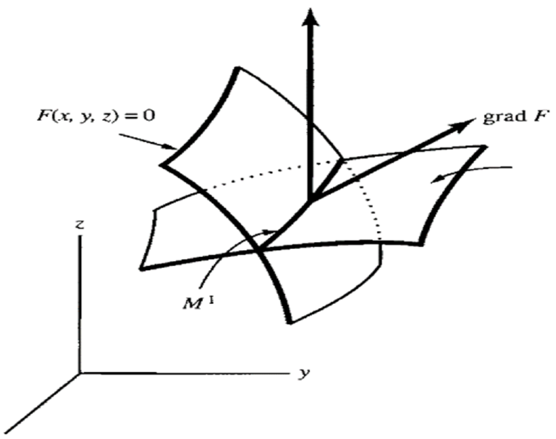

Fig 1.7: The locus of (n-1) dimensional sub manifold of $\mathbb{R}^{\boldsymbol{n}}$.

From Figure 1.7, here two surfaces *F*= 0 and *G*= 0 in $\mathbb{R}^3$ intersect to yield a curve M. The simplest case is one function *F* of *N* variables $(X^1, X^2, \ldots, X^n)$, If at each point of the locus *F* = *C* there is always at least one partial derivative that does not vanish, then the Jacobian (row) matrix $[\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}, \ldots\ldots\ldots, \frac{\partial F}{\partial x^n}]$ has rank 1.and we may conclude that this locus is indeed an (N - 1)-dimensional submanifold of $\mathbb{R}^{\boldsymbol{n}}$.

Given such a coordinate system $(U^1, U^2, \ldots, u^k)$, the restrictions of the coordinate functions $(U^1, U^2, \ldots\ldots, u^n)$, to $U \cap M$ provide a coordinate system on M around the Point m. The collection of all such coordinate systems constitutes smooth atlas on M. Suppose that we have r functions of n + r variables' F^{α} $(X^1, \dots \dots . X^{n+r})$, then $F^{\alpha}(X)$ C^{α}, at each point X_0 of the locus the jacobian Matrix is

$$\left(\frac{\partial F^{\alpha}}{\partial X^{i}}\right)\alpha = 1, \dots, r \text{ and } i = 1, \dots\dots\dots, n+r$$

has rank r. Then the equation $F^{\alpha} = C^{\alpha}$ define an n- dimensional submanifold of R^{n+r}.

1.4 Immersion and embedding

Let *f : M*→ *N* be smooth map and let dim *M* ≤ dim *N*.

(a) The map *f* is called an immersion of *M* into *N* if *f* *: $T_pM \to T_{f(p)}N$ is an injection (one to one), that is rank *f* * = dim *M*.

(b) The map *f* is called an embedding if *f* is an injection and an immersion. The image *f (M)* is called a **sub manifold** of *N*. [In practice, *f (M)* thus defined is diffeomorphic to *M*.

If *f* is an immersion, *f* *maps T_pM is isomorphic ally to an *m*-dimensional vector subspace of $T_{f(p)}N$since rank *f** = dim *M*, we also find Ker *f** = {0}.

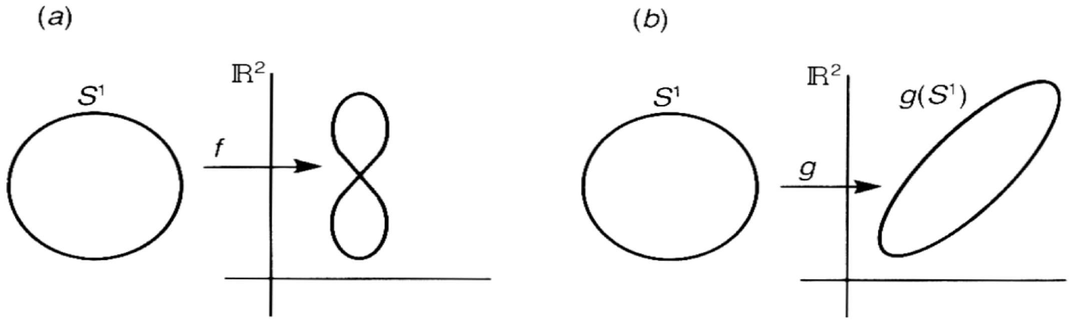

Fig 1.8 (a) An immersion *f* which is not embedding. (b) An embedding and the submanifold $g(S^1)$.

If *f* is an embedding, *M* is diffeomorphic to *f (M)*. Consider a map *f:* $S^1 \to R^2$.It is an immersion since a one-dimensional tangent space of S^1 is mapped by *f** to a subspace of $T_{f(p)}R^2$. The image *f* (S^1) is not a sub manifold of R^2 since *f* is not an injection. Clearly, an embedding is an immersion although the converse is not necessarily true. In the previous section, we occasionally mentioned the embedding of S^ninto R^{n+1} . Now this meaning is clear; if S^n is embedded by *f*: S^n→ R^{n+1} then is S^n diffeomorphic to *f* (S^n).

1.5 Topological Space:

If a topological space *M* is covered by a family of open sets $\{U_i\}$ each of which is **isomorphic** to an open set of H_m≡ { ($(X^1, X^2$,……, x^m)}∈ R^m/x^m|≥ 0}, *M* is Said to be a **manifold with a boundary .**The set of points which are mapped to Points with x^m= 0 is called the **boundary** of *M*, denoted by ∂*M*. The coordinates of ∂*M* may be given by $m-1$ numbers,{(X^1, X^2,…, x^m,0)}. Now we have to be careful when we define the smoothness. The map φ_i: $\emptyset_j(U_i \cap U_j)$ →$\varphi_i(U_i \cap U_j)$is defined on an open set of $H_m$in general,and ψ *j* is said to be smooth if it is C^∞in an open set of R^mwhich contains $\varphi_{ij}(U_i \cap U_j)$.Readers are encouraged to use their imagination since our definition is in Harmony with our intuitive notions about boundaries.

1.6 Connection

We would like to differentiate vector fields but as they take values in different vector spaces at different points, it is not so clear how to make difference quotients and so derivatives. A connection which should be thought of as a directional derivative for vector fields. The apparatus of vector bundles, principal bundles and connections on them plays an extraordinary important role in the modern differential geometry. Loosely speaking, this structure by itself is different only for developing analysis on the manifold, while doing geometry requires in addition some way to relate the tangent spaces at different points, i.e. a notion of parallel transport. An important example is provided by affine connections. For a surface in R^3 tangent planes at different points can be identified the flat nature of the ambient Euclidean space. In Riemannian geometry, the Levi-Civita connection serves a similar purpose**.**

Definition

A connection on *TM* is a bilinear map$(TM) \times \Gamma(TM) \rightarrow (TM)$

$$(\xi, X) \rightarrow \nabla_\xi X$$

Such *that* $\xi \in M_m$, X, $Y \in \Gamma(TM)$ *and* $f \in c^\infty(M)$.

Hence the three conditions are as follows:

1. $\nabla_\xi X \in M_m$,
2. $\nabla_\xi(fX) = (\xi f) X_m + f(m)\nabla_\xi X$.
3. $\nabla_X Y = \Gamma(TM)$.

Which also defined by c^∞*(M)-linear* in X and R-linear in Y and satisfies the product rule

$$\nabla_x(f Y) = (Xf)\, Y + f\nabla_x Y \quad \text{for all } f \in c^\infty(M).$$

Much of the power of Riemannian geometry comes from the fact that there is a canonical choice of Connection. Consider the following two desirable properties for a connection ∇ on *(M, g):*

1. ∇ is metric: $X_g(Y, Z) = g(\nabla_x Y, Z) + g(Y, \nabla_x Z)$.
2. ∇ is torsion free $\quad \nabla_x Y - \nabla_y X = [X, Y]$.

1.7 Connection Co-efficient

Take a chart (U, ϕ) with the coordinate x = ϕ(p) on *M*, and The Functions $\Gamma^{\mu}_{\nu\lambda}$ is called the connection **coefficients** by,

$$\nabla_V e_\mu \equiv \nabla_{eV} e_\mu = e_\lambda \Gamma^{\mu}_{\nu\lambda}$$

Where $\{e_\mu\} = \{\{\partial / \partial x^\mu\}$ is the coordinate basis in T_pM. The connection coefficients specify how the basis vectors change from point to point. Once the action of ∇on the basis vectors is defined, we can calculate the action of ∇ on any vectors.

Let $V = V^{\mu} e_{\mu}$ and $W = W^{v} e_{v}$ be elements of T_p *(M)*. Then

$$\nabla_v W = V^{\mu} \nabla_{e\mu}(W^{v} e_v) = V^{\mu}(e_{\mu}[W^{\mu}] e_v + W^{v} \nabla_{e\mu} e_v)$$

$$= V^{\mu} \left(\frac{\partial W^{\lambda}}{\partial x^{\mu}} + W^{v} \Gamma^{\lambda}_{\mu v}\right) e_{\lambda}$$

By definition, ∇ maps two Vectors V and W to a new vector, whose λ -th component is $V^{\mu} \nabla_{\mu} W^{\lambda}$.
where

$$\nabla_v W^{\lambda} \equiv \frac{\partial W^{\lambda}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\mu v} W^{v}$$

$\nabla_v W^{\lambda}$ is the λ-th component of a vector $\nabla_v W = \nabla_v W^{\lambda} e_{\lambda}$ and should not be confused with the covariant derivative of a component W^{λ}. $\nabla_v W$ is independent of the derivative of V. In this sense, the covariant derivative is a proper generalization of the directional derivative of functions to tensors.

1.8 Affine connection

Let *M* be a smooth manifold of dimension n,O_M be the set of smooth function and*Γ(TM)* be the vector space of vector field. An affine Connection on M is a map, denoted by $\nabla(nabla)$.an affine connection ∇is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM),$$

$$\text{or, } (X, Y) \rightarrow \nabla_X Y$$

Such that,

1. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
2. $\nabla_{(X_1+X_2)} Z = \nabla_{X_1} Y + \nabla_{X_2} Y.$
3. $\nabla_{(fX)} Y = f \nabla_X Y.$
4. $\nabla_X(f Y) = X[f] Y + f \nabla_X Y$

where $f \in \Gamma(M)$ *and* $X, Y, \in \Gamma(TM)$

Theorem 2:

Let (M, g) be a Riemannian manifold. Then there exists a unique torsion-free affine connection ∇ *on M* compatible with the Riemannian Metric g. This connection is characterized by the identity,

$$2g(\nabla_x Y,Z) = X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X, Y) + g([X, Y],Z) - g([X, Z], Y) - g([Y,Z],X)$$

for all smooth vector fields X, Y and Z on M.

Proof

Given smooth vector fields *X, Y* and *Z* on M, let *A (X, Y, Z)* be the smooth function on *M* defined by

$$A(X, Y, Z) = \frac{1}{2}(X[g(Y,Z)] + Y[g(X, Z)] - Z[g(X, Y)] + g([X, Y],Z) - g([X, Z], Y) - g([Y, Z],X)).$$

Then $A(X, Y, Z_1 + Z_2) = A(X, Y, Z_1) + A(X, Y, Z_2)$ for all smooth vector fields X, Y, Z_1 and Z_2 on *M*. Using the identities

$$[X, fZ] = f[X, Z] + X[f]Z \quad and \quad [Y, fZ] = f[Y, Z] + Y[f]Z$$

Here $A(X, Y, fZ) = fA(X, Y, Z)$ for all smooth real valued functions *f* and vector fields *X*, *Y and Z* on *M*. On applying Lemma to the transformation $Z \leftrightarrow A(X, Y, Z)$, we see that there is a unique vector field $\nabla_x Y$ on *M* with the property that $A(X, Y, Z) = g(\nabla_x Y, Z)$ for all smooth vector fields *X, Y* and *Z* on M.

Moreover

$$\nabla_{x_1+x_2} Y = \nabla_{x_1} Y + \nabla_{x_2} Y,\ \nabla_x(Y_1 + Y_2) = \nabla_x Y_1 + \nabla_x Y_2.$$

After calculations, we show that

$$g(\nabla_{fx} Y, Z) = A(fX, Y, Z) = f A(X, Y, Z) = g(f\nabla_x Y, Z),$$

$$g(\nabla_x (fY), Z) = A(X, fY, Z) = f A(X, Y, Z) + X[f]\, g(Y, Z)$$

$$= g(f\nabla_x Y + X[f]\, Y, Z).$$

for all smooth real-valued *functions for M, so that*

$$\nabla_{fx} Y = f\nabla_x Y, \text{ and } \nabla_x(fY) = f\nabla_x Y + X[f]\, Y.$$

These properties show that ∇ is indeed an affine connection on M. Moreover

$$A(X, Y, Z) - A(Y, X, Z) = g([X, Y], Z),$$

so that $\nabla_x Y - \nabla_y X = [X, Y]$. Thus, the affine connection ∇ is torsion-free. Also

$$g(\nabla_x y, Z) + g(Y, \nabla_x Z) = A(X, Y, Z) + A(X, Z, Y) = X[g(Y, Z)]$$

Showing that, the affine connection ∇ preserves the Riemannian metric. Finally suppose that ∇' is any torsion-free affine connection on M which preserves the Riemannian metric. Then

$$X[g(Y, Z)] = g(\nabla'_x Y, Z) + g(Y, \nabla'_x Z),$$

$$Y[g(X, Z)] = g(\nabla'_y X, Z) + g(X, \nabla'_y Z),$$

$$Z[g(X, Y)] = g(\nabla'_Z X, Y) + g(X, \nabla'_Z Y).$$

After calculation (using the fact that ∇' is torsion-free) shows that $A(X, Y, Z) = g(\nabla'_x Y, Z)$. Therefore $\nabla'_x Y = \nabla_x Y$ for all smooth vector fields X and Y on M.

This completes the proof of the theorem. □

1.9 Torsion Free Connection

Let ∇ be an affine connection on manifold *M*.

We have $\nabla(X, Y) = \nabla_X Y - \nabla_y X - [X, Y]$ For all $X, Y \in \Gamma(TM)$.

Is a tensor of rank *(1, 2)*.

That is, $T(fX, Y) = T(X, fY) = fT(X, Y)$.

Then we call T be the torsion tensor of ∇.

If $T=0$, we call ∇ Torsion free connection or a symmetric connection.

Example 6:

If r is a torsion-free affine connection on M. Now the restriction to the tangent spaces of M of the standard inner product (,) on R^k gives a Riemannian metric g on M. Moreover

$$g(\nabla_x Y, Z) + g(Y, \nabla_x Z) = (\partial_x Y, Z) + (Y, \partial_x Z) = X[(Y,Z)] = X[g(Y,Z)]$$

For all vector fields *X, Y* and *Z* on *M* that are everywhere tangential to M. We conclude that the affine connection ∇on M coincides with the Levi-Civita connection of the Riemannian manifold (*M, g*).

Lemma 1: ∇ is a torsion free connection if and only if $\Gamma^{\mu}_{v\lambda} = \Gamma^{\mu}_{v\lambda} - \Gamma^{\mu}_{\lambda v}$ for all v, λ.

Proof:

If ∇ is a torsion free connection then for any v, λ we get

$$T(e_v, e_\lambda) = \nabla_{e_v} e_\lambda - \nabla_{e_\lambda} e_v - [e_v, e_\lambda]$$

$$= \nabla_{e_v}^{e_\lambda} - \nabla_{e_\lambda}^{e_v} - [e_v, e_\lambda]$$

$$= \Gamma^{\mu}_{v\lambda} e_\mu - \Gamma^{\mu}_{\lambda v} e_\mu - 0$$

i.e $\Gamma^{\mu}_{v\lambda} = (\Gamma^{\mu}_{v\lambda} - \Gamma^{\mu}_{\lambda v}) - 0$

Since, $e_\mu \neq 0$ so that $\Gamma^{\mu}_{v\lambda} - \Gamma^{\mu}_{\lambda v} = 0$ or $\Gamma^{\mu}_{v\lambda} = \Gamma^{\mu}_{\lambda v}$.

Since *T* is a torsion tensor and $T(e_v, e_\lambda) = 0$ for all e_v, e_λ.

So, ∇ is a torsion free connection. □

Lemma 2: For all ∇ its torsion tensor T^{∇} is a tensor of rank (1,2).

Proof:

We need to show that $T^{\nabla}(fX,Y) = T^{\nabla}(X,fY) = f\,T^{\nabla}(X,Y)$ for all X ,Y $\in \Gamma(TM)$.

We know $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_y X - [X, Y]$

$$T^{\nabla}(fX, Y) = \nabla_{fx} Y - \nabla_y fX - [fX, Y]$$

$$= f\nabla_X Y - (Y(f)X + f\nabla_y X) - [fX, Y]$$

$$= f\nabla_X Y - Y(f)X - f\nabla_y X - [fX, Y]$$

Take any $g \in 0_M$, then

$[fX, Y]\,g = fX\,Y(g) - Y(fX(g))$

$= fX\,Y(g) - Y(f).X(g) - fY\,X(g)$

$= f(XY(g) - YX(g)) - Y(f).X(g)$

$= f[X, Y](g) - Y(f).X(g)$

$[fX, Y] = f[X, Y] - Y(f).X$

$T^{\nabla}(fX,Y) = f\nabla_X Y - Y(f)X - f\nabla_y X - [fX, Y] + y(f)X$

$= f(\nabla_X Y - \nabla_y X - [X, Y])$

$= fT^{\nabla}(X,Y)$

Again, we know

$$T^{\nabla}(X, Y) = -T^{\nabla}(Y, X)$$

$$T^{\nabla}(X, fY) = -T^{\nabla}(fY, X)$$

$$= -fT^{\nabla}(Y, X)$$

$$= -(-)fT^{\nabla}(Y, X) = fT^{\nabla}(Y, X)$$

So, T^{∇} is a tensor of rank (1, 2). □

1.10 The Levi – Civita Connection

We introduce the Levi-Civita connection ∇ of a Riemannian manifold (*M, g*). This is the most important fact of the general notion of a connection in a smooth vector bundle.

It is very important to note that the Levi-Civita connection is an intrinsic object on (*M, g*) i.e. only depending on the differentiable structure of the manifold and its Riemannian metric.

Definition 1:

Let (*M, g)* be a Riemannian manifold and let ∇ be an affine connection on*M*. The covariant derivative of *g* with respect to ∇ is a multi linear map. We say that $\nabla$is compatible with the Riemannian metric g.

$$\text{if } \nabla_g : \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \rightarrow O_M$$

$$(Z, X, Y) \rightarrow \nabla_Z g\ (X, Y) \text{ for all } X, Y, Z \in \Gamma(TM)$$

and

$$\nabla_Z\, g\ (X, Y) = Z\, g\ (X, Y) - g(\nabla_z X, Y) - g(X, \nabla_z Y)\ .$$

For all smooth vector fields *X, Y and Z on M*. The unique torsion-free affine Connection on M which preserves the Riemannian metric is known as the **Levi - Civita connection** on M.

Definition 2:

Let (*M, g*) be a Riemannian manifold then the map $\nabla: C^\infty(TM) \times C^\infty((TM) \rightarrow C^\infty((TM)$ given by

$$2\, g(\nabla_x Y, Z) = \{X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([Z, X], Y) + g([Z, Y], X) + g(Z, [X, Y])\}.$$

is called the **Levi - Civita connection** on *M*

Lemma 3:

Let ∇ be an affine connection on a Riemannian manifold *M* that is compatible with the Riemannian metric g. Let *V and W* be smooth vector fields along some smooth curve $\gamma: I \to M$ in M (where I denote some open interval in R). Then

$$\frac{d}{dt} g(V(t), W(t)) = g\left(\frac{DV(t)}{dt}, W(t)\right) + g\left(V(t), \frac{DW(t)}{dt}\right)$$

(Where *DV/dt* and *DW/dt* are the covariant derivatives of the vector fields V and W along the curve)

For all vector fields *X, Y* and *Z* on *M* that are everywhere tangential to M. We conclude that the affine connection ∇on M coincides with the Levi-Civita connection of the Riemannian manifold (M, g).

Theorem 4: *Let* (M, g)be a Riemannian manifold. Then the Levi-Civita connection ∇is a connection on the tangent bundle *TM* of *M*.

Proof: It follows from the fact that $g$is a tensor field that

$$g(\nabla_X(\lambda \cdot Y_1 + \mu \cdot Y_2), Z) = \lambda \cdot g(\nabla_X Y_1, Z) + \mu \cdot g(\nabla_x Y_2, Z)$$

and

$$g(\nabla_{Y_1 + Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z)$$

for all $\lambda, \mu \in$R and $X, Y_1, Y_2, Z \in C^\infty(TM)$.

Furthermore, we have for all $f \in C^\infty(M)$

$$2\, g(\nabla_X f Y, Z) = \{X(f \cdot g(Y, Z)) + f \cdot Y(g(X,Z)) - Z(f \cdot g(X, Y)) + f \cdot g([Z, X], Y) + g([Z, f \cdot Y], X) + g(Z, [X, f \cdot Y])$$

$$= \{X(f) \cdot g(Y,Z) + f \cdot X(g(Y,Z)) + f \cdot Y(g(X,Z)) - Z(f) \cdot g(X,Y) - f \cdot Z(g(X,Y)) + f \cdot g([Z,X],Y)$$

$$+g(Z(f)\cdot Y+f\cdot[Z,Y],X)+g(Z,X(f)\cdot Y+f\cdot[X,Y])\}$$

$$=2\cdot\{X(f)\cdot g(Y,Z)+f\cdot g(\nabla_X Y,Z)\}$$
$$=2\cdot g(X(f)\cdot Y+f\cdot\nabla_X Y,Z)$$

And

$$2\cdot g(\nabla_{f}.X^{Y},Z)=\{f\cdot X(g(Y,Z))+Y(f\cdot g(X,Z))-Z(f\cdot g(X,Y))$$
$$+g([Z,f\cdot X],Y)+f\cdot g([Z,Y],X)+g(Z,[f\cdot X,Y])\}$$

$$=\{f\cdot X(g(Y,Z))+Y(f)\cdot g(X,Z)+f\cdot Y(g(X,Z)$$
$$-Z(f)\cdot g(X,Y)-f\cdot Z\big(g(X,Y)\big)+g(Z(f)\cdot X,Y)+f.g([Z,X],Y)+f\cdot$$
$$g([Z,Y],X)+f\cdot g(Z,[X,Y])-g(Z,Y(f)\cdot X$$
$$=2\cdot f\cdot g(\nabla_X Y,Z)$$

This proves that ∇ is a connection on the tangent bundle on *TM* of *M* . □

Theorem 5: Let (*M, g)* be a Riemannian manifold. Then the Levi- Civita connection is the unique metric and torsion-free connection on the tangent bundle (TM, M, π).

Proof: The difference $g(\nabla_X Y,Z)-g(\nabla_Y X,Z)$ equals twice the skew-symmetric part (with respect to the pair (*X, Y*) of the right hand side of the equation in Definition 2. This is the same as

$$\frac{1}{2}\{g(Z,[X,Y])-g(Z,[Y,X])\}=g(Z,[X,Y])$$

This proves that the Levi-Civita connection is torsion-free.

The sum $g(\nabla_x Y, Z)+g(\nabla_X Z, Y)$ equals twice the symmetric part(with respect to the pair (*Y,Z*)) on the right hand side of Definition 2. This is exactly

$$=\frac{1}{2}\{X(g(Y,Z))+X(g(Z,Y))\}=X(g(Y,Z)).$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric *g* on *M*. □

Theorem 6: For all affine connection ∇ and metric g, then ∇_g is a tensor.

Proof: We need to show that ∇_g is a tensor. So, that

$$\nabla_{fZ}\, g(X, Y) = \nabla_Z\, g(fX, Y)$$
$$= \nabla_Z\, g(X, fY)$$
$$= f\,\nabla_Z\, g(X, Y) \quad \text{for all } X, Y, Z \in \Gamma(TM).$$

(a) $\nabla_{fZ}\, g(X, Y) = fZg(X, Y) - g(\nabla_{fZ} X, Y) - g(X, \nabla_{fZ} Y)$

$$= fZg(X, Y) - g(f\nabla_Z X, Y) - g(X, f\,\nabla_Z Y)$$

$$= fZg(X, Y) - fg(\nabla_Z X, Y) - fg(X, \nabla_Z Y)$$

$$= f\,[Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)]$$

$$= f\,\nabla_Z\, g(X, Y)$$

(b) $\nabla_Z\, g(fX, Y) = Zg(fX, Y) - g(\nabla_Z (fX), Y) - g(fX, \nabla_Z Y)$

$$= Zg(fX, Y) - g(f\nabla_Z X, Y) - g(X, f\,\nabla_Z Y)$$

$$= Zfg(X, Y) - g(Z(f), X + f\nabla_Z X, Y) - fg(X, \nabla_Z Y)$$

$$= Z(f).g(X, Y) + fZ.\, g(X, Y) - Z(f)\, g(X, Y) -$$

$$fg(\nabla_Z X, Y) - fg(X, \nabla_Z Y)$$

$$= f\,[Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)]$$

$$= f\,\nabla_Z\, g(X, Y)$$

(c) $\nabla_Z\, g(X, fY) = \nabla_Z\, g(fY, X)$ [Due to symmetry of g]

$$= f\nabla_Z\, g(Y, X) \quad \text{[by (b)]}$$

$$= f\nabla_Z\, g(X, Y)$$

Hence, ∇_g is a tensor. This completes the proof. □

Chapter Two

Group Theory with Lie Algebra

2.1 Introduction

The idea of groups is one that has evolved from some very intuitive concepts. We have acquired in our attempts of understanding Nature. One of these is the concept of mathematical structure. A set of elements can have a variety of degrees of structure. The set of natural numbers possesses a higher mathematical structure. In addition, of being naturally" ordered we can perform operations on it. We can do binary operations like adding or multiplying two elements and also unary operations like taking the square root of an element (in this case the result is not always in the set). The existence of an operation endows the set with a mathematical structure. In the case when this operation closes within the set, i.e. the composition of two elements is again an element of the set, the endowed structure has very nice properties. Differential form has applications to the field of group theory. Differential form and exterior algebra can be applied to Lie group, matrix group, Bi-invariant forms.

2.1.1 Group

We say two elements, g_1 and g_2 of a group commute with each other if their product is independent of the order, i.e., if $g_1g_2 = g_2g_1$. If all elements of a given group commute with one another then we say that this group is **Abelian**. The real numbers under addition or multiplication (without zero) form an abelian group. The cyclic groups Zn are abelian for any n. The symmetric group Sn is not abelian for n >2, but it is abelian for n = 2.

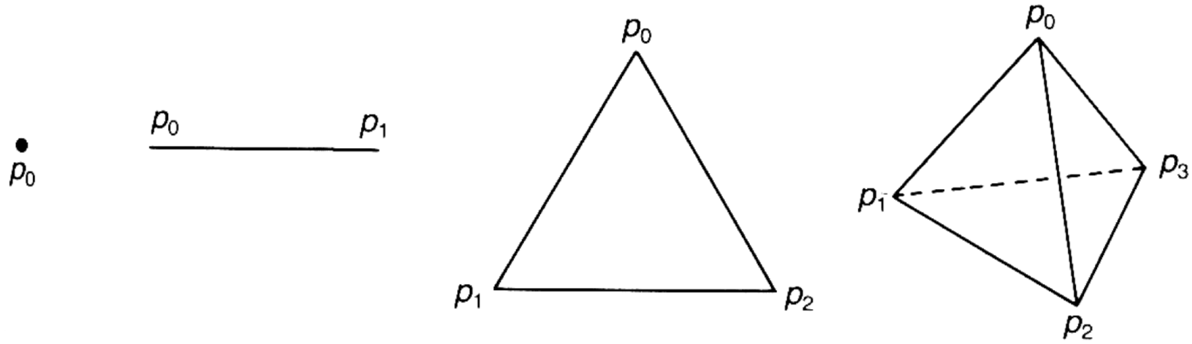

Figure 2.1: 0-, 1-, 2- and 3-simplexes

Simplexes are building blocks of a polyhedron. A 0-simplex $\langle P_0 \rangle$ is a point, or a vertex, and a 1-simplex $\langle P_0P_1 \rangle$ is a line, or an edge. A 2-simplex $\langle P_0P_1P_2 \rangle$ is defined to be a triangle with its interior included and a 3-simplex $\langle P_0P_1P_2P_3 \rangle$ is a solid tetrahedron.

Let us consider some groups of order two, i.e., with two elements. The elements 0 and 1 form a group under addition modulo 2. We have

$$0 + 0 = 0;\ 0 + 1 = 1;\ 1 + 0 = 1;\ 1 + 1 = 0 \qquad (1)$$

The elements 1 and -1 also form a group, but under multiplication. We have

$$1.1 = -1.(-1) = 1; 1.(-1) = (-1).1 = -1 \qquad (2)$$

The symmetric group of degree 2, S_2 has two elements as shown in figure below,

Figure 2.2: The elements of S_2

That satisfies

$$e.e = e,\ e.a = a.e = a, \quad a.a = e \qquad (3)$$

These three examples of groups are in fact different realizations of the same abstract group. If we make the identifications as shown in above. We see that the structure of these groups are the same. We say that these groups are isomorphic.

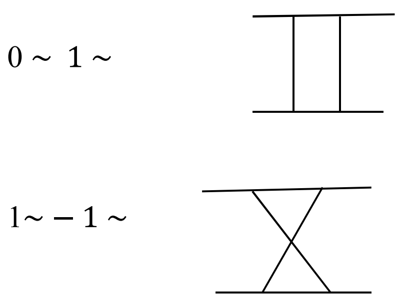

Figure 2.3: Isomorphism

Definition 1: Two groups G and G' are isomorphic if their elements canbe put into one-to-one correspondence which is preserved under the composition laws of the groups. The mapping between these two groups is called an **isomorphism**.

Definition 2: a group G being mapped into another group G' but not in a one-to-one manner, i.e. two or more elements of G are mapped into just one element of G'. If such mapping respects the product law of the groups we say they are homomorphic. The mapping is then called a **homomorphism** between G and G'.

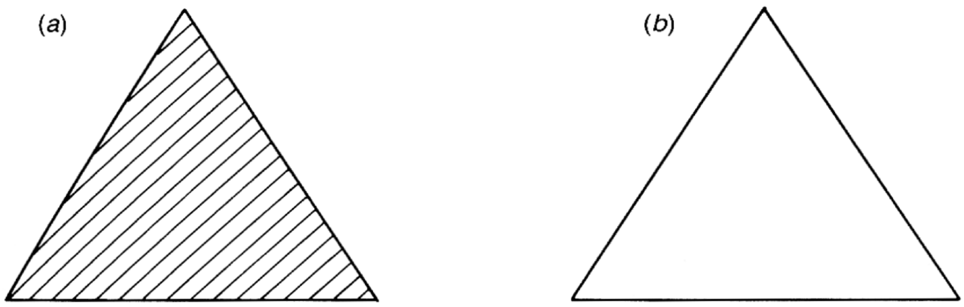

Figure 2.4**.** (*a*) is a solid triangle while (*b*) is the edges of a triangle without an interior.

From figure, Let *G* and G' be Abelian groups. A map *f: G*→ G' is said to be a homomorphism if

$$f(x+y)=f(x)+f(y)$$

Example 1: Consider the cyclic groups Z_6 with elements e, a,a^2,.......,a^5anda^6 = e, and $Z_6$with elements e' and b ($b^2 = e'$). The mapping σ: $Z_6 \rightarrow Z_2$ given by

$$\sigma(e) = \sigma(a^2) = \sigma(a^4) = e'$$

$$\sigma(e) = \sigma(a^2) = \sigma(a^4) = b$$

is a homomorphism between Z_6 and Z_2.

Analogously one can define mappings of a given group G into itself, i.e.,for each element g∈ G one associates another element g'. The one-to-onemappings which respect the product law of G are called **automorphisms** of G. Otherwise, an automorphism of G is an isomorphism of G onto itself.

In fact the above example is just a particular case of the automorphism of any abelian group where a given element is mapped into its inverse. If σ and σ' are two automorphisms of a group G, then the composition of both $\sigma\sigma'$ is also an automorphism of G. such composition is an associative operation. In addition, since automorphisms are one-to-one mappings, they are invertible. Therefore, if one considers the set of all automorphisms of a group G together with the identity mapping of G into G, one gets a group which is called the automorphism group of G.

2.1.2 Subgroup

A subset *H* of a group *G* which satisfies the group postulates under the same composition law used for *G*, is said to be a subgroup of *G*. The identity element and the whole group *G* itself are subgroups of *G*. They are called improper subgroups. All other subgroups of a group *G* are called proper subgroups. If *H* is a subgroup of *G*, and K a subgroup of *H* then K is a subgroup of *G*.

In order to find if a subset *H* of a group *G* is a subgroup we have to check only two of the four group postulates. We have to check if the product of any two elements of *H* is in *H* (closure) and if the inverse of each element of *H* is in *H*. The associatively property is guaranteed since the composition law is the same as the one used for *G*. As *G* has an identity element it follows from the closure and inverse element properties of *H* that this identity element is also in *H*. Although all elements of the centralizer commute with a given element g they do not have to commute among themselves and therefore it is not necessarily an Abelian subgroup of G.

Definition 3: The center of a group G is the set of all elements of G which commute with all elements of G. We could say that the center of G is the intersection of the centralizers of all elements of G. The center of a group G is a subgroup of G and it is Abelian, since by definition its elements have to commute with one another. In addition, it is an (Abelian) invariant subgroup.

2.1.3 Cosets

Given a group G and a subgroup H of G we can divide the group G into disjoint sets such that any two elements of a given set differ by an element of H multiplied from the right. That is, we construct the sets

$gH \equiv$ {All elements gh of G such that h is any element of H and g is a fixed element of G}

If $g = e$ the set eH is the subgroup H itself. All elements in a set gH are different, because if $gh_1 = gh_2$ then $h_1 = h_2$. Therefore, the numbers of elements of a given set gH is the same as the number of elements of the subgroup H. Also, an element of a set gH is not contained by any other set $g'H$ with $g' \neq g$. Because if $gh_1 = g'h_2$ then $g = g'h_2h_1'$ and therefore g would be contained in $g'H$ and consequently $gH \equiv g'H^1$. Thus, we have split the group G into disjoint sets, each with the same number of elements, and a given element $g \epsilon$ G belongs to one and only one of these sets.

2.1.4 Theorem

The order of a subgroup of a finite group is a divisor of the order of the group.

Proof

For a finite group, G of order m with a proper subgroup H of order n,
we can write,

$$m = kn$$

where k is the number of disjoint sets g*H*.

The set of elements g*H* are called left cosets of *H* in *G*. They are certainly not subgroups of *G* since they do not contain the identity element, except for the set $eH = H$. Analogously we could have split G into sets *H*g which are formed by elements of *G* which differ by an element of *H* multiplied from the left. The same results would be true for these sets. They are called right cosets of *H* in *G*. The set of left cosets of *H* in *G* is denoted *by G=H* and is called the left cosetspace. An element of G=H is a set of elements of G, namely *gH*.

Analogously the set of right cosets of H in G is denoted by H n G and it is called the right coset space.If the subgroup H of G is an invariant subgroup, then the left and right cosets are the same since g^{-1}Hg = H implies $gH = Hg$. In addition, the coset space G=H, for the case in which H is invariant, has the structure of a group and it is called the factor group or the quotient group. In order to show this, we consider the product of two elements of two different cosets. We get

$$gh_1g'h_2 = gg'g'^{-1}h_1g'h_2 = g\, g^{-1}h_3h_2$$

Where we have used the fact that H is invariant, and therefore there exist$h_3 \in$ H such that $g'^{-1}h_1g' = h_3$. Thus, we have obtained an element of a third coset, namely$gg'H$. If we had taken any other elements of the cosets *gH* and $g'H$, their product would produce an element of the same coset$gg'H$.Consequently, we can introduce, in a well-defined way, the product of elements of the coset space *G/H*, namely

$$gH\, g'H \equiv gg'H \tag{4}$$

The invariant subgroup H plays the role of the identity element since

$$(gH)\, H = H(gH) = gH \tag{5}$$

The inverse element is $g^{-1}H$ since

$$g'\mathrm{H}gH = g^{-1}gH = H = gH\, g^{-1}H$$

The associatively is guaranteed by the associatively of the composition law of the group *G*. Therefore, the coset space *G/H* and *H\G* is a group in the case, where H is an invariant subgroup. Such group is not necessarily asubgroup of *G* or *H*.

This completes the proof. □

2.1.5 Direct Products

We say a group G is the direct product of its subgroups H_1, H_2,……., H_n , denoted by

$$G = H_1 \otimes H_2 \otimes H_3 \otimes \ldots\ldots \otimes H_n$$

1. The elements of different subgroups commute.
2. Every element $g \in$ G can be expressed in one and only one way as

g = $h_1 h_2 \ldots \ldots . h_n$

Where h_i is an element of the subgroup H_i ,i = 1,2, ….,n

From these requirements, it follows that the subgroups H_i have only the identity *e* in common. Because if $f \neq e$ is a common element to $H_2$and H_5 say, then the element

$g = h_1 f h_3 h_4 f^{-1} h_6 \ldots \ldots h_n$ could be also written as $g = h_1 f^{-1} h_3 h_4 f h_6 \ldots \ldots h_n$

Where every subgroup H_i is an invariant subgroup of G, because if $h_i' \in H_i$, then

$$g^{-1} h_i' g = (h_1 h_2 \ldots \ldots . h_n)^{-1} h_i' (h_1 h_2 \ldots \ldots . h_n) = h_i^{-1} h_i' h_i \in H_i$$

Example 2: Consider the cyclic groups Z_6 with elements e, a,a^2,…....,a^5anda^6 = e. It can be written as the direct product of its subgroups

$H_1 = \{ e, a^2, a^4 \}$ and $H_2 = \{ e, a^3 \}$ since

$$e = e.e;\ a = a^4 a^3; a^2 = a^2 e;\ a^3 = e a^3;\ a^4 = a^4 e;\ a^5 = a^2 a^3 \qquad (6)$$

Therefore, we write $Z_6 = H_1 \otimes H_2$ (or $Z_6 = Z_3 \otimes Z_2$).

Given two groups G and G' we can construct another group by taking thedirect product of G G'as follows: the elements of $G'' = G \otimes G'$are formed by the pairs (g, g') where $g \in G$ and $g' \in G'$. The composition law for$G''$is defined by,

$$(g_1, g_1') \, (g_2, g_2') = (g_1 g_2, g_1' g_2')$$

2.1.6 Lie Groups

A Lie group is a manifold on which the group manipulations, product and inverse, are defined. It plays an extremely important role in the theory of fiber Bundles and also find vast applications in physics. Here we will work out the Geometrical aspects of Lie groups and Lie algebra. It is a group in the category of smooth manifolds. i.e., beside the algebraic properties this enjoys also differential geometric properties. The most obvious construction is that of a Lie algebra which is the tangent space at the unit endowed with the Lie bracket between left-invariant vector fields. Beside the structure theory there is also the wide field of representation.

Let G is a differentiable manifold which is endowed with a group structure such that the group operations.

(i) $G \times G \to G$, $(g_1, g_2) \to g_1 \cdot g_2$

(ii) $G \to G$, $g \leftrightarrow g^{-1}$

are differentiable. It can be shown that G has a unique analytic structure with which the product and the inverse operations are written as convergent power series. The unit element of a Lie group is written as e.

The dimension of a Lie Group G is defined to be the dimension of G as a manifold. The product symbol may be omitted and $g_1 \cdot g_2$ is usually written as $g_1 g_2$. If the product is commutative, namely $g_1 . g_2 = g_1 . g_2$, we often use the additive symbol + instead of the product symbol.

Definition 1: A Lie group G is a group and a smooth manifold such that group multiplication $G \times G \to G \; (x, y) \mapsto x \, y$ and group inversion $G \to G \; x \mapsto x^{-1}$ are smooth maps. If G and H are Lie groups, a Lie group homomorphism $\varphi: G \to H$ is a smooth mapping which is also a homomorphism of the abstract groups. If the mapping is a diffeomorphism, then φ is called an isomorphism. Much of the structure of Lie groups comes from the so-called left and right translations.

Definition 2: Let G be a Lie group, $s \in G$. The left translation by s is the map $L_s : G \to G$ is given by $L_s(t) = St$ for every $t \in G$. Right translations are defined analogously. The group structure implies that for every s, Ls and Rs are bijections with inverses L_s^{-1} and R_s^{-1} . The conditions in definition 1 imply that both of these maps (and their inverses) are smooth. Thus, left and right translations are diffeomorphisms of G onto itself.

Familiar examples of Lie groups include $\mathbb{R}^n$ under addition and the elementary matrix groups ($GL(n,R), SL(n,R), O(n), U(n)$, etc.). In fact, all of the matrix groups can be viewed as closed Lie subgroups of $GL(n;C)$**.** In particular, the circle group $S^1 \cong U(1)$is a Lie group. The product of Lie groups given the standard differentiable and group structures is again a Lie group.

Example 3: (a) ($\mathbb{R}^n, +$) is trivially an abelian Lie group.

(b) The general Linear Group GL(n) = {n× n invertible real Matrices} of a nontrivial Lie group.

Lemma: The Special Linear group SL(n) is a Lie Group.

Proof: Let f **:** $GL(n) \to \mathbb{R}$ is given by f(A) = det (A) is differentiable. The level set $f^{-1}(1)$ is given by,

SL(n) = {A ∈ $M_{n\times n}$ / det(A) = 1}

The Special linear group. The derivative of f is surjective at a point A∈ GL(n), making SL(n) into a Lie group. Such that

$$(df)_I(B) = \lim_{h\to 0} \frac{\det(I+hB)-\det(I)}{h} = \text{tr (B)}$$

Implying that

$$(df)_A(B) = \lim_{h\to 0}\frac{\det(A+hB)-\det(A)}{h}$$

$$= \lim_{h\to 0}\frac{\det(\mathrm{A})\det(I+hA^{-1}B)-\det(A)}{h}$$

$$= (\det \mathrm{A})\lim_{h\to 0}\frac{\det(I+hA^{-1}B)-1}{h}$$

$$= (\det \mathrm{A})(df)_I(A^{-1}B)$$

$$= (\det \mathrm{A})\,\mathrm{tr}(A^{-1}B)$$

Since $\det(\mathrm{A}) = 1$ for any $k \in \mathbb{R}$.We can take the matrix $B = \frac{k}{N}A$ to obtain $(df)_A(B) = \mathrm{tr}\,(\frac{k}{N}I)$ = K, therefore $(df)_A$ is surjective for every A ∈ SL(n).

Consequently, SL(n) is a sub manifold of GL(n). Therefore, the group multiplication and inversion are differentiable, so SL(n) is a Lie Group.

2.1.7 Smooth Mapping on Lie Group

We suppose that this Lie group operation which may be considered as a mapping

$$G \times G \to G$$

is smooth and also that the map $x \to x^{-1}$ on $G \to G$ is smooth.

With each element x in G there is associated a transformation L_x of G called left translation: $L_x^*\omega = \omega$ for all x in G.

Let denote the unit element of G. the left translation $L_x^{-1} = L_{x^{-1}}$ sends x to e. If ω is left invariant, $\omega = L_{x^{-1}}^*\omega$ is completely determined at x by its value ω_0 at e. If ω_0 is any given p-form at e then a left invariant form ω is defined by

$$\omega_x = L_{x^{-1}}^*\omega_0$$

Let n be the dimension of G. Since the space of one form at e is an n-dimensional linear space, there are exactly n linearly independent left invariant one forms on G. Let

$$\sigma^1, \dots, \sigma^n$$

be such a system. Any other left invariant one form is a linear combination of these with constant coefficients.

More generally, if ω is any left invariant p-form on G,

$$\omega = \sum c_H \sigma^H$$

Where the c_H are constants and$\sigma^H = \sigma^{h_1} \dots \sigma^{h_p}$. Any p-form ω can be expanded in this way and the coefficients c_H will in general be scalars on G, supposing ω left invariant forces each of these scalars to be left invariant. This means that each c_H takes the same value at each point of G, hence is constant.

Example 4: We shall determine the local structure of all one-dimensional groups.

Solution

Let t be a parameter on G, chosen so that $t = 0$ is the identity e. Let σ be a nontrivial left invariant one form; locally,

$$\sigma = f(t)dt \text{ never zero.}$$

We integrate σ to get a new parameter for G,

$$\int_0^t f(t)dt$$

Thus, we may assume we have started with a parameterization of a neighborhood of e by a single variable t such that

$$\sigma = dt$$

is a left invariant form.

We next express the group product analytically. The product of the point with coordinate s with that of coordinate t will have coordinate u given by

$$u = p(s,t)$$

with

$$p(s,0) = s, \qquad p(0,t) = t$$

according to $xe = x, ey = y$. In coordinates

$$L_s: t \to u = p(s,t)$$

The left invariance of σ, $L_s^*\sigma = \sigma$, means

$$dt = \frac{\partial p}{\partial t} dt$$

hence

$$\frac{\partial p}{\partial t} = 1, \quad p(s,t) = t + \varphi(s)$$

Setting $t = 0$:

$$s = p(s,0) = \varphi(s)$$

and so $$p(s,t) = s + t$$

Hence G is abelian (commutative).

2.1.8 The action of Lie groups on manifold

In physics, a Lie group often appears as the set of transformations acting on a Manifold. For example, S*0(*3*)* special orthogonal group is the group of rotations in R^3,while the Poincare's Group is the set of transformations acting on the Minkowski space-time. To study more general cases, we abstract the action of a Lie group *G* on a manifold *M*. We have already encountered this interaction between a group and geometry. We defined a flow in a manifold *M* as a map *σ:* R× *M* → *M*, in which R acts as an additive group.

Example

(a) A flow is an action of R on a manifold *M*. If a flow is periodic with a period *T*, it may be regarded as an action of U*(1)* real Orthogonal group or SO*(2)*on *M*. Given a periodic flow *σ (t, x)* with period *T* we construct a new action**,**

*σ (*exp *(2π*i*t/T), x)* ≡ *σ (t, x)* whose group *G* is U*(*1*)*.

(b) Let *M* ϵ *GL(n,* R*)* and let *x* ∈ R_n. The action of *GL(n,* R*)* on _*n* is defined by the usual matrix action on a vector.

$$\sigma : (M, x) = M \cdot x$$

The action of the subgroups of general linear group *GL(n,* R*)* is defined similarly. They may also action on a smaller space. For example, O*(no)*orthogonal group acts on $S^{n-1}(r)$, an *(n* − 1*)* sphere of radius *r*.

$$\sigma : O(n) \times S^{n-1}(r) \rightarrow s^{n-1}(r)$$

2.1.8 Lie-Algebra structure

Let *M* be a smooth manifold of dimension n, 0_M be the set of smooth function and *Γ(TM*) be the vector space of smooth vector fields is a bilinear map called a Lie-bracket or commutator , if

$$, [\, ,] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$$
$$(X, Y) \leftrightarrow [X, Y]$$

given by, $[X, Y] : 0_M \rightarrow 0_M$

$$f \leftrightarrow [X, Y]\, f := X(Y(f)) - Y(X(f))$$

Which satisfies

(i) $[X, Y](\alpha f + \beta g) = \alpha [X, Y] f + \beta [X, Y] g$

(ii) $[X, Y](fg) = f[X, Y] g + g[X, Y] f$ *for all* $f, g \in 0_M$ *and* $\alpha, \beta \in \mathbb{R}$

Theorem: The pair (Γ (TM), [,]) or (v, [,]) is a Lie algebra.

Proof: From the definition of Lie bracket [,] , so condition [x , y] = - [y , x] follows of it . Now we have to show that [[x , y] , z] + [[z , x] , y] + [[y , z] , x] = 0

For all x, y, z ∈ Γ (TM) then

L. H. S = $[[x, y], z] + [[z, x], y] + [[y, z], x]$

$= [xy - yx, z] + [zx - xz, y] + [yz - zy, x]$

$= (x\,y - y\,x)\,z - z(x\,y - y\,x) + (z\,x - x\,z)\,y - y\,(z\,x - x\,z) + (y\,z - z\,y)\,x - x(y\,z - z\,y)$

$= xyz - yxz - zxy + yzx + zxy - xyz - yzx + xyz + yzx - xzy - xyz + xzy$

$= 0$

= R. H. S

So that, The pair ($\Gamma(TM)$, [,]) or (v, [,]) is a Lie algebra. □

2.1.9 The Local Co-ordinate system of represent Lie Bracket [,] :

Let (U, φ) be a chart on a manifold M with co –ordinates ($x^1, x^2, \ldots \ldots .., x^n$)

Then

$$X = \sum_{i=1}^{n} x^i(x)\frac{\partial}{\partial x^i}$$

$$Y = \sum_{j=1}^{n} y^j(x)\frac{\partial}{\partial x^j}$$

Where, $x^i(x)$ and $y^j(x)$ are smooth functions for all g $(x^1, x^2, \ldots \ldots .., x^n)$

Let For all g $(x^1, x^2, \ldots \ldots .., x^n)$

$[x, y]\,g = x\,(y\,(g)) - y\,(x\,(g))$

$$= \sum_{i=1}^{n} X^i(x)\frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} y^j(x)\frac{\partial g}{\partial x^j}\right) - \sum_{j=1}^{n} y^j(x)\frac{\partial}{\partial x^j}\left(\sum_{j=1}^{n} X^i(x)\frac{\partial g}{\partial x^i}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^i\frac{\partial}{\partial x^i}\left(y^j\frac{\partial g}{\partial x^j}\right) - y^j\frac{\partial}{\partial x^j}\left(\sum_{j=1}^{n} x^i\frac{\partial g}{\partial x^i}\right)\right.$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^i\frac{\partial}{\partial x^i}\left(y^j\frac{\partial g}{\partial x^j}\right) - y^i\frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} x^j\frac{\partial g}{\partial x^j}\right)\right.$$

Just change for i & j only.

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^i\frac{\partial y^j}{\partial x^i}\frac{\partial g}{\partial x^j} + x^i y^j\frac{\partial^2 g}{\partial x^i\partial x^j} - y^i\frac{\partial x^j}{\partial x^i}\frac{\partial g}{\partial x^j} - y^i x^j\frac{\partial^2 g}{\partial x^i\partial x^j}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(X^i\frac{\partial y^j}{\partial x^i} - y^i\frac{\partial x^j}{\partial x^i}\right)\frac{\partial g}{\partial x^j}$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\left(X^i\frac{\partial y^j}{\partial x^i} - y^i\frac{\partial x^j}{\partial x^i}\right)\right)\frac{\partial}{\partial x^j}$$

which is an expression of Lie bracket or commutator in Local coordinate system of two vector fields.

Chapter Three

Fibre Bundles on Manifolds

3.1 Introduction

A fiber bundle is a topological space which looks locally like a direct product of two topological spaces. Many theories in physics such as general relativity and Gauge theories are described naturally in terms of fiber bundle. The apparatus of vector bundles, principal bundles and connections on them plays an extraordinary important role in the modern differential geometry. A smooth manifold always carries a natural vector bundle, the tangent bundle. Loosely speaking, this structure by itself is different only for developing analysis on the manifold, while doing geometry requires in addition some way to relate the tangent spaces at different points, i.e. a notion of parallel transport. direct product of two topological spaces. Many theories in physics such as general relativity and gauge theories are described naturally in terms of fibre bundles.

The use of fibre bundles when we "glued" together the tangent spaces at all points of a manifold M to form a new manifold TM that could be regarded as a bundle of vector spaces over M. The basic bundle with which TM is "associated" is the bundle of frames (i.e., sets of basis vectors for a tangent space) over M and as we shall see this can be used to give a very geometrical way of thinking about general tensor structures on manifold.

3.2 Basic Definitions

Definition 1

A bundle is a triple (E, π, M) where E and M are topological spaces and $\pi: E \to M$ is a continuous map. E is called the total space or bundle space and M is the base space. The map π is known as the projection map. The inverse image $\pi^{-1}(x)$, $x \epsilon M$, is the fibre over x. If for all $x \epsilon M$, $\pi^{-1}(x)$ is homeomorphic to a common space$F$, then F is known as the fibre of the bundle and the bundle is said to be a fibre bundle. This condition is always applied in the cases of interest to us and from now on we will assume that it is satisfied.

Examples 1

(a) A famous example of a fibre bundle is the Möbius band which is a twisted "strip" whose base space is the circleS^1. Thus, for example the fibre could be taken to be the closed interval [0, 1] but note that the total space E is not the product space $S^1 \times [0,1]$ (and neither is it homeomorphic to it). It can be represented as follows, where the two points "b" (respectively "c") on the two sides of the strip are not to be identified as below

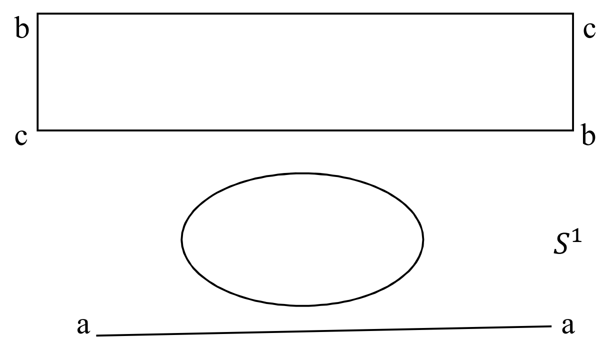

Figure: 3.1

(b) A famous example is the Klein bottle. This is rather difficult to draw as it cannot to be embedded in Euclidean 3-space, but it can be represented with the aid of the following diagram which shows how it is obtained from a cylinder by reflecting in the diameter $d - e$ and identifying:

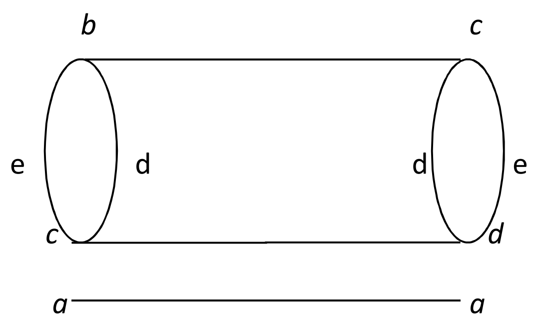

Figure: 3.2

Definition 2

(a) A bundle (E', π', M') is a sub-bundle of a bundle (E, π, M) if

(i) $E' \subset E$

(ii) $M' \subset M$

(iii) $\pi' = \pi_{|E}$

(b) Let N be a subspace of M. Then the restriction of (E, π, M) to N is defined to be the bundle $(\pi^{-1}(N), \pi_{|\pi^{-1}(N)}, N)$.

It should be noted that if (E, π, M) is a sub bundle of the product bundle $(M \times F, pr_1, M)$ then cross sections of the former have the form

$$s(x) = (x, \hat{s}(x))$$

Where $\hat{s}: M \to F$ is a function such that, $\forall\, x \in M, \big(x, \hat{s}(x)\big) \in E$.

Definition 3

A (differentiable) fibre bundle (E, π, M, F, G) consists of the following elements:

(i) A differentiable manifold E called the total space.

(ii) A differentiable manifold M called the base space.

(iii) A differentiable manifold F called the fibre (or typical fibre) space.

(iv) A surjection $\pi : E \to M$ called the projection. The inverse image $\pi^{-1}(p) \equiv F_p \cong F$ is called the fibre at p.

(v) A Lie group G called the structure group, which acts on F on the left.

(vi) A set of open covering $\{U_i\}$ of M with a diffeomorphism $\Phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\Phi_i(p, f) = p$. The map Φ_i is called the local trivialization since Φ_i^{-1} maps $\pi^{-1}(U_i)$ on to the direct product $U_i \times F$.

(vii) If we write $\Phi_i(p, f) = \Phi_{i,p}(f)$ the map $\Phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(p) = \Phi_{i,p}^{-1}\Phi_{j,p} : F \to F$ be an element of G. Then Φ_i and Φ_j are related by a smooth map $t_{ij}: U_i \cap U_j \to G$ as figure

$$\Phi_j(p, f) = \Phi_i\big(p, t_{ij}(p) f\big), \{t_{ij}\}$$

.

are called the transitions function

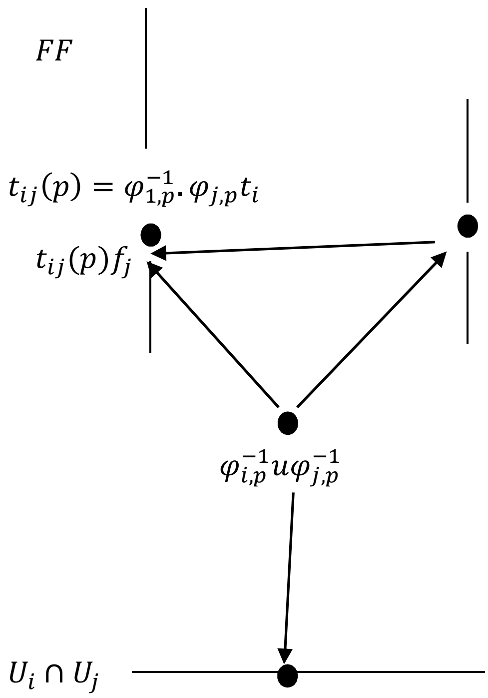

Figure: 3.2

The above definition is employed to define a coordinate bundle$(E, \pi, M, F, G, \{U_i\}, \{\Phi_i\})$. Two coordinate bundles $(E, \pi, M, F, G, \{U_i\}, \{\Phi_i\})$ and $(E, \pi, M, F, G, \{V_i\}, \{\psi_i\})$ are said to be equivalent if $(E, \pi, M, F, G, \{U_i\} \cup \{V_i\}, \{\Phi_i\} \cup \{\psi_i\})$ is again a coordinate bundles. In practical applications in physics, however we always employ a certain definite covering and make no distinction between a coordinate bundle and a fibre bundle.

Examples 2

(a) Let H be closed Lie subgroup of the Lie group G. Define the map $\pi: G \to G/H$by $\pi(g) \coloneqq gH$. Then $(G, \pi, G/H)$ is a bundle with fibre$H$. In general there will be no (smooth) cross sections for bundles of this type since a necessary condition for the existence of such sections is that G be diffeomorphic to $G/H \times H$ which is usually not true.

(b) The tangent bundle TS^n of the n-sphere S^n can be represented rather nicely as the specific sub bundle of $S^n \times \mathbb{R}^{n+1}$

$$E(TS^n) \approx \{(x,y) \in S^n \times \mathbb{R}^{n+1} \mid x.y = 0$$

Similarly, the normal bundle $v(S^n)$ of S^n is defined to be the set of all vectors in $\mathbb{R}^{n+1}$ that are normal to points on the sphere:

$$E\big(v(S^n)\big) := \{(x,y) \in S^n \times \mathbb{R}^{n+1} \mid \exists\, k \in \mathbb{R} \quad \text{s.t.}\, y = kx$$

A cross section of TS^n is of course simply a vector field on S^n; a cross section of $v(S^n)$ is called a normal field of S^n as a sub manifold of $\mathbb{R}^{n+1}$.

3.3 Fiber Bundle Construction Theorem

In mathematics, the fiber bundle construction theorem is a theorem which constructs a fiber bundle from a given base space, fiber and a suitable set of transition functions. The theorem also gives conditions under which two such bundles are isomorphic. The theorem is important in the associated bundle construction where one starts with a given bundle and surgically replaces the fiber with a new space while keeping all other data the same.

Formal statement

Let $X$and F be topological spaces and let G be a topological group with a continuous left action on$F$. Given an open cover $\{U_i\}$ of X and a set of continuous functions

$$t_{ij}: U_i \cap U_j \to G$$

defined on each nonempty overlap, such that the co cycle condition

$$t_{ik}(x) = t_{ij}(x) t_{jk}(x); \ \forall x \in U_i \cap U_j \cap U_k$$

holds, there exists a fiber bundle $E \to X$ with fiber F and structure group G that is trivializable over $\{U_i\}$ with transition functions t_{ij}.

Let E' be another fiber bundle with the same base space, fiber, structure group, and trivializing neighborhoods, but transition functions t'_{ij}. If the action of G on F is faithful, then E' and E are isomorphic if and only if there exist functions

$$t_i : U_i \to G$$

Such that $$t'_{ij}(x) = t_i(x)^{-1} t_{ij}(x) t_j(x);\ \forall x \in U_i \cap U_j$$

Taking t_i to be constant functions to the identity in G, we see that two fiber bundles with the same base, fiber, structure group, trivializing neighborhoods, and transition functions are isomorphic. A similar theorem holds in the smooth category, where X and Y are smooth manifolds, G is a Lie group with a smooth left action on Y and the maps t_{ij} are all smooth.

Construction

The proof of the theorem is constructive. That is, it actually constructs a fiber bundle with the given properties. One starts by taking the disjoint union of the product spaces $U_i \times F$

$$T = \coprod_{i \in I} U_i \times F = \{(i, x, y) : i \in I, x \in X, y \in F\}$$

and then forms the quotient by the equivalence relation

$$(j, x, y) \sim \big(i, x, t_{ij}(x).\ y\big);\ \forall x \in U_i \cap U_j\,, y \in F$$

The total space E of the bundle is $T/\sim$ and the projection $\pi: E \to X$ is the map which sends the equivalence class of (i, x, y) to x. The local trivializations

$\phi_i: \pi^{-1}(U_i) \to U_i \times F$

are then defined by $\phi_i^{-1}(x, y) = [(i, x, y)]$

3.4 Vector Bundles

A vector bundle is simply a rather special case of an associated bundle in which the fibre is a vector space.

Definition 4

(a) Ann-dimensional real (or complex) vector bundle (E, π, M) is a fibre bundle in which each fibre possesses the structure of an-dimensional real (or complex) vector space. Furthermore, for each $x \in M$ there must exist some neighborhood $U \subset M$ of x and a local trivialization $h: U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that, for all $y \in U$, $h: \{y\} \times \mathbb{R}^n \to \pi^{-1}(y)$ is a linear map.

(b) A vector bundle homorphism between a pair of vector bundles (E, π, M) and (E', π', M') is a bundle map (u, f) in which the restriction $u: E \to E'$ of to each fibre is a linear map.

(c) The space $\Gamma(E)$ of all cross sections of a vector bundle (E, π, M) is equipped with a natural module structure over the ring $C(M)$ of continuous, real valued functions on *M,* defined by:

(i) $(s_1 + s_2)(x) := s_1(x) + s_2(x) \quad \forall\, x \in M;\ s_1, s_2 \in \Gamma(E)$

(ii) $(\varphi s)(x) := \varphi(x) s(x) \quad \forall\, x \in M; s \in \Gamma(E)$

Examples 3

(a) The product space $M \times \mathbb{R}^n$ is a (trivial) vector bundle over *M*.

(b) The tangent bundle *TM* of a differentiable manifold is a real vector bundle whose dimension is equal to the dimension of *M*. Likewise the cotangent bundle T^*M.

(c) Let *M* be a smoothly embedded, *m*-dimensional sub-manifold of $\mathbb{R}^n$ for some *n*. Then the normal bundle of *M* in $\mathbb{R}^n$ is a $(n-m)$ −dimensional vector bundle $v(M)$ over *M* with total space:

$$E\big(v(M)\big) := \{(x, v) \in M \times \mathbb{R}^n \mid v.w = 0, w \in T_x M\}$$

and projection map $\pi: E\big(v(M)\big) \to M$ defined by $\pi(x, v) := x$.

Theorem

Let $\xi = (P, \pi, M)$ be principal $GL(n,\mathbb{R})$ bundle and let $GL(n,\mathbb{R})$ act on $\mathbb{R}^n$ in the usual way. Then the associated bundle $\xi[\mathbb{R}^n]$ can be given the structure of an n-dimensional real vector bundle.

Proof

The map $l_p: \mathbb{R}^n \to \pi_{\mathbb{R}^n}^{-1}(x)$, $l_p(v) \coloneqq [p, v]$ where $p \in \pi^{-1}(x)$ is a homeomorphism from $\mathbb{R}^n$onto $\pi_{\mathbb{R}^n}^{-1}(x)$. To give $\pi_{\mathbb{R}^n}^{-1}(x)$ a vector space structure, choose any $p \in \pi^{-1}(x)$, and define

(i) $l_p(v_1) + l_p(v_2) \coloneqq l_p(v_1 + v_2) \qquad \forall\, v_1, v_2 \in \mathbb{R}^n$

(ii) $\lambda l_p(v) \coloneqq l_p(\lambda v) \qquad \forall\, v \in \mathbb{R}^n, \forall\, \lambda \in \mathbb{R}$

If $p' \in \pi^{-1}(x)$ is any other choice such that $l_p(v') = l_p(v)$ for some $v' \in \mathbb{R}^n$, then

$$l_{p'}(v_1') + l_{p'}(v_2') = l_{p'}(v_1' + v_2') = l_{pq}(g^{-1}v_1 + g^{-1}v_2) \text{ for some } g \in GL(n,\mathbb{R})$$

$$l_{pq}(g^{-1}(v_1 + v_2)) = l_p(v_1 + v_2) = l_p(v_1) + l_p(v_2)$$

and hence the vector space structure is independent of the choice of $p \in \pi^{-1}(x)$.

To see that this bundle is locally trivial we define, as for any associated bundle, a map

$h': U \times \mathbb{R}^n \to \pi_{\mathbb{R}^n}^{-1}(U), h'(x, v) \coloneqq [h(x, e), v]$ where $h: U \times GL(n,\mathbb{R}) \to \pi^{-1}(U)$ is a trivializing map of P over $U \subset M$. It is clear that the restriction $h': \{x\} \times \mathbb{R}^n \to \pi_{\mathbb{R}^n}^{-1}(x)$ of the local trivializing map $h'$is linear for each $x \in U \subset M$.

This complete the proof of the theorem. □

3.5 Tangent and Cotangent Bundles

Definition 1

In mathematics, the tangent bundle of a smooth (differentiable) manifold, denoted by $\boldsymbol{TM}$ is the disjoint union of the tangent spaces of $\boldsymbol{M}$. That is,

$$TM = \bigcup_{x \in M} T_x M = \bigcup_{x \in M} \{x\} \times T_x M$$

where $T_x M$ denotes the tangent space to M at the point x. So an element of TM can be thought of as a pair(x, v) where x is a point in M and v is a tangent vector to M at x. There is a natural projection

$$\pi: TM \to M$$

defined by $\pi(x, v) = x$. This projection maps each tangent space $T_x M$ to the single point x.

The tangent bundle to a manifold is the prototypical example of a vector bundle (a fiber bundle whose fibers are vector spaces). A section of TM is a vector field on M, and the dual bundle to TM is the cotangent bundle, which is the disjoint union of the cotangent spaces of M. By definition, a manifold M is parallelizable if and only if the tangent bundle is trivial.

Definition 2

In Mathematics, the cotangent bundle of a smooth (differentiable) manifold, denoted by T^*M is the disjoint union of the tangent spaces of M. That is,

$$T^*M = \bigcup_{x \in M} T_x^* M = \bigcup_{x \in M} \{x\} \times T_x^* M$$

Where $\boldsymbol{T_x^* M}$ denotes the cotangent space to $\boldsymbol{M}$ at the point $\boldsymbol{x}$.

Example 4

i) The simplest example is that of $\mathbb{R}^n$.In this case the tangent bundle is trivial and isomorphic to $\mathbb{R}^{2n}$.

ii) Another simple example is the unit circle S^1, the tangent bundle of the circle is also trivial and isomorphic to $S^1 \times \mathbb{R}$.

Theorem

If M is an m-dimensional smooth manifold, then the tangent bundle TM is a $2m$-dimensional smooth manifold.

Proof

Let the projection map

$$\pi: TM \to M$$

be defined by

$$\pi(v) = x \ \ if \ \ v \in T_x M$$

Now for a chart $(U, \phi) \in \mathcal{A}'$ of M,we construct the pair $(\pi^{-1}(U),\ \phi)$,where

$$\phi: \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R}^m$$

given by

$$\phi(v) = (\phi(\pi(v)), \bar{\phi}(v))$$

$$= \left(\phi(x), \bar{\phi}(v)\right) \ \ if \ \ v \in T_x M$$

We claim that $(\pi^{-1}(U),\ \phi)$ is a chart of TM.

(1) ϕ is one – one:

Let $u, v \in T_x M$ such that $\phi(u) = \phi(v)$.We have to show that $u = v$.

There are two cases:

(Case-i): If u, v belong to the same tangent space i.e. $u, v \in T_xM$, then

$$\phi(u) := \left(\phi(x), \bar{\phi}(u)\right) \quad if\ u \in T_xM$$

and $\phi(v) := \left(\phi(x), \bar{\phi}(v)\right) \quad if\ v \in T_xM$

So, $\phi(u) = \phi(v) \Rightarrow \left(\phi(x), \bar{\phi}(u)\right) = \left(\phi(x), \bar{\phi}(v)\right)$

$\Rightarrow \bar{\phi}(u) = \bar{\phi}(v)$ [since by definition ϕ is an isomorphism]

$$\Rightarrow u = v$$

(Case-ii): If u, v belong to the different tangent spaces i.e. $u \in T_xM$ and $v \in T_yM$ with $x \neq y$, then

$$\phi(u) = \phi(v) \Rightarrow \left(\phi(x), \bar{\phi}(u)\right) = \left(\phi(y), \bar{\phi}(v)\right)$$

$\Rightarrow \phi(x) = \phi(y)$ [since ϕ is bijective $\bar{\phi}(u) = \bar{\phi}(v)$]

$\Rightarrow x = y$, which is a contradiction.

Hence case-(ii) cannot be happened.

(2) ϕ is onto:

Let $\left(a, \vec{h}\right) \in \phi(U) \times \mathbb{R}^m$ be any arbitrary pair. We want to find out some $v \in TM$ such that

$$\phi(v) = \left(a, \vec{h}\right)$$

If there is some $z \in M$, then we have

$$\phi(v) := \left(\phi(z), \bar{\phi}(v)\right) \quad if \quad v \in T_zM$$

Hence $\left(\phi(z), \bar{\phi}(v)\right) = \left(a, \vec{h}\right)$, from which we get

$\phi(z) = a$ and $\overline{\phi}(v) = \vec{h}$

So $z = \phi^{-1}(a\ )$ and $v = (\bar{\phi})^{-1}(\vec{h}\ )$.

Therefore, we are looking for that v is the vector,

$$v := (\bar{\phi})^{-1}(\vec{h}) \in T_{\phi^{-1}(a\ )}M$$

This vector mapped on $(a, \vec{h})$.

Now $\phi(v) = \phi\left((\bar{\phi})^{-1}(\vec{h})\right) := \left(\phi(\phi^{-1}(a\ )), \bar{\phi}\left((\bar{\phi})^{-1}(\vec{h})\right)\right)$

$$= (a, \vec{h})$$

Thus ϕ is onto.

(3) $\phi(\pi^{-1}(U)) = \phi(U) \times \mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$

Hence $(\pi^{-1}(U), \phi)$ is a chart ofTM.

Now, we take the collection of all charts $(\pi^{-1}(U), \phi)$ for all $(U, \phi) \in \mathcal{A}'$.We shall prove that

$\boldsymbol{\mathcal{B}} = \{(\pi^{-1}(U), \phi) /\ (U, \phi) \in \mathcal{A}'\}$ is a $2m$-dimensional smooth atlas on TM.

$(\boldsymbol{a})$ $\boldsymbol{\mathcal{B}}$ covers $\boldsymbol{TM}$:

We have to show that $TM = \cup\ (\pi^{-1}(U)\ )$

Now $\pi^{-1}(U) \subseteq TM \Rightarrow \cup\ \pi^{-1}(U) \subseteq TM$

But also $TM \subseteq \cup\ \pi^{-1}(U)$, because for every $v \in TM$, there exists some $x \in M$ such that

$v \in T_x M$. Hence $\phi(v) = x$.

But $\mathcal{A}'$ covers M. Thus there exists some charts $(U, \phi) \in \mathcal{A}'$ such that$x \in U$. That is

$$\pi(v) = x \in U$$

$$\Rightarrow\ v = \pi^{-1}\ (U)$$

Thus $v \in TM \Rightarrow v \in \pi^{-1}(U) \subseteq \cup \ \pi^{-1}(U)$

Hence, we conclude that

$$TM \subseteq \cup \, \pi^{-1}(U)$$

From above simplifications, we get,

$$TM = \cup \ \pi^{-1}(U)$$

Therefore $\mathcal{B}$ coversTM.

(b) The charts of $\boldsymbol{\mathcal{B}}$ are $\boldsymbol{C^{\infty}}$ compatible:

Let $(\pi^{-1}(U), \phi)$ and $(\pi^{-1}(V), \psi)$ be two charts of $\mathcal{B}$,which are constructed from (U, ϕ) and (V, ψ) respectively.

(b.1) Now $\phi\left(\pi^{-1}(U) \cap \ \pi^{-1}(V)\right) = \phi(\pi^{-1}(U \cap V))$

$$= \phi(U \cap V) \ \times \mathbb{R}^m \subseteq \ \mathbb{R}^m \times \ \mathbb{R}^m = \ \mathbb{R}^{2m}$$

Similarly, $\psi\left(\pi^{-1}(U) \cap \ \pi^{-1}(V)\right) = \ \psi(\pi^{-1}(U \cap V))$

$$= \psi(U \cap V) \ \times \mathbb{R}^m \subseteq \ \mathbb{R}^m \times \ \mathbb{R}^m = \ \mathbb{R}^{2m}$$

(b.2) We have to prove that

$$\psi o \ \phi^{-1} : \ \phi\left(\pi^{-1}(U \cap V)\right) \to \psi(\pi^{-1}(U) \ \cap \ \pi^{-1}(V))$$

$$\phi(U \cap V) \times \mathbb{R}^m \mapsto \ \psi(U \cap V) \times \mathbb{R}^m$$

is a C^{∞}diffeomorphismand

$$\phi o \ \psi^{-1}: \psi(U \cap V) \times \mathbb{R}^m \mapsto \ \phi(U \cap V) \times \mathbb{R}^m$$

is also a C^{∞}diffeomorphism.

Thus, we have

$$\phi \ o \ \pi^{-1}(U) \longrightarrow \ \phi(U) \times \mathbb{R}^m \subseteq \ \mathbb{R}^{2m}$$

Now $\forall \left(a, \vec{h}\right) \in \phi(U \cap V) \times \mathbb{R}^m,$ we have

$$(\psi o\, \phi^{-1})\left(a, \vec{h}\right) = \psi(\phi^{-1}\left(a, \vec{h}\right))$$

$$= \psi(v) \qquad [\because \ \phi(v) = \left(a, \vec{h}\right)]$$

$$= \psi\left((\bar{\phi})^{-1}\left(\vec{h}\right)\right) \qquad [\because \ (\bar{\phi})^{-1}\left(\vec{h}\right) \in T_{\phi^{-1}(a)}]$$

$$= \psi(\phi^{-1}(a)), \qquad \bar{\psi}(\,(\bar{\phi})^{-1}\left(\vec{h}\right)) \quad [\,by\ (4.2)]$$

$$= (\psi\ o\ \phi^{-1})(a), \qquad (\bar{\psi} o\ (\bar{\phi})^{-1}\left(\vec{h}\right))$$

$$= \left((\psi\ o\ \phi^{-1})(a), \qquad \left[D(\psi o \phi^{-1})\left(\phi\left(\phi^{-1}(a)\right)\right)\right]\left(\vec{h}\right)\right)$$

[Since there is a lemma that $\bar{\psi}\ o\ (\bar{\phi})^{-1} = D(\psi\ o\ \phi^{-1})\left(\phi\left(\phi^{-1}(a)\right)\right)$]

$$= ((\psi\ o\ \phi^{-1})(a), D(\psi\ o\ \phi^{-1})(a))\left(\vec{h}\right))$$

From which we conclude that $\psi\ o\ \phi^{-1}$ is smooth.

Similarly, $\phi\ o\ \psi^{-1}$ is also smooth.

Thus $\mathcal{B}$ is an C^∞ atlas on TM .Hence $\mathcal{B}$ defines on TM the structure of a $2m$-dimensional manifold.

Hence completes the proof of the theorem. □

3.5 Connections on Vector Bundles

A connection on a fiber bundle is a device that defines a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points.

If the fiber bundle is a vector bundle, then the notion of parallel transport is required to be linear. Such a connection is equivalently specified by a covariant derivative, which is an operator that can differentiate sections of that bundle along tangent directions in the base manifold. Connections in this sense generalize, to arbitrary vector bundles, the concept of a linear connection on the tangent bundle of a smooth manifold, and are sometimes known as linear connections. Nonlinear connections are connections that are not necessarily linear in this sense.

Definition

A connection on a vector bundle E is a map

$$D : \Gamma(E) \to \Gamma(T(M) \otimes E),$$

which satisfies the following conditions:

i) For any $s_1, s_2 \in \Gamma(E)$, when$D(s_1 + s_2) = Ds_1 + Ds_2$

ii) For $s \in \Gamma(E)$ and any $\alpha \in C^{\infty}(M)$, when$D(\alpha s) = d\alpha \otimes s + \alpha Ds)$

Suppose X is a smooth tangent vector fields on M and $s \in \Gamma(E)$. Let $D_X s = < X, Ds >$ where$<, >$ represents the pairing between $T(M)$ and $T^*(M)$. Then $D_X s$ is a section of E , called the absolute differential quotient or the covariant derivative of the section s along X.

Theorem 1

Suppose D is a connection on a vector bundle E, and $p \in M$. Then there exists a local frame field S in a coordinate neighborhood of p such that the corresponding connection matrix w is zero at p.

Proof

Choose a coordinate neighborhood $(U; u^i)$ of p such that $u^i(p) = 0,1 \leq i \leq m$.Suppose$S'$ is a local frame field on U with corresponding connection matrix $w^i = (w_{\alpha}^{\prime\beta})$, where

$$w_{\alpha}^{\prime\beta} = \sum_{i=1}^{m} \Gamma_{\alpha i}^{\prime\beta} u^i ,$$

and the $\Gamma_{\alpha i}^{\prime\beta}$ are smooth functions on U. Let

$$\alpha_\alpha^\beta = \delta_\alpha^\beta - \sum_{i=1}^{m} \Gamma'^\beta_{\alpha i}(p).u^i$$

Then $A = (\alpha_\alpha^\beta)$ is the identity matrix at p. Hence there exists a neighborhood $V \subset U$ of p such that A is non-degenerate in V. Thus $S = A.S'$

is a local frame field on V. Since $dA(p) = -w'(p)$

we can obtain, $$w(p) = (dA.A^{-1} + A.w'.A^{-1})(p)$$

$$= -w'(p) + w'(p) = 0$$

Thus S is the desired local frame field.

This complete the proof of the theorem. □

Theorem 2

Suppose *X* and *Y* are two arbitrary smooth tangent vector fields on the manifold *M*. Then

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

Here *D* is the connection on a vector bundle *E* of rank *q*.

Proof

Suppose $s \in \Gamma(E)$ has the local expression

$$s = \sum_{\alpha=1}^{q} \lambda^\alpha s_\alpha, \qquad \lambda^\alpha \in \mathbb{R}$$

Then $$D_X\, s = < X, Ds >$$

where$<,>$ represents the pairing between $T(M)$ and $T^*(M)$. Then $D_X s$ is a section of E , called the absolute differential quotient or the covariant derivative of the section s along X.

Since, $$D_X(\alpha s) = (X\alpha)s + \alpha D_X s$$

$$= (X\alpha)s + \alpha < X, Ds >$$

Now,

$$D_X s = D_X(\sum_{\alpha=1}^{q} \lambda^\alpha s_\alpha)$$

$$= X\sum_{\alpha=1}^{q} \lambda^\alpha s_\alpha + \sum_{\alpha=1}^{q} \lambda^\alpha < X, Ds_\alpha >$$

$$= \sum_{\alpha=1}^{q} (X\lambda^\alpha) + \sum_{\beta=1}^{q} \lambda^\beta < X, w_\beta^\alpha >)s_\alpha$$

And

$$D_Y D_X s = \sum_{\alpha=1}^{q} \{Y(X\lambda^\alpha) + \sum_{\beta=1}^{q} (X\lambda^\beta < Y, w_\beta^\alpha > + Y\lambda^\beta < X, w_\beta^\alpha$$

$$>) + \sum_{\beta=1}^{q} \lambda^\beta (Y < X, w_\beta^\alpha > + \sum_{\gamma=1}^{q} < X, w_\beta^\gamma >. < Y, w_\gamma^\alpha >)\}s_\alpha$$

similarly

$$D_X D_Y s = \sum_{\alpha=1}^{q} \{X(Y\lambda^\alpha) + \sum_{\beta=1}^{q} (Y\lambda^\beta < X, w_\beta^\alpha > + X\lambda^\beta < Y, w_\beta^\alpha$$

$$>) + \sum_{\beta=1}^{q} \lambda^\beta (X < Y, w_\beta^\alpha > + \sum_{\gamma=1}^{q} < Y, w_\beta^\gamma >. < X, w_\gamma^\alpha >)\}s_\alpha$$

Hence

$D_X D_Y s - D_Y D_X s$

$$= \sum_{\alpha=1}^{q} \{(XY - YX)\lambda^\alpha$$

$$+ \sum_{\beta=1}^{q} \lambda^\beta (Y < X, w_\beta^\alpha > +X < Y, w_\beta^\alpha > -X < Y, w_\beta^\alpha > -Y < X, w_\beta^\alpha > +X$$

$$< Y, w_\beta^\alpha > -Y < X, w_\beta^\alpha > + \sum_{\gamma=1}^{q} [< Y, w_\beta^\gamma >. < X, w_\gamma^\alpha > - < X, w_\beta^\gamma >. < Y, w_\gamma^\alpha$$

$$>])\} s_\alpha$$

$$= \sum_{\alpha=1}^{q} \{[X, Y]\lambda^\alpha + \sum_{\beta=1}^{q} \lambda^\beta (< [X, Y], w_\beta^\alpha > + < X \wedge Y, dw_\beta^\alpha - \sum_{\gamma=1}^{q} w_\beta^\gamma \wedge w_\gamma^\alpha >)\} s_\alpha$$

$$= D_{[X,Y]} s + \sum_{\alpha,\beta=1}^{q} \lambda^\beta < X \wedge Y, \Omega_\beta^\alpha > s_\alpha$$

$$=> D_X D_Y s - D_Y D_X s - D_{[X,Y]} s = \sum_{\alpha,\beta=1}^{q} \lambda^\beta < X \wedge Y, \Omega_\beta^\alpha > s_\alpha$$

Since

$$R(X, Y) = \sum_{\alpha,\beta=1}^{q} \lambda^\beta < X \wedge Y, \Omega_\beta^\alpha > s_\alpha$$

Therefore $\quad D_X D_Y s - D_Y D_X s - D_{[X,Y]} s = R(X, Y)s$

That is $R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$

This complete the proof of the theorem □

Chapter Four

Differential Form on Manifolds

4.1 Introduction

In mathematics, the exterior product or wedge product of vectors is an algebraic construction used in Euclidean geometry to study areas, volumes, and their higher-dimensional analogs. The exterior product of two vectors u and v, denoted by $u \wedge v$, is called a bivector. The magnitude of $u \wedge v$ can be interpreted as the area of the parallelogram with sides u and v, which in three-dimensions can also be computed using the cross product of the two vectors. Also like the cross product, the exterior product is anticommutative, meaning that $u \wedge v = -v \wedge u$ for all vectors u and v. More generally, the exterior product of any number k of vectors can be defined and is sometimes called a k-blade.

Definition 1

The exterior algebra $\Lambda(V)$ over a vector space V over a field$K$ is defined as the quotient algebra of the tensor algebra by the two-sided ideal$I$ generated by all elements of the form $x \otimes x$ such that $x \in V$. Symbolically,

$$\Lambda(V) \coloneqq T(V)/I$$

The wedge product $\wedge$ of two elements of $\Lambda(V)$ is defined by,

$$\alpha \wedge \beta = \alpha \otimes \beta \quad (mod\ I)$$

The exterior algebra was first introduced by Hermann Grassmann in 1844.

4.2 The exterior power

The k-th exterior power of V, denoted $\Lambda^k(V)$, is the vector subspace of $\Lambda(V)$ spanned by elements of the form $x_1 \wedge x_2 \wedge \ldots \wedge x_k$, $x_i \in V, i = 1,2, \ldots, k$

If $\alpha \in \Lambda^k(V)$, then α is said to be a k-multivector. If, furthermore, α can be expressed as a wedge product of k elements of V, then α is said to be decomposable.

For example, in $\mathbb{R}^4$, the following 2-multivector is not decomposable:

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4$$

This is in fact a symplectic form, since $\alpha \wedge \alpha \neq 0$.

Definition 2

In the mathematical fields of differential geometry and tensor calculus, differential forms are an approach to multivariable calculus that is independent of coordinates. Differential forms provide a better definition for integrands in calculus. For instance, $f(x)\,dx$ is a 1-form which can be integrated over an interval $[a, b]$ in the domain of f

$$\int_a^b f(x)dx$$

and similarly $f(x, y)\,dx + g(x, y)\,dy$ is a 1-form which has a line integral over any oriented curve γ in the domain of f and g

$$\int_\gamma f(x,y)dx + g(x,y)dy$$

Likewise, a 3-form $f(x, y, z)dxdydz$ represents something that can be integrated over a region of space. The modern notion of differential forms was pioneered by Élie Cartan, and has many applications, especially in geometry, topology and physics.

Differential forms provide an approach to multivariable calculus that is independent of coordinates.

Let U be an open set in $\mathbb{R}^n$. A differential 0-form is defined to be a smooth function f on U. If v is any vector in $\mathbb{R}^n$, then f has a directional derivative $\partial_v f$, which is another function on U whose value at a point $p \in U$ is the rate of change (at p) of f in the v direction:

$$(\partial_v f)(p) = \frac{d}{dt} f(p + tv)\ |_{t=0}$$

This notion can be extended to the case that v is a vector field on U by evaluating v at the point p in the definition.

In particular, if $v = e_j$ is the j-th coordinate vector then ∂_v f is the partial derivative of f with respect to the j-th coordinate function, i.e., $\partial f / \partial x^j$, where $x^1, x^2, \ldots, x^n$ are the coordinate functions on U. Partial derivatives depend upon the choice of coordinates: if new coordinates $y^1, y^2, \ldots, y^n$ are introduced, then

$$\frac{\partial f}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial f}{\partial y^i}$$

The first idea leading to differential forms is the observation that $\partial_v f(p)$ is a linear function of v:

$$(\partial_{v+w} f)(p) = (\partial_v f)(p) + (\partial_w f)(p)$$

$$(\partial_{cv} f)(p) = c(\partial_v f)(p)$$

for any vectors v, w and any real number c. This linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is denoted df_p and called the derivative of f at p. Thus $df_p(v) = \partial_v f(p)$.

Since any vector v is a linear combination $\sum v^j e_j$ of its components, df is uniquely determined by $df_p(e_j)$ for each j and each $p \in U$, which are just the partial derivatives of f on U. Thus df provides a way of encoding the partial derivatives of f. It can be decoded by noticing that the coordinates x^1, x^2..., x^n are themselves functions on U, and so define differential 1-forms dx^1, dx^2... dx^n. Since $\partial x^i / \partial x^j = \delta_{ij}$, the **Kronecker delta function**, it follows that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

The meaning of this expression is given by evaluating both sides at an arbitrary point p: on the right-hand side, the sum is defined "point wise", so that

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(dx^i)_p$$

Applying both sides to e_j, the result on each side is the j-th partial derivative of f at p. Since p and j were arbitrary, this proves the formula.

More generally, for any smooth functions g_i and h_i on U, we define the differential 1-form $\alpha = \sum_i g_i dh_i$ point wise by

$$\alpha_p = \sum_i g_i(p)(dh_i)_p$$

for each $p \in U$. Any differential 1-form arises this way, and by using (A) it follows that any differential 1-form α on U may be expressed in coordinates as

$$\alpha = \sum_{i=1}^{n} f_i dx^i$$

for some smooth functions f_i on U.

Given a differential 1-form α on U, when does there exist a function f on U such that $\alpha = df$. The above expansion reduces this question to the search for a function f whose partial derivatives $\partial f / \partial x^i$ are equal to n given functions f_i. For $n > 1$, such a function does not always exist: any smooth function f satisfies

$$\frac{\partial^2 f}{\partial x^i\, \partial x^j} = \frac{\partial^2 f}{\partial x^j\, \partial x^i}$$

so it will be impossible to find such an f unless

$$\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} = 0$$

for all i and j.

The skew-symmetry of the left hand side in i and j suggests introducing an anti-symmetric product $\wedge$ on differential 1-forms, the wedge product, so that these equations can be combined into a single condition

$$\sum_{i,j=1}^{n} \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j$$

where $$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

This is an example of a **differential 2**-form: the exterior derivative $d\alpha$ of

$$\alpha = \sum_{i=1}^{n} f_j dx^j$$

is given by

$$d\alpha = \sum_{j=1}^{n} df_j \wedge dx^j = \sum_{i,j=1}^{n} \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j$$

For each k, there is a space of differential k-forms, which can be expressed in terms of the coordinates as

$$\sum_{i_1, i_2, \ldots, i_k = 1}^{n} f_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

for a collection of functions $f_{i_1, i_{2..,...,} i_k}$

Differential forms can be multiplied together using the wedge product, and for any differential k-form α, there is a differential $(k + 1)$ −form $d\alpha$ called the exterior derivative of α.

Differential forms, the wedge product and the exterior derivative are independent of a choice of coordinates. Consequently, they may be defined on any smooth manifold *M*.

Theorem 1

Let M be a C^∞ manifold. Then the set $\mathcal{A}^k(M)$ of all forms on *M* can be naturally identified with that of all multilinear and alternating maps, as $C^\infty(M)$ modules from k-fold, direct product of $\mathfrak{X}(M)$to $C^\infty(M)$.

Proof

Suppose that a map $\widehat{\omega} : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to C^\infty(M)$ with the above conditions is given. First of all, we shall see that for arbitrary vector fields $X_i \in \mathfrak{X}$, the value $\overline{\omega}(X_1, \ldots, X_k)(p)$ at point *p* is determined depending only on the values$X_i(p)$of each vector field X_i at *p*. For that it is enough to linearly to show that if $X_i(p) = 0$ for some *i* then the above value is 0. For the sake of simplicity assume that $i = 1$ and $(U, x_1, \ldots, x_n)$ be a local coordinate system around p. Then we can write

$$X_1 = \sum_i f_i \frac{\partial}{\partial x_i}$$

on U with $f_i(p) = 0$. We choose an open neighborhood V of p such that $\bar{V} \subset U$ and a C^∞ function $h \in C^\infty(M)$ such that it is identically 1on *V* and *0* outside of *U*. let $Y_i = h\frac{\partial}{\partial x_i}$ then we have $Y_i \in \mathfrak{X}(M)$ and if we set $\hat{f}_i = hf_i$ then we have $\hat{f}_i \in C^\infty(M)$.

Now it is easy to see that

$$X_1 = \sum_i \hat{f}_i Y_i + (1 - h^2)X_1$$

Therefore, we have

$$\varpi(X_1, \dots, X_k)(p)$$

$$= \sum_i \hat{f}_i(p)\, \varpi(Y_1, X_2 \dots, X_k)(p) + (1 - h(p)^2)\varpi(X_1, \dots, X_k)(p) = 0$$

Thus, the theorem is proved. □

4.3 Wedge product

The wedge product of a *k*-form α and an *l*-form β is a $(k + l)$-form denoted $\alpha \Lambda \beta$. For example, if $k = l = 1$, then $\alpha \Lambda \beta$ is the 2-form whose value at a point p is the alternating bilinear form defined by

$$(\alpha \wedge \beta)_p(v, w) = \alpha_p(v)\beta_p(w) - \alpha_p(w)\beta_p(v)$$

for v, $w \in T_pM$. The wedge product is bilinear: for instance, if α, β, and γ are any differential forms, then

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

It is skew commutative meaning that it satisfies a variant of anti-commutativity that depends on the degrees of the forms: if α is a *k*-form and β is an *l*-form, then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

More generally, given 1-formω and, we may consider their product $\omega \wedge \eta$, which is a 2-form. This rules governing exterior algebra are:

1. (Distributive Law)

$$(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta), \omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2)$$

2. (Associative Law)

$$(f\omega) \wedge \eta = \omega \wedge (f\eta) = f(\omega \wedge \eta)$$

3. (Skew Symmetry) $\eta \wedge \omega = -\omega \wedge \eta$

Here $\omega, \eta, \omega_1, \omega_2, \eta_1,$ andη_2 are 1-form and f is a function.

4. $$g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta,\ g^*(f\omega) = (g^*f)(g^*\omega)$$

Here *f* is a scalar function and hence can be regarded as a 0-form. It's pull back g^*f by *g* is just the composite $f \circ g$.

5. (Exterior Differentiation)

If $\omega = f_1 dg_1 + f_2 dg_2 + \cdots + f_m dg_m$ then we have

$$d\omega = df_1 \wedge dg_1 + df_2 \wedge dg_2 + \cdots + df_m \wedge dg_m$$

6. ($d^2 = 0$). For each function$f$, $d^2 f \equiv d(df) = 0$.

7. (Product Rule) $d(f\omega) = df \wedge \omega + f d\omega$

8. $$f^*(d\omega) = d(f^*\omega)$$

Note that: Wedge products are intimately related to determinants.

The wedge product is alternating on elements of *V*, which means that $x \wedge x = 0$ for all $x \in V$. It follows that the product is also anti-commutative on elements of *V*, for supposing that $x, y \in V$,

$$0 = (x + y) \wedge (x + y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$$

Hence $x \wedge y = -y \wedge x$

Conversely, it follows from the anti-commutativity of the product that the product is alternating, unless *K* has characteristic two.

More generally, if x_1, x_2,...,x_k are elements of V, and σ is a permutation of the integers $[1,\dots,k]$, then

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \dots \wedge x_{\sigma(k)} = sgn(\sigma) x_1 \wedge x_2 \wedge \dots \wedge x_k$$

where $sgn\ (\sigma)$ is the signature of the permutation σ.

Theorem 2

Suppose $f: V \to W$ is a linear map. Then f^* commutes with the exterior product, that is, for any $\varphi \in \Lambda^r(W^*)$ and $\psi \in \Lambda^s(W^*)$,

$$f^*(\varphi \Lambda \psi) = f^*\varphi \Lambda f^*\psi.$$

Proof

Choose any$v_1, \dots, v_{r+s} \in V$. Then

$$f^*(\varphi \Lambda \psi)(v_1, \dots, v_{r+s}) = \varphi \Lambda \psi\ (f(v_1), \dots, f(v_{r+s}))$$

$$= \frac{1}{(r+s)!} \sum_{\sigma \in S\,(r+s)} sgn\sigma \,.\, \varphi\left(f(v_{\sigma(1)}), \dots, f(v_{\sigma(r)})\right).$$

$$\psi(f(v_{\sigma(r+1)}), \dots, f(v_{\sigma(r+s)}))$$

$$= \frac{1}{(r+s)!} \sum_{\sigma \in S\,(r+s)} sgn\sigma \,.\, f^*\varphi(v_{\sigma(1)}, \dots, v_{\sigma(r)}).$$

$$f^*\psi(v_{\sigma(r+1)}, \dots, v_{\sigma(r+s)})$$

$$= f^*\varphi \Lambda f^*\psi(v_1, \dots, v_{r+s}).$$

Therefore

$$f^*(\varphi \Lambda \psi) = f^*\varphi \Lambda f^*\psi.$$

This complete the proof of the theorem. □

Example 1

Express the 2-form $dx \wedge dy$ in polar coordinates.

Solution: We have $x = rcos\theta,\ y = rsin\theta$. Then we obtain

$$dx = cos\theta dr - rsin\theta d\theta \quad and\ dy = sin\theta dr + rcos\theta d\theta$$

Then $dx \wedge dy = (cos\theta dr - rsin\theta d\theta) \wedge (sin\theta dr + rcos\theta d\theta)$

$$= cos\theta . rcos\theta dr \wedge d\theta - rsin\theta . sin\theta d\theta \wedge dr$$

$$= rcos^2\theta dr \wedge d\theta + rsin^2\theta dr \wedge d\theta = rdr \wedge d\theta$$

4.4 Exterior Differentiation

In differential geometry, the exterior derivative extends the concept of the differential of a function, which is a 1-form, to differential forms of higher degree. Its current form was invented by Élie Cartan.

The exterior derivative d has the property that $d^2 = 0$ and is the differential (coboundary) used to define de Rham cohomology on forms. Integration of forms gives a natural homomorphism from the de Rham cohomology to the singular cohomology of a smooth manifold.

The exterior derivative of a differential form of degree k is a differential form of degree $k + 1$. There are a variety of equivalent definitions of the exterior derivative.

Definition 3

Suppose M is an m-dimensional smooth manifold. The bundle of exterior r-forms on M

$$\Lambda^r(M^*) = \bigcup_{p \in M} \Lambda^r\,(T_p^*)$$

is a vector bundle on M. Use $A^r(M)$ to denote the space of the smooth sections of the exterior bundle $\Lambda^r(M^*)$:

$$A^r(M) = \Gamma(\Lambda^r(M^*)).$$

$A^r(M)$ is a $C^\infty(M)$-module. The elements of $A^r(M)$ are called exterior differential r-forms on M. Therefore, an exterior differential r-form on M is a smooth skew-symmetric covariant tensor field of order r on M.

Similarly, the exterior form bundle $\Lambda(M^*) = \cup_{p\in M} \Lambda(T_p^*)$ is also a vector bundle on M. The elements of the apace of its sections $A(M)$ are called exterior differential forms on M. Obviously $A(M)$ can be expressed as the direct sum

$$A(M) = \sum_{r=0}^{m} A^r(M)$$

i.e., every differential form w can be written as

$$w = w^0 + w^1 + \cdots + w^m,$$

where w^i is an exterior differential i-form. The wedge product of exterior forms can be extended to the space of exterior differential form $A(M)$. Suppose $w_1, w_2 \in A(M)$. For any $p \in M$, let

$$w_1 \Lambda\, w_2(p) = w_1(p)\,\Lambda\, w_2(p),$$

where the right-hand side is a wedge product of two exterior forms.

It is obvious that $w_1 \Lambda\, w_2 \in A(M)$. The space $A(M)$ then becomes an algebra with respect to addition, scalar multiplication and the wedge product. Moreover, it is a graded algebra. This means that $A(M)$ is a direct sum of a sequence of vector space and the wedge product Λ defines a map

$$\Lambda : A^r(M) \times A^s(M) \to A^{r+s}(M),$$

where $A^{r+s}(M)$ is zero when $r + s > m$.

4.5 Exterior derivative of a function

If f is a smooth function, then the exterior derivative of f is the differential of f. That is, df is the unique one-form such that for every smooth vector field X, $df(X) = Xf$, where Xf is the directional derivative of f in the direction of X. Thus the exterior derivative of a function is a one-form.

4.6 Exterior derivative of a *k*-form

The exterior derivative is defined to be the unique **R**-linear mapping from k-forms to $(k+1)$-forms satisfying the following properties:

1. df is the differential of f for smooth functions f.
2. $d(df) = 0$ for any smooth function f.
3. $d\,(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta)$ where α is a p-form. That is to say, d is a derivation of degree 1 on the exterior algebra of differential forms.

The second defining property holds in more generality: in fact, $d(d\alpha) = 0$ for any k −form α.The third defining property implies as a special case that if f is a function and αa k-form, then

$d(f\alpha) = df \wedge \alpha + f \wedge d\alpha$ because functions are forms of degree 0.

4.7 Exterior derivative in local coordinates

Let $(x^1,..., x^n)$ be a local coordinate system First, the coordinate differentials $dx^1,..., dx^n$ form a basic set of one-forms within the coordinate chart. Given a multi-index $I = (i_1,...,i_k)$ with $1 \leq i_p \leq n$ for $1 \leq p \leq k$, the exterior derivative of a k-form

$$\omega = f_I dx^I = f_{i_1,i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \dots dx^{i_k}$$

over $\mathbb{R}^n$ is defined as

$$d\omega = \sum_{i=1}^{n} \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I$$

For general k-forms $\omega = \sum_I f_I dx_I$ (where the components of the multi-index I run over all the values in $\{1, \dots, n\}$).

Indeed, if $\omega = f_I\, dx_{i1} \wedge \dots \wedge dx_{ik}$, then $d\omega = d(f_I dx^{i_1} \wedge \dots \wedge dx^{i_k})$

$$= df_I \wedge \left(dx^{i_1} \wedge \dots \wedge dx^{i_k}\right) + f_I d(dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

$$= df_I \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} + \sum_{p=1}^{k} (-1)^{(p-1)f_I} dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}} \wedge d^2 x^{i_p} \wedge dx^{i_{p+1}} \dots \wedge dx^{i_k}$$

$$= df_I \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{i=1}^{n} \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Definition 4

Suppose M is an m-dimensional smooth manifold. Then there exists a unique map $d: A(M) \to A(M)$ such that $d(A^r(M)) \subset A^{r+1}(M)$ and such that d satisfies the following:

1) For any ω_1, $\omega_2 \in A(M)$, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

2) Suppose ω_1 is an exterior differential r-form. Then

$$d(\omega_1 \Lambda\, \omega_2) = d\omega_1 \Lambda\, \omega_2 + (-1)^r \omega_1 \Lambda\, d\omega_2.$$

3) If f is a smooth function on M, i.e., $f \in A^0(M)$, then df is precisely the differential of f.

4) If $f \in A^0(M)$, then $d(df) = 0$.

The map d defined above is called the exterior derivative.

Theorem 3

Let f be differentiable map, ϕ, ψ are 1-Form, then

i) $d(f\phi) = df \wedge \phi + fd\phi$

ii) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

Proof:

i) Let $\phi = \sum_{i=1}^{n} f_i \, dx_i$ then $f\phi = \sum_{i=1}^{n} f f_i \, dx_i$, so

$$d(f\phi) = d\left(\sum_{i=1}^{n} f f_i \, dx_i\right) = \sum_{i=1}^{n} d(f f_i) \, dx_i$$

$$= \sum_{i=1}^{n} (d(f) f_i \wedge dx_i + f d(f_i) \wedge dx_i)$$

$$= \sum_{i=1}^{n} d(f) f_i \wedge dx_i + \sum_{i=1}^{n} f d(f_i) \wedge dx_i$$

$$= df \wedge \phi + fd\phi$$

ii) Let $\phi = \sum_{i=1}^{n} f_i dx_i$, $\psi = \sum_{i=1}^{n} g_i \, dx_i$, then

$\phi \wedge \psi = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i \, g_j dx_i dx_j$ so it is sufficient to prove the statement for

$\phi = f dx_i$ and $\psi = g dx_j$and then extend linearly, so

$$d(\phi \wedge \psi) = d\big(f dx_i \wedge g dx_j\big) = d(f g dx_i dx_j) = d(fg) \wedge dx_i dx_j$$

$$= gd(f) dx_i dx_j + f d(g) dx_i dx_j$$

$$= d(f) dx_i \wedge g dx_j - f dx_i \wedge d(g) dx_j$$

$$= d(\phi) \wedge \psi) - \phi \wedge d\psi$$

Theorem 4 (Poincare's Lemma)

$d^2 = 0$, i.e., for any exterior differential formω, $d(d\omega) = 0$

Proof

Since d is a linear operator, we only prove the lemma when ω is a monomial. By the local properties of d, it is sufficient to assume that $\omega = a \wedge du^1 \wedge \ldots . \wedge du^r$

Hence $$d\omega = da \wedge du^1 \wedge \ldots . . \wedge du^r$$

Differentiating one more time and applying conditions (2) and (4), we have

$$d(d\omega) = d(da) \wedge du^1 \wedge \ldots \wedge du^r$$

$$-da \wedge \mathrm{d}(du^1) \wedge \ldots \ldots \wedge du^r$$

$$+ \ldots = 0$$

This complete the proof of the theorem □

Theorem 5

Suppose ω is a differential 1-form on a smooth manifold M. X and Y are smooth tangent vector fields on M. Then

$$< X \wedge Y , d\omega > = X < Y , \omega > - Y < X , \omega > - < [X , Y] , \omega >$$

Proof:

Given,

$$< X \wedge Y , d\omega > = X < Y , \omega > - Y < X , \omega > - < [X , Y] , \omega > \quad (3.2)$$

Since both sides of (3.2) are linear with respect to ω , we may assume that ω is a monomial

$\omega = g\, df$; where f and g are smooth functions on M

$$\Rightarrow d\omega = dg \wedge df$$

L.H.S: $< X \wedge Y ,\ d\omega >$

$$= < X \wedge Y , \quad dg \wedge df >$$

$$= \begin{vmatrix} < X , dg > & < X , df > \\ < Y , dg > & < Y , df > \end{vmatrix}$$

$$= \begin{vmatrix} Xg & Xf \\ Yg & Yf \end{vmatrix}$$

$$= \; Xg.Yf - Xf.Yg$$

R.H.S: $X < Y, \omega > -Y < X, \omega > -< [X, Y], \omega >$

$$= X < Y, gdf > -Y < X, gdf > -< [X, Y], g\, df >$$

$$= X(g\, Yf) - \; Y(g\, Xf) \; - \; g[X, Y] f$$

$$= Xg.Yf \; + \; g\, XYf - Yg.Xf - g\, YXf - g\, XYf \; + g\, YXf$$

$$= Xg.Yf - Xf.Yg$$

Therefore, L.H.S = R.H.S

This complete the proof of the theorem □

Theorem 6

Suppose $f: M \to N$ is a smooth map from a smooth manifold M to a smooth manifold N. Then the induced map $f^*: A(N) \to A(M)$ commutes with the exterior derivative d, that is,

$$f^* o\; d = d\; o\; f^*: A(N) \to A(M) \qquad (3.3).$$

Proof

Since both f^* and d are linear, we need only consider the operation of both sides of (3.3) on a monomial β.

First suppose β is a smooth function on N i.e., $\beta \in A^0(N)$. Choose any smooth tangent vector field X on M. Then

$$< X, f^*(d\beta) > \; = \; < f_* X, d\beta >$$

$$= \; f_* X(\beta)$$

$$= X(\beta \, o \, f)$$

$$= < X\, , d(f^*\beta) >.$$

Therefore $$f^*(d\beta) = d(f^*\beta).$$

Next suppose $\beta = u\, dv$, where u, v are smooth functions on N.

Then $$f^*(d\beta) = f^*(du \,\Lambda\, dv)$$

$$= f^*du \,\Lambda\, f^*dv$$

$$= d(f^*u) \,\Lambda\, d(f^*v)$$

$$= d(f^*\beta).$$

Now assume that (1) holds for exterior differential forms of degree $< r$. We need to show that it also holds for exterior differential r-forms. Suppose

$$\beta = \beta_1 \,\Lambda\, \beta_2,$$

where β_1 is a differential 1-form on N and β_2 is an exterior differential $(r-1)$ form on N. Then by the induction hypothesis we have

$$d \, o \, f^*(\beta_1 \,\Lambda\, \beta_2) = d(f^*\beta_1 \,\Lambda\, f^*\beta_2)$$

$$= d(f^*\beta_1) \,\Lambda\, f^*\beta_2 - f^*\beta_1 \,\Lambda\, d(f^*\beta_2)$$

$$= f^*(d\beta_1 \,\Lambda\, \beta_2) - f^*(\beta_1 \,\Lambda\, d\beta_2)$$

$$= f^* o \, d(\beta_1 \,\Lambda\, \beta_2).$$

This complete the proof of the theorem □

Example 2

Consider $\sigma = u\, dx^1 \wedge dx^2$ over a 1-form basis $dx^1, \ldots, dx^n$. The exterior derivative is:

$$d\sigma = d(u) \wedge dx^1 \wedge dx^2$$

$$= \left(\sum_{i=1}^{n} \frac{\partial u}{\partial x^i} dx^i \right) \wedge dx^1 \wedge dx^2$$

$$= \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x^i} dx^i \wedge dx^1 \wedge dx^2 \right)$$

The last formula follows easily from the properties of the wedge product. Namely $dx^i \wedge dx^i = 0$.

Example 3

For a 1-form $\sigma = u\, dx + v\, dy$ defined over $\mathbb{R}^2$. We have, by applying the above formula to each term (consider $x^1 = x$ and $x^2 = y$) the following sum,

$$d\sigma = \left(\sum_{i=1}^{2} \frac{\partial u}{\partial x^i} dx^i \wedge dx \right) + \left(\sum_{i=1}^{2} \frac{\partial v}{\partial x^i} dx^i \wedge dy \right)$$

$$= \left(\frac{\partial u}{\partial x} dx \wedge dx + \frac{\partial u}{\partial y} dy \wedge dx \right) + \left(\frac{\partial v}{\partial x} dx \wedge dy + \frac{\partial v}{\partial y} dy \wedge dy \right)$$

$$= 0 - \frac{\partial u}{\partial y} dy \wedge dx + \frac{\partial v}{\partial x} dx \wedge dy + 0$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx \wedge dy.$$

Example 4

Suppose the Cartesian coordinates in $\mathbb{R}^3$ are given by (x, y, z).

1) If f is a smooth function on $\mathbb{R}^3$, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The vector formed by its coefficients $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ is the gradient of f, denoted by $grad\, f$.

2) Suppose $a = Adx + Bdy + Cdz$, where A, B, C are smooth functions on $\mathbb{R}^3$. Then

$$da = dA \wedge dx + dB \wedge dy + dC \wedge dz$$
$$= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx +$$
$\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy.$

Let X be the vector (A, B, C), then the vector

$$\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \quad \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

formed by the coefficients of da is just the $curl$ of the vector field X, denoted by $curl\, X$.

3) Suppose $a = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$. Then

$$da = \left(\frac{\partial A}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= div\, X\, dx \wedge dy \wedge dz$$

where $div\, X$ means the divergence of the vector field $X = (A, B, C)$.

From theorems, two fundamental formulas in a vector calculus follow immediately. Suppose f is a smooth function on $\mathbb{R}^3$ and X is a smooth tangent vector field on $\mathbb{R}^3$. Then

$$\begin{cases} curl\,(\,grad\, f\,) = 0, \\ div\,(\,curl\, X\,) = 0. \end{cases}$$

Chapter Five

Existence of Kodaira Moduli Spaces

5.1 Introduction

Let X be a compact Sub manifold of a complex manifold Y with normal bundle $N_{X/Y}$, such that $\mathrm{H}^1(\mathrm{X}, N_{X/Y}) = 0$. In 1962 kodaira [4] proved that, such a sub manifolds X_t of Y with the moduli space M being dim M=$h^0(X, N_{X/Y})$is dimensional complex manifold. Moreover, there is a canonical isomorphism $k_t : T_tM \to H^0(X_t, N_t)$ which associates a global Section of the normal bundle N_t of the associated submanifold $X_t \hookrightarrow Y$, called the Kodaira map. The manifold (parameter space) M is called the Kodaira Moduli space. In Kodaira [5], showed that if X be a compact complex Legendre submanifold of a complex contact manifold Y with contact Line bundle L then $H^1(X, L_x) = 0$, where we see that L_x is the restriction of L on X,then X *is a* stable Legendre submanifold of *Y*. This generalizes the result in Kodaira's stable manifold. Again if $H^1(X, S_x) = 0$, where S_x is the restriction of *S* on *X*, then *X* is a stable isotropic submanifold of *Y*. This generalizes the result in Legendre and kodaira's stable submanifold.

5.1.1 Existence of Kodaira Moduli Spaces

Let *Y* and M be complex manifolds and let $\pi_1: Y \times M \to Y$ and $\pi_2: Y \times M \to Y$ be two natural projections. An analytic family of compact sub manifolds of the complex manifold *Y* with the moduli space *M* is a complex sub manifolds $F \to Y \times M$ such that the restriction of the projection $\tilde{\pi}_2$ on *F* is a proper regular map (regularity means that the rank of the differential of $v \equiv \pi_2 \mid_F: F \to M$ is equal to dim M at every point). Thus, the family *F* has double fibration structure.

$$Y \overset{u}{\leftarrow} F \overset{v}{\rightarrow} M$$

where $\mu = \pi_1 \mid_F$. For each $t \in M$ we say that the compact complex submanifolds, such that $Xt := \mu \circ V^{-1}(t) \to Y$ belong to the family. If $F \to Y \times M$ is an analytic family of compact submanifolds, then for any $t \in M$, there is a natural linear map,

$$K_t : T_tM \to H^0(X_t, N_{Xt|Y}),$$

From the tangent space at t to the vector space of global holomorphic sections of the normal bundle $N_{X_t|Y} = TY|_{X_t} / TX_t$ to the submanifold $X_t \hookrightarrow Y$.

An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds is called complete if the Kodaira map k_t is an isomorphism at each point t in the moduli space M. This moduli space M is called Kodaira moduli space. It is called maximal if for any other analytic family $\tilde{F} \hookrightarrow Y \times \widetilde{M}$ of compact complex submanifolds such that $\mu \circ V^{-1}\ (t) = \tilde{\mu} \circ \tilde{V}^{-1}(\tilde{t})$ for some points $t \in M$ and $\tilde{t} \in \widetilde{M}$, there is a neighborhood $\tilde{U} \subset \widetilde{M}$ of the point$\tilde{t}_0$ and a holomorphic map $f: \tilde{U} \to M$ such that $\tilde{\mu} \circ \tilde{V}^{-1}(\tilde{t}') = \mu \circ V^{-1}(f(\tilde{t}'))$ for every $\tilde{t}' \in \tilde{U}$.

Theorem 1: [4] If $X \hookrightarrow Y$ is a compact complex submainfold in complex manifold Y with normal bundle $N_{X|Y}$ such that $H^1(X, N_{X|Y}) = 0$, then X belongs to the complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex submanifolds with the moduli space M being a $h^0(X, N_{X|Y})$ dimensional complex manifold.

5.1.2. Complex Contact Manifolds

Definition 1. A complex contact manifold is a pair (Y, D) consisting of a (2n+1)-dimensional complex manifold Y and a rank 2n-holomorphic sub bundle $D \subset \mathcal{T}Y$ of the holomorphic tangent bundle to y such that the Frobenious form

$$\phi: D \times D \to \mathcal{T}Y/D$$

$$(v, w) \to [v, w] \bmod D$$

is non degenerate. Define the contact line bundle $L := \mathcal{T}Y/D$, on Y by the exact sequence

$$0 \to D^{2n} \to \mathcal{T}Y^{2n+1} \overset{\theta}{\to} L \to \theta,$$

where θ is the tautological projection and D=ker θ.But we may also think of θ(in a trivialisation of L) as a line bundle-valued 1-form $\theta \epsilon H^0(Y, \Omega^1 Y \otimes L)$, and so attempt to form its exterior derivative $d\theta$.we can easily verify that the maximal non-degeneracy of the distribution D is equivalent to the above defined "twisted" 1-form satisfies the condition

$$\theta \wedge (d\theta)^n \neq 0.$$

Definition 2. A compact complex p-dimensional sub manifold$X^p \hookrightarrow Y^{2n+1}$ of a complex contact manifold Y^{2n+1} is called isotropic if

$$\mathcal{T}X \subset D \mid x.$$

An isotropic sub manifold of maximal possible dimensions n is called a Legendre submanifold. The normal bundle N_{XlY} of any Legendre submanifold X $\hookrightarrow$Y isomorphic to $J^1 L_x$ [8] where L_X =L/$_X$, and therefore fits into the exact sequence

$$0 \rightarrow \Omega^1 X \otimes L_x \rightarrow N_{X|_Y} \overset{pr}{\rightarrow} L_x \rightarrow 0.$$

5.2 Existence of Legendre Moduli Spaces

Let *Y* be a complex contact manifold. An analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact submanifolds of *Y* is called an analytic family of compact Legendre submanifolds if, for any point $\in M$, the corresponding subset $X_t := \mu \circ V^{1}(t) \rightarrow Y$ is a Legendre submanifolds. The parameter space *M* is called a Legendre moduli space. Let $F \hookrightarrow Y \times M$ be a family of compact Legendre sub manifolds. If $F \hookrightarrow Y \times M$ is an analytic family of compact Legendre sub manifolds, it is also an analytic family of complex sub manifolds in the sense of Kodaira and thus, for each $t \in M$, there is a linear map

$$K_t : T_t M \rightarrow H^0 (X_t , N_{Xt|Y}).$$

Definition 1

The analytic family $F \rightarrow Y \times M$ of compact Legendre sub manifolds is complete at a point $t \in M$ if the composition

$$s_t : T_t M \overset{kt}{\rightarrow} H^0 (X_t, N_{Xt|Y}) \overset{pr}{\rightarrow} H^0 (X_t, L_{Xt})$$

provides an isomorphism between the tangent space to M at the point t and the vector space of global sections of the contact line bundle over X_t. The analytic family $F \to Y \ x \ M$ is called complete if it is complete at each point of the moduli apace M.

Lemma 1

If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre sub manifolds is complete at a point $t_0 \in M$, then there is an open neighborhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.

Definition 2:

An analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre submanifolds is maximal at appoint $t_0 \in M$, if for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ of compact complex Legendre submanifolds such that $\mu^0 \ V^{-1}(t_0) = \tilde{\mu}^0 \tilde{V}^{-1}(\tilde{t}_0)$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists neighborhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f: \tilde{U} \to M$ such that $f(\tilde{t}_0) = \tilde{t}_0$ and $\tilde{\mu} \ o \ \tilde{V}^{-1}(\tilde{t}_0) = \mu \ o \ V^{-1} (f(\tilde{t}_0))$ for each $t \in \tilde{U}$. The family $F \hookrightarrow Y \times M$ is called maximal if it is maximal at each point t in the moduli space M.

Lemma 2:

If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre sub manifolds is complete at a point $t_0 \in M$, then it is maximal at the point .The map $s_t : T_t M \to H^0 \ (X_t, L_X)$studies by Lemma 1 and Lemma 2 will play a fundamental role, our study of the reach geometric structure induced canonically on moduli spaces of complete and maximal analytic families of compact Legendre sub manifolds described by the following theorem.

Theorem 1:

Let X be a compact complex Legendre sub manifolds of a complex contact manifold Y with contact line bundle L. If $H^1(X, L_x) = 0$ then, there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre sub manifolds containing X with Legendre moduli space M, is a $H^0 \ (X, L_x)$ dimensional complex manifold.

This theorem is proved by working in local coordinates adapted to the contact structure and expanding the defining functions of nearby compact Legendre submanifolds in terms of local coordinates on the moduli space *M*. This is much in the spirit of the original proof of Kodaira's theorem of the existence, completeness and maximality of compact submanifolds of complex manifolds. The essential difference from the Kodaira case is that the infinite sequence of obstructions to agreements on overlaps of formal power series is situated now in $H^1(X, L_x)$ rather than in $H^1(X, N_{X}|_{Y})$.

5.3 Existence of Isotropic Moduli Spaces

Let *Y* be a complex contact manifold. An analytic family F↪Y ×M of compact submanifold of the complex manifold Y is called an analytic family of isotropic submanifolds, if for any $t \in M$,the corresponding subset X_t=μ $o\ v^{-1}(t) \hookrightarrow Y$ is an isotropic sub manifold. We will use the symbol$\{X_t \hookrightarrow Y \mid t \in M\}$ to denote an analytic family of isotropic sub manifold, the parameter Space M is called isotropic moduli Space. Let $X = X_{t_0}$ for some $t_0 \in M$. If $X^p \hookrightarrow y^{2n+1}$ is an isotropic sub manifold, then each point in *X* has a neighborhood U in Y such that the contact structure in a suitable trivialization of L over *U* , is

$$\theta = d\omega^0 + \sum_{\bar{a}=p+1}^{n} \omega^{\tilde{a}}\, d\omega^{\bar{\bar{a}}} + \sum_{a=1}^{p} \omega^a\, dz^a$$

and *X* in *U* is given by

$$\omega^0 = \omega^a = \omega^{\bar{a}} = \omega^{\bar{\bar{a}}}=0.$$

There exists an adopted coordinate covering $\{U_i\}$ of a tubular neighborhood of inside Y.As a consequence one can always choose local coordinator functions $(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, z_i^a)$,in U_i where $\bar{a}, \bar{\bar{a}} = 1, \ldots, p$ such that the contact structure in U_i is represented by

$$\theta_i = dw_i^0 + \sum_{a=1}^{p} \underbrace{\omega_i^{\bar{a}} \omega_i^{\bar{a}}}_{(n-p)-terms} + \sum_{a=1}^{p} \underbrace{\omega_i^{\bar{a}} dz_i^a}_{p-terms}$$

With $U_i \cap X$ given by $\omega_i^0 = \omega_i^a = \omega_i^{\bar{a}} = \omega_i^{\bar{\bar{a}}} = 0$,

And

$$\theta_i|_{U_i \cap U_j} = A_{ij} \theta_j|_{U_i \cap U_j} \tag{1}$$

For some nowhere vanishing holomorphic functions $A_{ij}(\omega_i \omega_j)$.They satisfy the condition

$$A_{ik} = A_{ij} A_{jk}$$

On every triple intersection $U_i \cap U_j$, the coordinates $\omega_i^A := (\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}})$ and Z_i^a are holomorphic functions $\omega_j^B := (\omega_j^0, \omega_j^b, \omega_j^{\bar{b}}, \omega_j^{\bar{\bar{b}}})$ and Z_j^b,

$$\begin{cases} \omega_i^0 = f_{ij}^0\left(\omega_j^B, z_j^b\right) \\ \omega_i^a = f_{ij}^a\left(\omega_j^B, z_j^b\right) \\ \omega_i^{\bar{a}} = f_{ij}^{\bar{a}}\left(\omega_j^B, z_j^b\right) \\ \omega_i^{\bar{\bar{a}}} = f_{ij}^{\bar{\bar{a}}}\left(\omega_j^B, z_j^b\right) \\ z_i^a = g_{ij}^a(\omega_j^B, z_j^b) \end{cases} \Leftrightarrow \begin{cases} \omega_i^A = f_{if}^A(\omega_j^B, z_j^b) \\ z_i^a = g_{ij}^a(\omega_j^B, z_j^b) \end{cases}$$

With $f_{ij}^A\left(0, z_j^b\right) = 0$,where A=0,a,$\bar{a}, \bar{\bar{a}}$

Equation (1) puts the following constrains on gluing functions.

$$A_{ij} = \frac{\partial f_{ij}^0}{\partial \omega_j^o} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^o} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^o} \tag{2}$$

$$O = \frac{\partial f_{ij}^0}{\partial \omega_j^a} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^a} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^a} \tag{3}$$

$$O = \frac{\partial f_{ij}^0}{\partial \omega_j^{\bar{a}}} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^{\bar{a}}} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^{\bar{a}}} \tag{4}$$

$$A_{ij}\omega_j^{\bar{a}} = \frac{\partial f_{ij}^0}{\partial \omega_j^{\bar{\bar{a}}}} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^{\bar{\bar{a}}}} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^{\bar{\bar{a}}}} \tag{5}$$

$$A_{ij}\omega_j^{a} = \frac{\partial f_{ij}^0}{\partial \omega_j^{a}} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial Z_j^{a}} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial Z_j^{a}} \tag{6}$$

Which express the fact that the chosen coordinate charts μ_i are glutted by the contactomorphisms. For any point t in a sufficient small co-0rdinate neighborhood $M_0 \subset M$ of t_0 with coordinate functionst^α, α = 1,…, m = dim M , the associated isotropic sub manifold $X_t = \mu \, o \, v^{-1}(t)$

is expressed in the domain U_i by equations of the form [5]

$$\omega_i^A = \emptyset_i^A \left(Z_i^a, t^\alpha \right), \ \alpha = 0, \mathrm{a}, \bar{a}, \bar{\bar{a}}$$

Definition 1: The bundle S_x is defined to be the kernel of the canonical projection

$$p: N_{X}|_{Y} \to J^1 L_x.$$

i.e, it is defined by the exact sequence

$$0 \to S_x \to N_{X}|_{Y} \to J^1 L_x \to 0.$$

Lemma 1: X_t is isotropic if and only

$$\phi_i^a(Z_i, \text{t}) = - \frac{\partial\phi_i^o(Z_i, t)}{\partial Z_i^a} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\bar{a}}\ (z_i\,, \text{t})\, \frac{\partial\phi_i^{\bar{\bar{b}}}(Z_i\,, t)}{\partial z_i^a}$$

Proof: Let $X^p \hookrightarrow Y^{2n+1}$be an isotropic submanifold in complex contact manifold Y. An arbitrary X_t , deformation of X inside Y, is given by

$$\begin{cases} \omega_i^0 = \phi_i^o\ (z_i\,, \text{t}) \\ \omega_i^a = \phi_i^a\ (z_i\,, \text{t}) \\ \omega_i^{\bar{a}} = \phi_i^{\bar{a}}\ (z_i\,, \text{t}) \\ \omega_i^{\bar{\bar{a}}} = \phi_t^{\bar{\bar{a}}}\ (z_i\,, \text{t}) \end{cases}$$

$$\Rightarrow \omega_i^A = \phi_i^A\ (\text{z}_i\,, \text{t})$$

Then, $\left\{\frac{\partial\phi_i^A}{\partial t}\,|o\right\}$ is a global section of $N_{X|Y}$. X_t is isotropic if and only if $\theta_t = d\omega_t^o + \omega_i^{\bar{a}} d\omega_i^{\bar{\bar{a}}} + \omega_t^a dz_i^a$ vanishes on X_t. Then

$$\text{O} = \theta_i|x_t = d\phi_i^o\ (z_{i,}\ t) + \phi_i^{\bar{a}}(z_{i,}\ t)\ d\ \phi_i^{\bar{\bar{a}}}(z_{i,}\ t) + \phi_i^a(z_{i,}\ t) dz_i^a$$

$$= \frac{\partial\phi_i^o(z_{i,}\,t)}{\partial z_i^a}\partial z_i^a + \phi_i^{\bar{a}}\big(z_{i,}\ t\big)\frac{\partial\phi_i^{\bar{\bar{a}}}}{\partial z_i^b}\ dz_i^b + \phi_i^a(z_{i,}\ t)\ dz_i^a$$

$$= [\phi_i^a(z_{i,}\ t) + \frac{\partial\phi_i^o(z_{i,}\,t)}{\partial z_i^a} + \sum_{\bar{b}=p+1}^{n} \phi_t^{\bar{b}}(z_{i,}\ t)\frac{\partial\phi_i^{\bar{\bar{b}}}(z_{i,}\,t)}{\partial z_i^a}]\ \partial z_i^a$$

Thus we obtain

$$\phi_i^a(z_{i,}\ t) = - \frac{\partial\phi_i^o(z_{i,}\,t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \phi_t^{\bar{b}}(z_{i,}\ t)\frac{\partial\phi_i^{\bar{\bar{b}}}(z_{i,}\,t)}{\partial z_i^a} \qquad (7)$$

Where $\phi_i^A\big(z_{i,}\ t\big)$ is a holomorphic function of z_i^a and t, which satisfy the boundary condition $\phi_i^A\big(z_{i,}\ t\big) = 0$ for t= t_o . Hence the proof is completed.

Theorem 2: If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre sub manifolds is complete at a point $t_0 \in M$, then there is an open neighborhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.

Proof: For any point t in a sufficient small coordinate neighborhood $M_0 \subset M$ of t_0

With coordinate functiont^{α}, α = 1,…, m = dim M , the associated sub manifold X_t is expressible in the domain U_t by the equations of the for

$\omega_j^A = \emptyset_i^A (z_i^{\alpha}, t^{\alpha}),$

Where, $\emptyset_i^A (z_i^a, t) = -\frac{\partial \emptyset_i^0 (z_i^a, t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \emptyset_i^{\bar{b}} (z_i^a, t) \frac{\partial \emptyset_i^{\bar{b}} (z_i^a, t)}{\partial z_i^a}$. (by lemma 1)

Let $F_0 := v^{-1} (M_0)$ be covered by coordinate neighbourhood W_i with coordinate functions $(z_i^{\alpha}, t^{\alpha})$.Take any vector field v on M_0 , and apply the corresponding 1 order differential operator v = $\sum_{\alpha=1}^{n} v^{\alpha} \partial_{\alpha}$to each function $\emptyset_i^A (z_i, t)$, then the result is a collection of vector valued holomorphic functions

$$\sigma_i^A = \sum_{\alpha=1}^{m} v^{\alpha} \frac{\partial \emptyset_i^A (z_i, t)}{\partial t^{\alpha}}$$

Defined respectively on W_i on the intersection $W_i \cap W_j$ they are related to each other by the rule

$$\sigma_i^A = \sum_{B=0}^{n} F_{ijB}^A \sigma_j^B ,$$

Where the gluing functions are

$F_{ijB}^A = \frac{\partial f_{ij}^A}{\partial W_j^B} / W_j = \emptyset_i^A (z_j, t) - \sum_{\alpha=1}^{n} \frac{\partial \emptyset_i^A}{\partial z_i^a} |z_i^a = g_{ij\,(\emptyset_i^A,\ z_j)}^a \frac{\partial g_{ij}^B}{\partial W_j^B} / W_j = \emptyset_j (z_j, t)$

And thus, represent a global section of the normal bundle N_F over F_0. Therefore the kodaira

map K_t is given by

$$K_t : T_t M \rightarrow H^0 (X_t , N_{X_t|Y}).$$

$$\sum_{\alpha=1}^{n} v^{\alpha}\partial_{\alpha} \mapsto \{v^{\alpha}\partial_{\alpha}\ \emptyset_i^A(z_{i\,,}t)\}.$$

To prove the completeness, we have to show that the sequence

$$0 \to H^0(X_t, S_{x_t}) \to K_t(T_t M) \to H^0(X_t, L_{x_t}) \to 0,$$

is exact at points t sufficiently close to t_0.

Note that $\{ K_t(\frac{\partial}{\partial t^{\alpha}}) \}$ are sections of L_{x_t} at X_t while $\{ K_t(\frac{\partial}{\partial t^{\alpha'}}) \}$ are sections of S_{x_t} at X_t

With $\alpha = 1,\ldots, m = \dim H^0(X_t, L_{x_t})$ and $\alpha' = 1,\ldots,\dim H^0(X_t, S_{x_t})$.

We know that these form a basis at $t=t_0$. Hence they must form a basis for all t with $|t - t_0| < \in_1$,

Where $\in_1 \in R^+$ is a sufficiently small positive number. Thus

$$\dim \operatorname{Im} K_t(T_t M) \geq \dim(H^0(X_{t0}, L_{x_{t0}})) + \dim(H^0(X_{t0}, S_{x_{t0}})),$$

for all $|t - t_0| < \in_1$. On the other hand , the upper semi –continuity principle [3] implies

$$\dim \operatorname{Im} K_t(T_t M) \leq \dim(H^0(X_{t0}, L_{x_{t0}})) + \dim(H^0(X_{t0}, S_{x_{t0}})),$$

for all $|t - t_0| < \in_2$. for some $\in_2 \in R^+$, therefore if the exact sequence

$$0 \to H^0(X_t, S_{x_t}) \to K_t(T_t M) \to H^0(X_t, L_{x_t}) \to 0,$$

Is exact at t_0 , then it is also exact for all $|t - t_0| < \min(\in_1, \in_2)$

This completes the proof.

5.4 Existence Theorem of Isotropic Submanifolds

If X $\hookrightarrow$ Y is compact isotropic submanifold in a complex contact manifold Y. Then its normal bundle $N_{X|Y}$ fits into an extension. If H^1 (X, L_x) = H^1 (X, S_X) =0, then there exists a complete analytic family $\{X_t \hookrightarrow Y | t \in M\}$ of isotropic submanifold, such that

- X_{t_o} = X for some $t_o \in$ M,
- $them$oduli space M is smooth
- dim M = h^o (X, L_X) + h^o (X, S_X)
- the tangent space T_tM, t ∈ M fits into the extension

$$0 \rightarrow H^o(X_{t,}\ S_{X_{t})\rightarrow}k_{t,}\ (T_{t,}\ \text{M}) \rightarrow H^o(X_{t,}\ X_{t,}\ S_{X_t}) \rightarrow 0$$

Proof: Let $(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, Z_i^a)$ be a coordinate system on Y which is adopted to the isotropic character of the embedding $X \hookrightarrow Y$ by previous discussion . Assume $\{ X_t \rightarrow Y | \, t \in M \}$ is a family of compact complex isotropic submanifold in complex contact manifold. Let a family can be described by $\phi_i^o(z_i, \text{t}), \phi_i^a(z_i, \text{t}), \phi_i^{\bar{a}}(z_i, \text{t}), \phi_i^{\bar{\bar{a}}}(z_i, \text{t})$, which solve the equations in $U_i \cap U_j$.

$$\phi_i^o(z_i, \text{t}) = f_{ij}^0\big(\emptyset_j^0(z_i, \text{t}), \emptyset_j^a(z_i, \text{t}), \emptyset_j^{\bar{a}}(z_i, \text{t}), \emptyset_j^{\bar{\bar{a}}}(z_i, \text{t}), Z_j\big)$$
$$\phi_i^a(z_i, \text{t}) = f_{ij}^0\big(\emptyset_j^0(z_i, \text{t}), \emptyset_j^a(z_i, \text{t}), \emptyset_j^{\bar{a}}(z_i, \text{t}), \emptyset_j^{\bar{\bar{a}}}(z_i, \text{t}), Z_j\big)$$
$$\phi_i^{\bar{a}}(z_i, \text{t}) = f_{ij}^0\big(\emptyset_j^0(z_i, \text{t}), \emptyset_j^a(z_i, \text{t}), \emptyset_j^{\bar{a}}(z_i, \text{t}), \emptyset_j^{\bar{\bar{a}}}(z_i, \text{t}), Z_j\big)$$
$$\phi_i^{\bar{\bar{a}}}(z_i, \text{t}) = f_{ij}^0\big(\emptyset_j^0(z_i, \text{t}), \emptyset_j^a(z_i, \text{t}), \emptyset_j^{\bar{a}}(z_i, \text{t}), \emptyset_j^{\bar{\bar{a}}}(z_i, \text{t}), Z_j\big)$$
$$z_i^a = g_{ij}^a\big(\emptyset_j^0(z_i, \text{t}), \emptyset_j^a(z_i, \text{t}), \emptyset_j^{\bar{a}}(z_i, \text{t}), \emptyset_j^{\bar{\bar{a}}}(z_i, \text{t}), Z_j\big)$$

By the equation which given

$$\phi_i^a(z_{i,}\ t) = - \frac{\partial \phi_i^o(z_{i,} t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \phi_t^{\bar{b}}(z_{i,}\ t) \frac{\partial \phi_i^{\bar{\bar{b}}}(z_{i,} t)}{\partial z_i^a} \qquad (7)$$

With equation (7) We know that $N_{X|Y}$ fits into a diagram of isotropic manifolds. There exists a canonical morphism of sheaves of Abelian groups $\alpha :\ L_x\, j^1\, L_x$ which in our local coordinates is given explicitly by,

$$\{\ \phi_i^o(z_i\,, \mathrm{t})\ \} \ \rightarrow \ \begin{Bmatrix} \phi_i^o(z_i\,,\mathrm{t}) \\ \frac{\partial \phi_i^o(z_{i,}\,t)}{\partial z_i^a} \end{Bmatrix}.$$

Define a sub sheaf of Abelian groups in the sheaves $N_{X|Y}$ as $\widehat{N}_{X|Y} := {p_y}^{-1}\left(\ \alpha\ (L_x\,)\right), where\ \ p_y$: $N_{X|Y} \rightarrow\ j^1\, L_x$ is the canonical epimorphism. By construction $\widehat{N}_{X|Y}$ fits into an exact sequence

$$0 \rightarrow S_x \rightarrow \overline{N}_{X\,|_Y} \rightarrow L_x \rightarrow 0$$

The long exact sequence associated with the above sequence gives

$$0 \rightarrow H^0\ (X,\, S_x) \rightarrow H^0\ (X\,, \overline{N}_{X\,|_Y}) \rightarrow\ H^0(\,X, L_x) \rightarrow H^1(\,X, S_x) \rightarrow \ldots\ldots$$

By assumption $H^1(\,X, S_x) = 0$. Hence, we have an exact sequence of vector spaces,

$$0 \rightarrow H^0\ (X,\, S_x\,) \rightarrow H^0\ (X\,, \overline{N}_{X\,|_Y}) \rightarrow\ H^0(\,X, L_x) \rightarrow 0,$$

Implying

$$\dim H^0\ (X\,, \overline{N}_{X\,|_Y}) =\ dimH^0(X\,, S_x\,) +\ \ dimH^0(X\,, L_x\,) := m$$

Let $\emptyset_\alpha$, $\alpha = 1\,, \ldots\ldots, m$ be a basis of the global sections of $\overline{N}_{X\,|_Y}$. In our coordinate system, each $\emptyset_\alpha$can be represented by a 0-cocycle

$$\emptyset_\alpha = \begin{Bmatrix} \theta_{\alpha\, i}^{o} \\ \frac{\partial \theta_{\alpha i}^{o}}{\partial z_i^a} \\ \theta_{\alpha\, i}^{\bar{a}} \\ \phi_{\alpha\, i}^{\bar{\bar{a}}} \end{Bmatrix} = \{\ \theta_{\alpha\, i}^{A}\ \},\ \ \mathrm{A}{=}0,\mathrm{a}, \bar{a}, \bar{\bar{a}}$$

In $U_i \cap\ U_j$,   we have

$$\theta_{\alpha\, i}^{A}(z) = F_{ijB}^{A}(Z)\theta_{\beta j}^{B}(\mathrm{Z}), \qquad\qquad \mathrm{Z} = (\ 0, z_i) \qquad (8)$$

Where the matrix-valued functions are given by,

$$F_{ijB}^{A} = \begin{bmatrix} A_{ij} & o & & \\ \frac{\partial f_{ij}^{a}}{\partial \omega_j^{o}}|X & \frac{\partial f_{ij}^{a}}{\partial \omega_j^{b}}|X & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \frac{\partial f_{ij}^{\bar{a}}}{\partial \omega_j^{o}}|X & \frac{\partial f_{ij}^{\bar{a}}}{\partial \omega_j^{\bar{b}}}|X & \frac{\partial f_{ij}^{\bar{a}}}{\partial \omega_j^{o}}|X & \frac{\partial f_{ij}^{\bar{a}}}{\partial \omega_j^{o}}|X \\ \frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial \omega_j^{o}}|X & \frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial \omega_j^{\bar{\bar{b}}}}|X & \frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial \omega_j^{o}}|X & \frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial \omega_j^{o}}|X \end{bmatrix}$$

Define,

$$\emptyset_i^{A}(z_i\,, t) = \begin{bmatrix} \emptyset_i^{0}(z_i\,, t) \\ \emptyset_i^{a}(z_i\,, t) \\ \emptyset_i^{\bar{a}}(z_i\,, t) \\ \emptyset_i^{\bar{\bar{a}}}(z_i\,, t) \end{bmatrix}$$

Where equation (1) holds. Let ϵ be a small positive number. In order to prove the theorem,

We have to to find the holomorphic functions $\emptyset_i^{A}(Z\,, t)$, in $z_i = (z_i^1, \dots \dots, z_i^n)$ and in

$t = (t^1, \dots \dots, t^m)$, $|z_i| < 1$,$|t|<\epsilon$ with $|\emptyset_i^{A}(Z\,, t)| < 1$, such that

$$\emptyset_i^{A}(\, g_{ij}^{a}\left(\emptyset_j^{B}(z_{j\,,}t), z_j\right), t\,) \;=\; (\, f_{ij}^{A}\;(\,\emptyset_j^{B}(z_j\,, t\,)\,, z_j) \tag{9}$$

Where A=0, a,$\bar{a}$, $\bar{\bar{a}}$with equation (7) and the boundary conditions

$$\emptyset_i^{A}(Z\,, 0) = 0 \tag{10}$$

And

$$\frac{\partial \phi_i^A(z_{i,} t)}{\partial t^\alpha} |t = 0 = \theta_{\alpha i}^A \text{ (Z)}, \qquad z = (0, \ z_i) \tag{11}$$

are satisfied. If we succeed in solving all these equations for functions $\{\emptyset_i^A(Z\ ,t)\}$ which are holomorphic in t in some neighborhood$U \subset C^q$ of the origin, then the boundary conditions will guarantee that the resulting analytic family $F \rightarrow Y \times U$ is complete at t = 0 and hence by theorem 2 is complete in some neighborhood $M \subseteq U$ of the origin. Therefore, we need to Solve the equations (9) to (11) to prove the theorem. We shall do it in three stages.

Stage 1: (Simplifications of the basic system of equations). Firstly it is sufficient to solve the equation (9), which correspond to A=0,a,$\bar{a}, \bar{\bar{a}}$ that the holomorphic functions$\{\emptyset_i^A(Z\ ,t)\}$,

Satisfy on overlaps $X \cap U_i \cap U_j$. Then, denoting

$$A_b^q\!: = [\sum_{A=0}^n \frac{\partial g_{ij}^a}{\partial w_j^A}\frac{\partial \emptyset_j^A}{\partial z_j^b} + \frac{\partial g_{ij}^a}{\partial z_j^b}]\ |w_j^A = \emptyset_j^A\ (z_j, t)$$

and using equations (2) – (6), we obtain [17]

$$\sum_{a=1}^n \frac{\partial \emptyset_i^o}{\partial z_i^a} A_b^a = [\sum_{a=1}^n \frac{\partial \emptyset_i^a}{\partial z_i^a} (\sum_{A=0}^n \frac{\partial g_{ij}^a}{\partial w_j^A}\frac{\partial \emptyset_j^A}{\partial z_j^b} + \frac{\partial g_{ij}^a}{\partial z_j^b}\ ]\ |w_j^A\!= \emptyset_j^A\ (z_j, t)$$

$$= [\sum_{A=0}^n \frac{\partial f_{ij}^0}{\partial w_j^A}\frac{\partial \emptyset_j^A}{\partial z_j^b} + \frac{\partial f_{ij}^a}{\partial z_j^b}\ ]\ |w_j^A\!= \emptyset_j^A\ (z_j, t)$$

$$= -\sum_c f_{ij}^c A_b^c - \sum_{\bar{c}} f_{ij}^{\bar{c}} \frac{\partial f_{ij}^{\bar{\bar{c}}}}{\partial z_j^a} A_b^a$$

which implies

$$\sum_{a=1}^{n}(\frac{\partial \emptyset_i^0}{\partial z_i^a} + \sum_{\bar{c}} f_{ij}^{\bar{c}} \frac{\partial f_{ij}^{\bar{\bar{c}}}}{\partial z_i^a})\ A_b^a = -\sum_{a=1}^{n} f_{ij}^{c}\ A_b^c \quad (12)$$

Since the Jacobian of the coordinate transformation

$$\det\frac{\partial(w_i^0 w_i^a w_i^{\bar{a}} w_i^{\bar{\bar{a}}} z_i^a)}{\partial(w_j^0 w_j^b w_j^{\bar{b}} w_j^{\bar{\bar{b}}} z_i^b)}|X = \frac{\partial f_{ij}^0}{\partial w_j^0}|x^{det}\ (\frac{\partial f_{ij}^a}{\partial w_j^b})|x^{det}\ (\frac{\partial f_{ij}^{\bar{a}}}{\partial w_j^{\bar{b}}})\ |x^{det}\ (\frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial w_{ij}^{\bar{\bar{b}}}})|x^{det} A_b^a|t = o$$

is nowhere zero on X, the matrix A_b^a is non-degenerate at t= o and hence is non-degenerate for all t in some small neighborhood $U^!$ Of the zero in $₵^m$, Then equation (12) implies

$$(-\frac{\partial \emptyset_i^o}{\partial z_i^a} - \sum_{\bar{a}} f_{ij}^{\bar{a}} \frac{\partial f_{ij}^{\bar{\bar{a}}}}{\partial z_i^a})|z_i = g_{ij}\ (\emptyset_j^B\ (z_j, t), z_j) = f_{ij}^a \big| w_j^A = \emptyset_j^A\ (z_j, t)'$$

i.e., it implies that the equation (9) with A=a is automatically satisfied. Thus we have to solve the equations (9) for A= o, $\bar{a}$, $\bar{\bar{a}}$ with boundary conditions (10) – (11).

Stage 2: (existence of formal Solutions) In what follows we write the power series expansion of an arbitrary- holomorphic function *P(t)* in t^1,…, t^mdefined on a neighbourhood of the origin in the form *P(t)* = $p_0(t) + p_1(t) + \cdots + p_q(t)$+...,where each term $p_q(t)$denotes a homogeneous polynomial of degree q in t^1,…, t^m, and denote it by $p^{|q|}(t) = p_0(t) + p_1(t) + \cdots + p_q(t)$. If *Q(t)*is another holomorphic function in t, we write P(t) $\equiv_q$ *Q(t)* if $p^{|q|}(t) = Q^{|q|}(t)$.

Now we expand each component $\emptyset_i^A(Z, t)$of $\emptyset_i(Z_i, t)$ into a power series

$$\emptyset_i^A(Z_i, t) = \emptyset_{i|1}^A(Z_i, t) + \ldots + \emptyset_{i|q}^A(Z_i, t) + \ldots$$

In t^1 ….., t^m, and write

$$\emptyset_{i|q}^A(Z_i, t) = (\emptyset_{i|q}^1(Z_i, t), \ldots\ \emptyset_{i|q}^A(Z_i, t) \ldots \emptyset_{i|q}^P(Z_i, t),$$

$$\emptyset_i^{A[q]}(Z_i, t) = \emptyset_{i|1}^A(Z_i, t) + \ldots + \emptyset_{i|q}^A(Z_i, t).$$

Then the equality (9) is reduced to the system of congruence's

$$\emptyset_i^{A[q]}(g_{ij}^{a}(\emptyset_j^{B[q]}(Z_i, t)Z_j), t) \equiv_q f_{ij}^{A}(\emptyset_j^{B[q]}(Z_i, t), Z_j), \quad q=1,2,3\ldots.. \tag{13}$$

We note that the congruence $(13)_1$ is equivalent to

$$\emptyset_{i|1}^{A}(Z_i, t) = F_{ijB^{(z)}}^{A}.\emptyset_{j|1}^{B}(Z_i, t), \; z = (O,\ Z_i) = (O,\ Z_j).$$

first we shall construct the polynomials $\emptyset_i^{A[q]}(Z_i, t)$ by induction on q. In view of the boundary Conditions (10) - (11) , we define

$$\emptyset_{i|1}^{A}(Z_i, t) = \sum_{\alpha} \theta_{\alpha i}^{A}(Z) t^{\alpha} ,$$

It is clear by (8) that the linear forms $\emptyset_{i|1}^{A}(Z_i, t), t \in I$, satisfy

Assume that the polynomials $\emptyset_{i|1}^{A}(Z_i, t), t \in I$, satisfying (13)q are already determined for an integer $q \geq 1$. For the sake of simplicity we write , $\emptyset_j^{A[q]}(t) = \emptyset_j^{A[q]}(Z_j, t), f_{ij}^{A}(\omega_j^{B}) = f_{ij}^{A}(\omega_j^{B}, Z_j), f_{kj}^{A}(\omega_j^{B}) = f_{kj}^{A}(\omega_j^{B}, Z_j), g_{ij}^{a}(\omega_j^{B}) = g_{ij}^{a}(\omega_j^{B}) = g_{ij}^{a}(\omega_j^{B}, Z_j)$ and we set

$$\Psi_{ij}^{A}(Z_i, t) =_q (\emptyset_j^{A[q]}(Z_j, t)|_{z_i^a = g_{ij}^a}\left(\emptyset_j^{B[q]}(Z_j, t), Z_j\right) - f_{ij}^{A}(\omega_j^{B}, Z_j)|_{\omega_j^B = \emptyset_j^{B[q]}(Z_j, t)} \tag{14}$$

Where $\Psi_{ij}^{A}(Z_i, t)$is a homogeneous polynomial of degree q+1 in t^1,……….., t^mwhose coefficients are vector valued holomorphic functions of Z_j, $|Z_j| < 1$, $|g_{ij}(0, Z_j)| < 1$, and that

$$\Psi_{ij}^{A}(Z_i, t) \equiv_{q+1} \emptyset_i^{A[q]}\left(g_{ij}\left(\emptyset_j^{B[q]}(t)\right), t\right) - f_{ij}^{A}\left(\emptyset_j^{B[q]}(t)\right) \tag{15}$$

We define

$$\Psi_{ij}^{A}(Z_i, t) = \Psi_{ij}^{A}(Z_i, t), for\ z = (0, Z_j) \in U_i \cap U_j .$$

We have the equality [3]

$$\Psi_{ij}^{A}(Z_i, t) = \Psi_{ij}^{A}(Z_i, t) + F_{iKB}^{A}(z) . \Psi_{Kj}^{B}(Z_j, t), \qquad for\ z \in U_i \cap U_j \cap U_k \tag{16}$$

Now we have to prove that the 1- cocycle $\{\Psi^{A}_{ij}(Z_i , t)\}$ takes values in $\bar{N}_{X}|_{Y}$ rather than in $N_{X}|_{Y}$.

We get by definition

$$\Psi^{0}_{ij}(Z_i , t) =_{q+1} \emptyset^{0[q]}_{j}(Z_j , t)|_{z^{a}_{i} = g_{ij}\left(\emptyset^{B[q]}_{j}(Z_i ,t), Z_j\right)} - f^{0}_{ij}(\omega^{B}_{j}, Z_j)|_{\omega^{B}_{j} = \emptyset^{B[q]}_{j}(Z_j ,t)} \quad (17)$$

and $$\Psi^{a}_{ij}(Z_i , t) =_{q+1} \emptyset^{a[q]}_{j}(Z_j , t)|_{z^{a}_{i} = g_{ij}\left(\emptyset^{B[q]}_{j}(Z_i ,t), Z_j\right)} - f^{a}_{ij}(\omega^{B}_{j}, Z_j)|_{\omega^{B}_{j} = \emptyset^{B[q]}_{j}(Z_j ,t)} \quad (18)$$

then $\{\Psi^{a}_{ij}(Z_i , t)\}$ represents a cohomology class in $H^{1}(X , N_{X}|_{Y})$ if and only if

$$\Psi^{a}_{ij}(Z_i , t) = \frac{\partial \Psi^{0}_{ij}(Z_i ,t)}{\partial z^{b}_{j}} (A^{-1})^{b}_{a}$$

or

$$\frac{\partial \Psi^{0}_{ij}(Z_i , t)}{\partial z^{b}_{j}} = -\sum_{\alpha} \Psi^{a}_{ij}(Z_j , t) A^{a}_{b}$$

To prove this, differentiate (17) with respect to Z^{b}_{j} and using equation (2) – (6) and (18) with Lemma 1, we obtain [17]

$$\frac{\partial \Psi^{0}_{ij}}{\partial z^{b}_{j}} = \frac{\partial \emptyset^{0[q]}_{ij}(Z_i ,t)}{\partial z^{a}_{j}} |_{z^{a}_{i} = g_{ij}\left(\emptyset^{B[q]}_{j}(Z_i ,t), Z_j\right)} \left(\frac{\partial g^{a}_{ij}}{\partial w^{A}_{j}} \frac{\partial \emptyset^{A}_{ij}}{\partial z^{b}_{j}} + \frac{\partial g^{a}_{ij}}{\partial z^{b}_{j}}\right)$$

$$- \frac{\partial f^{0}_{ij}}{\partial \omega^{B}_{j}} |_{\omega^{B}_{j} =} |_{\omega^{B}_{j} = \left(\emptyset^{B[q]}_{j}(Z_i ,t),\right)} \frac{\partial \emptyset^{B[q]}_{j}}{\partial z^{b}_{j}} - \frac{\partial f^{0}_{ij}}{\partial z^{b}_{j}} |_{\omega^{B}_{j} = \left(\emptyset^{B[q]}_{j}(Z_i ,t),\right)}$$

$$= - \emptyset^{a[q]}_{i} A^{a}_{b} - \sum_{\bar{b}} \emptyset^{\bar{b}[q]}_{i} \frac{\partial \emptyset^{\bar{b}[q]}_{i}}{\partial z^{a}_{j}} A^{a}_{b} + \sum_{a} f^{a}_{ij} |_{\omega^{B}_{j} = \left(\emptyset^{B[q]}_{j}(Z_i ,t),\right)} A^{a}_{b} + \sum_{\bar{b}} \emptyset^{\bar{b}[q]}_{i} \frac{\partial \emptyset^{\bar{b}[q]}_{i}}{\partial z^{a}_{i}} A^{a}_{b}$$

$$= - \sum_{a} \Psi^{a}_{ij}(Z_j , t) A^{a}_{b}$$

Hence

$$\frac{\partial \Psi_{ij}^{0}\,(Z_i\,,t)}{\partial\, z_j^{b}} = -\sum_{\alpha} \Psi_{ij}^{a}\,\big(Z_j\,,t\big)A_b^{a}$$

From the exact sequence

$$0 \to S_x \to \overline{N}_{X}|_{Y} \to L_x \to 0$$

It follows that

$$0 \to H^0\,(X, S_X) \to H^0\,(X, \overline{N}_{X}|_{Y}) \to H^0(\,X, L_x) \to H^1(\,X, S_x) \to \ldots\ldots$$

As $H^1\,(X, S_x) = H^1(\,X, L_x) = 0$, and hence we get $H^1\,(X, \overline{N}_{X}|_{Y}) = 0$. Therefore there exists a collection $\{\, \emptyset_{i|q+1}^{A}(Z\,,t)\}$ of homogeneous polynomial $\emptyset_{i|q+1}^{A}(Z\,,t)$of degree q+1 in $t^1,\ldots, t^m$ whose coefficients are holomorphic functions of z defined on $U_i$and taking values in $\overline{N}_{X}|_{Y}$such that

$$\emptyset_{ij}^{A}\,(Z\,,t) = F_{ijB}^{A}(z)\,\emptyset_{j|q+1}^{B}\,(Z\,,t) - \emptyset_{i|q+1}^{A}\,(Z\,,t)f\ or\ z\,\epsilon\, U_i \cap U_j \tag{19}$$

Considering the coefficients of $\emptyset_{i|q+1}^{A}\,(Z\,,t)$ as functions of the local coordinate Z_i of z , we write

$\emptyset_{i|q+1}^{A}\,(Z_i\,,t)$ for $\emptyset_{i|q+1}^{A}\,(Z\,,t)$,then the form (19) is written in the below as

$$\Psi_{ij}^{A}\,\big(Z_j\,,t\big) = F_{ijB}^{A}(z)\,\emptyset_{j|q+1}^{B}\,\big(Z_j\,\,,t\big) - \emptyset_{i|q+1}^{A}\,\big(g_{ij}\,(\,0, Z_j\,), t\big) \tag{20}$$

Now we define

$$\emptyset_{i}^{A[q+1]}(Z_i\,,t) = \emptyset_{i}^{A[q]}(Z_i\,,t) + \emptyset_{i|q+1}^{A}\,(z_i\,,t),\ \ \mathrm{i}\,\epsilon\, I,$$

Then writing $\emptyset_{j}^{A[q+1]}(t)$ for $\emptyset_{j}^{A[q]}\big(Z_j\,,t\big)$ we have

$$\emptyset_j^{A[q+1]}(g_{ij}\,(\emptyset_j^{B[q+1]}(t)),\,\mathrm{t}) \equiv_{q+1} \emptyset_i^{A[q]}\left(g_{ij}\left(\emptyset_j^{B[q]}(t)\right),t\right) + \emptyset_{i|q+1}^{A}\,(g_{ij}\,(\,0,Z_j\,),t),$$

$$f_{ij}^{A}\left(\emptyset_j^{B[q+1]}(t)\right) \equiv_{q+1} f_{ij}^{A}\left(\emptyset_j^{B[q]}(t)\right) + F_{ijB}^{A}(z)\,\emptyset_{j|q+1}^{B}\left(Z_j\ ,t\right)$$

Consequently, we obtain from (15) and (16), the congruence

$$\emptyset_i^{A[q+1]}\left(g_{ij}\left(\emptyset_j^{B[q+1]}(t)\right),t\right) \equiv_{q+1} f_{ij}^{A}\left(\emptyset_j^{B[q+1]}(t)\right).$$

This completes our inductive construction of the polynomials $\emptyset_i^{A[q]}(Z_i\,,t)$, $i \in I$, satisfying (13).

Thus, setting

$$\emptyset_i^{A}(Z_i\,,t) = \emptyset_{i|q+1}^{A}\,(Z_i\,,t) + \ldots + \emptyset_{i|q}^{A}\,(Z_i\ ,t) + \ldots$$

We obtain a formal power series $\emptyset_i^{A}(Z_i\,,t)\,, i \in I$, in t^1,……, t^m whose coefficients are vector – valued holomorphic functions of Z_i, $|Z_i| < 1$,which satisfies the equations (9) – (11).

Stage 3: (convergence) There is an arbitrariness involved in the construction of the formal power series $\emptyset_i^{A}(Z_i\,,t)$for each $q \geq 1$ the 0-cochain $\{\ \emptyset_{i|q+1}^{A}\,(Z_i\,,t)\}$ whose image under the coboundary map is the 1-cocycle $\{\Psi_{ij}^{A}\left(Z_j\,,t\right)\}$ is defined up to addition of a global holomorphic section of $\overline{N}_{X/Y}$ over *X*.

Now we want to use this freedom to assume convergence of the formal constructions. The idea is to estimate each holomorphic function involved in the construction of $\emptyset_i^{A}(Z_i\,,t)$ and show that under appropriate choices of $\{\ \emptyset_{i|q+1}^{A}\,(Z_i\,,t)\}$, q= 1,2,….., all the resulting power series $\{\,\emptyset_i^{A}(Z_i\,,t)\}$ are majorized by an obviously convergent series.

$$A(t) = \frac{a}{16b}\sum_{p=1}^{\infty}\frac{b^n}{n^2}(\,t_1 + \ t_2 + t_3 + \cdots + \ t_m)^n$$

Where a and b are some positive constants. Fortunately, what positive really counts on this stage are compactness of X and analyticity of all functions involved in the construction. Therefore, all the estimates obtained by kodaira [4] carry over verbatim to our case. We conclude that polynomials $\emptyset^{A}_{i|q+1}(Z_i\,,t)$ can be chosen in such a way that the power series $\emptyset^{A}_{i}(Z_i\,,t)\}$ converges for $|t| < \epsilon$,where ϵ is some number. Therefore, this completes the proof.

5.5 Interconnections among Isotropic, Legendre, and Kodaira moduli Spaces

If $X \rightarrow Y$ is a complex submanifold, there is an exact sequence of vector bundles

$$0 \rightarrow N^{*} \rightarrow \Omega^{1}Y/X \rightarrow \Omega^{1}X \rightarrow 0$$

Which induces natural embedding $\mathbb{P}(N^{*}) \rightarrow \mathbb{P}(\Omega^{1}Y)$ of total spaces of the associated projective zed bundles. The manifold $\hat{Y} = \mathbb{P}(\Omega^{1}Y)$ caries a natural contact structure such that the constructed embedding $\mathbb{P}(N^{*}) \rightarrow \hat{Y}$ is an isotropic as well as a Legendre one [5], indeed, the contact distribution $D \subset T\hat{Y}$at each point $y' \epsilon\ \hat{Y}$ consists of those tangent vectors $V_{\acute{y}} \epsilon\ T_{\acute{y}}\hat{Y}$ which

Satisfy the equation <ŷ, $T_{*}(V_{\hat{y}}\ )$ > = 0, where $\tau\ :\ \hat{Y} \rightarrow Y$ is a natural projection and angular brackets denote the paring of 1-forms and vectors at $\tau(\hat{y}) \epsilon\ Y$. Since the submanifold $\acute{X} \subset \hat{Y}$consists precisely of those projective classes of 1-forms in $\Omega^{1}Y/X$ which vanish when restricted on TX. we conclude that

$$TX \acute{\subset} D/_{\acute{x}}$$

Chapter Six

Stability of Isotropic Moduli Spaces

6.1 Introduction

Without references to Differential Geometry or twister theory a solution of a certain moduli problem was solved by Kodaira [5] in 1962. kodaira's initial data is pair, X↪Yconsisting of compact complex submanifold X of a complex manifold Y. He showed that if the normal bundle $N_{X}|_{Y}$of the initial submanifold X↪Y is such that H^1(X,$N_{X}|_{Y}$)=0,then the moduli set M has two properties :first, it is a manifold with dim=$h^0(X, N_{XlY)}$; second a tangent vector at any point $t \epsilon M$ can be realized canonically as a global section of the normal bundle $N_t$of the associated sub manifold $X_t \to Y, i.e.,$ there is a canonical isomorphism k_t:T_tM→ $H^0(X_t, N_t)$,called the kodaira map.the manifold (parameter space) M is called the kodaira moduli space, In 1995 Merkulov [6] proved a theorem on completeness of moduli spaces of compact Isotropic submanifolds in complex contact manifolds.

6.1.1 Kodaira Relative Deformation Theory

Let Y and M be complex manifolds and let $\pi_1: Y \times M \to Y$ and $\pi_2: Y \times M \to M$ be two natural projections. An analytic family of compact submanifolds of the complex manifold Y with the moduli space M is complex sub manifold $F \to Y \times M$ such that the restriction of the projection π_2 on F is proper regular map (regularity means that the rank of the differential of $V \equiv \pi_2|_{F}: F \to M$ is equal to dim M at every point. Thus, the family F has double fib ration structure

$$Y \overset{u}{\leftarrow} F \overset{v}{\rightarrow} M,$$

Where $\mu = \pi_2|_{F}$.For each tϵM we say that the compact complex sub manifolds

$$X_t := \mu\, o v^{-1}(t) \hookrightarrow Y$$

Belong to the family. If F↪ $Y \times M$is an analytic family of compact submanifolds, then, for any t ϵ M,there is a natural linear map,

$$k_t := \mu\ T_tM \rightarrow H^0(X_t, N_{XlY}),$$

From the tangent space at t to the vector space of global holomorphic sections of the normal bundle $N_{X}|_{Y}$=$TY|\ _{X_t}/TX_t$ to the sub manifold $X_t \hookrightarrow Y$. An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds is called complete if the Kodaira map k_t is an isomorphism at each point t in the moduli space *M*. This moduli space *M* is called kodaira moduli space

6.1.2 Complex Contact Manifolds

Definition 1. A compact complex m-dimensional submanifold $X^m \hookrightarrow Y^{2n+1}$ of a complex contact manifold Y^{2n+1} is called isotropic if $\mathcal{T}p \subset D|\ p$.

An isotropic sub manifold of maximal possible dimensions n is called a Legendre submanifold. The normal bundle N_{ZlW} of any Legendre submanifold Z→W isomorphic J^1L_x whereL_Z=L$|\ _Z$,and therefore, fits into the exact sequence

$$0 \rightarrow \Omega^1 Z \otimes L_Z \rightarrow N_{Z/W} \overset{tr}{\rightarrow} L_Z \rightarrow 0.$$

Definition 2. Let *X* be an isotropic submanifold of a complex contact manifold *(Y, D)*. Let

$$TX^{\perp} = \{\, Z\ \epsilon\ D\ |d\theta\ (Z\ ,W) = 0\ for\ all\ W\ \epsilon\ TX\}$$

Then $TX \subseteq TX^{\perp}$and the bundle S_x is defined by $S_x\ =\ TX^{\perp}/TX$.

6.1.3 Families of Complex Legendre Submanifolds

Let *Y* be a complex contact manifold. An analytic family $\{X_t \hookrightarrow Y|\ t \in M\}$ of compact submanifolds of *Y* is called an analytic family of compact Legendre submanifolds if, for any point t$\in$ M, the corresponding subset $X_t := \mu^0\ V^1(t) \rightarrow Y$ is a Legendre submanifolds. The parameter space *M* is called a Legendre moduli space.

Let $F \to Y \times M$ be a family of compact Legendre submanifolds. If $F \to Y \times M$ is an analytic family of compact Legendre submanifolds, it is also an analytic family of complex submanifolds in the sense of kodaira and thus, for each t $\in M,$ there is a linear group.

$$K_t: T_tM \to H^0 (X_t, N_{X_t|Y}).$$

6.1.4 Families of complex Isotropic Submanifolds

Let y be a complex contact manifold. An analytic family F $\hookrightarrow$Y×M of compact submanifold of the complex manifold Y is called an analytic family of isotropic submanifolds, if for any $t\epsilon M$,the corresponding subset X_t=μ $ov^{-1}(t) \hookrightarrow Y$ is an isotropic submanifold. Use is made of the symbol

$\{X_t \hookrightarrow Y|$ t $\epsilon M\}$ to denote an analytic family of isotropic sumanifold, then each point in X has a neighborhood U in Y such that the contact structure in a suitable trivialisation of L over U, is

$$\theta = d\omega^0 + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}}\, d\omega^{\bar{\bar{a}}} + \sum_{a=1}^{p} \omega^a\, dz^a$$

And X in U is given by

$$\omega^0 = \omega^a = \omega^{\bar{a}} = \omega^{\bar{\bar{a}}}=0.$$

There exists an adopted coordinate covering $\{U_i\}$ of a tubular neighborhood of inside Y. As a consequence one can always choose local coordinator functions.

$(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, z_i^a)$, in U_i where $\bar{a}, \bar{\bar{a}} = 1, \ldots., p$such that the contact structure U_i is represted by

$$\theta = d\omega^0 + \sum_{a=1}^{p} \underbrace{\omega_i^{\bar{a}}\omega_i^{\bar{a}}}_{(n-p)-terms} + \sum_{a=1}^{p} \underbrace{\omega_i^{\bar{a}} dz_i^a}_{p-terms}$$

With $U_i \cap X$ given by$\omega_i^0 = \omega_i^a = \omega_i^{\bar{a}} = \omega_i^{\bar{\bar{a}}} = 0$,

And

$$\theta_i|_{U_i \cap U_j} = A_{ij}\theta_j|_{U_i \cap U_j} \qquad (1)$$

For some nowhere vanishing holomorphic functions $A_{ij}(\omega_i\omega_j)$.They satisfy the condition

$$A_{ik} = A_{ij}A_{jk}$$

On every triple intersection$U_i \cap U_j$,the coordinates $\omega_i^A := (\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}})$ and Z_i^a are holomorphic functions $\omega_j^B := (\omega_j^0, \omega_j^b, \omega_j^{\bar{b}}, \omega_j^{\bar{\bar{b}}})andZ_j^b$,

$$\begin{cases} \omega_i^0 = f_{ij}^0\left(\omega_j^B, z_j^b\right) \\ \omega_i^a = f_{ij}^a\left(\omega_j^B, z_j^b\right) \\ \omega_i^{\bar{a}} = f_{ij}^{\bar{a}}\left(\omega_j^B, z_j^b\right) \\ \omega_i^{\bar{\bar{a}}} = f_{ij}^{\bar{\bar{a}}}\left(\omega_j^B, z_j^b\right) \\ z_i^a = g_{ij}^a(\omega_j^B, z_j^b) \end{cases} \Leftrightarrow \begin{cases} \omega_i^A = f_{if}^A(\omega_j^B, z_j^b) \\ z_i^a = g_{ij}^a(\omega_j^B, z_j^b) \end{cases}$$

With $f_{ij}^A\left(0, z_j^b\right) = 0$,where A=0, a, $\bar{a}, \bar{\bar{a}}$

6.1.5 Legendre Moduli Space with Completeness & Maximality:

Let Y be a complex contact manifold. An analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact submanifolds of Y is called an analytic family of compact Legendre submanifolds if, for any point t $\in M$, the corresponding subset $X_t := \mu^0\, V^{1}\;(t) \hookrightarrow Y$ is a Legendre submanifolds. The parameter space M is called a Legendre moduli space.

Let $F \hookrightarrow Y \times M$ be a family of compact Legendre sub manifolds. If $F \hookrightarrow Y \times M$ is an analytic family of compact Legendre sub manifolds, it is also an analytic family of complex sub manifolds in the sense of Kodaira (1962), for each $t \in M$, there is a linear map

$$K_t \colon T_t M \rightarrow H^0\,(X_t, N_{X_t|Y}).$$

We say that the family F is complete at a point t $\in M$ if the composition

$$s_t : T_t M \overset{kt}{\rightarrow} H^0\,(X_t,\, N_{X_t\,|Y}) \overset{pr}{\rightarrow} H^0\,(X_t,\, L_{X_t})$$

provides an isomorphism between the tangent space to M at the point t and the vector space of global sections of the contact line bundle over X_t. The analytic family $F \hookrightarrow Y \times M$ is called complete if it is complete at each point of the moduli apace M.

Lemma 1: If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre sub manifolds is complete at a point $t_0 \in M$, then there is an open neighborhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.

Definition 2:

An analytic family F $\hookrightarrow$Y× M of compact complex Legendre submanifolds is maximal at appoint $t_0 \in M$, if for any other analytic family $\tilde{F} \hookrightarrow Y\times\tilde{M}$ of compact complex Legendre submanifolds such that $\mu^0\, V^{-1}(t_0) = \tilde{\mu}^0\tilde{V}^{-1}(\tilde{t}_0)$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists neighborhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f\colon \tilde{U} \hookrightarrow M$ such that $f(\tilde{t}_0) = \tilde{t}_0$ and $\tilde{\mu}^0\tilde{V}^{-1}(\tilde{t}_0) = \mu \circ V^1\,(f(\tilde{t}_0))$ for each $t \in \tilde{U}$.The family $F \hookrightarrow Y \times M$ is called maximal if it is maximal at each point t in the moduli space M.

Lemma 2: If an analytic family $F \hookrightarrow Y \times M$ of compact complex Legendre sub manifolds is complete at a point $t_0 \in M$, then it is maximal at the point t_0. The map

$$s_t : T_t M \rightarrow H^0\,(X_t,\, L_X)$$

studies by Lemma1 & Lemma 2 will also play a fundamental role. Our study of the reach geometric structure induced canonically on moduli spaces of complete and maximal analytic families of compact Legendre sub manifolds described by the following theorem.

Theorem 2: [6] Let X be compact complex Legendre sub manifold of a complex contact manifold Y with contact line bundle L. If $H^1(X, L_x) = 0$,then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y/\ t \in M\}$ of compact Legendre sub manifolds containing X with Legendre moduli space M, is a $H^0(X, L_x)$ –dimensional complex manifold.

This theorem is proved by working in local coordinates adapted to the contact structure and expanding the defining functions of nearby compact Legendre sub manifolds in terms of local coordinates on the moduli space *M.* This is much in the spirit of the original proof of kodaira's theorem of the existence, completeness and maximality of compact sub manifolds of complex manifolds, the essential difference from the Kodaira case is that the infinite sequence of obstructions to agreements on; overlaps of formal power series is situated now in $H^1(X, L_x)$ rather than in $H^1(X, N_{X/Y})$.

6.1.6 Isotropic Moduli Spaces with Compleness & Maximality

Let Y be a complex contact manifold, and let F $\hookrightarrow Y \times M$ be an analytic family of compact complex isotropic submanifolds. The latter is also an analytic family of compact complex submanifolds in the sense of kodaira and thus, for each t $\in M$, there is a canonical linear map

$$K_t: T_t M \to H^0(X_t, N_{Xt|Y}).$$

The exact sequence

$$0 \to S_{x_t} \to N_{x_t/Y} \to J^1 L_{x_t} \to 0.$$

Has an expansion as follows,

$$\begin{array}{c} 0 \\ \downarrow \\ \Omega^1 X \otimes L_x \\ \downarrow \\ 0 \to S_{x_t} \to N_{x_t/Y} \to J^1 L_{x_t} \to 0 \\ \downarrow \\ L_{x_t} \\ \downarrow \\ 0 \end{array}$$

Hence, there is a canonical map represented by a dashed arrow

$$0$$

$$\downarrow$$

$$H^0\left(X_t, \Omega^1 X \otimes L_{x_t}\right)$$

$$\downarrow$$

$$0 \to H^0(X_t, S_{x_t}) \to H^0(X_t, N_{X_t|Y}) \to \left(N_{x_t} \to J^1 L_{x_t}\right) \to 0$$

$$\downarrow$$

$$H^0(X_t, S_{x_t})$$

$$\downarrow$$

$$0$$

Thus, there is a canonical sequence of linear spaces,

$$0 \to H^0(X_t, S_{x_t}) \to H^0(X_t, N_{X_t|Y} \to \left(N_{x_t} \to J^1 L_{x_t}\right) \to 0$$

which is not in general.

6.2 Stability of Compact Complex Legendre Sub manifolds:

A family of complex contact manifolds is by definition a quadruple $(Y, S, D, \overline{\omega})$, consisting of complex manifolds Y and $S,$ a holomorphic submersion $\overline{\omega} : y \to S$ and a maximality non-integrable codomain 1 vector Subbundle, D $\subset ker\overline{\omega}$, of the bundle of $\overline{\omega}$- vertical tangent vectors, therefore, each fibre $y_s = \pi^{-1}(s), s \in S,$ is a complex contact manifold with contact line bundle L_s isomorphic to $(ker\overline{\omega}/D)|_{y_{s.}}$ The manifold $S$is often called a parameter space. For any $s \in S$ there is a canonical linear map,

$$\rho_s : T_s S \to H^1(Y_s, L_s)$$

According to Kodaira [5], if (Y, L) is a compact complex contact manifold with $H^1(Y, L) = 0$ then there exists a complete analytic family $(Y, S, D, \overline{\omega})$ of contact manifolds such that

(i) each fibre y_s is compact,

(ii) $Y = y_{s_0}$ for some $s_0 \in S,$ and

(iii) The map $\rho_s : T_s S \rightarrow H^1 (Y_s , L_s)$is an isomorphism for each s $\in M$.

In the present context it is more suitable to call a family of complex contact manifolds $(Y, S, D, \overline{\omega})$ simply a complex contact fibre manifold and denote by y. then a submanifold $X \rightarrow Y$ is called a complex Legendre fibre submanifold if the restriction of $\overline{\omega}$ to X defines a holomorphic submersion $\overline{\omega} : X \rightarrow S$ whose fibres are compact, then X is called a complex Legendre fibre submanifold with compact fibres.

Definition : A compact complex Legendre Submanifold X of a complex contact manifold Y is called stable if for any complex fibre manifold y such that $\overline{\omega}^{-1}(s_0) = Y$ for some point $s_0 \in S$. There exists a neighbourhood U of s_0 in S and a complex Legendre fibre submanifold X $\subset y|_v$ with compact fibres such thet $X \cap Y = X$.

Let $X \hookrightarrow Y$ be a compact complex Legendre Submanifold and Y a complex contact fibre manifold such that $\overline{\omega}^{-1}(s_0) = Y$ for some point $s_0 \in S$. Then the normal bundle $N_{X|Y}$ of X↪Y fits into the exact sequence,

$$0 \rightarrow N_{X|Y} \rightarrow N_{X|Y} \rightarrow C^p \otimes 0_x \rightarrow 0,$$

Where $N_{X|Y}$ is the normal bundle of $X \rightarrow Y$ and $\rho = \dim S$, Therefore

The quotient bundle

$N \rightarrow N_{X|Y} / i(\Omega^1 X \otimes L_X)$ has the extension structure

$$0 \rightarrow L_x \rightarrow N \rightarrow C^p \otimes 0_x \rightarrow 0.$$

Theorem 1: Let $X \rightarrow Y$ be a compact complex Legendre Submanifold and Y a complex contact fibre manifold such that $\overline{\omega}^{-1}(s_0) = Y$ for some point $s_0 \in S$. If $H^1 (X, L_x) = 0$,then there exists an analytic family of compact complex submanifolds $\{X_t \hookrightarrow y | \, t \in M\}$ such that each X_t is a Legendre submanifold of Y_s for some $s \in S$ and such that there is a canonical isomorphism $T_s S \rightarrow H^0 (X_t , N_t)$ for all $t \in M$.

Theorem 2: Let X be a Complex compact Legendre sub manifold of a complex contact manifold (Y, L) .If $H^0(X, L_x) = 0$, then X is a stable Legendre submanifold of Y.

Proof: Let Ya complex contact fibre manifold such that $\varpi^{-1}(s_0) = Y$ for some point $s_0 \in S$,by the Previous theorem, there exists an analytic family of compact sub manifolds $\{X_t \to y | t \in M\}$ and a holomorphic submersion $\zeta: M \to S$, such that, for each $t \in M$, the manifold X_t is a compact complex Legendre submanifold $Y_{\zeta(t)}$of then ,for a sufficiently small polycylinder neighborhood U of s_0 in S one may find an immersed polycylinder $W \subset M$ such that the restriction of ζdefines a diholomorphic map $\zeta: W \to U$, then the union X = $\cup_{s\in U} X_{\zeta-1(s)}$ forms a Legendre fibre submanifold of y with compact fibres and $X \cap Y = X$.
Hence completes the proof.

6.3 Stability of Fiber Structures on Legendre Moduli Spaces

In what fellows, we assume that all manifolds under consideration are paracompact and connected. By a complex fibre manifold we shall mean a complex manifold *M* together with a holomorphic map p of *M* onto a complex manifold B such that the rank of the Jacobian of p at each point of M is equal to the dimension of B. We call B the base space of the complex fibre manifold M. Moreover, for any point $u \in B$, we denote the inverse image $p^{-1}(u)$ of u by W_u and call it the fibre of *M* over u. Obviously each fibre $W_u = p^{-1}(u)$ is a complex (analytic) submanifold of *M*. We denote the complex fibre manifold *M* by the triple (*M,B, p)* when we want to indicate the map *p* and the base space *B* explicitly. Note that, in case the fibres W_u, $u \in B$, are compact, the triple (*M, B, p)* forms a complex analytic family of compact complex manifolds. For any subdomain *N*of *B* we call the complex fibre manifold (p^{-1} *(N, Np*)) the restriction of *M* to *N* and denote it by the symbol $M|_N$. Let V be a complex submanifold of $M|_N$ such that *p (V) = N*. We call *V* a fibre submanifold of the complex fibre manifold $M|_N$, if and only if the triple *(V, N, P)* forms a complex fibre manifold. If, moreover, each fibre $V_u = V \cap W_u$, $u \in N$, or *V* is compact, we call *V* a fibre submanifold with compact fibers of the complex fibre manifold M $|_N$. Consider a complex submanifold *X* of a complex manifold *Y*.

Definition 3:

We call *X* a stable submanifold of *Y* if and only if, for any complex fibre manifold *(M, B, p)* such that $p^{-1}(0) = Y$ for a point o *B*, there exist a neighborhood *N* of *o* in *B* and a fibre submanifold *V* with compact fibres of the complex fibre manifold $M|_N$ such that V ∩ Y = X.

Theorem 3:

Let $X \rightarrow Y$ be a compact complex submanifold in complex manifold *Y* and ψ be the sheaf over *X* of germs of holomorphic sections of the normal bundle of *X* in *Y* such that $H^1(X, \psi) = 0$, then *X* is a stable submanifold of *Y*

Let *Y* be a complex manifold. By a fibre structure on *Y* we shall mean a pair (*B, p)* of a complex manifold *B* and a holomorphic map *p* of *Y* onto *B* such that the triple *(Y, B, p)* forms a complex fibre manifold. An analytic family of compact complex manifolds is, by definition, a complex fibre manifold with compact fibres. Let $M = (M, B, \omega)$ be an analytic family of compact complex manifolds. By an analytic family of fibre structures on the analytic family *M* we shall mean a pair ($\mathcal{B}$, *P*) of an analytic family $\mathcal{B}$ = ($\mathcal{B}$, S,π) of compact complex manifolds and a holomorphic map P of M onto $\mathcal{B}$ such that π P= ω and such that the triple (M, $\mathcal{B}, P$) forms a complex fibre manifold. For each point s of B we set $Y_s = \omega^{-1}(s)$, $B_s = \pi^{-1}(s)$ and denote by p_s the restriction of the map $\mathcal{P}$ to the submanifold Y_s of M. Obviously the pair (B_s, p_s)defines a fibre structure on Y_s. We call (B_s, p_s) the fibre over *s* of the family ($\mathcal{B}$, P).

Definition 4:

A fibre structure *(B , p)* on a compact complex manifold *Y* is said to be stable if and only if, for any analytic family $M= (M, B, \omega)$ of compact complex manifolds such that $\omega^{-1}(o) = Y$ for a point $o \in B$, there exist a neighborhood *N* of *o* in *B* and an analytic family ($\mathcal{B}$, P) of fibre structures on the analytic family $M|_N$ of which the fibre(B_o, P_o) over o coincides with *(B , p)*.

Definition 5:

A compact complex manifold *X* is said to be regular if and only if the first cohomology group $H^1(X, O)$ of *X* with coefficients in the sheaf *O* over *X* of germs of holomorphic functions vanishes.

Theorem 4:

Let *(Y, B, p)* be a compact complex fibre manifold. If all fibres $p^{-1}(b)$, b ∈ B, are regular, then the fibre structure *(B, p)* on *Y* is stable.

6.4 Stability of Compact Complex Isotropic Sub manifolds

Theorem 5: Let (Y, D) be complex contact manifold and $X \subset Y$ be a isotropic submanifold of Y with contact line bundle L. Then there is a short exact sequence

$$0 \to S_x \to N_{X|_Y} \to J^1L_x \to 0.$$

Proof: Consider a particular I-form θ that represents the contact structure. Let $p \epsilon X, Z \epsilon T_pX$ be a vector in the normal bundle and $Q \epsilon T_pY$. Then there are two equations

$$f(P) = \theta(Q), \qquad d\theta(Z, Q) = Z(f)/p$$

That uniquely determines the 1-jet on X at P of a function f. Consider rescaling $\theta \mapsto g\theta$ where *g* is a function on *Y*. If we set $\theta^{\wedge} = g\theta$ and $f^{\wedge} = gf$ then we have

$$\grave{\theta}(Q) = g\theta(Q) = gf(p) = \acute{f}|p)$$

$$d\check{\theta}(Z, Q) = (dg \wedge \theta)(Z, Q) + gd\theta(Z, Q)$$

$$= dg(Z)\,\theta(Q) - dg(Q)\theta(Z) + gZ(f)|p$$

$$= Z(g)\,f(p) - 0 + g\,Z(f)|p$$

$$= Z(gf)/p$$

$$= Z(f)/p$$

Since $T_p X \subseteq T_p X^{\perp} \subset D$ then $Z \epsilon D$ so that $\theta(Z) = 0.$ Therefore this elementary calculation shows that the above two conditions are satisfied by *gf* and so we conclude that we have defined a map $N_{X|Y} \rightarrow J^1 L_x$. Furthermore it is clear that the kernel is T$X^{\perp}$ / TX. Hence the proof is completed.

Theorem 6: If an analytic family F $\hookrightarrow$Y × $\breve{M}$ be any analytic family of compact complex isotropic sub manifolds such that$X_t = \breve{\mu} \circ \tilde{V}^{-1}(\tilde{t}_0)$ for some point for a point $\tilde{t}_0 \in \widetilde{M}$, Let { μ_t}

Be a covering of *Y* by coordinate charts with coordinate functions $\{\omega_t^A, Z_t^A\}$ such that

$$d\omega_i^0 + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}} \, d\omega^{\bar{\bar{a}}}|_{U_i \cap U_j} + \sum_{a=1}^{p} \omega_i^a dz_i^a|_{U_i \cap U_j}$$

$$= A_{ij} \, (d\omega_i^a + \textstyle\sum_{\bar{a}=p+1}^{n} \omega^a \, d\omega^a|_{U_i \cap U_j} + \sum_{a=1}^{p} \omega_i^u dz_i^u|_{U_i \cap U_j})$$

For some non-vanishing holomorphic functions $A_{ij}\left(\omega_j \, , Z_j\right)$, and the isotropic submanifold X_t is given in each intersection $X_t \cap U_t$ by equations $\omega_i^A = 0$. Define $\emptyset_i^A(Z_i\, , t\,)$ by

$$\emptyset_i^A(z_i\, , t) = \begin{bmatrix} \emptyset_i^0(z_i\, , t) \\ \emptyset_i^a(z_i\, , t) \\ \emptyset_i^{\bar{a}}(z_i\, , t) \\ \emptyset_i^{\bar{\bar{a}}}(z_i\, , t) \end{bmatrix}$$

and $\overline{\emptyset}_i^A(z_i\, , t)$ by

$$\widetilde{\emptyset}_i^A(z_i\, , t) = \begin{bmatrix} \overline{\emptyset}_i^0(z_i\, , \tilde{t}) \\ \overline{\emptyset}_i^a(z_i\, , \tilde{t}) \\ \overline{\emptyset}_i^{\bar{a}}(z_i\, , \tilde{t}) \\ \overline{\varphi}_i^{\bar{\bar{a}}}(z_i\, , \tilde{t}) \end{bmatrix}$$

Then, for sufficiently small neighborhoods U and $\tilde{U}$ of points t $\in M$ and $\tilde{t}_0 \in \tilde{M}$, the sub manifolds $v^{-1}U \mapsto Y \times U$ and $\tilde{v}^{-1}U \mapsto Y \times \tilde{u}$ are given respectively by using lemma 1.

$$\emptyset_i^a(z_i, t) = \frac{\partial \emptyset_i^a(z, t)}{\partial z_i^a} + \sum_{\bar{b}=p+1}^{n} \emptyset_i^{\bar{b}}(z_i, t) \frac{\partial \emptyset_i^{\bar{\bar{b}}}(Z_i, t)}{\partial z_i^a}$$

and

$$\tilde{\emptyset}_i^a(z_i, \tilde{t}) = \frac{\partial \tilde{a}_i^a(z, t)}{\partial z_i^a} + \sum_{\bar{b}=p+1}^{n} \emptyset_i^{\bar{b}}(z_i, t) \frac{\partial \emptyset_i^{\bar{\bar{b}}}(Z_i, t)}{\partial z_i^a}$$

Where t = ($t^1, \dots\dots, t^m$), $m = \dim M$ and $\tilde{t}$ = ($\tilde{t}^1, \dots\dots, \tilde{t}^i$), m = dim $\tilde{M}$

, are coordinates on U and $\tilde{U}$ respectively, and $\emptyset_i^a(z_i, t)$ and $\tilde{\emptyset}_i^a(z_i, \tilde{t})$ are some Holomorphic functions. We may assume without loss of generality that coordinate functions $t^1, \dots\dots, t^m$ vanish at $t_0 \in U$ while coordinate functions $\tilde{t}$ = ($\tilde{t}^1, \dots\dots, \tilde{t}^i$), vanish at $\tilde{t_0} \in \tilde{U}$.

To prove this theorem, we have to construct a holomorphic map $f : \tilde{U} \to U$ such that $f(\tilde{t_0}) = t_0$ and

$$\emptyset_i^{\tilde{A}}(z_i, t) = \emptyset_i^{A}(z_i, f(\tilde{t})) \qquad (1)$$

For all $\tilde{t}$ in some sufficiently small neighbourhood of $\tilde{t_0}$.

Let us first prove the existence of a unique formal power series $f(\tilde{t})$ satisfying the equation. For this purpose, we introduce the following notations. If *P(S)* is a power series in variables

S = ($S^1, \dots\dots, S^K$), we write

$$P(S) = p_0(S) + p_1(S) + \cdots \dots\dots + p_q(S) + \dots\dots$$

Where each term $p_q(S)$ is a homogeneous polynomial in $S^1, \dots\dots, S^K$ of degree q and denote

It by $P|^q|\ (s)$ *is* the polynomial.,

$$P|^q|\ (s) = p_0(S) + p_1(S) + \cdots \dots\dots + p_q(S)\ .$$

If *Q(s)* is another power series in *s*. we write *P(S)* $\equiv Q|^q|\ (s)$

We look for a solution of equation (1) in the form of a formal power Series

$f(\check{t}) = f_1(\tilde{t}) + f_2(\tilde{t}) + f_q(\tilde{t}) + \ldots\ldots\ldots\ldots\ldots$

Then equations (1) reduce to the system of congruencies

$$\overline{\emptyset}_i^1(z_i, \tilde{t}) = \emptyset_i^A(z_i, f^{|q|}(\tilde{t})) \qquad \mathrm{I} \in I,\ \mathrm{q} = 1, 2 \tag{2}$$

We shall first construct polynomials $f^{|q|}(\tilde{t}))$ by induction on q, Let

$$\emptyset_i^A(Z_i, t) = \emptyset_{t/1}^A(Z_i, t) + \emptyset_{\frac{t}{2}}^A(Z_i, t) + \cdots$$

Be the power series expansion of $\emptyset_i^A(Z_i, t)$ in $t^1, \ldots \ldots, t^m$. By hypothesis the family $F \mapsto Y \times M$ is complete at $t_0 \in M$. By the previous theorem , the sequence is

$$0 \to H^0(X, S_X) \to K_{t_0}(T_{t_0}M) \to H^0(X, L_x) \to 0$$

is exact. On the other hand, if there exists a sheaf of Abelian groups $\widetilde{N}_{X}|_{Y}$ that fits into there exact sequence.

$$0 \to H^0(X, S_X) \to H^0(X, \widetilde{N}_{X}|_{Y}) \to H^0(X, L_x) \to 0,$$

Moreover, the Kodaira map K_{t_0} maps exactly $T_{t_0}M$. To the space of the global sections of $\widetilde{N}_{X}|_{Y}$. Thus, we have an isomorphism

$$K_{t_0}: T_{t_0}M \to H^0(X_t, \widetilde{N}_{X_{t/Y}}),$$

According to the local coordinate description of the map K_{t_0}, given in the proof of theorem, this means that the collection of 0-cocycles $\{\frac{\partial \emptyset_{t/1}^A(z_i, t)}{\partial t^\alpha}\}$, $\alpha = 1, \ldots \ldots, m$ represents a basis of the Vector space $H^0(X, \widetilde{N}_{X_{t/Y}})$. Since

$$\widetilde{\emptyset}_{t/1}^A(z_i, \tilde{t}) = \sum_{\gamma=1}^{t} \frac{\partial \widetilde{\emptyset}_t^A(z_i, t)}{\partial \tilde{t}^\gamma}\Big|_{t=0} \tilde{t}^\gamma$$

And each 1-cochain $\{ \frac{\partial \widetilde{\emptyset}_t^A(z_i,t)}{\partial \tilde{t}^{\gamma}} |_{t=0} \}, \gamma = 1, \ldots, I =$ dim $\widetilde{M}$ represents a global section of $\widetilde{N}_{X}|_{Y}$ over X, we conclude that the collection $\{ \widetilde{\emptyset}_{t/1}^A(z_i, \tilde{t}) \}$ may be interpreted as a homogeneous polynomial in $\tilde{t}$ of degree 1 with coefficients in $H^0(X, \widetilde{N}_X|_Y)$,

Therefore, we can decompose

$$\widetilde{\emptyset}_{t/1}^A(z_i, \tilde{t}) = \sum_{\alpha=1}^{m} f_1^{\mu}(\tilde{t}) \frac{\partial \widetilde{\emptyset}_{t/1}^A(z_i,t)}{\partial \tilde{t}^{\alpha}}$$

Where coefficients $f_1(\tilde{t})$ are linear vector valued functions of $\tilde{t}^1, \ldots, \tilde{t}^I$. Thuswhich means that the functions $f_1(\tilde{t})$ satisfy the congruences $(2)_q$

Assume that the Polynomials$f_1^{|q|}(\tilde{t})$ satisfying $(2)_q$are already constructed. Define a homogeneous polynomial $\omega_i^A(z_i, \tilde{t})$ in $\tilde{t}$of degree q+1 by the congruence,

$$\omega_i^A(z_i, \tilde{t}) \equiv_{q+1} \widetilde{\emptyset}_i^A(z_i, \tilde{t}) - \widetilde{\emptyset}_i^A\left(z_i, f^{|q|}(\tilde{t})\right)$$

From the known equalities

$$\emptyset_i^A\left(g_{ij}(\emptyset_i^B(z_i, \mathrm{t}), z_i), t\right) = f_B^A(\emptyset_i^B(z_i, \mathrm{t}), z_i),$$

$$\widetilde{\emptyset}_i^A(\left(g_{ij}(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i\right), t\right) = f_B^A\left(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i\right),$$

Where

$$\emptyset_i^a(z_i, \mathrm{t}) = \frac{\partial \emptyset_i^0(Z_i,t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \phi_t^{\bar{b}}(z_i, t) \frac{\partial \phi_i^{\bar{\bar{b}}}(z_i, t)}{\partial z_i^a}$$

and

$$\widetilde{\emptyset}_i^a(z_i, \mathrm{t}) = \frac{\partial \widetilde{\emptyset}_i^0(Z_i,t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \widetilde{\emptyset}_t^b(z_i, t) \frac{\partial \widetilde{\emptyset}_i^b(z_i, t)}{\partial z_i^a},$$

We find

$$\omega_i^A \frac{(z_i,\tilde{t})}{z_i} = g_{ij}(0, z_i) \equiv_{q+1} \omega_i^A \frac{(z_i,\tilde{t})}{z_i} = g_{ij}(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i)$$

$$\equiv_{q+1} [\widetilde{\emptyset}_i^a(z_i, \tilde{t}) - \emptyset_i^A\left(z_i, f^{|q|}(\tilde{t})\right)] z_i = g_{ij}(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i)$$

$$\equiv_{q+1} [\widetilde{\emptyset}_i^a(z_i, \tilde{t})|_{z_i} = g_{ij}(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i) - \emptyset_i^A\left(z_i, f^{|q|}(\tilde{t})\right)]/z_i = g_{ij}(\widetilde{\emptyset}_t^B\left(z_i, f^{|q|}(\tilde{t})\right), z_i)$$

$$\equiv (q+1) f_q^A\left(\widetilde{\emptyset}_i^B(z_i, \tilde{t}), z_i\right) - f_q^A(\widetilde{\emptyset}_i^B\left(z_i, f^{|q|}(\tilde{t})\right), z_i) \equiv_{q+1} \frac{\partial f_B^A}{\partial \omega_i^B}|_{\omega_i=0} \omega_i^B(z_i, \mathrm{t}).$$

The letter congruence means that the collection $\widetilde{\omega}_i^A(z_i, \tilde{t})$ is a homogeneous polynomial in $\tilde{t} = (\tilde{t}^1, \dots\dots, \tilde{t}^l)$ of degree q+1 with coefficients in $H^0(X, \widetilde{N}_X|_Y)$. We now see that $\{\widetilde{\omega}_i^A(z_i, \tilde{t})\}$ takes Values in fact in $\widetilde{N}_X|_Y$. This requires to show that

$$\omega_i^a(z_i, \tilde{t}) = -\frac{\partial \omega_i^0(z_i, t)}{\partial z_i^a}$$

Hence by definition,

$$\omega_i^0(z_i, \tilde{t}) \equiv_{q+1} \widetilde{\emptyset}_i^0(z_i, \tilde{t}) - \widetilde{\emptyset}_i^0(z_i, f^{|q|}(\tilde{t})) \tag{3}$$

And

$$\omega_i^a(z_i, \tilde{t}) \equiv_{q+1} \widetilde{\emptyset}_i^a(z_i, \tilde{t}) - \widetilde{\emptyset}_i^a(z_i, f^{|q|}(\tilde{t})) \tag{4}$$

(4)

Differentiate (3) with respect to

$$\frac{\partial \omega_i^0(z_i, \tilde{t})}{\partial z_i^a} = \frac{\partial \widetilde{\emptyset}_i^0(z_i, \tilde{t})}{\partial z_i^a} - \frac{\partial \widetilde{\emptyset}_i^0(z_i, f^{|q|}(\tilde{t}))}{\partial z_i^a}$$

Equation (4) implies

$$\omega_i^a(z_i, \tilde{t}) = -\frac{\partial \omega_i^0(z_i, t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \widetilde{\emptyset}_t^b(z_i, t)\frac{\partial \widetilde{\emptyset}_i^b(z_i, t)}{\partial z_i^a} + \frac{\partial \widetilde{\emptyset}_i^a(z_i, f^{|q|}(\tilde{t}))}{\partial z_i^a}$$

$$+ \sum_{\bar{b}=p+1}^{n} \widetilde{\emptyset}_t^b(z_i, t)\frac{\partial \widetilde{\emptyset}_i^a(z_i, f^{|q|}(\tilde{t}))}{\partial z_i^a}$$

As $\widetilde{\emptyset}_i^a(z_i, \tilde{t}) =_q \widetilde{\emptyset}_i^a(z_i, \tilde{t})$, $\widetilde{\emptyset}_i^a(z_i, \tilde{t}) =_q \emptyset_i^a(z_i, \tilde{t})$ and degree $\widetilde{\emptyset}_i^a, \widetilde{\emptyset}_i^a \geq 1$, degree $\widetilde{\emptyset}_i^a, \widetilde{\emptyset}_i^a \geq 1$, so the second and fourth terms of equation (5) cancel out by induction assumption, yielding

$$\omega_i^a(z_i, \tilde{t}) = -\left(\frac{\partial \widetilde{\emptyset}_i^0(z_i, \tilde{t})}{\partial z_i^a} - \frac{\partial \widetilde{\emptyset}_i^a(z_i, f^{|q|}(\tilde{t}))}{\partial z_i^a} \right.$$

Hence

$$\omega_i^a(z_i, \tilde{t}) = -\frac{\partial \widetilde{\omega}_i^0(z_i, \tilde{t})}{\partial z_i^a}$$

Therefore, { $\omega_i^A(z_i\,,\tilde{t})$} represents a global section of bundle $\widetilde{N}_X|_Y$ so that we can decompose again over the basis section { $\omega_{t/1}^A(z_i\,,\tilde{t})$}

$$\omega_i^A(z_i\,,\tilde{t}) = \sum_{\bar{b}=p+1}^{n} f_{q+1}^{a}\,(\tilde{t})\,\frac{\partial\widetilde{\emptyset}_{t/1}^{A}(z_{i,}\tilde{t})}{\partial t^{\alpha}}$$

Where, the coefficient$f_{q+1}(\tilde{t})$ are vector valued homogeneous polynomials in $\tilde{t}^1, \ldots \ldots, \tilde{t}^I$

Of degree q+1, defining

$$f^{|q+1|}(\widetilde{t}) = , f^{|q|}(\widetilde{t}) + , f^{|q+1|}(\widetilde{t})\ ,$$

We have

$$\widetilde{\emptyset}_i^A\big(z_{i,}\tilde{t}\big) \equiv_{q+1} \widetilde{\emptyset}_i^A\big(z_i\,, f^{|q|}(\widetilde{t})\big) + \omega_i^A(z_i\,,\tilde{t})$$

$$\equiv_{q+1} \widetilde{\emptyset}_i^A\big(z_i\,, f^{|q+1|}(\widetilde{t})\big).$$

This completes our inductive construction of the polynomials $f^{|q|}(\widetilde{t})$ satisfying equations $(2)_q$.

The convergence of the resulting formal power series

$f(\check{t}) = f_1\,(\tilde{t}) + f_2(\tilde{t}) + f_q\,(\tilde{t}) +$……………..

for all $\tilde{t}$ in some open neighborhood of the origin in $\mathbb{C}^1$ follows from estimates by Kodaira [22], which carry over verbatim to our case, this completes the proof.

Conclusion

The object of our study is the set M, of all possible holomorphic deformations of an isotropic sub manifolds *X* inside Y remain isotropic.

We define a subshef of Abelian groups in the sheaves $N_{X/Y}$ as $\widetilde{N}_{X/Y} := pr^{-1}(\alpha(L_x))$,

Where $pr: N_{X/Y} \rightarrow j^1 L_X$is the canonical epimorphism and $\alpha: L_x \rightarrow j^1 L_X$, which is the main object,We show that that the infinite sequence of obstructions to deforming an isotropic sub manifolds X of Y is controlled by the cohomologics $H^0(X, \widetilde{N}_{X/Y})$ and $H^1(X, \widetilde{N}_{X/Y})$. The only (though very essential) novelty in our case is that the infinite sequence of obstructions to agreements on overlaps of formal power series is situated now in the cohomology group $H^1(X, \widetilde{N}_{X/Y})$ rather than in $H^1(X, L_x)$. For the case of Legendre submanifold X of Y.

Here, $\widetilde{N}_{X/Y} \equiv L_x$ which implies $H^0(X, \widetilde{N}_{X/Y}) \equiv H^0(X, L_X)$ and therefore that generalizes Merkulov's result to the isotropic submanifold. If $H^1(X, \widetilde{N}_{X|Y}) = 0$ then there exists a complete and maximal analytic family $\{ X_t \mapsto Y \setminus t \in M \}$ of compact complex isotropic submanifold X_t of Complex contact manifold *Y* with the moduli spaces *M*, called isotropic moduli space, being a $H^0(X, \widetilde{N}_{X|Y})$- dimensional smooth complex manifold, there are strong indications that the isotropic moduli spaces will play a pivotal role in the twister theory of G- Structures with restricted invariant torsion.

References

[1] Sigmundur Gudmundsson, An introduction to Riemannian Geometry, Lund University, version 1.0303 - 17 March 2016

[2] Isaac Chavel, Riemannian Geometry, A modern introduction, 2nd Edition.

[3] Leonor Godinho and Jos´eNat´ario, *An Introduction to Riemannian Geometry with Applications to Mechanics and Relativity*, Lisbon, 2004

[4] Kodaira, K., Complex manifolds and deformation of Complex Structures, Heidelberg New York, Springer, 1986

[5] Kodaira, K., A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann Math*,75(1962), pp.146-162

[6] Merkulov, S. A., Existence and geometry of Legendre moduli spaces, Math. Z, 226(1997), pp.211.

[7] Mikio Nakahara, *Geometry, Topology and Physics,*Second Edition, Institute of Physics PublishingBristol and Philadelphia

[8] R.W. R. DARLING, *Differential Forms and Connections,* University of South Florida

[9] Clifford Henry Taubes, *Differential Geometry, Bundles, Connections, Matrices and Curvature*, Oxford Graduates Texts in Mathematics

[10] Peng Zhao, *Riemannian Geometry and General Relativity*, July 10, 2009

[11] Theodore Frankel, *The Geometry of Physics,* An Introduction, 2nd Edition

[12] John M. Lee, *Riemannian Manifolds,* An Introductions to Curvature, Springer

[13] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations, Lawrence Berkeley Laboratory*, University of California, Berkeley, California1984

[14] Luiz Agostinho Ferreira, *Lecture Notes on Lie Algebras and Lie Groups*, Instituto de F__sica de S~ao Carlos - IFSC/USP, Universidade de S~ao Paulo, Caixa Postal 369, CEP 13560-970S~ao Carlos-SP, Brasil, August – 2011

[15] Jean Gallier and Jocelyn Quaintance*, Notes on Differential Geometry and Lie Groups, Department of Computer and Information Science, University of Pennsylvania Philadelphia, PA 19104, USA*

[16] A. Besse, *Einstein manifolds*, Springer-Verlag, 1987

[17]Ali, M. S, *Moduli spaces of compact complex isotropic submanifolds in complex contact manifold*, PhD Thesis, Department of Mathematics, University of Glasgow, UK,2003

[18] Arnold, V.I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978 [Russian: Nauka, Moscow, 1974].

[19] T.N. Bailey and R.J. Baston (eds.), *Twistors in Mathematics and Physics*, Cambridge University Press, 1990

[20] R.J. Baston and M.G. Eastwood, *The Penrose transform, its interaction with representation theory*, Oxford University Press, 1989.

[21]Merkulov, S. A., Geometry of kodaira moduli spaces, *Proc.Amer. Math*.Soc. 124(5) (1996), pp.1499-1506.

[22] Kodaira, K., Complex Manifolds and Deformation of complex Structures, Heidelberg New York,Springer, 1986.

[23]Merkulov, S. A., *Moduli of Complex Legendre submanifolds of complex contact manifolds*, Mathematical Research Letters, (1),717-727, (1994).

[24] Le Brun, C., Thickness and conformal gravity, Comm Math Phys 139(1), 1-43, (1991).

[25] Merkulov, S. A., *Affine Connections on involute G-Structures.* In geometry and Physics (Aarhus, 1995) Volume 184 of lecture notes on pure and App. Math,261-274, Dekker, New York,1997.

[26] Ali, M. S., Existence of compact complex isotropic submanifolds in complex contact manifolds.The proceeding of Edinburgh Mathematical Society, Edinburgh, UK,49 pp. 551-565.2006

[27] Ali, M. S., A Completeness theorem of compact complex isotropic submanifolds in complex contact manifolds, GANIT. The journal of Bangladesh Mathematical Society,23, pp, 33-40,2003.

[28]Ali, M. S. and Ahmed K. M., Relative universal deformation of complex structures, Dhaka University journal of Science, Dhaka, September 2005.