Study of the Cosmological Model

M. Phil Thesis

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Session: 2003-2004

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April-2014

Certificate

This is to certify that the thesis entitled "Study of the Cosmological Model" is prepared by Md. Abdul Alim Miah under my direct supervision and this thesis or a part of it has not been presented or published for any degree elsewhere previously.

I recommended this thesis for awarding the M. Phil degree.

Acknowledgement

First of all, I would like to express my gratefulness to the almighty Allah who has created the universe for His endless mercy to complete the tedious thesis.

I would like to express my deepest gratitude to my honourable thesis supervisor, **Prof. Dr. Md. Nurul Islam,** Department of Mathematics, University of Dhaka for his excellent supervision, constructive guidance, proper advice, inspiration and earnest assistance throughout the tedious thesis.

I would like express my gratitude to all the honourable teachers of Department of Mathematics, University of Dhaka.

I also express my gratefulness to the authors whose books and papers help me to prepare the tedious thesis.

Abstract

The thesis consists of ten chapters. In this thesis chapter one, chapter two, chapter three, chapter four, chapter five, chapter six, chapter seven, chapter nine, chapter ten are expository and the chapter eight is our contribution.

In chapter one, we have presented Historical Cosmology, Rotating Galaxies, Inertial Frames and the Cosmological Principal, Galactic and Extragalactic Astronomy and the Cosmic Scale.

In chapter two, we have established Classical Tests of General Relativity, Black Holes, Falling into a Black Hole and Hawking Radiation.

In chapter three, we have presented Old and New Inflation, Chaotic Inflation and the Inflation as Quintessence.

In chapter four, we have established the Standard Hot Big Bang Model, CMB and the Surface of Last Scattering and the COBE Satellite.

In chapter five, we have established Schwarzschild Solution, Removing the Singularity of Schwarzschild Solution and here we also have discussed Crucial Tests in Relativity such as the Advance of Perihelion of the Mercury Planet, Gravitational Deflection of Light Rays and Shift in Spectral Lines.

In chapter six, we have presented Equivalence of Mass and Energy, Maxwell's equations and Energy Momentum Tensor $T^{\mu\upsilon}$ and its Physical Significance.

In chapter seven, we have presented Robertson-Walker Metric and Calculating R_{00}, R_{11}, R_{22}, R_{33} from Robertson-Walker Line Element and we have established Friedmann Model from Robertson-Walker Line Element such as Flat Model, Closed Model and Open Model. Here, we also have presented Einstein's Line Element-its properties, de-Sitter's Line Element-its properties and Similarity and Difference between Einstein and de-Sitter's Line Element.

In section 8.1 of chapter eight, we have presented Huge Viscous Bianchi Type-1 Cosmological Model for Barotropic Fluid and Decaying Λ with Time and here we have observed the volume expansion θ, the Hubble's parameter H, the pressure p, the deceleration parameter q, the matter energy density ρ and the cosmological parameter Λ on evolution of the universe at large time. In this chapter in section 8.2, we have presented Bianchi Type-1 Cosmological Model for Fluid Distribution and Expanding Universe and here we have observed the volume expansion θ, the Hubble's Parameter H, the pressure p, the deceleration parameter q and the matter energy density ρ on evolution of the universe at large time. In this chapter in section 8.3, we have presented Phenomenology and Accelerating Universe with Time Variable Λ and here we have observed the parameter ω, the decelerating parameter q, the pressure p, the matter energy density ρ and the cosmological parameter Λ on the phenomenological evolution of the universe at large time. This is our contributory chapter.

In chapter nine, we have established the Conception of Albert Einstein about Hubble's Cosmology, the Conception of Stephen Hawking about Hubble's Cosmology, Hubble's Law, Hubble's Time and Radius, Hubble's Constant and the Changing Views of Hubble about Cosmology.

In chapter ten, we have discussed First Frame, Second Frame, Third Frame, Fourth Frame, Fifth Frame and Sixth Frame of the Early Universe.

Declaration

I hereby declare that the thesis entitled "Study of the Cosmological Model" is prepared by me under the supervision of honourable **Prof. Dr. Md. Nurul Islam,** Department of Mathematics, University of Dhaka and this thesis or a part of it has not been submitted for any degree elsewhere previously.

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Chapter One

Cosmology from Newton to Hubble

1.1 Introduction [5]:

The history of ideas on the structure and origin of the universe shows that humankind has always put itself at the centre of creation. As astronomical evidence has accumulated, these anthropocentric convictions have had to be abandoned one by one. From the natural idea that the solid earth is at rest and the celestial objects all rotate around us, we have come to understand that we inhabit an average sized planet orbiting an average sized sun, that the solar system is in the periphery of a rotating galaxy of average size, flying at hundreds of kilometers per second towards an unknown goal in an immense universe, containing billions of similar galaxies.

Cosmology aims to explain the origin and evolution of the entire contents of the universe, the underlying physical processes and thereby to obtain a deeper understanding of the laws of physics assumed to hold throughout the universe. Unfortunately, we have only one universe to study, the one we live in and we cannot make experiments with it, only observations. This puts serious limits on what we can learn about the origin. If there are other universes, we will never know. Although the history of cosmology is long and fascinating, we shall not trace if in detail, nor any further back than Newton, accounting only for those ideas which have fertilized modern cosmology directly or which happened to be right although they failed to earn timely recognition. In the early days of cosmology when little was known about the universe, the field was really just a branch of philosophy.

Having a rigid earth to stand on is a very valuable asset. How can we describe motion except in relation to a fixed point? Important understanding has come from the study of inertial systems, in uniform motion with respect to one another. From the work of Einstein on inertial systems, the theory of special relativity was born. We discuss inertial frames and see how expansion and contraction are natural consequences of the homogeneity and isotropy of the universe.

A classic problem is why the night sky is dark and not blazing like the disc of the sun, as simple theory in the past would have it. We describe Newton's theory of gravitation which is the earliest explanation of a gravitational force. We shall modernize it by introducing Hubble's law into it. In face, we shall see that this leads to a cosmology which already contains many features of current Big Bang cosmologies.

1.2 Historical Cosmology [1], [5]:

At the time of Isaac Newton (1642-1727) the heliocentric universe of Nicolaus Copernicus (1473-1543), Galileo Galilei (1564-1642) and Johannes Kepler (1571-1630) had been accepted because no sensible description of the motion of the planets could be found if the earth was at rest at the centre of the solar system. Humankind was thus dethroned to live on an average sized planet orbiting around an average sized sun.

The stars were understood to be suns like ours with fixed positions in a static universe. The Milky Way had been resolved into an accumulation of faint stars with the telescope of Galileo. The anthropocentric view still persisted, however in locating the solar system at the centre of the universe.

Newtonian Cosmology (1642-1727):

The first theory of gravitation appeared when Newton published his Philosophiae Naturalis Principia Mathematica in 1687. With this theory he could explain the empirical laws of Kepler that the planets moved in elliptical orbits with the sun at one of the focal points. An early success of this theory came when Edmund Halley (1656-1742) successfully predicted that the comet sighted in 1456, 1531, 1607 and 1682 would return in 1758. Actually, the first observation confirming the heliocentric theory came in 1727 when James Bradley (1693-1762) discovered the aberration of starlight and explained it as due to the changes in the velocity of the earth in its annual orbit. In our time, Newton's theory of gravitation still suffices to describe most of planetary and satellite mechanics and it constitutes the non-relativistic limit of Einstein's relativistic theory of gravitation.

Newton considered the stars to be suns evenly distributed throughout infinite space in spite of the obvious concentration of stars in the Milky Way. A distribution is called homogeneous if it is uniformly distributed and it is called isotropic, if it has the same properties in all spatial directions. Thus in a homogeneous and isotropic space the distribution of matter would look the same to observers located anywhere, no point would be preferential. Each local region of an isotropic universe contains information which remains true also on a global scale. Clearly, matter introduces lumpiness which grossly violates homogeneity on the scale of stars but on some larger scale isotropy and homogeneity may still be a good approximation. Going one step further, one may postulate what is called the cosmological principle or sometimes the Copernican principle. The universe is homogeneous and isotropic in three dimensional space, has always been so and will always remain so.

It has always been debated whether this principle is true and on what scale. On the galactic scale visible matter is lumpy and on larger scales galaxies form gravitationally bound clusters and narrow strings separated by voids. But galaxies also appear to form loose groups of three to five or more galaxies. Several surveys have now reached agreement that the distribution of these galaxy groups appears to be homogeneous and isotropic within a sphere of 170 Mpc radius. This is an order of magnitude larger than the supercluster to which our galaxy and our local galaxy group belong and which is centred in the constellation of Virgo.

Based on his theory of gravitation, Newton formulated a cosmology in 1691. Since all massive bodies attract each other, a finite system of stars distributed over a finite region of space should collapse under their mutual attraction. But this was not observed, in fact the stars were known to have had fixed positions since antiquity and Newton sought a reason for this stability. He concluded, erroneously that the self gravitation within a finite system of stars would be compensated for by the attraction of a sufficient number of stars outside the system, distributed evenly throughout infinite space. However, the total number of stars could not be infinite because then their attraction would also be infinite, making the static universe unstable. It was understood only much later that the addition of external layers of stars would have no influence on the dynamics of the interior. The right conclusion is that the universe cannot be static, an idea which would have been too revolutionary at the time.

Newton's contemporary and competitor Gottfried Wilhelm Von Leibnitz (1646-1716) also regarded the universe to be spanned by an abstract infinite space but in contrast to Newton he maintained that the stars must be infinite in number and distributed all over space, otherwise the universe would be bounded and have a centre, contrary to contemporary philosophy. Finiteness was considered equivalent to boundedness and infinity to unboundedness.

1.3 Rotating Galaxies [5], [6]:

The first description of the Milky Way as a rotating galaxy can be traced to Thomas Wright (1711-1786) who wrote An Original Theory or New Hypothesis of the universe in 1750, suggesting that the stars are all moving the same way and not much deviating from the same plane, as the planets in their heliocentric motion do round the solar body.

Wright's galactic picture had a direct impact on Immanuel Kant (1724-1804). In 1755 Kant went a step further, suggesting that the diffuse nebulae which Galileo had already observed could be distant galaxies rather than nearby clouds of incandescent gas. This implied that the universe could be homogeneous on the scale of galactic distances in support of the cosmological principle.

Kant also pondered over the reason for transversal velocities such as the movement of the moon. If the Milky Way was the outcome of a gaseous nebula contracting under Newton's law of gravitation, why was all movement not directed towards a common centre? Perhaps there also existed repulsive forces of gravitation which would scatter bodies onto trajectories other than radial ones and perhaps such forces at large distances would compensate for the infinite attraction of an infinite number of stars? It is noted that the idea of a contracting gaseous nebula constituted the first example of a non-static system of stars but at galactic scale with the universe still static.

Kant thought that he had settled the argument between Newton and Leibnitz about the finiteness or infiniteness of the system of stars. He claimed that either type of system embedded in an infinite space could not be stable and homogeneous and thus the question of infinity was irrelevant.

The infinity argument was not properly understood until Bernhard Riemann (1826-1866) pointed out that the world could be finite yet unbounded, provided the geometry of the space had a positive curvature, however small. On the basis of Riemann's geometry, Albert Einstein (1879-1955) subsequently established the connection between the geometry of space and the distribution of matter.

Kant's repulsive force would have produced trajectories in random directions but all the planets and satellites in the solar system exhibit transversal motion in one and the same direction. This was noticed by Pierre Simon de Laplace (1749-1827) who refuted Kant's hypothesis by a simple probabilistic argument in 1825, the observed movements were just too improbable if they were due to random scattering by a repulsive force. Laplace also showed that the large transversal velocities and their direction had their origin in the rotation of the primordial gaseous nebula and the law of conservation of angular momentum. Thus no repulsive force is needed to explain the transversal motion of the planets and their moons, no nebula could contract to a point and the moon would not be expected to fall down upon us.

1.4 Inertial Frames and the Cosmological Principle [5]:

Newton's first law, the law of inertia, states that a system on which no forces act is either at rest or in uniform motion. Such systems are called inertial frames. Accelerated or rotating frames are not inertial frames. Newton considered that at rest and in motion implicitly referred to an absolute space which was unobservable but which had a real existence independent of humankind. Mach rejected the notion of an empty, unobservable space and only Einstein was able to clarify the physics of motion of observers in inertial frames.

It may be interesting to follow a non-relativistic argument about the static or non-static nature of the universe which is a direct consequence of the cosmological principle.

Consider an observer 'A' in an inertial frame who measures the density of galaxies and their velocities in the space around him. Because the distribution of galaxies is observered to be homogeneous and isotropic on very large scales, he would see the same mean density of galaxies (at one time t) in two different directions r and r'.

$$\rho_A(r,t) = \rho_A(r',t)$$

Another observer 'B' in another inertial frame looking in the direction r from her location would also see the same mean density of galaxies,

$$\rho_B(r',t) = \rho_A(r,t)$$

The velocity distributions of galaxies would also look the same to both observers, in fact in all directions, for instance in the r' direction,

$$\upsilon_B(r',t) = \upsilon_A(r',t)$$

Suppose that the B frame has the relative velocity $\upsilon_A\ (r'',t)$ as seen from the A frame along the radius vector $r'' = r - r'$. If all velocities are non-relativistic, i.e., small compared with the speed of light, we can write

$$\upsilon_A(r',t) = \upsilon_A(r - r'', t) = \upsilon_A(r,t) - \upsilon_A(r'',t)$$

This equation is true only if $\upsilon_A(r, t)$ has a specific form, it must be proportional to r,

$$\upsilon_A(r,t) = f(t)r \qquad \text{.............................} \quad (1)$$

where $f(t)$ is an arbitrary function.

Let this universe start to expand. From the vantage point of A (or B equally well, since all points of observation are equal), nearby galaxies will appear to recede slowly.

But in order to preserve uniformity, distant ones must recede faster, in fact their recession velocities must increase linearly with distance. That is the content of equation (1).

If $f(t) > 0$, the universe would be seen by both observers to expand, each galaxy having a radial velocity proportional to its radial distance r. If $f(t) < 0$, the universe would be seen to contract with velocities in the reversed direction. Thus we have seen that expansion and contraction are natural consequences of the cosmological principle. If $f(t)$ is a positive constant and then equation (1) is Hubble's law.

Actually, it is somewhat misleading to say that the galaxies recede when rather, it is space itself which expands or contracts. This distinction is important when we come to general relativity. A useful lesson may be learned from studying the limited gravitational system consisting of the earth and rockets launched into space. This system is not quite like the previous example because it is not homogeneous and because the motion of a rocket or a satellite in earth's gravitational field is different from the motion of galaxies in the gravitational field of the universe. Thus to simplify the case, we only consider radial velocities and we ignore earth's rotation. Suppose the rockets have initial velocities low enough to make them fall back onto earth. The rocket earth gravitational system is then closed and contracting, corresponding to $f(t) < 0$.

When the kinetic energy is large enough to balance gravity, our idealized rocket becomes a satellite, staying above earth at a fixed height. This corresponds to the static solution $f(t) = 0$ for the rocket earth gravitational system.

If the launch velocities are increased beyond about 11 kms^{-1}, the potential energy of earth's gravitational field no longer suffices to keep the rockets bound to earth. Beyond this speed, called the second cosmic velocity by rocket engineers, the rockets escape for good. This is an expanding or open gravitational system, corresponding to $f(t) > 0$.

The static case is different if we consider the universe as a whole. According to the cosmological principle, no point is preferred and therefore there exists no centre around which bodies can gravitate in steady state orbits. Thus the universe is either expanding or contracting, the static solution being unstable and therefore unlikely.

1.5 Galactic and Extragalactic Astronomy[5]:

Newton should also be credited with the invention of the reflecting telescope, he even built one and the first one of importance was built a century later by William Herschel (1738-1822). With this instrument, observational astronomy took a big leap forward. Herschel and his son John could map the nearby stars well enough in 1785 to conclude correctly that the Milky Way was a disc shaped star system. They also concluded erroneously that the solar system was at its centre but many more observations were needed before it was corrected. Herschel made many important discoveries, among them the planet Uranus and some 700 binary stars whose movements confirmed the validity of Newton's theory of gravitation outside the solar system. He also observed some 250 diffuse nebulae which he first believed were distant galaxies but which he and many other astronomers later considered to be nearby incandescent gaseous clouds belonging to our Galaxy. The main problem was then to explain why they avoided the directions of the galactic disc, since they were evenly distributed in all other directions.

The view of Kant that the nebulae were distant galaxies was also defended by Johann Heinrich Lambert (1728-1777). He came to the conclusion that the solar system along, with the other stars in our Galaxy, orbited around the galactic centre, thus departing from the heliocentric view. The correct reason for the absence of nebulae in the galactic plane was only given by Richard Anthony Proctor (1837-1888) who proposed the presence of interstellar dust. The arguments for or against the interpretation of nebulae as distant galaxies nevertheless raged throughout the 19th century because it was not understood how stars in galaxies more luminous than the whole galaxy could exist, these were observations of supernovae. Only in 1925 did Edwin Powell Hubble (1889-1953) resolve the conflict indisputably by discovering

Cepheid's and ordinary stars in nebulae and by determining the distance to several galaxies, among them the celebrated M31 galaxy in the Andromeda. Although this distance was off by a factor of two, the conclusion was qualitatively correct.

In spite of the work of Kant and Lambert, the heliocentric picture of the Galaxy or almost heliocentric since the sun was located quite close to Herschel's galactic centre. A decisive change came with the observations in 1915-1919 by Harlow Shapley (l895-1972) of the distribution of globular clusters hosting 10^5 to 10^7 stars. He found that perpendicular to the galactic plane they were uniformly distributed but along the plane these clusters had a distribution which peaked in the direction of the Sagittarius. This defined the centre of the galaxy to be quite far from the solar system and we are at a distance of about two-thirds of the galactic radius. Thus the anthropocentric world picture received its second blow and not the last one, if we count Copernicus's heliocentric picture as the first one. It is noted that Shapley still believed our galaxy to be at the centre of the astronomical universe.

1.6 The Cosmic Scale[5]:

The size of the universe is unknown and unmeasurable but if it undergoes expansion or contraction, it is convenient to express distances at different epochs in terms of a cosmic scale R(t) and denote its present value $R_o = R(t_o)$. The value of R(t) can be chosen arbitrarily, so it is often more convenient to normalized it to its present value and thereby define a dimensionless quantity, the cosmic scale factor.

$$a(t) = R(t) / R_o \quad \text{......................................(1)}$$

The cosmic scale factor affects all distances, for instance the wave length λ of light emitted at one time t and observed as λ_o at another time t_o.

$$\frac{\lambda_0}{R_0} = \frac{\lambda}{R(t)}$$

Let us find an approximation for $a(t)$ at times $t < t_o$ by expanding it to first order time differences.

$$a(t) \approx 1 - \dot{a}_o (t_o - t) \text{ ....(2)}$$

using the notation a_0 for $\dot{a}$ (t_o) and r=c(t_o-t) for the distance to the source. The cosmological red shift can be approximated by

$$z = \frac{\lambda_o}{\lambda} - 1 = a^{-1} - 1 = \dot{a}_o \frac{r}{c} \text{ ....(3)}$$

Thus $\frac{1}{1+z}$ is a measure of the scale factor a (t) at the time when a source emitted the red shifted radiation. We get from equation from (3) and equation $Z = H_o \frac{r}{c}$.

We obtain

$$\dot{a}_o = \frac{\dot{R}_o}{R_o} = H_o \text{ ....(4)}$$

Chapter Two
Gravitational Phenomena

2.1 Classical Tests of General Relativity [5], [7], [9]:

The classical testing ground of theories of gravitation, Einstein's among them is celestial mechanics within the solar system. Ideally one should consider the full many body problem of the solar system, a task which one can readily characterize as impossible. Already the relativistic two body problem presents extreme mathematical difficulties. Therefore, all the classical tests treated only the one body problem of the massive sun influencing its surroundings.

The earliest phenomenon requiring general relativity for its explanation was noted in 1859, 20 years before Einstein's birth. The French astronomer Urban Le Verrier (1811-1877) found that something was wrong with the planet Mercury's elongated elliptical orbit. As the innermost planet it feels the solar gravitation is very strong but the orbit is also perturbed by the other planets. The total effect is that the elliptical orbit is non-stationary, it precesses slowly around the sun. The locus of Mercury's orbit nearest the sun, the perihelion, advances 574" per century. This is calculable using Newtonian mechanics and Newtonian gravity but the result is only 531", 43" too little. Le Verrier, who had already successfully predicted the existence of Neptune from perturbations in the orbit of Uranus, suspected that the discrepancy was caused by a small undetected planet inside Mercury's orbit which he named Vulcan. That prediction was never confirmed. With the advent of general relativity the calculations could be remade. This time the discrepant 43" were successfully explained by the new theory which thereby gained credibility. This counts as the first one of three classical tests of general relativity. For details on this test as well as on most of the subsequent tests.

Also, the precessions of venus and earth have been put to similar use and within the solar system many more consistency tests have been done, based on measurements of distances and other orbital parameters.

The second classical test was the predicted deflection of a ray of light passing near the sun. The third classical test was the gravitational shift of atomic spectra, first observed by John Evershed in 1927. The frequency of emitted radiation makes atoms into clocks. In a strong gravitational field these clocks run slower, so the atomic spectra shift towards lower frequencies. Evershed observed the line shifts in a cloud of plasma ejected by the sun to an elevation of about 72000 km above the photosphere and found an effect only slightly larger than that predicted by general relativity. Modern observations of atoms radiating above the photosphere of the sun have improved on this result, finding agreement with theory at the level of about 2.1×10^{-6}. Similar measurements have been made in the vicinity of more massive stars such as Sirius.

Since then, many experiments have studied the effects of changes in a gravitational potential on the rate of a clock or on the frequency of an electromagnetic signal. Clocks have been put in towers or have travelled in rockets and satellites. The so called fourth test of general relativity which was conceived by *I.I* Shapiro in 1964 and carried out successfully in 1971 and later, deserves a special mention. This is based on the prediction that an electromagnetic wave suffers a time delay when traversing an increased gravitational potential.

The fourth test was carried out with the radio telescopes at the Haystack and Arecibo observatories by emitting radar signals towards Mercury, Mars and notably, Venus, through the gravitational potential of the sun. The round trip time delay of the reflected signal was compared with theoretical calculations. Further refinement was achieved later by posing the Viking Lander on the Martian surface and having it participate in the experiment by receiving and retransmitting the radio signal from earth. This experiment found the ratio of the delay observed to the delay predicted by general relativity to be 1.000 ± 0.002.

It is noted that the expansion of the universe and Hubble's linear law are not tests of general relativity. Objects observed at wave lengths ranging from radio to gamma rays are close to isotropically distributed over the sky. Either we are close to a centre of spherical symmetry, an anthropocentric view or the universe is close to homogeneous.

2.2 Black Holes[5]:

The Schwarzachild Metric:

Suppose that we want to measure time t and radial elevation r in the vicinity of a spherical star of mass M in isolation from all other gravitational influences. Since the gravitational field varies with elevation, these measurements will surely depend on r. The spherical symmetry guarantees that the measurements will be the same on all sides of the star and thus they are independent of θ and φ. The metric does not then contain $d\theta$ and dφ terms. Let us also consider that we have stable conditions that the field is static during our observations, so that the measurements do not depend on t.

The metric is then not flat but the 00 time-time component and the 11 space-space component must be modified by some functions of r. Thus it is of the form

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 \quad \text{.............................} (1)$$

where B(r) and A(r) have to be found by solving of the Einstein equations.

Far away from the star the space-time is fiat. This gives us the asymptotic conditions

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \quad \text{...................................}(2)$$

Newtonian limit of g_{00} is known. Here B(r) plays the role of g_{00}, thus we have

$$B(r) = 1 - \frac{2GM}{c^2 r} \quad \text{..} (3)$$

To obtain A(r) from the Einstein equations is more difficult and we shall not go to the trouble of deriving it. The exact solution found by Karl Schwarzachild (1873-1916) in 1916 preceded any solution found by Einstein himself. The result is simply

$$A(r) = B(r)^{-1} \quad \text{..} (4)$$

These functions clearly satisfy the asymptotic conditions (2).

Let us introduce the concept of Schwarzschild radius r_c for a star of mass M, defined by $B(r_c) = 0$. It follows that

$$r_c = \frac{2GM}{c^2} \quad \text{.................................} (5)$$

The physical meaning of r_c is the following. Consider a body of mass m and radial velocity υ attempting to escape from the gravitational field of the star. To succeed, the kinetic energy must overcome the gravitational potential. In the non-relativistic case the condition for this is

$$\frac{1}{2} m\upsilon^2 \geq GMm/r \quad \text{........................} (6)$$

The larger the ratio M/r of the star, the higher the velocity required to the escape. Ultimately, in the ultra-relativistic case when $\upsilon = c$, only light can escape. At that point a non-relativistic treatment is no longer justified. Nevertheless, it just so happens that the equality in equation (6) fixes the radius of the star correctly to be precisely r_c as defined above. Because nothing can escape the interior of r_c, not even light, John A. Wheeler coined the term black hole for it in 1967. It is noted that the escape velocity of objects on earth is 11 kms^{-1}, on the sun, it is 2.2×10^6 kmh^{-1} but on a black hole, it is c.

This is the simplest kind of a Schwarzschild black hole and r_c defines its event horizon. Inserting r_c into the functions A and B the Schwarzschild metric becomes

$$d\tau^2 = \left(1 - \frac{r_c}{r}\right) dt^2 - \left(1 - \frac{r_c}{r}\right)^{-1} \frac{dr^2}{c^2} \quad \text{....................} (7)$$

2.3 Falling Into a Black Hole [5]:

The Schwarzschild metric has very fascinating consequences. Consider a spacecraft approaching a black hole with apparent velocity $\upsilon = dr/dt$ in the fixed frame of an outside observer. Light signals from the spacecraft travel on the light cone, $d\tau = 0$, so that

$$\frac{dr}{dt} = c\left(1 - \frac{r_c}{r}\right) \text{ .... (8)}$$

Thus the spacecraft appears to slow down with decreasing r, finally coming to a full stop as it reaches $r = r_c$.

No information can ever reach the outside observer beyond the event horizon. The reason for this is the mathematical singularity of dt in the expression

$$c\,dt = \frac{dr}{1 - r_c / r} \text{ .... (9)}$$

The time intervals dt between successive crests in the wave of the emitted light become longer, reaching infinite wave length at the singularity. Thus the frequency ν of the emitted photons goes to zero and the energy E = hν of the signal vanishes. One cannot receive signals from beyond the event horizon because photons cannot have negative energy. Thus the outside observer sees the spacecraft slowing down and the signals red shifting until they cease completely.

The pilot in the spacecraft uses local co-ordinates, so he sees the passage into the black hole entirely differently. If he started out at distance r_o with velocity dr/dt = 0 at time t_o, he will have reached position r at proper time τ which we can find by integrating dτ in equation (7) from 0 to τ.

$$\int_0^\tau \sqrt{d\tau^2} = \tau = \int_{r_o}^{r} \left[\frac{1 - r_c / r}{(dr/dt)^2} - \frac{1}{c^2(1 - r_c / r)} \right]^{1/2} dr \text{ .... (10)}$$

The result depends on dr/dt which can only be obtained from the equation of motion. The pilot considers that he can use Newtonian mechanics, so he may take

$$\frac{dr}{dt} = c\sqrt{\frac{r_c}{r}}$$

The result is then

$$\tau \propto (r_o - r)^{3/2} \text{ ....(11)}$$

However, many other expressions for dr/dt also make the integral in equation (10) converge.

Thus the singularity at r_c does not exist to the pilot, his comoving clock shows finite time when he reaches the event horizon. Once across r_c the spacecraft reaches the centre of the black hole rapidly. For a hole of mass $10M_\odot$ this final passage lasts about 10^{-4}s. The fact that the singularity at r_c does not exist in the local frame of the spaceship indicates that the horizon at r_c is a mathematical singularity and not a physical singularity. The singularity at the horizon arises because we are using, in a region of extreme curvature, co-ordinates most appropriate for flat or mildly curved space-time. Alternate co-ordinates, more appropriate for the region of a black hole and in which the horizon does not appear as a singularity were invented by Eddington (1924) and rediscovered by Finkelstein (1958).

Although this spacecraft voyage is pure science fiction, we may be able to observe the collapse of a supernova into a black hole. Just as for the spacecraft, the collapse towards the Schwarzschild radius will appear to take a very long time. Towards the end of it, the ever red shifting light will fade and finally disappear completely.

It is noted from the metric equation (7) that inside r_c the time term becomes negative and the space term positive, thus space becomes timelike and time spacelike. The implications of this are best understood if one considers the shape of the light cone of the spacecraft during its voyage. Outside the event horizon the future light cone contains the outside observer who receives signals from the spacecraft. Nearer r_c the light cone narrows and the slope dr/dt steepens because of the approaching singularity in expression on the right hand side of equation (8). The portion of the future space-time which can receive signals therefore diminishes.

Since the time and space axes have exchanged positions inside the horizon, the future light cone is turned inwards and no part of the outside spare-time is included in the future light cone. The slope of the light cone is vertical at the horizon. Thus it defines, at the same time, a cone of zero opening angle around the original time axis and a cone of 180^0 around the final time axis, encompassing the full space-time of the black bole. As the spacecraft approaches the centre, dt/dr decreases, defining a narrowing opening angle which always contains the centre.

2.4 Radiation of Hawking [5], [10]:

Stephen Hawking has shown that although no light can escape from black holes, they can nevertheless radiate if one takes quantum mechanics into account. It is a property of the vacuum that particle-antiparticle pairs such as $e^- e^+$ are continuously created out of nothing, to disappear in the next moment by annihilation which is the inverse process. Since energy cannot be created or destroyed, one of the particles must have positive energy and the other one an equal amount of negative energy. They form a virtual pair, neither one is real in the sense that it could escape to infinity or be observed by us.

In a strong electromagnetic field the electron e^- and the positron e^+ may become separated by a Compton wave length λ of the order of the Schwarzschild radius r_c.

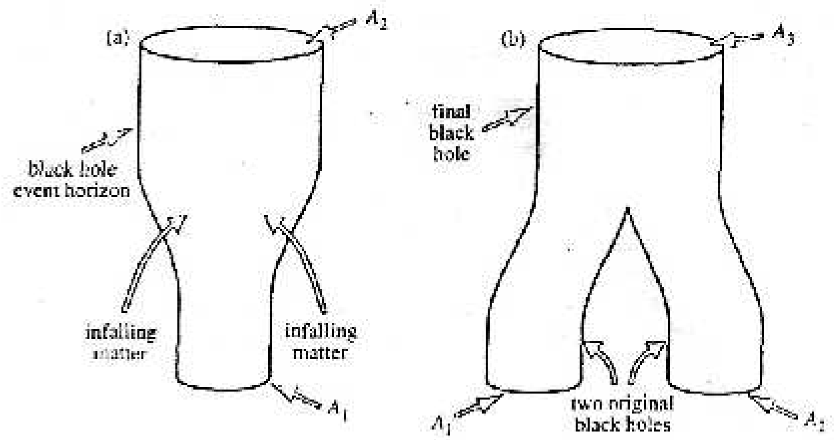


Fig. 2.1

Figure-2.1: When we throw matter into a black hole, or allow two black holes to merge, the total area of the event horizons will never decrease, (a) $A_2 \geq A_1$, (b) $A_3 \geq A_1 + A_2$.

Hawking has shown that there is a small but finite probability for one of them to tunnel through the barrier of the quantum vacuum and escape the black hole horizon as a real particle with positive energy, leaving the negative energy particle inside the horizon of the hole. Since energy must be conserved the hole loses mass in this process, a phenomenon called Hawking radiation. The timescale of complete evaporation is

$$t = 10Gy\tau\left(\frac{M}{10^{12}\kappa g}\right)^3 \text{ .... (16)}$$

Thus small black holes evaporate fast, whereas heavy ones may have lifetimes exceeding the age of the universe. The analogy with entropy can be used even further. A system in thermal equilibrium is characterized by a unique temperature T throughout. When Hawking applied quantum theory to black holes, he found that the radiation emitted from particle-antiparticle creation at the event horizon is exactly thermal. The rate of particle emission is as if the hole were a black body with a unique temperature proportional to the gravitational field on the horizon, the Hawking temperature,

$$T_H = \frac{1}{8\pi GM} = 6.15\times10^{-8}\frac{M_\odot}{M}K \text{ ....(17)}$$

Chapter Three
Cosmic Inflation

3.1 Introduction:

The standard FLRW Big Bang model describes an adiabatically expanding universe, having a beginning of space and time with nearly infinite temperature and density. This model has, as so far presented, been essentially a success story. But the Big Bang assumes very problematic initial conditions for instance where did the 10^{90} particles which make up the visible universe come from? We are now going to correct that optimistic picture and present a remedy of cosmic inflation.

3.2 Old and New Inflation [5], [11] :

The earliest time after the Big Bang, we can meaningfully consider is Planck time t_p because earlier than that the theory of gravitation must be married to quantum field theory, a task which has not yet been mastered. Let us assume that the r_p sized universe then was pervaded by a homogeneous scalar classical field φ, the inflation field and that all points in this universe were causally connected. The idea with inflation is to provide a mechanism which blows up the universe so rapidly and to such an enormous scale that the causal connection between its different parts is lost, yet they are similar due to their common origin. This should solve the horizon problem and dilute the monopole density to acceptable values as well as flatten the local fluctuations to near homogeneity.

Guth's Scenario:

Let us try to make this idea more quantitative. Suppose that the mass m_φ of the inflation carrying the field φ was much lighter than the Planck mass m_p,

$$0 < m_\varphi << m_p \qquad (1)$$

so that the inflation can be considered to be massless. In fact, the particle symmetry at Planck time is characterized by all fields except the inflation field being exactly massless. Only when this symmetry is spontaneously broken in the transition to a lower temperature phase do some particles become massive.

Let us introduce the potential $V(\varphi, T)$ of the scalar field at temperature T. Its φ dependence is arbitrary but we could take it to be a power function of φ. Suppose that the potential at time t_p has a minimum at a particular value φ_p. The universe would then settle in that minimum, given enough time and the value φ_p would gradually pervade all of space-time. It would be difficult to observe such a constant field because it would have the same value to all observers, regardless of their frame of motion. Thus the value of the potential $V(\varphi_p, T_p)$ may be considered as a property of the vacuum.

Suppose that the minimum of the potential is at $\varphi_p = 0$ in some region of space-time and it is non-vanishing.

$$|V(0, T_p)| > 0 \quad \text{.................................(2)}$$

An observer moving along a trajectory in space-time would notice that the field fluctuates around its vacuum expectation value

$$\langle \varphi_p \rangle = 0$$

and the potential energy consequently fluctuates around the mean vacuum-energy value

$$\langle V(0, T_p) \rangle > 0$$

This vacuum energy contributes a repulsive energy density to the total energy density in Friedmann's equation, acting just as dark energy or as a cosmological constant, if we make the identification

$$\frac{1}{3} 8\pi G \langle V_0 \rangle = \frac{1}{3} \lambda \quad \text{................................ (3)}$$

where $V_0 = V(0, 0)$ is a temperature-independent constant.

Inflation occurs when the universe is dominated by the inflation field φ. We shall restrict our considerations to theories with a single inflation field. Inflationary models assume that there is a moment when this domination starts and subsequently drives the universe into a de-Sitter like exponential expansion in which $T \simeq 0$. Alan Guth in 1981, named this an inflationary universe.

The timescale for inflation is

$$H = \sqrt{\frac{8\pi G}{3}\langle V_0 \rangle} \propto \frac{\sqrt{hc}}{M_P} \simeq (10^{-34} s)^{-1} \text{.......................} (4)$$

Clearly the cosmic inflation cannot go on forever if we want to arrive a our present slowly expanding Friedmann-Lemaitre universe. Thus there must be a mechanism to halt the exponential expansion, a graceful exit. The freedom we have to arrange this is in the choice of the potential function V(φ,T) at different temperature T.

3.3 Chaotic Inflation [5], [12], [13] :

Initial Conditions:

Guth's model made the rather specific assumption that the universe started out with the vacuum energy in the false minimum φ = 0 at time t_p. However, Linde pointed out that this value as well as, any other fixed starting value, is as improbable as complete homogeneity and isotropy because of the quantum fluctuations at Planck time. Instead, the scalar field may have had some random starting value φ_a which could be assumed to be fairly uniform across the horizon of size N_p^{-1} changing only by an amount

$$\Delta\varphi_a \simeq M_p << \varphi_a \text{................................} (1)$$

Regions of higher potential would expand faster and come to dominate. With time the value of the field would change slowly until it finally reached φ_o at the true minimum $V(\varphi_0)$ of the potential.

But causally connected spaces are only of size M_p^{-1} so even the metric of space-time may be fluctuating from open to closed in adjacent spaces of this size. Thus the universe can be thought of as a chaotic foam of causally disconnected bubbles in which the initial conditions are different and which would subsequently evolve into different kinds of universes. Only one bubble would become our universe and we could never get any information about the other ones.

According to Heisenberg's uncertainty relation, at a timescale $\Delta t = \hbar/M_p c^2$ the energy is uncertain *by* an amount

$$\Delta E > \frac{\hbar}{\Delta t} = M_p c^2 \text{ .... (2)}$$

Let us for convenience work in units common to particle physics where ħ=c=1. Then the energy density is uncertain by the amount

$$\Delta\rho = \frac{\Delta E}{(\Delta r)^3} = \frac{\Delta E}{(\Delta t)^3} = N_p^4 \text{ .... (3)}$$

Thus there is no reason to assume that the potential $V(\varphi_a)$ would be much smaller than M_p^4. We may choose a general parametrization for the potential,

$$V(\varphi) = \frac{k\varphi^n}{nM_p^{n-4}} \approx M_p^4 \text{ ....(4)}$$

where $n > 0$ and $0 < \kappa << 1$. This assumption ensures that $V(\varphi_a)$ does not rise too steeply with φ. For $n = 4$ it then follows that

$$\varphi_a \simeq \left(\frac{4}{k}\right)^{1/4} M_p >> M_p \text{ .... (5)}$$

when the free parameter κ is chosen to be very small.

A large number of different models of inflation have been studied in the literature. Essentially they differ in their choice of potential function.

3.4 The Inflation as Quintessence[5]:

We have met two cases of scalar fields causing expansion, the inflation field acting before t_{GUT} and the quintessence field describing present day dark energy. It would seem economical if one and the same scalar field could do both jobs. Then the inflation field and quintessence would have to be matched at some time later than t_{GUT}. This seems quite feasible since on the one hand, the initially dominating inflation potential V(φ) must give way to the background energy density $\rho_r + \rho_m$ as the universe cools and on the other hand, the dark energy density must have been much smaller than the background energy density until recently. Recall that quintessence models are constructed to be quite insensitive to the initial conditions.

On the other hand, nothing forces the identification of the inflation and quintessence fields. The inflationary paradigm in no way needs nor predicts quintessence.

In the previously described models of inflation, the inflation field φ settled to oscillate around the minimum $V(\varphi = 0)$ at the end of inflation. Now, we want the inflation energy density to continue a monotonic roll down toward zero, turning ultimately into a minute but non-vanishing quintessence tail. The global minimum of the potential is only reached in a distant future, $V(\varphi \rightarrow \infty) \rightarrow 0$. In this process the inflation does not decay into a thermal bath of ordinary matter and radiation because it does not interact with particles at all, it is said to be sterile. A sterile inflation field avoids violation of the equivalence principle, otherwise the interaction of the ultralight quintessence field would correspond to a new long range force. Entropy in the matter fields comes from gravitational generation at the end of inflation rather than from decay of the inflation field.

The task is then to find a potential V(φ) such that it has two phases of accelerated expansion from t_p to t_{end} at the end of inflation and from a time $t_F \approx t_{GUT}$ when the instant on field freezes to a constant value until now, t_O. Moreover, the inflation energy density must decrease faster than the background energy density, equalling it at some time t_* when the field is φ_* and thereafter remaining subdominant to the energy density of the particles produced at t_{end}. Finally it must catch up with a tracking potential at some time during matter domination, $t > t_{eq}$.

The mathematical form of candidate potentials is of course very complicated and it would not be very useful to give many examples here. However, it is instructive to follow through the physics requirements on φ and V(φ) from inflation to present.

Chapter Four
Cosmic Microwave Background

4.1 Introduction [5]:
The cosmic microwave background (CMB) which is a consequence of the hot Big Bang and the following radiation dominated epoch was discovered in 1964. The hot Big Bang also predicts that the Cosmic Microwave Background (CMB) radiation should have a blackbody spectrum.

4.2 The Standard Hot Big Bang Model [8]:
Cosmology is a scientific attempt to answer fundamental questions of mythical proportion. How did the universe come to be? How did it evolve? How will it end? Over the past century progress has been made towards answering these questions and has resulted in a standard hot Big Bang model describing the evolution of the universe. This model provides a consistent framework into which all the relevant cosmological data seem to fit and is the dominant paradigm against which all new ideas are tested.

The Big Bang model of the universe is based on the following observations.

(i) The universe is expanding

(ii) On the largest scales the universe is isotropic and homogeneous

(iii) The universe is filled with microwave photons coming from all directions

(iv) The universe is composed of $\approx$75% hydrogen and $\approx$25% helium

Each observation and its implications are discussed separately below.

(i) The universe is expanding:
In the early part of this century cosmology came of age when the nebulae were found to be galaxies external to our Milky Way Galaxy and that curiously, they were all receding from us. This universal recession was interpreted as the expansion of the universe and was codified in Hubble's law $\upsilon = Hd$. That is, the recession velocity υ of a galaxy is proportional to its distance d from us and H is Hubble's constant. The implications of an expanding universe were profound. The universe could no longer be considered static. It must have been smaller, denser and hotter in the past and this implies a finite age for the universe. So far no objects older than 20 billion years have

been found. The dark night sky is also evidence for the finite age of the observable universe(Harrison 1987). A finite age however carries with it the notion of a creation event or a Big Bang. The origin of the universe, t = 0, seems to be the Achilles heel of the model. Is there a singularity at t = 0? What happens before that? Strictly speaking however, the Big Bang origin of the universe at t =0 is not part of the Big Bang model.

(ii) On the largest scales the universe is isotropic and homogeneous :

The universe looks the same in all directions and the matter seems to be smoothly distributed. Observations of the CMB, the x-ray background and deep radio surveys provide solid evidence for the isotropy. Tests of homogeneity are more difficult since they require three dimensional information. The largest galaxy red shift surveys may be beginning to see homogeneity however the assumption of homogeneity has been based as much on mathematical convenience as observational evidence. Einstein's equations have a simple isotropic and homogeneous solution known as Friedmann's equation. It is the dynamical equation relating a universal scale factor R to the matter in the universe. This allows us to write Hubble's law without reference to galaxies, $\dot{R} = H\,R$ where the dot indicates differentiation with respect to time and we take R=l. On small scales the universe is not isotropic and homogeneous. There are lots of small scale structures in the universe (galaxies, galaxy clusters, voids, superclusters) and any model of the universe needs to explain how they got there. Gravitational collapse of initially small density in homogeneities is the standard Big Bang explanation. This idea is not far fetched since we are now falling at about 630 km/s towards the largest local overdensity.

(iii) The universe is filled with microwave photons coming from all directions:

This sea of Photons is known as the cosmic microwave background (CMB) radiation. The photon wave lengths are about as big as these letters and there are about 415 of them in every cubic centimeter of the universe. This article is about measurements of these photons and their slightly anisotropic distribution.

(iv)The universe is composed of ≈ 75% hydrogen and ≈25% helium:

This is true for the visible baryonic matter, i.e., stars, dust and gas. In the Big Bang model this three to one H/He ratio was fixed during an epoch of nucleosynthesis in the early universe. Big Bang nucleosynthesis (BBN) occurred within the first few minutes after the Big Bang. The agreement of BBN predictions for the light nuclei abundances H, D, ^{3}He, ^{4}He and ^{7}Li provides the earliest solid evidence supporting the Big Bang model.

The trace amounts of other elements were cooked up in stellar kitchens at much lower red shift. Since BBN occurred during the first three minutes after the Big Bang, it is often said that the Big Bang model has been tested that far back. In addition to this visible baryonic matter, there is much evidence that some kind of dark matter lurks about the outlying parts of galaxies and galaxy clusters are orbiting too fast to be contained by the gravitational potential wells of the visible matter alone.

In the 1980's theorists devised an important extension of the standard Big Bang model called inflation. Inflation adds an early period of exponential expansion to the history of the universe and also provides a mechanism (quantum fluctuations) for the production of the initially small density inhomogeneities needed for gravitational instability to induce structure formation. The exponential expansion solves two initial condition problems of the standard Big Bang model:

(1) The horizon problem , the CMB has the same temperature in opposite directions yet the gas in those directions has never been in causal contact.

(2) The flatness problem, the density of the universe is near the critical value of a flat universe yet the Friedmann equation tells us that the initial deviation from flatness would have had to be unbelieveably small for this to be the case today. Inflation solves these problems, it permits opposite sides of the observable universe to be in causal contact before inflation and the expansion flattens any pre-inflationary curvature, yielding Ω=1. There are alternatives to the hot Big Bang model. For example Layzer (1992) who advocates a cold Big Bang model.

4.3 CMB and the Surface of Last Scattering [8]:

The observable universe is expanding and cooling. Therefore, in the past it was hotter and smaller. The CMB is the after glow of thermal radiation left over from this hot early epoch in the evolution of the universe. The CMB is a bath of photons coming from every direction. These are the oddest photons one can observe and they contain information about the universe at red shifts much larger than the red shifts of galaxies or quasars. The CMB is thus unique tool for probing the early universe.

The prediction of the existence and the temperature of a CMB in 1948 (Alpher & Herman 1948) followed by its detections in 1964 (Penzias & Wilson 1965, Dicke et al. 1965) provides possibly the strongest evidence for the Big Bang. The CMB detection began the search for the determination of its exact spectrum and level of anisotropy. A CMB of truly cosmic origin is expected to have blackbody spectrum and to be extremely isotropic.

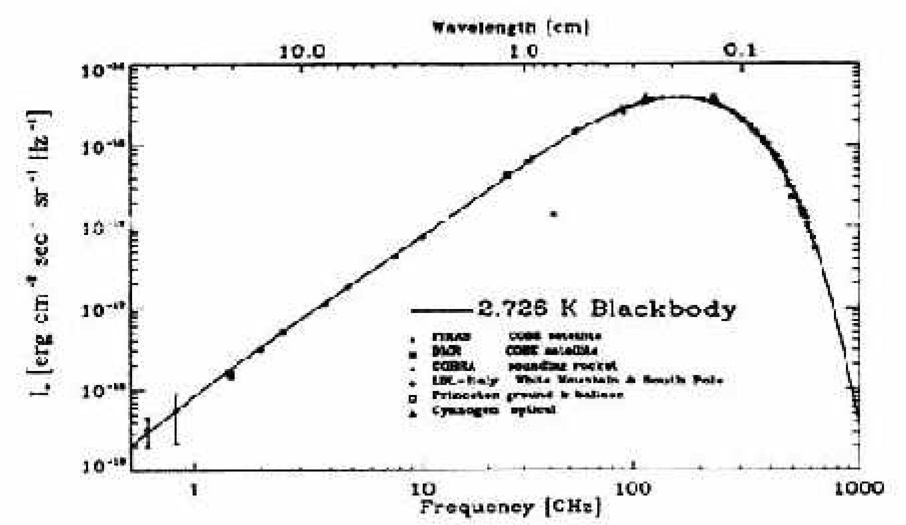


Fig. 4.1

Figure 4.1: CMB Spectrum Measurements. The spectrum of the CMB has been measured over 3 decades in frequency and found to be consistent with a blackbody at T_0 = 2.726 K.

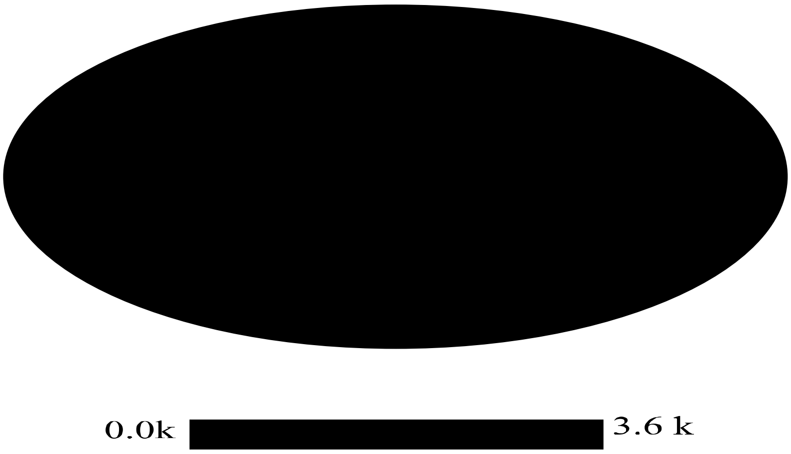


Fig. 4.2

Figure 4.2: CMB isotropy. DMR two-year 53 GHz full-sky map in galactic co-ordinates. The CMB is very well approximated by an isotropic blackbody.

COBE observations show that the CMB is very well approximated by an isotropic blackbody. The recently published COBE FIRAS result is that the CMB has the spectrum of a blackbody at a temperature of $T_0 = 2.726 \pm 0.01$K (95% CL) (Mather et al. 1994). Figure 4.1 shows how well the FIRAS measurements along with many other measurements agree with a blackbody spectrum at T_0 =2.726 K. Figure 4.2 displays the DMR 53 GHz two- year map. Not only is the CMB blackbody, it is also highly isotropic.

At approximately 10^{-2} seconds after the Big Bang of the universe was about 10^{10} k and nucleosynthesis began. Recombination or decoupling occurred 3,00000 years later when the universe had cooled down enough to allow the free electrons and protons to combine to form neutral hydrogen. This neutralization of the plasma allowed photons to free stream in all directions.

Before recombination the universe was an opaque fog of free electrons, afterwards it was transparent. The boundary is called the cosmic photosphere or the surface of last scattering. As its name implies, the surface of last scattering is where the CMB photons were Thomson scattered for the last time before arriving in our detectors (Figure 4.3). Except for the tiny contribution of one Lyman-α photon per hydrogen atom, the CMB photons were not produced at this time, they were only scattered.

The surface of last scattering can be described by several parameters. Here, we derive the red shift z_{ls},the temperature T_{ls} and the time t_{ls} of last scattering. As the universe expands it cools. The CMB photons get red shifted and their blackbody temperature goes down. The fact that a red shifted blackbody remains a blackbody can be shown using the Lorentz invariance of I_v / v^3 or equivalently the mean photon occupation number. The expansion red shift z is defined by

$$1+z=\frac{\lambda_0}{\lambda_e}=\frac{1}{R(t_e)} \quad \text{.........................(1)}$$

where λ is photon wave length e and o stand for emitted and observed respectively. Since the temperature scales as $T \propto 1/R \propto 1+z$, the temperature as a function of red shift is

$$T(z)=T_0(1+z) \quad \text{.......................................(2)}$$

Recombination occurs when the CMB temperature has dropped to the point when there are no longer enough high energy photons in the CMB to keep hydrogen ionized; $\gamma + H \leftrightarrow e^- + p$. Although the ionization potential of hydrogen is 13.6eV (T~10^5 k) recombination occurs at T$\approx$3000 k. The high photon to proton ratio ($\eta \approx 10^9$) allows the high energy tail of the Planck distribution to keep the comparatively small number of hydrogen atoms ionized until this much lower temperature. The Saha equation describes this balance between the ionizing photons and the ionized and neutral hydrogen. As the temperature decreases an increasing Boltzmann factor suppresses ionization while the large photon to proton ratio, η, maintains it. Recombination occurs when we have

$$e^{-\frac{\lambda}{kT}} \sim \eta \quad \text{.. (3)}$$

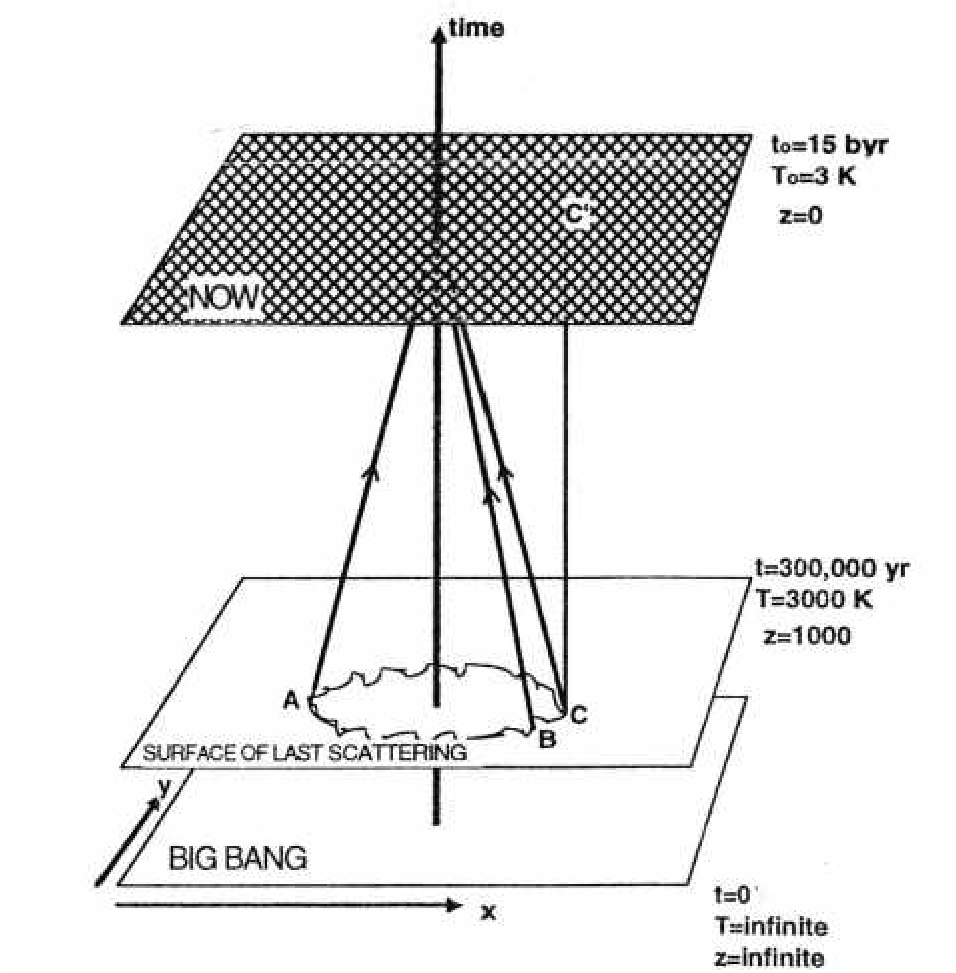


Fig. 4.3

Figure 4.3: Space-time and the surface of last scattering. The time axis is the world line of the stationary observer who is currently located at the apex of the light cone. CMB photons travel from the wavy circle in the surface of last scattering along the surface of the light cone to the observer. Points A and C are in opposite sides of the sky.

where λ is the ionization potential. The large value of η allows T to get as low as 8000k. In addition, trapped Lyman-α photons keep munch of the neutral hydrogen in an excited state making it easier to ionize. The result is that recombination occurs at $T_{ls} \approx 3000$ k. Equation (2) then yields $z_{ts} \approx 1000$.

We get the time of last scattering using the time dependence of the scale factor R in the matter dominated regime. Inserting $\rho \propto R^{-3}$ in the Friedman equation yields $R(t) \propto t^{2/3}$. Thus

$$\frac{T_{ls}}{T_0} \approx \frac{3000}{3} \approx \left(\frac{t_0}{t_{ts}}\right)^{2/3} \text{.................................(4)}$$

Therefore, if the present age of universe if t_0~10×10^9 years, then t_{ls}~3×10^5 years after the Big Bang. Thus the CMB photons have come to us from the surface of last scattering which can be described by the temperature, red shift and time

$$T_{ls} \approx 3000K \text{..(5)}$$

$$z_{ls} \approx 1000 \text{...(6)}$$

$$t_{ls} \approx 3\times10^5 \text{ years.......................................(7)}$$

The surface of last scattering is at a fixed time and temperature after the Big Bang. $t_{ls} =$ constant and $T_{ls}=$ constant. Thus equation (2) leads to the interesting conclusion that the surface of last scattering is receding from us with an ever increasing red shift, $z_{ls} \propto \frac{1}{T_0(t)}$.

The size of a causally connected region on the surface of last scattering is important because it determines the size over which astrophysical processes can occur. A causally connected Hubble's patch at last scattering is L_H = $3d_{ls}(1+z_{ls})$~ $200h^{-1}$Mpc which subtends an angular size θ_H

$$\theta_H \approx 1^0 \Omega_0^{1/2} \left(\frac{Z_{ls}}{1000}\right)^{-1/2} \text{ .... (8)}$$

Since the DMR beam averages over patches approximately 7^0 across, the smallest spots detected by DMR at the surface of last scattering are well into the causally disconnected $\theta > \theta_H$ regime.

The thickness of the surface of last scattering is $\Delta z \approx 80$ which corresponds to a length $\Delta L \approx 7\Omega^{-1/2}h^{-1}$ Mpc or an angular size of $\Delta\theta \approx 8'\Omega^{-1/2}$ (Kaiser & Silk 1986). Anisotropies on scales smaller than about 8' are suppressed because they are superimposed on each other over the finite path length of the photon in the surface. It is possible that high red shift sources of ultraviolet photons reionized the hydrogen or kept it from recombining. This reionization increases the effective thickness of the surface of last scattering and suppresses anisotropies on scales larger than 8'. For example for reionization $z_{reion} > 200$, anisotropies at scales less than ~ 1^0 are suppressed while for $z_{reion} > 20$, anisotropies at scales than ~ 5^0 are suppressed (Bartlett and Stebbins 1991, Bond 1995). It is noted that the DMR results (scales > 7^0) are uneffected by this reionization suppression.

4.4 The COBE Satellite [8]:

Orbit:

COBE is NASA's first cosmology satellite and was launched successfully November 18, 1989 on a Delta rocket (Boggess et al. 1992). The primary goals of the COBE satellite are to measure the CMB spectrum and anisotropy and measure the diffuse infrared background from primordial objects forming in the early universe. The three instruments designed to achieve these goals are the Far-Infrared Absolute Spectrophtometer (FIRAS), the Differential Microwave Radiometer (DMR) and the Diffuse Infrared Background Experiment (Figure 4.4). Secondary goals of the mission include measuring radiation from our local environment such as interplanetary dust, interstellar electrons, starlight and other Galactic emission. These local sources can mask and limit the accuracy of the cosmological measurements.

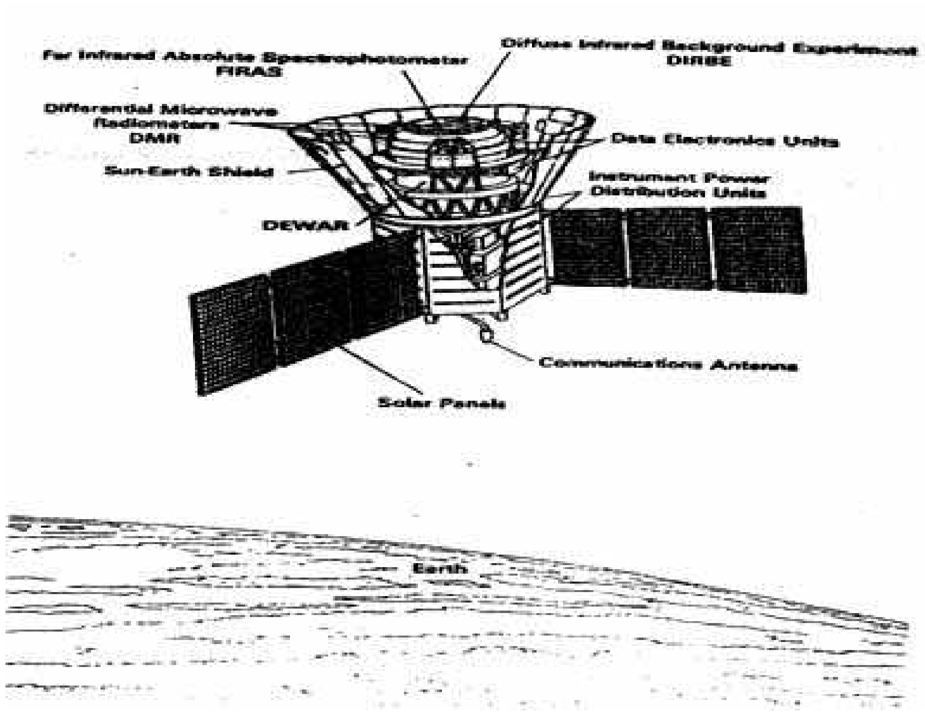


Fig. 4.4

Figure 4.4: COBE satellite in orbit. COBE is about the size of a large van and weights 2270 kg (Boggess et al. 1992).

COBE is in a 900 km altitude, near terminator orbit with a 103 minute period. The combination of the 99^0 inclination of the orbit, 900 km altitude and the quadrupole moment of the earth results in a torque on the plane of the orbit causing it to precess $\approx 1^0$ per day. This precession rate keeps the orbital plane as close to perpendicular as possible to the earth-sun line. The relative positions of the earth, sun and orbital plane during the winter and summer solstices are displayed (Figure4.5).

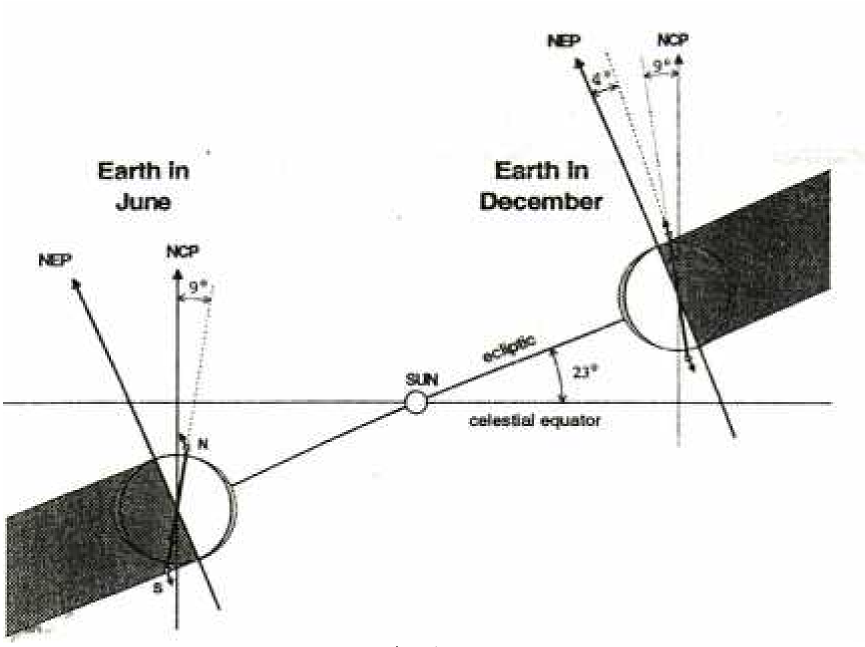


Fig. 4.5

Figure 4.5: COBE's Orbit. The Earth is shown at both the winter and summer solstices. COBE's orbit is the thick line inclined 9^0 from the north celestial pole (NCP).

The satellite spins with a 73 second period (o.815 rpm). The spin axis points away from the earth and 92^0-94^0 away from the sun. This spinning, orbiting and processing, combined with the fairly large 7^0 FWHM beams, enables the DMR to sample the entire sky in ≈ 5 months.

Chapter Five

Schwarzschild Solution

5.1 Deduce Schwarzschild Solution.

Let us consider the line element is

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In matrix form

$$g_{\mu\upsilon} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

where $g_{00} = A(r), g_{11} = -B(r), g_{22} = -r^2, g_{33} = -r^2\sin^2\theta$

and the other terms are zero.

Now, the inverse of $g_{\mu\upsilon}$ is $g^{\mu\upsilon} = \frac{1}{|g_{\mu\upsilon}|}$ (Adjoint matrix of $g_{\mu\upsilon}$)

$$|g_{\mu\upsilon}| = \begin{vmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{vmatrix}$$

$$= -A(r)B(r)r^4\text{Sin}^2\theta$$

Adjoint matrix of $g_{\mu\upsilon}$ $= \begin{pmatrix} -B(r)r^4\text{Sin}^2\theta & 0 & 0 & 0 \\ 0 & A(r)r^4\text{Sin}^2\theta & 0 & 0 \\ 0 & 0 & A(r)B(r)r^2\text{Sin}^2\theta & 0 \\ 0 & 0 & 0 & A(r)B(r)r^2 \end{pmatrix}$

$$\therefore g^{\mu\upsilon} = \frac{1}{-A(r)B(r)r^4 Sin^2\theta}\begin{pmatrix} -B(r)r^4 Sin^2\theta & 0 & 0 & 0 \\ 0 & A(r)r^4 Sin^2\theta & 0 & 0 \\ 0 & 0 & A(r)B(r)r^2 Sin^2\theta & 0 \\ 0 & 0 & 0 & A(r)B(r)r^2 \end{pmatrix}$$

$$\text{or, } g^{\mu\upsilon} = \begin{pmatrix} \frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & -\frac{1}{B(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 Sin^2\theta} \end{pmatrix} = \begin{pmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{pmatrix}$$

where $g^{00} = \frac{1}{A(r)}, g^{11} = -\frac{1}{B(r)}, g^{22} = -\frac{1}{r^2}, g^{33} = -\frac{1}{r^2 \sin^2\theta}$

and the other terms are zero.

Here $x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi$ and $(0, 1, 2, 3) = (t, r, \theta, \varphi)$

We know the Christoffel symbol of second kind is

$$\Gamma^{\mu}_{\upsilon\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\upsilon,\lambda} + g_{\sigma\lambda,\upsilon} - g_{\upsilon\lambda,\sigma})$$

The non-zero Christoffel symbols are

$$\Gamma^0_{01} = \frac{1}{2} g^{0\sigma}(g_{\sigma 0,1} + g_{\sigma 1,0} - g_{01,\sigma})$$

$$= \frac{1}{2} g^{00}(g_{00,1} + g_{01,0} - g_{01,0})$$

$$= \frac{1}{2}\frac{1}{A(r)}\frac{\partial}{\partial r}\{A(r)\}$$

$$= \frac{A'}{2A}$$

$$\therefore \Gamma^0_{10} = \frac{A'}{2A}$$

$$\Gamma^1_{00} = \frac{1}{2} g^{1\sigma}(g_{\sigma 0,0} + g_{\sigma 0,0} - g_{00,\sigma})$$

$$= \frac{1}{2} g^{11}(g_{10,0} + g_{10,0} - g_{00,1})$$

$$= -\frac{1}{2}\frac{1}{B(r)}[-\frac{\partial}{\partial r}\{A(r)\}]$$

$$= \frac{A'}{2B}$$

$$\Gamma_{11}^{1} = \frac{1}{2} g^{1\sigma} (g_{\sigma 1,1} + g_{\sigma 1,1} - g_{11,\sigma})$$

$$= \frac{1}{2} g^{11} (g_{11,1} + g_{11,1} - g_{11,1})$$

$$= -\frac{1}{2B(r)} [-\frac{\partial}{\partial r} \{B(r)\}]$$

$$= \frac{B'}{2B}$$

$$\Gamma_{22}^{1} = \frac{1}{2} g^{1\sigma} (g_{\sigma 2,2} + g_{\sigma 2,2} - g_{22,\sigma})$$

$$= \frac{1}{2} g^{11} (g_{12,1} + g_{12,1} - g_{22,1})$$

$$= -\frac{1}{2} \frac{}{B(r)} \{-\frac{\partial}{\partial r} (-r^2)\}$$

$$= \frac{-2r}{2B}$$

$$= -\frac{r}{B}$$

$$\Gamma_{33}^{1} = \frac{1}{2} g^{1\sigma} (g_{\sigma 3,3} + g_{\sigma 3,3} - g_{33,\sigma})$$

$$= \frac{1}{2} g^{11} (g_{13,1} + g_{13,1} - g_{33,1})$$

$$= -\frac{1}{2} \frac{}{B(r)} \left\{-\frac{\partial}{\partial r} \left(-r^2 \sin^2 \theta\right)\right\}$$

$$= \frac{-2r \sin^2 \theta}{2B(r)}$$

$$= \frac{-r \sin^2 \theta}{B}$$

$$\Gamma_{12}^{2} = \frac{1}{2} g^{2\sigma} (g_{\sigma 1,2} + g_{\sigma 2,1} - g_{12,\sigma})$$

$$= \frac{1}{2} g^{22} (g_{21,2} + g_{22,1} - g_{12,2})$$

$$= \frac{1}{2} \left(-\frac{1}{r^2}\right) \frac{\partial}{\partial r} (-r^2)$$

$$= \frac{2r}{2r^2}$$

$$= \frac{1}{r}$$

$$\therefore \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{2\sigma}(g_{\sigma 3,3} + g_{\sigma 3,3} - g_{33,\sigma})$$

$$= \frac{1}{2} g^{22}(g_{23,3} + g_{23,3} - g_{33,2})$$

$$= \frac{1}{2}(-\frac{1}{r^2})\{-\frac{\partial}{\partial\theta}(-r^2 \sin^2\theta)\}$$

$$= \frac{-r^2.2\sin\theta\cos\theta}{2r^2}$$

$$= -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \frac{1}{2} g^{3\sigma}(g_{\sigma 1,3} + g_{\sigma 3,1} - g_{13,\sigma})$$

$$= \frac{1}{2} g^{33}(g_{31,3} + g_{33,1} - g_{13,3})$$

$$= \frac{1}{2}(-\frac{1}{r^2 \sin^2\theta})\{\frac{\partial}{\partial r}(-r^2 \sin^2\theta)\}$$

$$= \frac{2r\sin^2\theta}{2r^2\sin^2\theta}$$

$$= \frac{1}{r}$$

$$\therefore \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{3\sigma}(g_{\sigma 2,3} + g_{\sigma 3,2} - g_{23,\sigma})$$

$$= \frac{1}{2} g^{33}(g_{32,3} + g_{33,2} - g_{23,3})$$

$$= \frac{1}{2}(-\frac{1}{r^2 \sin^2\theta})\frac{\partial}{\partial\theta}(-r^2 \sin^2\theta)$$

$$= \frac{2r^2\sin\theta\cos\theta}{2r^2\sin^2\theta}$$

$$= \cot\theta$$

$$\therefore \Gamma^3_{32} = \cot\theta$$

Now, $R_{\mu\upsilon} = \Gamma^{\sigma}_{\mu\sigma,\upsilon} - \Gamma^{\sigma}_{\mu\upsilon,\sigma} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\upsilon} - \Gamma^{\rho}_{\mu\upsilon}\Gamma^{\sigma}_{\rho\sigma}$

$$R_{00} = \Gamma^{\sigma}_{0\sigma,0} - \Gamma^{\sigma}_{00,\sigma} + \Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0} - \Gamma^{\rho}_{00}\Gamma^{\sigma}_{\rho\sigma}$$

$$\Gamma^{\sigma}_{0\sigma,0} = \Gamma^0_{00,0} + \Gamma^1_{01,0} + \Gamma^2_{02,0} + \Gamma^3_{03,0}$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

and $\Gamma^{\sigma}_{00,\sigma} = \Gamma^0_{00,0} + \Gamma^1_{00,1} + \Gamma^2_{00,2} + \Gamma^3_{00,3}$

$$= 0 + \frac{\partial}{\partial r}\left(\frac{A'}{2B}\right) + 0 + 0$$

$$= \frac{1}{2}\left(\frac{BA'' - A'B'}{B^2}\right)$$

$$= \frac{A''}{2B} - \frac{A'B'}{2B^2}$$

and $\Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0} = \Gamma^0_{0\sigma}\Gamma^{\sigma}_{00} + \Gamma^1_{0\sigma}\Gamma^{\sigma}_{10} + \Gamma^2_{0\sigma}\Gamma^{\sigma}_{20} + \Gamma^3_{0\sigma}\Gamma^{\sigma}_{30}$

$$= \Gamma^0_{00}\Gamma^0_{00} + \Gamma^0_{01}\Gamma^1_{00} + \Gamma^0_{02}\Gamma^2_{00} + \Gamma^0_{03}\Gamma^3_{00}$$

$$+ \Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{01}\Gamma^1_{10} + \Gamma^1_{02}\Gamma^2_{10} + \Gamma^1_{03}\Gamma^3_{10}$$

$$+ \Gamma^2_{00}\Gamma^0_{20} + \Gamma^2_{01}\Gamma^1_{20} + \Gamma^2_{02}\Gamma^2_{20} + \Gamma^2_{03}\Gamma^3_{20}$$

$$+ \Gamma^3_{00}\Gamma^0_{30} + \Gamma^3_{01}\Gamma^1_{30} + \Gamma^3_{02}\Gamma^2_{30} + \Gamma^3_{03}\Gamma^3_{30}$$

$$= 0 + \frac{A'}{2A}.\frac{A'}{2B} + 0 + 0$$

$$+ \frac{A'}{2B}.\frac{A'}{2A} + 0 + 0 + 0$$

$$+ 0 + 0 + 0 + 0$$

$$+ 0 + 0 + 0 + 0$$

$$= \frac{A'^2}{4AB} + \frac{A'^2}{4AB}$$

$$= \frac{A'^2}{2AB}$$

and $\Gamma^{\rho}_{00}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^0_{00}\Gamma^{\sigma}_{0\sigma} + \Gamma^1_{00}\Gamma^{\sigma}_{1\sigma} + \Gamma^2_{00}\Gamma^{\sigma}_{2\sigma} + \Gamma^3_{00}\Gamma^{\sigma}_{3\sigma}$

$$=\Gamma^0_{00}(\Gamma^0_{00}+\Gamma^1_{01}+\Gamma^2_{02}+\Gamma^3_{03})$$

$$+\Gamma^1_{00}(\Gamma^0_{10}+\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13})$$

$$+\Gamma^2_{00}(\Gamma^0_{20}+\Gamma^1_{21}+\Gamma^2_{22}+\Gamma^3_{23})$$

$$+\Gamma^3_{00}(\Gamma^0_{30}+\Gamma^1_{31}+\Gamma^2_{32}+\Gamma^3_{33})$$

$$=\frac{A'}{2B}(\frac{A'}{2A}+\frac{B'}{2B}+\frac{1}{r}+\frac{1}{r})$$

$$=\frac{A'}{2B}(\frac{A'}{2A}+\frac{B'}{2B}+\frac{2}{r})$$

$$=\frac{A'^2}{4AB}+\frac{A'\ B'}{4B^2}+\frac{A'}{Br}$$

$$R_{00}=0-\frac{A''}{2B}+\frac{A'B'}{2B^2}+\frac{A'^2}{2AB}-\frac{A'^2}{4AB}-\frac{A'B'}{4B^2}-\frac{A'}{Br}$$

$$=\frac{A'B'}{4B^2}-\frac{A''}{2B}+\frac{A'^2}{4AB}-\frac{A'}{Br}$$

$$=-\frac{A''}{2B}+\frac{A'}{4B}(\frac{A'}{A}+\frac{B'}{B})-\frac{A'}{Br}$$

Now, $R_{11}=\Gamma^\sigma_{1\sigma,1}-\Gamma^\sigma_{11,\sigma}+\Gamma^\rho_{1\sigma}\Gamma^\sigma_{\rho 1}-\Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma}$

$$\Gamma^\sigma_{1\sigma,1}=\Gamma^0_{10,1}+\Gamma^1_{11,1}+\Gamma^2_{12,1}+\Gamma^3_{13,1}$$

$$=\frac{\partial}{\partial r}\left(\frac{A'}{2A}\right)+\frac{\partial}{\partial r}\left(\frac{B'}{2B}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\right)$$

$$=\frac{1}{2}\left(\frac{A.A''-A'.A'}{A^2}\right)+\frac{1}{2}\left(\frac{BB''-B'.B'}{B^2}\right)-\frac{1}{r^2}-\frac{1}{r^2}$$

$$=\frac{A''}{2A}-\frac{A'^2}{2A^2}+\frac{B''}{2B}-\frac{B'^2}{2B^2}-\frac{2}{r^2}$$

and $\Gamma^\sigma_{11,\sigma}=\Gamma^0_{11,0}+\Gamma^1_{11,1}+\Gamma^2_{11,2}+\Gamma^3_{11,3}$

$$=0+\frac{\partial}{\partial r}\left(\frac{B'}{2B}\right)+0+0$$

$$=\frac{1}{2}\left(\frac{BB''-B'.B'}{B^2}\right)$$

$$=\frac{B''}{2B}-\frac{B'^2}{2B^2}$$

and $\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1} = \Gamma^{0}_{1\sigma}\Gamma^{\sigma}_{01} + \Gamma^{1}_{1\sigma}\Gamma^{\sigma}_{11} + \Gamma^{2}_{1\sigma}\Gamma^{\sigma}_{21} + \Gamma^{3}_{1\sigma}\Gamma^{\sigma}_{31}$

$$= \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{12}\Gamma^{2}_{21} + \Gamma^{3}_{13}\Gamma^{3}_{31}$$

$$= \frac{A'}{2A}.\frac{A'}{2A} + \frac{B'}{2B}.\frac{B'}{2B} + \frac{1}{r}.\frac{1}{r} + \frac{1}{r}.\frac{1}{r}$$

$$= \frac{A'^2}{4A^2} + \frac{B'^2}{4B^2} + \frac{2}{r^2}$$

and $\Gamma^{\rho}_{11,\sigma}\,\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{0}_{11}\Gamma^{\sigma}_{0\sigma} + \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{11}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{11}\Gamma^{\sigma}_{3\sigma}$

$$= 0 + \Gamma^{1}_{11}(\Gamma^{0}_{10} + \Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13}) + 0 + 0$$

$$= \frac{B'}{2B}(\frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r})$$

$$= \frac{B'}{2B}.(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r})$$

$$= \frac{A'B'}{4AB} + \frac{B'^2}{4B^2} + \frac{B'}{Br}$$

$$\therefore R_{11} = \frac{A''}{2A} - \frac{A'^2}{2A^2} + \frac{B''}{2B} - \frac{B'^2}{2B^2} - \frac{2}{r^2} - \frac{B''}{2B} + \frac{B'^2}{2B^2} + \frac{A'^2}{4A^2} + \frac{B'^2}{4B^2} + \frac{2}{r^2} - \frac{A'B'}{4AB} - \frac{B'^2}{4B^2} - \frac{B'}{Br}$$

$$= \frac{A''}{2A} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{Br}$$

$$= \frac{A''}{2A} - \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{Br}$$

$$R_{22} = \Gamma^{\sigma}_{2\sigma,2} - \Gamma^{\sigma}_{22,\sigma} + \Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2} - \Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma}$$

Now, $\Gamma^{2}_{2\sigma,2} = \Gamma^{0}_{20,2} + \Gamma^{1}_{21,2} + \Gamma^{2}_{22,2} + \Gamma^{3}_{23,2}$

$$= 0 + 0 + 0 + \frac{\partial}{\partial\theta}(\cot)$$

$$= -\cos ec^2\theta$$

and $\Gamma^{\sigma}_{22,\sigma} = \Gamma^{0}_{12,0} + \Gamma^{1}_{22,1} + \Gamma^{2}_{22,2} + \Gamma^{3}_{23,3}$

$$= 0 + \frac{\partial}{\partial r}\left(\frac{-r}{B}\right) + 0 + 0$$

$$= -\left(\frac{B - rB'}{B^2}\right)$$

$$= \frac{-B}{B^2} + \frac{rB'}{B^2}$$

$$= -\frac{1}{B} + \frac{rB'}{B^2}$$

and $\Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2} = \Gamma^{0}_{2\sigma}\Gamma^{\sigma}_{02} + \Gamma^{1}_{2\sigma}\Gamma^{\sigma}_{12} + \Gamma^{2}_{2\sigma}\Gamma^{\sigma}_{22} + \Gamma^{3}_{2\sigma}\Gamma^{\sigma}_{32}$

$$= 0 + \Gamma^{1}_{22}\Gamma^{2}_{12} + \Gamma^{2}_{21}\Gamma^{1}_{22} + \Gamma^{3}_{23}\Gamma^{3}_{32}$$

$$= 0 + (\frac{-r}{B}).\frac{1}{r} + \frac{1}{r}(\frac{-r}{B}) + \cot\theta.\cot\theta$$

$$= -\frac{1}{B} - \frac{1}{B} + \cot^2\theta$$

$$= -\frac{2}{B} + \cot^2\theta$$

and $\Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{0}_{22}\Gamma^{\sigma}_{0\sigma} + \Gamma^{1}_{22}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{22}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{22}\Gamma^{\sigma}_{3\sigma}$

$$= 0 + \Gamma^{1}_{22}(\Gamma^{0}_{10} + \Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13}) + 0 + 0$$

$$= -\frac{r}{B}(\frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r})$$

$$= -\frac{r}{B}(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r})$$

$$= -\frac{rA'}{2AB} - \frac{rB'}{2B^2} - \frac{2}{B}$$

$$\therefore R_{22} = -\cos ec^2\theta + \frac{1}{B} - \frac{rB'}{B^2} + \cot^2\theta - \frac{2}{B} + \frac{rA'}{2AB} + \frac{rB'}{2B^2} + \frac{2}{B}$$

$$= \frac{1}{B} - (\cos ee^2\theta - \cot^2\theta) + \frac{rA'}{2AB} - \frac{rB'}{2B^2}$$

$$= \frac{1}{B} - 1 + \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B})$$

$$R_{33} = \Gamma^{\sigma}_{3\sigma,3} - \Gamma^{\sigma}_{33,\sigma,} + \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3} - \Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma}$$

$$\Gamma^{\sigma}_{3\sigma,3} = \Gamma^{0}_{30,3} + \Gamma^{1}_{31,3} + \Gamma^{2}_{32,3} + \Gamma^{3}_{33,3}$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

and $\Gamma^{\sigma}_{33,\sigma} = \Gamma^{0}_{33,0} + \Gamma^{1}_{33,1} + \Gamma^{2}_{33,2} + \Gamma^{3}_{33,3}$

$$= 0 + \frac{\partial}{\partial r}\left(\frac{-r\sin^2\theta}{B}\right) + \frac{\partial}{\partial\theta}(-\sin\theta\cos\theta) + 0$$

$$= -\sin^2\theta\frac{\partial}{\partial r}\left(\frac{r}{B}\right) - \frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$= -Sin^2\theta\left(\frac{B-rB'}{B^2}\right)-\{\sin\theta(-\sin\theta)+\cos\theta.\cos\theta\}$$

$$= \sin^2\theta\left(\frac{rB'-B}{B^2}\right)+\sin^2\theta-\cos^2\theta$$

$$= \frac{rB'\sin^2\theta}{B^2}-\frac{\sin^2\theta}{B}+\sin^2\theta-\cos^2\theta$$

and $\Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3} = \Gamma^{0}_{3\sigma}\Gamma^{\sigma}_{03}+\Gamma^{1}_{3\sigma}\Gamma^{\sigma}_{13}+\Gamma^{2}_{3\sigma}\Gamma^{\sigma}_{23}+\Gamma^{3}_{3\sigma}\Gamma^{\sigma}_{33}$

$$= 0+\Gamma^{1}_{33}\Gamma^{3}_{13}+\Gamma^{2}_{33}\Gamma^{3}_{23}+\Gamma^{3}_{31}\Gamma^{1}_{33}+\Gamma^{3}_{32}\Gamma^{2}_{33}$$

$$= \left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right)+(-\sin\theta\cos\theta).\cot\theta+\left(\frac{1}{r}\right)\left(\frac{-r\sin^2\theta}{B}\right)+(\cot\theta)(-\sin\theta\cos\theta)$$

$$= -\frac{2\sin^2\theta}{B}-2\sin\theta\cos\theta\frac{\cos\theta}{\sin\theta}$$

$$= -\frac{2\sin^2\theta}{B}-2\cos^2\theta$$

and $\Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{0}_{33}\Gamma^{\sigma}_{0\sigma}+\Gamma^{1}_{33}\Gamma^{\sigma}_{1\sigma}+\Gamma^{2}_{33}\Gamma^{\sigma}_{2\sigma}+\Gamma^{3}_{33}\Gamma^{\sigma}_{3\sigma}$

$$= 0+\Gamma^{1}_{33}(\Gamma^{0}_{10}+\Gamma^{1}_{11}+\Gamma^{2}_{12}+\Gamma^{3}_{13})+\Gamma^{2}_{33}\Gamma^{3}_{23}+0$$

$$= \frac{-r\sin^2\theta}{B}\left(\frac{A'}{2A}+\frac{B'}{2B}+\frac{1}{r}+\frac{1}{r}\right)+(-\sin\theta\cos\theta).\cot\theta$$

$$= \frac{-r\sin^2\theta}{B}\left(\frac{A'}{2A}+\frac{B'}{2B}+\frac{2}{r}\right)-\cos^2\theta$$

$$= \frac{-rA'\sin^2\theta}{2AB}-\frac{rB'\sin^2\theta}{2B^2}-\frac{2\sin^2\theta}{B}-\cos^2\theta$$

$$\therefore R_{33} = 0-\frac{rB'\sin^2\theta}{B^2}+\frac{\sin^2\theta}{B}-\sin^2\theta+\cos^2\theta-\frac{2\sin^2\theta}{B}-2\cos^2\theta+\frac{rA'\sin^2\theta}{2AB}$$

$$+\frac{rB'\sin^2\theta}{2B^2}+\frac{2\sin^2\theta}{B}+\cos^2\theta$$

$$= \frac{\sin^2\theta}{B}-\frac{rB'\sin^2\theta}{2B^2}+\frac{rA'\sin^2\theta}{2AB}-\sin^2\theta$$

$$= \frac{\sin^2\theta}{B}-\sin^2\theta+\frac{rA'\sin^2\theta}{2AB}-\frac{rB'\sin^2\theta}{2B^2}$$

$$= \sin^2\theta\left(\frac{1}{B}-1+\frac{rA'}{2AB}-\frac{rB'}{2B^2}\right)$$

$= \sin^2\theta\{\frac{1}{B} - 1 + \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B})\}$

$= R_{22}\sin^2\theta$

Hence, in empty space

$$R_{00} = \frac{-A''}{2B} + \frac{A'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rB} = 0 \qquad \text{.....................} (1)$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rB} = 0 \qquad \text{.....................} (2)$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) = 0 \qquad \text{.....................} (3)$$

$$R_{33} = R_{22}\sin^2\theta \qquad \text{...} (4)$$

Multiplying equation (1) by $\frac{B}{A}$, we have

$$-\frac{A''}{2A} + \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{Ar} = 0 \text{ ....} (5)$$

Adding equation (2) and (5), we have

$$-\frac{A'}{Ar} - \frac{B'}{Br} = 0$$

$$\Rightarrow \frac{A'}{Ar} + \frac{B'}{Br} = 0$$

$$\Rightarrow \frac{1}{r}(\frac{A'}{A} + \frac{B'}{B}) = 0$$

$$\Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0$$

$$\Rightarrow A'B + B'A = 0$$

$$\Rightarrow d(AB) = 0$$

Integrating the above equation, we have

AB = c^2 where c^2 is integrating constant.

Hence c^2=AB

$\therefore A = \frac{c^2}{B}$ and $B = \frac{c^2}{A}$

Again $A'B + AB' = 0$

$$\Rightarrow A'B = -AB'$$

$$\Rightarrow B' = \frac{-A'B}{A}$$

$$= \frac{-A'}{A} . \frac{c^2}{A}$$

$$= \frac{-A'c^2}{A^2}$$

Now, putting the value of B and B' in equation (3), we have

$$\frac{A}{c^2} - 1 + \frac{r}{\frac{2c^2}{A}} \left(\frac{A'}{A} + \frac{A'c^2}{A^2} \times \frac{A}{c^2} \right) = 0$$

$$\Rightarrow \frac{A}{c^2} + \frac{rA}{2c^2} \left(\frac{A'}{A} + \frac{A'}{A} \right) = 1$$

$$\Rightarrow \frac{A}{c^2} + \frac{rA}{2c^2} \times \frac{2A'}{A} = 1$$

$$\Rightarrow \frac{A}{c^2} + \frac{A'r}{c^2} = 1$$

$$\Rightarrow \frac{A + A'r}{c^2} = 1$$

$$\Rightarrow A + A'r = c^2$$

$$\Rightarrow d(Ar) = c^2$$

Integrating the above equation, we have

$\Rightarrow Ar = c^2 r + c^2 k$ where c^2k is constant.

$$\Rightarrow Ar = c^2 (r + k)$$

$$\Rightarrow A = c^2 \left(1 + \frac{k}{r}\right)$$

Again $B = \frac{c^2}{A} = \frac{c^2}{c^2 \left(1 + \frac{k}{r}\right)} = \frac{1}{1 + \frac{k}{r}} = \left(1 + \frac{k}{r}\right)^{-1}$

Now, putting the value of A and B in the equation

$$ds^2 = Adt^2 - Bdr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

$$\therefore ds^2 = c^2 \left(1 + \frac{k}{r}\right) dt^2 - \left(1 + \frac{k}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad \text{.............................. (6)}$$

We put $K = \frac{-2MG}{c^2}$ in equation (6), we get

$$\therefore ds^2 = c^2 \left(1 - \frac{2MG}{c^2 r}\right) dt^2 - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad \text{................ (7)}$$

Now, putting $c^2=1$ and $m = \frac{MG}{c^2}$ in equation (7), we get

$$\therefore ds^2 = (1-\frac{2m}{r})dt^2 - (1-\frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

This is the Schwarzschild solution for empty space-time outside a spherical body of mass M.

5.2 Removing the Singularity of Schwarzschild Solution:

We know the Schwarzschild solution is

$$ds^2 = (1-\frac{2m}{r})dt^2 - (1-\frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 .... (1)$$

If we put *r= 2m* in equation (1), then we cannot assume the behavior of the Schwarzschild solution. For Schwarzschild solution, we have three cases in singularity and these are

r = 2m, r > 2m and r < 2m

We can remove the singularity by some co-ordinate transformation. A particle falling radial inwards appear to continue beyond the threshold at *r = 2m,* although, as we have seen and observer viewing its fall always sees it before it passes the threshold. These two observation suggest that some odd things happen at *r = 2m*. However, the co-ordinates (t, r, θ, φ) are inadequate for discussing what happen at *r = 2m* and beyond so we introduce new co-ordinates which are valid for $r \le 2m$.

5.3 Crucial Tests in Relativity:

The following are known as crucial tests in relativity

(i) The Advance of Perihelion of the Mercury Planet.

(ii) Gravitational Deflection of Light Rays.

(iii) Shift in Spectral Lines.

We shall now discuss these one by one:

(i) The Advance of Perihelion of the Mercury Planet:

The differential equation of the path of a planet is

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2 ....(1)$$

with $r^2 \frac{d\varphi}{ds} = h$

As a first approximation the small term $3mu^2$ can be neglected, so that we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2}$$

The solution of above equation is

$$u=\frac{m}{h^2}[1+e\cos(\varphi-\omega)] \text{ ....(2)}$$

where e and ω are constants of integration giving eccentricity and longitude of the perihelion. Putting this first approximation on the R.H.S of (1), we obtain

$$\frac{d^2u}{d\varphi^2}+u=\frac{m}{h^2}+3m.\frac{m^2}{h^4}[1+e\cos(\varphi-\omega)]^2$$

$$=\frac{m}{h^2}+\frac{3m^3}{h^4}+\frac{6m^3}{h^4}e\cos(\varphi-\omega)+\frac{3m^3e^2}{h^4}\cos^2(\varphi-\omega)$$

Of the additional terms, the only one term which can produce and effect within the range of observation is the term

$$\frac{3m^3e}{h^4}\cos(\varphi-\omega)$$

The particular integral of this term is

$$\frac{1}{1+D^2}.\frac{6m^3e}{h^4}\cos(\varphi-\omega)=\frac{6m^3e}{h^4}.\frac{1}{1+D^2}.\cos(\varphi-\omega)$$

$$=\frac{6m^3e}{h^4}.\frac{\varphi}{2}\sin(\varphi-\omega)=\frac{3m^3e}{h^4}\varphi\sin(\varphi-\omega)$$

Here, we are using the formula $\frac{1}{1+D^2}\cos x=\frac{x}{2}\sin x.$ As a second approximation, the complete solution of (1) is

$$u=\frac{m}{h^2}[1+e\cos(\varphi-\omega)]+\frac{3m^3e}{h^4}\varphi\sin(\varphi-\omega), by (2)$$

$$=\frac{m}{h^2}+\frac{me}{h^2}\left[\cos(\varphi-\omega)+\frac{3m^2\varphi}{h^2}\sin(\varphi-\omega)\right]$$

Taking $3m^2\varphi/h^2=\delta\omega$ *and observing that*
$\sin\delta\omega=\delta\omega, \cos\delta\omega=1 \therefore$ *Since* $\delta\omega$ *is very small*,

$$u=\frac{m}{h^2}+\frac{me}{h^2}[\cos\delta\omega\cos(\varphi-\omega)+\sin\delta\omega\sin(\varphi-\omega)]$$

$$=\frac{m}{h^2}+\frac{me}{h^2}\cos(\varphi-\omega-\delta\omega)$$

$$Or, u=\frac{m}{h^2}+\frac{me}{h^2}\cos(\varphi-\omega-\delta\omega) \text{ with } \delta\omega=\frac{3m^2\varphi}{h^2} \text{ ....(3)}$$

This is the required solution of (1). When a plant moves round the sun through one revolution, the perihelion of the planet advances a fraction of revolution equal to

$$\frac{\delta\omega}{\varphi}=\frac{3m^2}{h^2}=\frac{3m^2}{ml}$$

$$=\frac{3m^2}{ma(1-e^2)}=\frac{3m}{a(1-e^2)} \qquad \left[\therefore l=a\left(1-e^2\right)\right]$$

$$i.e.\frac{\delta\omega}{\varphi}=\frac{3m}{a(1-e^2)} \qquad \text{.....(4)}$$

[On using the well known formula of area $h^2 = ml$]

From equation (4) $\delta\omega=\dfrac{3m\varphi}{a(1-e^2)}$ .....(5)

By Kepler's third law, $T=\dfrac{2\pi}{\sqrt{m}}a^{3/2}$

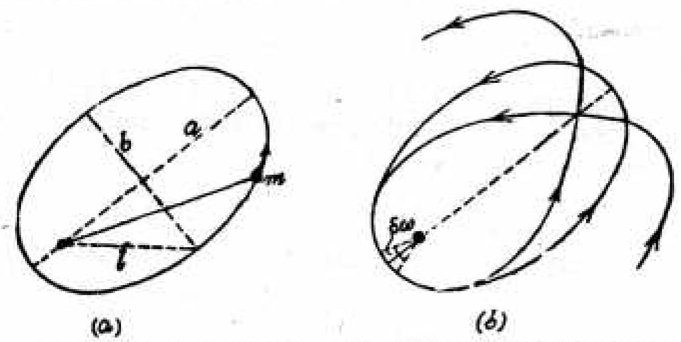
From which, $m=4\pi^2a^3/T^2$

Using this in (5) $\delta\omega=\dfrac{3\varphi}{a(1-e^2)}.\ \dfrac{4\pi^2a^3}{T^2}$

$$\Rightarrow \delta\omega=\frac{12\pi^2a^2\varphi}{T^2(1-e^2)}$$

Taking velocity of light into consideration $\delta\omega=\dfrac{12\pi^2a^2\varphi}{c^2T^2(1-e^2)}$

Taking $\varphi=2\pi, \delta\omega=\dfrac{24\pi^3a^2}{c^2T^2(1-e^2)}$ .....(6)



(a) (b)

(a) Closed elliptical planetary orbit (predicted by Newtonian theory). (b) Planetary orbit with a slow rotation of perihelion (predicted by Einstein's theory).

Fig. 2.1

T being time period. Thus the relativistic theory leads to an advance of perihelion of a planetary orbit. That is to say, this theory leads to planetary orbit with a slow rotation of perihelion instead of to be perfectly closed elliptical orbits of the Newtonian theory. The advance of perihelion, given by (6), is very small for all the planets except Mercury, in which it is appreciable. For Mercury, e = 0.2, a = 0.6 $\times 10^8$ Kilometres.

c = 3×10^{10} cms. per second, T = 88 days = 88×24×3600 second.

Number of revolutions per century = $\dfrac{365\times100}{88}$

$$\delta\omega = \frac{24\pi^3\times(0.6\times10^8\times10^5)^2}{(3\times10^{10})^2\,(88\times24\times3600)^2\,(1-0.04)}\times\frac{365\times100}{88}\ radians.$$

But π radian = 180^0 = 180×3600 seconds.

Hence, advance of perihelion in case of Mercury is

$$\frac{24\times(180\times3600)^3\times(0.6\times10^{13})^2}{9\times10^{20}\times(88\times24\times3600)\times0.96}\times\frac{365\times100}{88}=43\ \sec onds.$$

Thus the predicted value of advance of perihelion in case of Mercury is 43 seconds per century and the observed value is 43.5 seconds. Hence, the agreement is satisfactory.

(ii) Gravitational Deflection of Light Rays (Bending of Light Rays):

Treatment in general theory of relativity, we consider the deflection of a light ray in the gravitational field of the sun. According to the general theory of relativity the track of a light ray is given by geodesic equations with the added condition ds=0. It means the differential equations of a planetary orbit such as

$$\frac{d^2u}{d\varphi^2}+u=\frac{m}{h^2}+3mu^2 \quad ....(1)$$

with $r^2\dfrac{d\varphi}{ds}=h$ ....(2)

is also applicable to the path of light ray.

If we put ds = 0 in (2), then we get h = ∞, we put h = ∞ in equation (1), we get

$$\frac{d^2u}{d\varphi^2}+u=3mu^2 \quad \text{...(3)}$$

Thus, the track of a light ray is given by (3)

Neglecting the small term $3mu^2$ as a first approximation, we have

$$\frac{d^2u}{d\varphi^2}+u=0$$

The solution of this is

u = Acos φ+ Bsin φ ....(4)

Initial conditions are φ = 0, $\frac{du}{d\varphi}=0$

and φ = 0, u = $\frac{1}{R}$

Subjecting equation (4) to these conditions

$$\frac{1}{R}=A+B.0=A$$

$$0=\frac{du}{d\varphi}=-A\sin\varphi+B\cos\varphi=-A.0+B.1=B$$

i.e., A = $\frac{1}{R}, B=0$

Substituting these values in equation (4)

$$u=\frac{1}{R}\cos\varphi$$

Putting this approximation of the R.H.S. of equation (3),

$$\frac{d^2u}{d\varphi^2}+u=\frac{3m}{R^2}\cos^2\varphi$$

The particular integral of $\frac{3m\cos^2\varphi}{R^2}$ is

$$\frac{1}{1+D^2}.\frac{3m}{R^2}\cos^2\varphi=\frac{3m}{R^2}.\frac{1}{1+D^2}\left(\frac{1+\cos2\varphi}{2}\right)$$

$$=\frac{3m}{2R^2}.\frac{1}{1+D^2}(e^0+\cos2\varphi)=\frac{3m}{2R^2}\left[\frac{e^0}{1+0}+\frac{\cos2\varphi}{1-2^2}\right]$$

$$=\frac{3m}{2R^2}\left(1-\frac{\cos2\varphi}{3}\right)=\frac{m}{2R^2}(3-\cos2\varphi)$$

$= \frac{m}{2R^2}\left(3\cos^2\varphi + 3\sin^2\varphi - \cos^2\varphi + \sin^2\varphi\right)$

$= \frac{m}{R^2}\left(\cos^2\varphi + 2\sin^2\varphi\right) = \frac{m}{r^2R^2}\left(r^2\cos^2\varphi + 2r^2\sin^2\varphi\right)$

As a second approximation, the complete solution of equation (3) is

$$\frac{1}{r} = u = \frac{1}{R}\cos\varphi + \frac{m(r^2\cos^2\varphi + 2r^2\sin^2\varphi)}{r^2R^2}$$

Multiplying by rR,

Changing this into cartesian co-ordinates which may be assumed to be valid in the nearly Euclidean space surrounding the sun, we obtain

$$R = x + \frac{m(x^2 + 2y^2)}{R\sqrt{(x^2 + y^2)}}$$

or, $x = R - \frac{m(x^2 + 2y^2)}{R\sqrt{(x^2 + y^2)}}$ .... (5)

The first approximation is

$$\frac{1}{r} = u = \frac{1}{R}\cos\varphi$$

or, R= r cosφ, or $x = R$ .... (6)

From equation (5) and (6), it is clear that the second term = $\frac{m(x^2 + 2y^2)}{R\sqrt{(x^2 + y^2)}}$ in equation (5) shows a deviation from the path x = R. Asymptotes to (5) are obtained by taking y very large compared to x so that asymptotes to (5) are

$$x = R - \frac{m}{R}(\pm 2y)$$

$$x = R + \frac{2my}{R} \; and \; x = R - \frac{2my}{R}$$

i.e., $y = \frac{Rx}{2m} - \frac{R^2}{2m}$

and $y = -\frac{Rx}{2m} + \frac{R^2}{2m}$

Let α be the angle between these asymptotes so that

$$\tan\alpha = \frac{\frac{R}{2m} - \left(-\frac{R}{2m}\right)}{1 + \left(\frac{R}{2m}\right)\left(-\frac{R}{2m}\right)} = \frac{4mR}{4m^2 - R^2}$$

or, $\tan\alpha = \dfrac{4mR}{4m^2 - R^2}$

Then $\sin\alpha = \dfrac{4mR}{4m^2 + R^2}$

Since $5m^2 << R^2$ and hence neglected.

$$\sin\alpha = \frac{4mR}{R^2} = \frac{4m}{R}$$

$$\alpha = \frac{4m}{R}$$

For α is very small and so $\sin\alpha = \alpha$

For a light ray grazing sun's limb.

$$\alpha = \frac{4m}{R} = \frac{4 \times 1.47}{697000} = 1.75 \sec onds$$

i.e., deflection = 1.75 seconds

This prediction can be verified by observations at the time of eclipse on the apparent positions of the stars.

Treatment in Newtonian Theory:

Let a light ray emitted from a star be moving parallel to y-axis and be passing through the mass m at a distance x = R

The acceleration in x-direction is given by

$$\frac{d^2x}{dt^2} = -\frac{m}{r^2} \cdot \frac{x}{r} = -\frac{mx}{(x^2 + y^2)^{\frac{3}{2}}} \qquad \text{......................... (7)}$$

For a light ray moving parallel to y-axis,

$\dfrac{dy}{dt}$ = 1 in gravitational units $\dfrac{d^2y}{dt^2} = 0$

$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dy}.\frac{dy}{dt}\right)=\frac{d^2x}{dy^2}\left(\frac{dy}{dt}\right)^2+\frac{dx}{dy}.\frac{d^2y}{dt^2}=\frac{d^2x}{dy^2}.1^2+\frac{dx}{dy}.0=\frac{d^2x}{dy^2}$$

or, $\frac{d^2x}{dt^2}=\frac{d^2x}{dy^2}$

Using this in equation (7), we have

$$\frac{d^2x}{dy^2}=-\frac{mx}{(x^2+y^2)^{\frac{3}{2}}}=-\frac{mR}{(R^2+y^2)^{\frac{3}{2}}};\ For\ x=R$$

or, $\frac{d^2x}{dy^2}=-\frac{mR}{(R^2+y^2)^{\frac{3}{2}}}$

Integrating the above equation w.r.t.y, we have

$$\frac{dx}{dy}=-\int\frac{mRdy}{(R^2+y^2)^{\frac{3}{2}}}=-mR\int\frac{R\sec^2\theta d\theta}{R^3\sec^3\theta};\ Let\ y=R\tan\theta$$

$$=-\frac{m}{R}\int\cos\theta d\theta=-\frac{m}{R}\sin\theta+c$$

or, $\frac{dx}{dy}=-\frac{m}{R}\sin\theta+c=-\frac{my}{R\sqrt{(R^2+y^2)}}+c$.................(8)

or, Integrating the above equation, we have

$$x=-\frac{m}{R}\sqrt{(R^2+y^2)}+cy+c_1 \qquad ....(9)$$

Subjecting equation (8) and (9) to the conditions

$$\frac{dx}{dy}=0,\qquad x=R,\qquad y=0$$

we obtain c = 0 and R= - m + c_1

i.e., c = 0 and c_1 = m + R.

Now, equation (9) becomes

$$x=R+\left[m-\frac{m}{R}\sqrt{(R^2+y^2)}\right] \quad .... (10)$$

This is the equation of the path of a light ray according to Newtonian theory. The second term $m-\frac{m}{R}\sqrt{(R^2+y^2)}$ shows deviation from the path x = R

Asymptotes to equation (10) are obtained by taking y very large compared with x so that asymptotes are

$$x = R + m - \frac{m}{R}(\pm y)$$

or, $y = \frac{Rx}{m} - \frac{R}{m}(R + m)$

and $y = -\frac{Rx}{m} + \frac{R}{m}(R + m)$

let β be the angle between these asymptotes, then

$$\tan\beta = \frac{\frac{R}{m} - (-\frac{R}{m})}{1 + \frac{R}{m}\left(-\frac{R}{m}\right)} = \frac{2mR}{m^2 - R^2}$$

So that $\sin\beta = \frac{2mR}{m^2 + R^2}$

Using the fact that β is very small and

$m^2 << R^2$

and hence m^2 is neglected.

Therefore, $\beta = \frac{2mR}{R^2} = \frac{2m}{R}$

But $\alpha = \frac{4m}{R} = 2(\frac{2m}{R}) = 2\beta$

or, $\alpha = 2\beta$

This shown that the deflection on the path of a light ray due relativistic field is twice that predicted by Newtonian theory.

(iii) Shift in Spectral Lines:

We consider the shift in spectral lines of light emitted by an atom situated in a gravitational field when this light is observed on the surface of the earth. Atoms of sodium vibrate with uniform frequency. Let ds be interval between the beginning and the end of one vibration and dt the corresponding periodic time. Consider as observer who is moving with sodium atoms. Let the atom be momentarily at rest in co-ordinate system (r, θ, φ, t) so that, by Schwarzschild line clement,

$ds^2 = (1 - \frac{2m}{r})\, dt^2$; For dr, dθ, dφ = 0 .... (1)

From which we get

$$\frac{dt}{ds}=\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}=1+\frac{m}{r} \text{ up to first approximation}$$

or, $\frac{dt}{ds}=1+\frac{m}{r}$

We compare the periodic time of sodium atom at two places

(i) On the surface of the sun.

(ii) On the surface of the earth.

Let dt and dt′ be the periodic times of a sodium atom on the surfaces of the sun and the earth respectively. Then

$$\frac{\lambda+\delta\lambda}{\lambda}=\frac{dt}{ds}=1+\frac{m}{r}\text{..(2)}$$

$$\frac{dt'}{ds'}=1 \quad \text{...(3)}$$

$\frac{\lambda+\delta\lambda}{\lambda}$ is the ratio of the observed wave lengths of a light ray corresponding to a spectral line which originates on the surface of the sun. Using the fact that *ds* remains invariant under arbitrary co-ordinate transformation, we obtain

$$\frac{dt'}{ds}=1\text{...................(4)}, \quad by\,(3)$$

From equation (2) and (4), we have

$$\frac{\lambda+\delta\lambda}{\lambda}=\frac{dt}{dt'}=1+\frac{m}{r}$$

or, $\frac{\delta\lambda}{\lambda}=\frac{m}{r}$

This is the required expression for the shift in spectral lines. If the spectral line originates on the surface of the sun, then

$$\frac{\delta\lambda}{\lambda}=2.12\times10^{-6}$$

This prediction has been confirmed after the experiment of Adams and St. John.

Chapter Six
Relativistic Mechanics

6.1 Equivalence of Mass and Energy ($E=mc^2$)[3]:

From work-energy theorem, the kinetic energy of a moving body is equal to the work done by the external force that imparts the velocity to the body from rest. If F is the force acting on the body; then work done by the force on body in raising its velocity from

$v=0$ to $v=v$ is given by

$$W=\int_0^v F\,.ds$$

Kinetic energy of the body, $E_k=T(say)=W=\int_0^v F\,.ds$

$$=\int_0^v F.\frac{ds}{dt}dt$$

$$=\int_0^v F.vdt$$

$$=\int_0^v \frac{dp}{dt}.vdt$$ (Since F=rate of change of momentum= $\frac{dp}{dt}$)

$$=\int_0^v v.\frac{d}{dt}(mv)dt$$ (Since $p=mv$; m being the mass of moving body)

$$=\int_0^v v.d(mv)$$

But from the relation of variation of mass with velocity, the mass of body in motion

$m=\frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}, m_0$ being rest mass of body

$$\therefore E_k=T=\int_0^v vd\left(\frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}v\right)$$

$$=m_0\int_0^v vd\left[v\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$$

$$= m_0 \int_0^v v\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} dv - v\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2} \times\left(-\frac{2v}{c^2}\right)dv\right]$$

$$= m_0 \int_0^v v\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right]dv$$

$$= m_0 \int_0^v \frac{vdv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$\therefore E_k = m_0 \int_1^{\left(1-\frac{v^2}{c^2}\right)} \frac{\left(-\frac{c^2}{2}\right)dy}{y^{3/2}}$$

Let $y = 1-\frac{v^2}{c^2}$

$or, \frac{dy}{dv} = o - \frac{2v}{c^2}$

$or, -2vdv = c^2 dy$

$\therefore vdv = -\frac{c^2}{2}dy$

When v = 0 then y = 1

When v = v then y = $1-\frac{v^2}{c^2}$

$$= \frac{-m_0c^2}{2}\left[\frac{y^{-1/2}}{-1/2}\right]_1^{1-\frac{v^2}{c^2}}$$

$$= m_0c^2\left[\frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}-1\right]$$

$$= \left[\frac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}-m_0\right]c^2$$

$$= (m-m_0)c^2 \qquad \text{.......................} (2)$$

This is relativistic expression for kinetic energy. Thus the kinetic energy of moving body is equal to gain in mass times the squire of a speed of light. Therefore

m_0c^2 may be regarded as rest energy of a body of rest mass m_0. This rest energy may be considered as internal energy of the body. Then total energy (E)

i.e., $E =$ rest energy +relativistic kinetic energy

$$= m_0c^2 + (m - m_0)c^2$$

$$= m_0c^2 + mc^2 - m_0c^2$$

$$\therefore E = mc^2$$

This is well known Einstein mass energy relation which states a universal equivalence between mass and energy.

6.2 Deduce Maxwell's Equations.

Maxwell's four Equations are

(1) $\nabla.\underline{E} = 4\pi\rho$ **(2)** $Curl\,\underline{H} = \frac{1}{c}\left(\frac{\partial \underline{E}}{\partial t} + 4\pi \underline{J}\right)$

(3) $Curl\,\underline{E} = \frac{-1}{c}\frac{\partial \underline{H}}{\partial t}$ **(4)** $\nabla.\underline{H} = O$

where $\underline{E}$= Electric field, $\underline{H}$ = Magnetic field, ρ = Charge density, $\underline{J}$ = Current density.

The Maxwell's equations in the relativistic form are

$$\frac{\partial F_{\mu\upsilon}}{\partial x^{\sigma}} + \frac{\partial F_{\upsilon\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\upsilon}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial F^{\mu\upsilon}}{\partial x^{\mu}} = \frac{-4\pi}{c} J^{\upsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$

We have the electromagnetic field vectors

$$F^{\mu\upsilon} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}$$

$$and\ F_{\mu\upsilon} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix}$$

We put $\upsilon = 0$ in equation (2), we have

$$\frac{\partial F^{\mu 0}}{\partial x^{\mu}} = \frac{-4\pi}{c} J^0$$

or, $$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = -\frac{4\pi}{c}(c\rho)$$

Now, putting the values of $F^{00}, F^{10}, F^{20}, F^{30};$ we have

$$0 - \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = -4\pi\rho$$

Or, $$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho$$

Or, $$(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}).(\underline{i}E_x + \underline{j}E_y + \underline{k}E_z) = 4\pi\rho$$

$\therefore \nabla.\underline{E} = 4\pi\rho$ which is the Maxwell's equation (1)

Again we put $\upsilon = 1$ in equation (2) and we get

$$\frac{\partial F^{\mu 1}}{\partial x^{\mu}} = -\frac{4\pi}{c} J^1$$

$$\Rightarrow \frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = -\frac{4\pi}{c} J_x$$

$$\Rightarrow \frac{1}{c}\frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} + \frac{\partial H_y}{\partial z} = -\frac{4\pi}{c} J_x$$

$$\Rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{c}(4\pi J_x + \frac{\partial E_x}{\partial t}) \qquad \text{.........................(a)}$$

Now, we put $\upsilon = 2$ in equation (2) and we get

$$\frac{\partial F^{\mu 2}}{\partial x^{\mu}} = -\frac{4\pi J^2}{c}$$

$$\Rightarrow \frac{\partial F^{02}}{\partial x^0} + \frac{\partial F^{12}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{32}}{\partial x^3} = -\frac{4\pi}{c} J_y$$

$$\Rightarrow \frac{\partial E_y}{c\partial t}+\frac{\partial H_z}{\partial x}+\frac{\partial}{\partial y}.0+\frac{\partial}{\partial z}(-H_x)=-\frac{4\pi}{c}J_y$$

$$\Rightarrow \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\frac{1}{c}(4\pi J_y+\frac{\partial E_y}{\partial t}) \qquad \text{.........................} (b)$$

Again we put $\upsilon = 3$ in equation (2) and we get

$$\frac{\partial F^{\mu 3}}{\partial x^{\mu}}=-\frac{4\pi J^3}{c}$$

$$\Rightarrow \frac{\partial F^{03}}{\partial x^0}+\frac{\partial F^{13}}{\partial x^1}+\frac{\partial F^{23}}{\partial x^2}+\frac{\partial F^{33}}{\partial x^3}=-\frac{4\pi}{c}J_z$$

$$\Rightarrow \frac{\partial E_z}{c\partial t}+\frac{\partial}{\partial x}(-H_y)+\frac{\partial H_x}{\partial y}+0=-\frac{4\pi}{c}J_z$$

$$\Rightarrow \frac{\partial Hy}{\partial x}-\frac{\partial H_x}{\partial y}=\frac{1}{c}(4\pi J_z+\frac{\partial E_z}{\partial t}) \qquad \text{.........................} (c)$$

Now , (a) x$\underline{i}$ + (b) x$\underline{j}$ + (c) x$\underline{k}$ and we get

$$\underline{i}(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z})+\underline{j}(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x})+\underline{k}(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y})=\frac{1}{c}\{4\pi(\underline{i}J_x+\underline{j}J_y+\underline{k}J_z)+\underline{i}\frac{\partial E_x}{\partial t}+\underline{j}\frac{\partial E_y}{\partial t}+\underline{k}\frac{\partial E_z}{\partial t}\}$$

$\therefore Curl\,\underline{H}=\frac{1}{c}\left(\frac{\partial \underline{E}}{\partial t}+4\pi\underline{J}\right)$ which is the second Maxwell's equation.

We have

$$\frac{\partial F_{\mu\upsilon}}{\partial x^{\sigma}}+\frac{\partial F_{\upsilon\sigma}}{\partial x^{\mu}}+\frac{\partial F_{\sigma\mu}}{\partial x^{\upsilon}}=0 \qquad \text{...} (1)$$

Now, we put $\mu=0,\upsilon=2,\sigma=3,$ in equation (1) and we get

$$\frac{\partial F_{02}}{\partial x^3}+\frac{\partial F_{23}}{\partial x^0}+\frac{\partial F_{30}}{\partial x^2}=0$$

$$\Rightarrow -\frac{\partial E_y}{\partial z}+\frac{\partial H_x}{c\partial t}+\frac{\partial E_z}{\partial y}=0$$

$$\therefore \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{1}{c}.\frac{\partial H_x}{\partial t} \qquad \text{...} (a')$$

We put $\mu=1,\upsilon=0\,and\;\sigma=3$ in equation (1) and we get

$$\frac{\partial F_{10}}{\partial x^3}+\frac{\partial F_{03}}{\partial x^1}+\frac{\partial F_{31}}{\partial x^0}=0$$

$$\Rightarrow \frac{\partial E_x}{\partial z}+\frac{\partial(-E_z)}{\partial x}+\frac{\partial H_y}{c\,\partial t}=0$$

$$\therefore \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c}.\frac{\partial H_y}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b')$$

Lastly we put $\mu = 0, \upsilon = 1, \sigma = 2$ in equation (1) and we have

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0$$

$$\Rightarrow -\frac{\partial E_x}{\partial y} + \frac{\partial H_z}{c\partial t} + \frac{\partial E_y}{\partial x} = 0$$

$$\therefore \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c}.\frac{\partial H_z}{\partial t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c')$$

Apply (a') x$\underline{i}$ + (b') x$\underline{j}$ + (c') x$\underline{k}$

$$\underline{i}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) + \underline{j}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \underline{k}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) = -\frac{1}{c}(\underline{i}\frac{\partial H_x}{\partial t} + \underline{j}\frac{\partial H_y}{\partial t} + \underline{k}\frac{\partial H_z}{\partial t})$$

$\therefore Curl\underline{E} = -\frac{1}{c}\frac{\partial \underline{H}}{\partial t}$ which is the Maxwell's equation (3)

Now, putting $\mu = 1,\ \upsilon = 2,\ \sigma = 3$ in equation (1) and we get

$$\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial H_z}{\partial z} + \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

$$\Rightarrow \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

$\therefore \nabla.\underline{H} = 0$ which is the Maxwell's equation (4)

6.3 Energy Momentum Tensor $T^{\mu\upsilon}$ and its Physical Significance:

The electromagnetic field vector is defined by

$$F^{\mu\upsilon} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}$$

$$and\ F_{\mu\upsilon} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix}$$

$$\eta^{\mu\upsilon} = \eta_{\mu\upsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now, $F_{\alpha\beta}F^{\alpha\beta} = F_{0\beta}F^{0\beta} + F_{1\beta}F^{1\beta} + F_{2\beta}F^{2\beta} + F_{3\beta}F^{3\beta}$

$$= F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03}$$
$$+ F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13}$$
$$+ F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23}$$
$$+ F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33}$$
$$= 0 + (-E_x).E_x + (-E_y).E_y + (-E_z).E_z$$
$$+ E_x(-E_x) + 0 + H_z.H_z + (-H_y)(-H_y)$$
$$+ E_y(-E_y) + (-H_z)(-H_z) + 0 + H_x.H_x$$
$$+ E_z(-E_z) + H_y.H_y + (-H_x)(-H_x) + 0$$
$$= 2\left(-E_x^2 - E_y^2 - E_z^2 + H_x^2 + H_y^2 + H_z^2\right)$$
$$= 2\left(-E^2 + H^2\right)$$

The energy momentum tensor $T^{\mu\upsilon}$ is defined as

$$T^{\mu\upsilon} = \frac{1}{4\pi}\left(-F^{\mu\alpha}F_{\alpha}^{\upsilon} + \frac{1}{4}\eta^{\mu\upsilon}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$= \frac{1}{4\pi}\left\{- F^{\mu\alpha}\eta^{\upsilon\sigma}F_{\sigma\alpha} + \frac{1}{4}\eta^{\mu\upsilon}2(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{- F^{\mu\alpha}\eta^{\upsilon\sigma}F_{\sigma\alpha} + \frac{1}{2}\eta^{\mu\upsilon}(-E^2 + H^2)\right\}$$

$$T^{00} = \frac{1}{4\pi}\left\{- F^{0\alpha}\eta^{0\sigma}F_{\sigma\alpha} + \frac{1}{2}\eta^{00}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{- F^{0\alpha}\eta^{0\sigma}F_{\sigma\alpha} + \frac{1}{2}\eta^{00}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{- F^{0\alpha}\eta^{00}F_{0\alpha} + \frac{1}{2}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{- F^{00}F_{00} - F^{01}F_{01} - F^{02}F_{02} - F^{03}F_{03} + \frac{1}{2}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{0 - E_x(-E_x) - E_y(-E_y) - E_z(-E_z) + \frac{1}{2}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{E_x^2 + E_y^2 + E_z^2 + \frac{1}{2}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left\{E^2 + \frac{1}{2}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left(\frac{E^2}{2} + \frac{H^2}{2}\right)$$

$= \frac{1}{8\pi}(E^2 + H^2)$ which is the energy density of electromagnetic field.

$$T^{01} = \frac{1}{4\pi}\left\{- F^{0\alpha}\eta^{1\sigma}F_{\sigma\alpha} + \frac{1}{2}\eta^{01}(-E^2 + H^2)\right\}$$

$$= \frac{1}{4\pi}\left(-F^{0\alpha}\eta^{11}F_{1\alpha} + 0\right) \qquad \because \eta^{01} = 0$$

$$= \frac{1}{4\pi}\left(F^{00}F_{10} + F^{01}F_{11} + F^{02}F_{12} + F^{03}F_{13}\right)$$

$$= \frac{1}{4\pi}\left\{0 + 0 + E_y H_z + E_z(-H_y)\right\}$$

$$= \frac{1}{4\pi}\left(E_y H_z - E_z H_y\right)$$

$= x$-component of $\frac{1}{4\pi}(\underline{E}\Lambda\underline{H}) = \mathrm{T}^{10}$

$$T^{02}=\frac{1}{4\pi}\left\{-F^{0\alpha}\eta^{2\sigma}F_{\sigma\alpha}+\frac{1}{2}\eta^{02}(-E^2+H^2)\right\}$$

$$=\frac{1}{4\pi}\left(F^{0\alpha}F_{2\alpha}+0\right) \qquad \because \eta^{22}=-1, \eta^{02}=0$$

$$=\frac{1}{4\pi}\left(F^{00}F_{20}+F^{01}F_{21}+F^{02}F_{22}+F^{03}F_{23}\right)$$

$$=\frac{1}{4\pi}\{0+E_x(-H_z)+0+E_z.H_x\}$$

$$=\frac{1}{4\pi}\left(E_zH_x-E_xH_z\right)$$

$= y$-component of $\frac{1}{4\pi}\left(\underline{E}\Lambda\underline{H}\right)=T^{20}$

Similarly, $T^{03}=\frac{1}{4\pi}\left(E_xH_y-E_yH_x\right)$

$= z$- component of $\frac{1}{4\pi}\left(\underline{E}\Lambda\underline{H}\right)=T^{30}$

Thus $\left(T^{01},T^{02},T^{03}\right)=\left(T^{10},T^{20},T^{30}\right)=\frac{p}{\rho}$ where $\rho=\frac{c(\underline{E}\times\underline{H})}{4\pi}, c=1$

Now, $T^{11}=\frac{1}{4\pi}\left\{-F^{1\alpha}\eta^{1\sigma}F_{\sigma\alpha}+\frac{1}{2}\eta^{11}(-E^2+H^2)\right\}$

$$=\frac{1}{4\pi}\left\{F^{1\alpha}F_{1\alpha}-\frac{1}{2}(-E^2+H^2)\right\}$$

$$=\frac{1}{4\pi}\left\{F^{10}F_{10}+F^{11}F_{11}+F^{12}F_{12}+F^{13}F_{13}-\frac{1}{2}(-E^2+H^2)\right\}$$

$$=\frac{1}{4\pi}\left\{\left(-E_x\right).E_x+0+H_z.H_z+(-H_y)(-H_y)-\frac{1}{2}(-E^2+H^2)\right\}$$

$$=\frac{1}{4\pi}\left\{-E_x^2+H_z^2+H_y^2-\frac{1}{2}(-E^2+H^2)\right\}$$

$$=\frac{1}{4\pi}\left(-E_x^2+H_z^2+H_y^2+\frac{E^2}{2}-\frac{H^2}{2}\right) \qquad \left[\because H^2=H_x^2+H_y^2+H_z^2 \quad \therefore H_y^2+H_z^2=H^2-H_x^2\right]$$

$$=\frac{1}{4\pi}\left(-E_x^2-H_x^2+H^2+\frac{E^2}{2}-\frac{H^2}{2}\right)$$

$$=-\frac{1}{4\pi}\left(E_x^2+H_x^2\right)+\frac{1}{8\pi}(E^2+H^2)$$

Similarly, $T^{22} = -\frac{1}{4\pi}\left(E_y^2 + H_y^2\right) + \frac{1}{8\pi}(E^2 + H^2)$

and $T^{33} = -\frac{1}{4\pi}\left(E_z^2 + H_z^2\right) + \frac{1}{8\pi}(E^2 + H^2)$

Now, $T^{12} = \frac{1}{4\pi}\left\{-F^{1\alpha}\eta^{2\sigma}F_{\sigma\alpha} + \frac{1}{2}\eta^{12}(-E^2 + H^2)\right\}$

$= \frac{1}{4\pi}\left(F^{1\alpha}F_{2\alpha} + 0\right) \qquad \because \eta^{12} = 0, \eta^{22} = 1$

$= \frac{1}{4\pi}\left(F^{10}F_{20} + F^{11}F_{21} + F^{12}F_{22} + F^{13}F_{23}\right)$

$= \frac{1}{4\pi}\left\{(-E_x)E_y + 0 + 0 + (-H_y)H_x\right\}$

$= -\frac{1}{4\pi}(E_x.E_y + H_x.H_y) = T^{21}$

Similarly, $T^{23} = -\frac{1}{4\pi}\left(E_yE_z + H_yH_z\right) = T^{32}$

and $T^{31} = -\frac{1}{4\pi}\left(E_zE_x + H_zH_x\right) = T^{13}$

Thus the general solution is

$$T^{ij} = -\frac{1}{4\pi}\left(E^iE^j + H^iH^j\right) + \frac{1}{8\pi}\delta^{ij}(E^2 + H^2)$$

where $\delta^{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$

This general solution is called Maxwell's stress tensor.

The physical significance is as follows:

$T^{00} = \frac{1}{8\pi}(E^2 + H^2)$ which is the energy density of electromagnetic field.

$T^{i0} = T^{0j} = \frac{1}{4\pi}(\underline{E}\Lambda\underline{H}) = \frac{P}{\rho}$ where $\rho = \frac{c}{4\pi}(\underline{E}\Lambda\underline{H})$; pointing vector.

$$T^{ij} = -\frac{1}{4\pi}(E^iE^j + H^iH^j) + \frac{1}{8\pi}\delta^{ij}(E^2 + H^2)$$

where $\delta^{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$

Which is the Maxwell's stress tensor.

Chapter Seven
Cosmology

7.1 Introduction:

Cosmology is the branch of science concerned with the study of the universe. It is probably the oldest of sciences, as questions posed by cosmologists been pondered through the ages. How big the universe? How old is it? How did it come into being? At the same time, cosmology ranks among the most modern and dynamic of the physical sciences, as astounding advances in both theory and experiment change our view of the universe.

Cosmology:

The earliest records of rational attempts to describe the motion of the stars and planets date back to classical antiquity. By 200 BC, the accepted view of the universe was that of an exact sphere, with the distant stars and planets executing perfect circular motions about the earth. Although much of Greek scholarship was lost during the dark ages, this view of the universe merged with Christian theology and survived up to the sixteenth century AD.

Cosmology underwent a major paradigm shift during the European Renaissance, when it was first suggested that the planets revolve around the sun rather than the earth. With the invention of the telescope, support for the copernican view was provided by the astronomical observations of scientists such as Tycho Brahe, Galileo and Keplar despite the objections of the Church. In particular, Kepler discovered that the motion of the planets about the sun could be neatly described in terms of well known mathematical curves of orbits. However, the nature of the force responsible for this planetary motion remained a mystery.

Cosmology is the academic discipline that seeks to understand the origin, evolution, structure and ultimate fate of the universe at large, as well as the natural laws that keep it in order. Modern cosmology is dominated by the Big Bang theory, which brings together observational astronomy and particle physics.

Although the word cosmology is recent, the study of the universe has a long history involving science, philosophy, esotericism and religion. Related studies include cosmology which focuses on the origin of the universe and cosmography which maps the features of the universe. Cosmology is also connected to astronomy. However, they are contrasted in that while the former is concerned with the universe in its entirety, the latter deals with individual celestial objects.

In 1687, Isaac Newton (1642-1727) published a number of universal laws that described all known motion, the culmination of his life work. In particular, Newton postulated a law of gravity that predicted and attractive force between any two bodies due to their mass. This law successfully accounted for the known motion of falling bodies. Even better, the same law accurately predicted the Keplerian orbits of the planets about the sun. In this manner, Newton gave the first physical explanation for both terrestrial and celestial gravity, showing them to be of common origin.

German physicist Albert Einstein (1879-1955) published his papers on Relativity Theory between 1905 and 1917. He became internationally noted after 1919 and was awarded the Nobel Prize in 1921. Einstein emigrated to the USA when Hitler came to power in Germany. Einstein said, “Relativity teaches us the connection between the different descriptions of one and the same reality.” In his usual humble way, Einstein explained how he reinvented physics? I sometimes ask myself how it came about that I was the one to develop the theory of relativity. The reason, I think, is that a normal adult stops to think about problems of space and time. These are things which he has thought about as a child. But intellectual development was retarded, as a result of which I began to wonder about space and time only when I had already grown up.

This view of relativity, that there are different realities, has been picked up unanimously by the public and hence, has taken on a far greater meaning than that of the original scientific theory, the focus of which was strictly speaking on mechanics and electrodynamics. This astonishing success was at least in part due to Einstein’s personality. He understood himself as a philosopher as much as a scientist and he was ready to discuss philosophical issues at any time, particularly matters involving relativity. The philosophical aspect of relativity forced people to think differently about the universe.

An outstanding feature of special relativity is its mass-energy relation which is expressed in the well known formula, $E = mc^2$.

Einstein derived this relation in an attempt to reconcile Maxwell's electromagnetic theory with the conservation of energy and momentum. Maxwell said that light carries a momentum which is to say that a wave carries an amount of energy.

The mass-energy relation tells us that any change in the energy level of an object necessarily involves a change in the object's mass and vice-versa. The most dramatic consequences of this law are observed in nature, for example in nuclear fission and fusion processes, in which stars like the sun emit energy and lose mass. The same law also applies to the forces set free in the detonation of an atomic bomb.

7.2 Deduce Robertson-Walker Metric.

The Robertson-Walker line element is fundamental in the standard models of cosmology. The mathematical framework in which the Robertson-Walker metric occurs is that of general theory of relativity.

To us the universe appears to be homogeneous and isotropic on a sufficiently large scale. It is unlikely that we are in a special position in the universe. This leads us to the assumption of the cosmological principle which state roughly speaking that the universe looks the same from all positions in space at a particular time and that all directions in space at any point are equivalent. To define the moment of time which is valid globally, we proceed as follows. We introduce a series of non-intersecting space like hyper surfaces. All galaxies lies on such a hyper surface in such a way that the surface of simultaneity of the local Lorentz frame of any galaxy coincides locally with the hyper surfaces. Thus the 4-velocity of a galaxy is orthogonal to the space like hyper surface. All these hyper surfaces can be labeled by a parameter which may be taken as the proper time of any galaxy. Hence, each hyper surfaces defines a moment of time. Now, let t denotes the synchronized proper time of all galaxies and introduce co-ordinates $(x^1, x^2, x^3) =$ constant for all galaxies.

Then the space time metric can be written as,

$ds^2 = dt^2 - h_{ij} dx^i dx^j ; i, j = 1,2,3$ where $h_{ij} = h_{ij}(t, x^1, x^2, x^3)$. The spatial distance $d\sigma^2$ of any two nearby galaxies on the same hyper surfaces t = constant at (x^1, x^2, x^3) and $(x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3)$ is $d\sigma^2 = h_{ij} \Delta x^i \Delta x^j$.

Let us consider the triangle formed by three nearby galaxies at same particular time and the triangle formed by the same galaxies at same letter time. By the cosmological principle, the second triangle must be similar to the first and the magnification factor must be independent at the position of the triangle in the 3-space. This means that time can be enter $d\sigma^2$ only through a common factor $R^2(t)$ in order that the ratios of small distances may be the same at all times. Thus $ds^2 = dt^2 - R^2(t) r_{ij} dx^i dx^j ; i, j = 1,2,3$ where $r_{ij} = r_{ij}(x^1, x^2, x^3)$.

The 3-space $dl^2 = r_{ij} dx^i dx^j$ is homogeneous, isotropic and independent of time. Hence, this must be a space of constant curvature. The 3-dimensional curvature tensor ${}^{(3)}R^i_{jkl}$ of such a space can be expressed in terms of r_{ij} alone. From the symmetric properties of ${}^{(3)}R^i_{ijkl}$; it can be expressed as ${}^{(3)}R_{ijkl} = K(r_{ik} r_{jl} - r_{il} r_{jk})$, where k is a constant. It can be verified that the 3-dimensional curvature tensor ${}^{(3)}R^i_{jkl}$ of the space $dl^2 = r_{ij} dx^i dx^j$ has the above form if r_{ij} are chosen so that the space-time metric is as follows,

$$ds^2 = dt^2 - \frac{R^2(t)(dx^2 + dy^2 + dz^2)}{\left\{1 + \frac{k}{4}(x^2 + y^2 + z^2)\right\}^2}$$

With the transformation

$$x = R' \sin\theta \cos\varphi$$

$$y = R' \sin\theta \sin\varphi \,\&\, z = R' \cos\theta$$

Which is

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dR'^2 + R'^2(d\theta^2 + \sin^2\theta d\varphi^2)}{(1 + \frac{k}{4}R'^2)^2}\right\}$$

Again by the transformation

$$r = \frac{R'}{1 + \frac{k}{4} R'^2}$$

This becomes

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$

This is the Robertson- Walker metric.

7.3 Calculating R_{00}, R_{11}, R_{22}, R_{33} from Robertson-Walker Line Element:

We know the Robertson-Walker line element is

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$

In matrix form

$$g_{\mu\upsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -R^2(t)r^2 & 0 \\ 0 & 0 & 0 & -R^2(t)r^2\sin^2\theta \end{bmatrix}$$

$$= \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

where $g_{00} = 1, g_{11} = -\frac{R^2(t)}{1-kr^2}, g_{22} = -R^2(t)r^2, g_{33} = -R^2(t)r^2\sin^2\theta$

and the other terms are zero.

Now, the inverse of $g_{\mu\upsilon}$ is $g^{\mu\upsilon} = \frac{1}{|g_{\mu\upsilon}|}$ (Adjoint matrix of $g_{\mu\upsilon}$)

$$|g_{\mu\upsilon}|=\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & \dfrac{-R^2(t)}{1-kr^2} & 0 & 0\\ 0 & 0 & -R^2(t)r^2 & 0\\ 0 & 0 & 0 & -R^2(t)r^2\sin^2\theta\end{vmatrix}$$

$$=\frac{R^6(t)r^4\sin^2\theta}{kr^2-1}$$

$$\text{Adjoint matrix of } g_{\mu\upsilon}=\begin{bmatrix}\dfrac{R^6(t)r^4\sin^2\theta}{kr^2-1} & 0 & 0 & 0\\ 0 & R^4(t)r^4\sin^2\theta & 0 & 0\\ 0 & 0 & -\dfrac{R^4(t)r^2\sin^2\theta}{kr^2-1} & 0\\ 0 & 0 & 0 & -\dfrac{R^4(t)r^2}{kr^2-1}\end{bmatrix}$$

$$g^{\mu\upsilon}=\frac{kr^2-1}{R^6(t)r^4\sin^2\theta}\begin{bmatrix}\dfrac{R^6(t)r^4\sin^2\theta}{kr^2-1} & 0 & 0 & 0\\ 0 & R^4(t)r^4\sin^2\theta & 0 & 0\\ 0 & 0 & -\dfrac{R^4(t)r^2\sin^2\theta}{kr^2-1} & 0\\ 0 & 0 & 0 & -\dfrac{R^4(t)r^2}{kr^2-1}\end{bmatrix}$$

$$\text{Or, } g^{\mu\upsilon}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \dfrac{kr^2-1}{R^2(t)} & 0 & 0\\ 0 & 0 & -\dfrac{1}{R^2(t)r^2} & 0\\ 0 & 0 & 0 & -\dfrac{1}{R^2(t)r^2\sin^2\theta}\end{bmatrix}$$

$$=\begin{pmatrix}g^{00} & g^{01} & g^{02} & g^{03}\\ g^{10} & g^{11} & g^{12} & g^{13}\\ g^{20} & g^{21} & g^{22} & g^{23}\\ g^{30} & g^{31} & g^{32} & g^{33}\end{pmatrix}$$

where $g^{00}=1, g^{11}=\dfrac{kr^2-1}{R^2(t)}, g^{22}=-\dfrac{1}{R^2(t)r^2,} g^{33}=-\dfrac{1}{R^2(t)r^2\sin^2\theta}$

and the other terms are zero.

Here $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$ and (0, 1, 2, 3) = (t, r, θ, φ)

We know the Christoffel symbol of second kind is

$$\Gamma^{\mu}_{\upsilon\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\upsilon,\lambda} + g_{\sigma\lambda,\upsilon} - g_{\upsilon\lambda,\sigma})$$

The non-zero Christoffel symbols are

$$\Gamma^0_{11} = \frac{1}{2} g^{00}(g_{01,1} + g_{01,1} - g_{11,0})$$

$$= \frac{1}{2}.1\left[-\frac{\partial}{\partial t}\{-\frac{R^2(t)}{1-kr^2}\}\right]$$

$$= \frac{R(t)R'(t)}{1-kr^2}$$

$$\Gamma^0_{22} = \frac{1}{2} g^{00}(g_{02,2} + g_{02,2} - g_{22,0})$$

$$= \frac{1}{2}.1[-\frac{\partial}{\partial t}\{-R^2(t)r^2\}]$$

$$= R(t).R'(t)r^2$$

$$\Gamma^0_{33} = \frac{1}{2} g^{00}(g_{03,3} + g_{03,3} - g_{33,0})$$

$$= \frac{1}{2}.1\left[-\frac{\partial}{\partial t}\left\{-R^2(t)r^2\sin^2\theta\right\}\right]$$

$$= R(t)R'(t)r^2\sin^2\theta$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11}(g_{10,1} + g_{11,0} - g_{01,1})$$

$$= \frac{1}{2}\{\frac{kr^2-1}{R^2(t)}\}\frac{\partial}{\partial t}\left\{-\frac{R^2(t)}{1-kr^2}\right\}$$

$$= \frac{1}{2}\{\frac{kr^2-1}{R^2(t)}\}\frac{2R(t)R'(t)}{(kr^2-1)}$$

$$= \frac{R'(t)}{R(t)}$$

$$\therefore\ \Gamma^1_{10} = \frac{R'(t)}{R(t)}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1})$$

$$= \frac{1}{2}\left\{\frac{kr^2 - 1}{R^2(t)}\right\}\frac{\partial}{\partial r}\left\{-\frac{R^2(t)}{1-kr^2}\right\}$$

$$= \frac{1}{2}(kr^2 - 1)\{-\frac{\partial}{\partial r}(1-kr^2)^{-1}\}$$

$$= \frac{1}{2}(kr - 1)\{-(-1)(1-kr^2)^{-1-1}(-2kr)\}$$

$$= \frac{1}{2}(kr^2 - 1)\{-\frac{2kr}{(1-kr^2)^2}\}$$

$$= \frac{kr}{1-kr^2}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}(g_{12,2} + g_{12,2} - g_{22,1})$$

$$= \frac{1}{2}\{\frac{kr^2 - 1}{R^2(t)}\}\left[-\frac{\partial}{\partial r}\{-R^2(t)r^2\}\right]$$

$$= \frac{1}{2}\{\frac{kr^2 - 1}{R^2(t)}\}R^2(t).2r$$

$$= -(1-kr^2)r$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}(g_{13,3} + g_{13,3} - g_{33,1})$$

$$= \frac{1}{2}\{\frac{kr^2 - 1}{R^2(t)}\}\ [-\frac{\partial}{\partial r}\{-R^2(t)r^2\sin^2\theta\}]$$

$$= \frac{1}{2}\{\frac{kr^2 - 1}{R^2(t)}\}.R^2(t).2r.\sin^2\theta$$

$$= -(1-kr^2)r\sin^2\theta$$

$$\Gamma^2_{02} = \frac{1}{2} g^{22}(g_{20,2} + g_{22,0} - g_{02,2})$$

$$= \frac{1}{2}\{-\frac{1}{R^2(t)r^2}\}\frac{\partial}{\partial t}\{-R^2(t)r^2\}$$

$$= \frac{2R(r)R'(t)r^2}{2R^2(t)r^2}$$

$$=\frac{R'(t)}{R(t)}$$

$$\therefore\ \Gamma^2_{20}=\frac{R'(t)}{R(t)}$$

$$\Gamma^2_{12}=\frac{1}{2}g^{22}(g_{21,2}+g_{22,1}-g_{12,2})$$

$$=\frac{1}{2}\{-\frac{1}{R^2(t)r^2}\}\frac{\partial}{\partial r}\{-R^2(t)r^2\}$$

$$=\frac{1}{2}\frac{2rR^2(t)}{R^2(t)r^2}$$

$$=\frac{1}{r}$$

$$\therefore\Gamma^2_{21}=\frac{1}{r}$$

$$\Gamma^2_{33}=\frac{1}{2}g^{22}(g_{23,3}+g_{23,3}-g_{33,2})$$

$$=\frac{1}{2}\{-\frac{1}{R^2(t)r^2}\}\left[-\frac{\partial}{\partial\theta}\{-R^2(t)r^2\sin^2\theta\}\right]$$

$$=-\frac{R^2(t)r^2.2\sin\theta\cos\theta}{2R^2(t)r^2}$$

$$=-\sin\theta\cos\theta$$

$$\Gamma^3_{03}=\frac{1}{2}g^{33}(g_{30,3}+g_{33,0}-g_{03,3})$$

$$=\frac{1}{2}\{(-\frac{1}{R^2(t)r^2\sin^2\theta})\}\frac{\partial}{\partial t}\{-R^2(t)r^2\sin^2\theta\}$$

$$=\frac{1}{2}\frac{2R(t)R'(t)r^2\sin^2\theta}{R^2(t)r^2\sin^2\theta}$$

$$=\frac{R'(t)}{R(t)}$$

$$\therefore\Gamma^3_{30}=\frac{R'(t)}{R(t)}$$

$$\Gamma^3_{13}=\frac{1}{2}g^{33}(g_{31,3}+g_{33,1}-g_{13,3})$$

$$= -\frac{1}{2}\frac{1}{R^2(t)r^2\sin^2\theta}\frac{\partial}{\partial r}\{-R^2(t)r^2\sin^2\theta\}$$

$$= \frac{1}{2}\frac{1}{R^2(t)r^2\sin^2\theta}\{R^2(t)2r\sin^2\theta\}$$

$$=\frac{1}{r}$$

$$\therefore \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{23} = \frac{1}{2}g^{33}(g_{32,3} + g_{33,2} - g_{23,3})$$

$$= -\frac{1}{2}\frac{1}{R^2(t)r^2\sin^2\theta}\frac{\partial}{\partial\theta}\{-R^2(t)r^2\sin^2\theta\}$$

$$= \frac{R^2(t)r^2.2\sin\theta\cos\theta}{2R^2(t)r^2\sin^2\theta}$$

$$=\cot\theta$$

$$\therefore \Gamma^3_{32} = \cot\theta$$

Now, $R_{\mu\upsilon} = \Gamma^{\sigma}_{\mu\sigma,\upsilon} - \Gamma^{\sigma}_{\mu\upsilon,\sigma} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\upsilon} - \Gamma^{\rho}_{\mu\upsilon}\Gamma^{\sigma}_{\rho\sigma}$

$$R_{00} = \Gamma^{\sigma}_{0\sigma,0} - \Gamma^{\sigma}_{00,\sigma} + \Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0} - \Gamma^{\rho}_{00}\Gamma^{\sigma}_{\rho\sigma}$$

$$\Gamma^{\sigma}_{0\sigma,0} = \Gamma^0_{00,0} + \Gamma^1_{01,0} + \Gamma^2_{02,0} + \Gamma^3_{03,0}$$

$$= 3\frac{\partial}{\partial t}\left\{\frac{R'(t)}{R(t)}\right\}$$

$$= 3\left\{\frac{R(t)R''(t) - R'(t)R'(t)}{R^2(t)}\right\}$$

$$= 3\left\{\frac{R(t)R''(t) - R'^2(t)}{R^2(t)}\right\}$$

and $\Gamma^{\sigma}_{00,\sigma} = \Gamma^0_{00,0} + \Gamma^1_{00,1} + \Gamma^2_{00,2} + \Gamma^3_{00,3}$

$$= 0+0+0+0$$

$$= 0$$

and $\Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0} = \Gamma^0_{0\sigma}\Gamma^{\sigma}_{00} + \Gamma^1_{0\sigma}\Gamma^{\sigma}_{10} + \Gamma^2_{0\sigma}\Gamma^{\sigma}_{20} + \Gamma^3_{0\sigma}\Gamma^{\sigma}_{30}$

$$= \Gamma^0_{00}\Gamma^0_{00} + \Gamma^0_{01}\Gamma^1_{00} + \Gamma^0_{02}\Gamma^2_{00} + \Gamma^0_{03}\Gamma^3_{00}$$

$$+ \Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{01}\Gamma^1_{10} + \Gamma^1_{02}\Gamma^2_{10} + \Gamma^1_{03}\Gamma^3_{10}$$

$$+\Gamma^2_{00}\Gamma^0_{20}+\Gamma^2_{01}\Gamma^1_{20}+\Gamma^2_{02}\Gamma^2_{20}+\Gamma^2_{03}\Gamma^3_{20}$$

$$+\Gamma^3_{00}\Gamma^0_{30}+\Gamma^3_{01}\Gamma^1_{30}+\Gamma^3_{02}\Gamma^2_{30}+\Gamma^3_{03}\Gamma^3_{30}$$

$$=0+0+0+0$$

$$+0+\frac{R'(t)}{R(t)}\frac{R'(t)}{R(t)}+0+0$$

$$+0\;+0+\frac{R'(t)}{R(t)}\frac{R'(t)}{R(t)}+0$$

$$+0+0+0+\frac{R'(t)}{R(t)}\frac{R'(t)}{R(t)}$$

$$=\frac{R'^2(t)}{R^2(t)}+\frac{R'^2(t)}{R^2(t)}+\frac{R'^2(t)}{R^2(t)}$$

$$=\frac{3R'^2(t)}{R^2(t)}$$

and $\Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma}=\Gamma^0_{00}\Gamma^\sigma_{0\sigma}+\Gamma^1_{00}\Gamma^\sigma_{1\sigma}+\Gamma^2_{00}\Gamma^\sigma_{2\sigma}+\Gamma^3_{00}\Gamma^\sigma_{3\sigma}$

$$=\Gamma^0_{00}\left(\Gamma^0_{00}+\Gamma^1_{01}+\Gamma^2_{02}+\Gamma^3_{03}\right)+\Gamma^1_{00}\left(\Gamma^0_{10}+\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13}\right)$$

$$+\Gamma^2_{00}\left(\Gamma^0_{20}+\Gamma^1_{21}+\Gamma^2_{22}+\Gamma^3_{23}\right)+\Gamma^3_{00}\left(\Gamma^0_{30}+\Gamma^1_{31}+\Gamma^2_{32}+\Gamma^3_{33}\right)$$

$$=0+0+0+0$$

$$=0$$

$$\therefore R_{00}=\frac{3R(t)R''(t)-3R'^2(t)}{R^2(t)}-0+\frac{3R'^2(t)}{R^2(t)}-0$$

$$=\frac{3R(t)R''(t)-3R'^2(t)}{R^2(t)}+\frac{3R'^2(t)}{R^2(t)}$$

$$=\frac{3R(t)R''(t)-3R'^2(t)+3R'^2(t)}{R^2(t)}$$

$$=\frac{3R(t)R''(t)}{R^2(t)}$$

$$=\frac{3R''(t)}{R(t)}$$

$$=\frac{3\ddot{R}}{R}$$

Now, $R_{11}=\Gamma^\sigma_{1\sigma,1}-\Gamma^\sigma_{11,\sigma}+\Gamma^\rho_{1\sigma}\Gamma^\sigma_{\rho 1}-\Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma}$

$$\Gamma^{\sigma}_{1\sigma,1} = \Gamma^{0}_{10,1} + \Gamma^{1}_{11,1} + \Gamma^{2}_{12,1} + \Gamma^{3}_{13,1}$$

$$= 0 + \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\right)$$

$$= \frac{(1-kr^2)k - kr(0-2kr)}{(1-kr^2)^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$= \frac{k - k^2r^2 + 2k^2r^2}{(1-kr^2)^2} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$= \frac{k + k^2r^2}{(1-kr^2)^2} - \frac{2}{r^2}$$

and $\Gamma^{\sigma}_{11,\sigma} = \Gamma^{0}_{11,0} + \Gamma^{1}_{11,1} + \Gamma^{2}_{11,2} + \Gamma^{3}_{11,3}$

$$= \frac{\partial}{\partial t}\left(\frac{RR'}{1-kr^2}\right) + \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right) + 0 + 0$$

$$= \frac{RR'' + R'R'}{1-kr^2} + \frac{(1-kr^2)k - kr(0-2kr)}{(1-kr^2)^2}$$

$$= \frac{RR''}{1-kr^2} + \frac{R'^2}{1-kr^2} + \frac{k - k^2r^2 + 2k^2r^2}{(1-kr^2)^2}$$

$$= \frac{RR''}{1-kr^2} + \frac{R'^2}{1-kr^2} + \frac{k + k^2r^2}{(1-kr^2)^2}$$

and $\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1} = \Gamma^{0}_{1\sigma}\Gamma^{\sigma}_{01} + \Gamma^{1}_{1\sigma}\Gamma^{\sigma}_{11} + \Gamma^{2}_{1\sigma}\Gamma^{\sigma}_{21} + \Gamma^{3}_{1\sigma}\Gamma^{\sigma}_{31}$

$$= \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{0}_{11}\Gamma^{1}_{01} + \Gamma^{0}_{12}\Gamma^{2}_{01} + \Gamma^{0}_{13}\Gamma^{3}_{01}$$

$$+ \Gamma^{1}_{10}\Gamma^{0}_{11} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{1}_{12}\Gamma^{2}_{11} + \Gamma^{1}_{13}\Gamma^{3}_{11}$$

$$+ \Gamma^{2}_{10}\Gamma^{0}_{21} + \Gamma^{2}_{11}\Gamma^{1}_{21} + \Gamma^{2}_{12}\Gamma^{2}_{21} + \Gamma^{2}_{13}\Gamma^{3}_{21}$$

$$+ \Gamma^{3}_{10}\Gamma^{0}_{31} + \Gamma^{3}_{11}\Gamma^{1}_{31} + \Gamma^{3}_{12}\Gamma^{2}_{31} + \Gamma^{3}_{13}\Gamma^{3}_{31}$$

$$= 0 + \frac{R(t)R'(t)}{1-kr^2}.\frac{R'(t)}{R(t)} + 0 + 0$$

$$+ \frac{R'(t)}{R(t)}.\frac{R(t)R'(t)}{1-kr^2} + \left(\frac{kr}{1-kr^2}\right)\left(\frac{kr}{1-kr^2}\right) + 0 + 0$$

$$+ 0 + 0 + \frac{1}{r}.\frac{1}{r} + 0$$

$$+ 0 + 0 + 0 + \frac{1}{r}.\frac{1}{r}$$

$$= \frac{R'^2}{1-kr^2} + \frac{R'^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2}$$

$$= \frac{2R'^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2}$$

and $\Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{0}_{11}\Gamma^{\sigma}_{0\sigma} + \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{11}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{11}\Gamma^{\sigma}_{3\sigma}$

$$= \Gamma^0_{11}\left(\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}\right) + \Gamma^1_{11}\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)$$

$$+ \Gamma^2_{11}\left(\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) + \Gamma^3_{11}\left(\Gamma^0_{30} + \Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right)$$

$$= \frac{R(t)R'(t)}{1-kr^2}\left\{0 + \frac{R'(t)}{R(t)} + \frac{R'(t)}{R(t)} + \frac{R'(t)}{R(t)}\right\} + \frac{kr}{1-kr^2}\left(0 + \frac{kr}{1-kr^2} + \frac{1}{r} + \frac{1}{r}\right) + 0 + 0$$

$$= \frac{R(t)R'(t)}{1-kr^2} . \frac{3R'(t)}{R(t)} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2kr}{r(1-kr^2)}$$

$$= \frac{3R'^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2k}{1-kr^2}$$

$$\therefore R_{11} = \frac{k+k^2r^2}{(1-kr^2)^2} - \frac{2}{r^2} - \frac{RR''}{1-kr^2} - \frac{R'^2}{1-kr^2} - \frac{k+k^2r^2}{(1-kr^2)^2}$$

$$+ \frac{2R'^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2} - \frac{3R'^2}{1-kr^2} - \frac{k^2r^2}{(1-kr^2)^2} - \frac{2k}{1-kr^2}$$

$$= -\frac{RR''}{1-kr^2} - \frac{2R'^2}{1-kr^2} - \frac{2k}{1-kr^2}$$

$$R_{11} = -\left(\frac{RR'' + 2R'^2 + 2k}{1-kr^2}\right)$$

Now, $R_{22} = \Gamma^{\sigma}_{2\sigma,2} - \Gamma^{\sigma}_{22,\sigma} + \Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2} - \Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma}$

$$\Gamma^{\sigma}_{2\sigma,2} = \Gamma^0_{20,2} + \Gamma^1_{21,2} + \Gamma^2_{22,2} + \Gamma^3_{23,2}$$

$$= 0 + 0 + 0 + \frac{\partial}{\partial\theta}(\cot\theta)$$

$$= -\mathrm{cosec}^2\theta$$

and $\Gamma^{\sigma}_{22,\sigma} = \Gamma^0_{22,0} + \Gamma^1_{22,1} + \Gamma^2_{22,2} + \Gamma^3_{22,3}$

$$= \frac{\partial}{\partial t}\{R(t)R'(t)r^2\} + \frac{\partial}{\partial r}\{-(1-kr^2)r\} + 0 + 0$$

$$= \{R'(t)R'(t) + R(t)R''(t)\}r^2 + \frac{\partial}{\partial r}(-r + kr^3)$$

$$= \{R'^2(t) + R(t)R''(t)\}r^2 - 1 + 3kr^2$$

$$= (R'^2 + RR'')r^2 + 3kr^2 - 1$$

and $\Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2} = \Gamma^{0}_{2\sigma}\Gamma^{\sigma}_{02} + \Gamma^{1}_{2\sigma}\Gamma^{\sigma}_{12} + \Gamma^{2}_{2\sigma}\Gamma^{\sigma}_{22} + \Gamma^{3}_{2\sigma}\Gamma^{\sigma}_{32}$

$$= \Gamma^0_{20}\Gamma^0_{02} + \Gamma^0_{21}\Gamma^1_{02} + \Gamma^0_{22}\Gamma^2_{02} + \Gamma^0_{23}\Gamma^3_{02}$$

$$+ \Gamma^1_{20}\Gamma^0_{12} + \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^1_{23}\Gamma^3_{12}$$

$$+ \Gamma^2_{20}\Gamma^0_{22} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} + \Gamma^2_{23}\Gamma^3_{22}$$

$$+ \Gamma^3_{20}\Gamma^0_{32} + \Gamma^3_{21}\Gamma^1_{32} + \Gamma^3_{22}\Gamma^2_{32} + \Gamma^3_{23}\Gamma^3_{32}$$

$$= 0 + 0 + RR'r^2.\frac{R'}{R} + 0 + 0 + 0 + \{-(1-kr^2)r\}.\frac{1}{r} + 0$$

$$+ \frac{R'}{R}.RR'r^2 + \frac{1}{r}\{-(1-kr^2)r\} + 0 + 0 + 0 + 0 + 0 + \cot\theta.\cot\theta$$

$$= R'^2r^2 - 1 + kr^2 + R'^2R^2 - 1 + kr^2 + \cot^2\theta$$

$$= 2R'^2r^2 + 2kr^2 + \cot^2\theta - 2$$

$$\Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^0_{22}\Gamma^{\sigma}_{0\sigma} + \Gamma^1_{22}\Gamma^{\sigma}_{1\sigma} + \Gamma^2_{22}\Gamma^{\sigma}_{2\sigma} + \Gamma^3_{22}\Gamma^{\sigma}_{3\sigma}$$

$$= \Gamma^0_{22}(\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}) + \Gamma^1_{22}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13})$$

$$+ \Gamma^2_{22}(\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}) + \Gamma^3_{22}(\Gamma^0_{30} + \Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33})$$

$$= 0 + RR'r^2.\frac{R'}{R} + RR'r^2.\frac{R'}{R} + RR'r^2.\frac{R'}{R} + \{-(1-kr^2)r\}.(0 + \frac{kr}{1-kr^2} + \frac{1}{r} + \frac{1}{r}) + 0 + 0$$

$$= 3R'^2r^2 - kr^2 - 2(1-kr^2)$$

$$= 3R'^2r^2 - kr^2 - 2 + 2kr^2$$

$$= 3R'^2r^2 + kr^2 - 2$$

$$\therefore R_{22} = -cosec^2\theta - R'^2r^2 - RR''r^2 - 3kr^2 + 1 + 2R'^2r^2 + 2kr^2 + cot^2\theta - 2$$

$$-3R'^2r^2 - kr^2 + 2$$

$$= -(\operatorname{cosec}^2\theta - \cot^2\theta) - RR''r^2 - 2R'^2r^2 - 2kr^2 + 1$$

$$= -1 - RR''r^2 - 2R'^2r^2 - 2kr^2 + 1$$

$$= -(RR'' + 2R'^2 + 2k)r^2$$

Now, $R_{33} = \Gamma^{\sigma}_{3\sigma,3} - \Gamma^{\sigma}_{33,\sigma} + \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3} - \Gamma^{\sigma}_{33}\Gamma^{\sigma}_{\rho\sigma}$

$$\Gamma^{\sigma}_{3\sigma,3} = \Gamma^0_{30,3} + \Gamma^1_{31,3} + \Gamma^2_{32,3} + \Gamma^3_{33,3}$$

$$= 0 \ + 0 \quad + 0 \quad + 0$$

$$= 0$$

and $\Gamma^{\sigma}_{33,\sigma} = \Gamma^{0}_{33,0} + \Gamma^{1}_{33,1} + \Gamma^{2}_{33,2} + \Gamma^{3}_{33,3}$

$$= \frac{\partial}{\partial t}(RR'r^2 \sin^2\theta) + \frac{\partial}{\partial r}\{-r(1-kr^2)\sin^2\theta\} + \frac{\partial}{\partial\theta}(-\sin\theta\cos\theta) + 0$$

$$= (RR'' + R'R')r^2 \sin^2\theta + (-1+3kr^2)\sin^2\theta + \sin^2\theta - \cos^2\theta$$

$$= RR''r^2 \sin^2\theta + R'^2 r^2 \sin^2\theta - \sin^2\theta + 3kr^2 \sin^2\theta + \sin^2\theta - \cos^2\theta$$

$$= RR''r^2 \sin^2\theta + R'^2 r^2 \sin^2\theta + 3kr^2 \sin^2\theta - \cos^2\theta$$

and $\Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3} = \Gamma^{0}_{3\sigma}\Gamma^{\sigma}_{03} + \Gamma^{1}_{3\sigma}\Gamma^{\sigma}_{13} + \Gamma^{2}_{3\sigma}\Gamma^{\sigma}_{23} + \Gamma^{3}_{3\sigma}\Gamma^{\sigma}_{33}$

$$= \Gamma^{0}_{30}\Gamma^{0}_{03} + \Gamma^{0}_{31}\Gamma^{1}_{03} + \Gamma^{0}_{32}\Gamma^{2}_{03} + \Gamma^{0}_{33}\Gamma^{3}_{03}$$

$$+ \Gamma^{1}_{30}\Gamma^{0}_{13} + \Gamma^{1}_{31}\Gamma^{1}_{13} + \Gamma^{1}_{32}\Gamma^{2}_{13} + \Gamma^{1}_{33}\Gamma^{3}_{13}$$

$$+ \Gamma^{2}_{30}\Gamma^{0}_{23} + \Gamma^{2}_{31}\Gamma^{1}_{23} + \Gamma^{2}_{32}\Gamma^{2}_{23} + \Gamma^{2}_{33}\Gamma^{3}_{23}$$

$$+ \Gamma^{3}_{30}\Gamma^{0}_{33} + \Gamma^{3}_{31}\Gamma^{1}_{33} + \Gamma^{3}_{32}\Gamma^{2}_{33} + \Gamma^{3}_{33}\Gamma^{3}_{33}$$

$$= \ 0 + 0 + 0 + \ R(t)R'(t)r^2 \sin^2\theta . \frac{R'(t)}{R(t)}$$

$$+ 0 + 0 + 0 \ - r(1-kr^2)\sin^2\theta . \frac{1}{r}$$

$$+ 0 + 0 + 0 + \ (-\sin\theta\cos\theta).\cot\theta$$

$$+ \ \frac{R'(t)}{R(t)}.R(t)R'(t)r^2 \sin^2\theta + \frac{1}{r}\{-(1-kr^2)r\sin^2\theta\}$$

$$+ \ \cot\theta(-\sin\theta\cos\theta) + 0$$

$$= R'^2(t)r^2 \sin^2\theta + kr^2 \sin^2\theta - \sin^2\theta - \sin\theta\cos\theta.\frac{\cos\theta}{\sin\theta} + R'^2(t)r^2 \sin^2\theta$$

$$+ kr^2 \sin^2\theta - \sin^2\theta + \frac{\cos\theta}{\sin\theta}(-\sin\theta\cos\theta)$$

$$= \ 2R'^2(t)r^2 \sin^2\theta + 2kr^2 \sin^2\theta - 2\sin^2\theta - 2\cos^2\theta$$

$$= \ 2R'^2(t)r^2 \sin^2\theta + 2kr^2 \sin^2\theta - 2(\sin^2\theta + \cos^2\theta)$$

$$= \ 2R'^2(t)r^2 \sin^2\theta + 2kr^2 \sin^2\theta - 2$$

and $\Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{0}_{33}\Gamma^{\sigma}_{0\sigma} + \Gamma^{1}_{33}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{33}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{33}\Gamma^{\sigma}_{3\sigma}$

$$= \Gamma^{0}_{33}(\Gamma^{0}_{00} + \Gamma^{1}_{01} + \Gamma^{2}_{02} + \Gamma^{3}_{03}) + \Gamma^{1}_{33}(\Gamma^{0}_{10} + \Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13})$$

$$+\Gamma^2_{33}(\Gamma^0_{20}+\Gamma^1_{21}+\Gamma^2_{22}+\Gamma^3_{23})+\Gamma^3_{33}(\Gamma^0_{30}+\Gamma^1_{31}+\Gamma^2_{32}+\Gamma^3_{33})$$

$$= R(t)R'(t)r^2\sin^2\theta\{\frac{R'(t)}{R(t)}+\frac{R'(t)}{R(t)}+\frac{R'(t)}{R(t)}\}-(1-kr^2)r\sin^2\theta(\frac{kr}{1-kr^2})$$

$$-(1-kr^2)r\sin^2\theta.\frac{2}{r}+(-\sin\theta\cos\theta)(0+0+0+\cot\theta)+0$$

$$=R(t)\,R'(t)r^2\sin^2\theta.3\frac{R'(t)}{R(t)}-kr^2\sin^2\theta-2\sin^2\theta+2kr^2\sin^2\theta-\sin\theta\cos\theta.(\frac{\cos\theta}{\sin\theta})$$

$$= 3R'^2(t)r^2\sin^2\theta+kr^2\sin^2\theta-2\sin^2\theta-\cos^2\theta$$

$$\therefore R_{33}= 0-RR''r^2\sin^2\theta-R'^2r^2\sin^2\theta-3kr^2\sin^2\theta+\cos^2\theta+2R'^2(t)r^2\sin^2\theta$$

$$+2kr^2\sin^2\theta-2-3R'^2(t)r^2\sin^2\theta-kr^2\sin^2\theta+2\sin^2\theta+\cos^2\theta$$

$$= -RR''r^2\sin^2\theta-2R'^2r^2\sin^2\theta-2kr^2\sin^2\theta+2(\sin^2\theta+\cos^2\theta)-2$$

$$= -(RR''+2R'^2+2k)r^2\sin^2\theta+2-2$$

$$= -(RR''+2R'^2+2k)r^2\sin^2\theta$$

7.4 Friedmann Model from Robertson-Walker Line Element:

We know the Robertson-Walker line element is

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\varphi^2)\right\} \quad \text{.................................. (1)}$$

In matrix form

$$g_{\mu\upsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -R^2(t)r^2 & 0 \\ 0 & 0 & 0 & -R^2(t)r^2\sin^2\theta \end{bmatrix}$$

$$= \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

where $g_{00}=1, g_{11}=-\frac{R^2(t)}{1-kr^2}, g_{22}=-R^2(t)r^2, g_{33}=-R^2(t)r^2\sin^2\theta$

and the other terms are zero.

Now, the inverse of $g_{\mu\upsilon}$ is $g^{\mu\upsilon} = \frac{1}{|g_{\mu\upsilon}|}$ (Adjoint matrix of $g_{\mu\upsilon}$)

$$|g_{\mu\upsilon}| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -R^2(t)r^2 & 0 \\ 0 & 0 & 0 & -R^2(t)r^2\sin^2\theta \end{vmatrix}$$

$$= \frac{R^6(t)r^4\sin^2\theta}{kr^2-1}$$

$$\text{Adjoint matrix of } g_{\mu\upsilon} = \begin{bmatrix} \frac{R^6(t)r^4\sin^2\theta}{kr^2-1} & 0 & 0 & 0 \\ 0 & R^4(t)r^4\sin^2\theta & 0 & 0 \\ 0 & 0 & -\frac{R^4(t)r^2\sin^2\theta}{kr^2-1} & 0 \\ 0 & 0 & 0 & -\frac{R^4(t)r^2}{kr^2-1} \end{bmatrix}$$

$$g^{\mu\upsilon} = \frac{kr^2-1}{R^6(t)r^4\sin^2\theta} \begin{bmatrix} \frac{R^6(t)r^4\sin^2\theta}{kr^2-1} & 0 & 0 & 0 \\ 0 & R^4(t)r^4\sin^2\theta & 0 & 0 \\ 0 & 0 & -\frac{R^4(t)r^2\sin^2\theta}{kr^2-1} & 0 \\ 0 & 0 & 0 & -\frac{R^4(t)r^2}{kr^2-1} \end{bmatrix}$$

$$\text{Or, } g^{\mu\upsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{kr^2-1}{R^2(t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{R^2(t)r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{R^2(t)r^2\sin^2\theta} \end{bmatrix}$$

$$= \begin{pmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{pmatrix}$$

where $g^{00} = 1, g^{11} = \frac{kr^2 - 1}{R^2(t)}, g^{22} = -\frac{1}{R^2(t)r^2,} g^{33} = -\frac{1}{R^2(t)r^2 \sin^2\theta}$

and the other terms are zero.

The non-zero Christoffel symbols are

$$\Gamma^0_{11} = \frac{RR'}{1-kr^2}, \quad \Gamma^0_{22} = RR'r^2, \quad \Gamma^0_{33} = RR'r^2 \sin^2\theta$$

$$\Gamma^1_{01} = \frac{R'}{R} = \Gamma^1_{10}, \; \Gamma^1_{11} = \frac{kr}{1-kr^2}, \; \Gamma^1_{22} = -(1-kr^2)r, \; \Gamma^1_{33} = -(1-kr^2)r\sin^2\theta$$

$$\Gamma^2_{02} = \frac{R'}{R} = \Gamma^2_{02}, \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21}, \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{R'}{R}, \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31}, \Gamma^3_{23} = \cot\theta = \Gamma^3_{32}$$

We know that

$$R_{\mu\upsilon} = \Gamma^\sigma_{\mu\sigma,\upsilon} - \Gamma^\sigma_{\mu\upsilon,\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\upsilon} - \Gamma^\rho_{\mu\upsilon}\Gamma^\sigma_{\rho\sigma}$$

which gives

$$R_{00} = \frac{3R''}{R}$$

$$R_{11} = -\left(\frac{RR'' + 2R'^2 + 2k}{1-kr^2}\right)$$

$$R_{22} = -(RR'' + 2R'^2 + 2k)r^2$$

$$R_{33} = -(RR'' + 2R'^2 + 2k)r^2 \sin^2\theta$$

And $R_{\mu\upsilon} = 0 \; for \, \mu \neq \upsilon$

With c=1, we know $u^\mu u_\mu = 1$

For stress tensor $T_{\mu\upsilon}$, we know

$T_{\mu\upsilon} = (\rho+p)\, u_\mu\, u_\upsilon - p g_{\mu\upsilon}$

$T_{\mu\upsilon}\, g^{\mu\upsilon} = (\rho+p)\, u_\mu\, u_\upsilon\, g^{\mu\upsilon} - p g_{\mu\upsilon} g^{\mu\upsilon}$

$\Rightarrow T = (\rho+p) - 4\,p$ [since $u^\mu = u_\upsilon g^{\mu\upsilon}$ and $u_\mu = u^\upsilon g_{\mu\upsilon}$ and $g_{\mu\upsilon}\, g^{\mu\upsilon} = 4$]

$= \rho - 3p$ .... (2)

In our moving co-ordinate system

$u^{\mu} = \delta_0^{\mu}$

So that $u^{\mu} = g_{\mu\upsilon}\delta_0^{\upsilon} = g_{\mu 0} = \delta_{\mu}^{0}$

Hence, $T_{\mu\upsilon} = (\rho + p)\delta_{\mu}^{0}\delta_{\upsilon}^{0} - pg_{\mu\upsilon}$

From the above equation, we can write

$$T_{\mu\upsilon} - \frac{1}{2}Tg_{\mu\upsilon} = (\rho + p)\delta_{\mu}^{0}\delta_{\upsilon}^{0} - pg_{\mu\upsilon} - \frac{1}{2}(\rho - 3p)g_{\mu\upsilon}$$ [Using equation (2)]

$$= (\rho + p)\delta_{\mu}^{0}\delta_{\upsilon}^{0} - pg_{\mu\upsilon} - \frac{1}{2}\rho g_{\mu\upsilon} + \frac{3}{2}pg_{\mu\upsilon}$$

$$= (\rho + p)\delta_{\mu}^{0}\delta_{\upsilon}^{0} - \frac{1}{2}\rho g_{\mu\upsilon} + \frac{1}{2}pg_{\mu\upsilon}$$

$$= (\rho + p)\delta_{\mu}^{0}\delta_{\upsilon}^{0} - \frac{1}{2}(\rho - p)g_{\mu\upsilon}$$

From the above equation, we have

$$T_{00} - \frac{1}{2}Tg_{00} = (\rho + p)\delta_{0}^{0}\delta_{0}^{0} - \frac{1}{2}(\rho - p)g_{00} \quad [setting \mu = \upsilon = 0 and g_{00} = 1]$$

$$= (\rho + p) - \frac{1}{2}(\rho - p)$$

$$= \rho + p - \frac{1}{2}\rho + \frac{p}{2}$$

$$= \frac{1}{2}\rho + p + \frac{p}{2}$$

$$= \frac{\rho + 2p + p}{2}$$

$$= \frac{1}{2}(\rho + 3p)$$

$$T_{11} - \frac{1}{2}Tg_{11} = (\rho + p)\delta_{1}^{0}\delta_{1}^{0} - \frac{1}{2}(\rho - p)g_{11}$$

$$= 0 - \frac{1}{2}(\rho - p).\frac{-R^2}{1 - kr^2} \qquad [\because g_{11} = \frac{-R^2}{1 - kr^2}]$$

$$= \frac{1}{2}(\rho - p).\frac{R^2}{1 - kr^2}$$

$$T_{22}-\frac{1}{2}Tg_{22}=(\rho+p)\delta_2^0\delta_2^0-\frac{1}{2}(\rho-p)g_{22}$$

$$=0-\frac{1}{2}(\rho-p)\{-R^2(t)r^2\} \qquad [\because g_{22}=-R^2(t)r^2]$$

$$=\frac{1}{2}(\rho-p)R^2(t)r^2$$

$$T_{33}-\frac{1}{2}Tg_{33}=(\rho+p)\delta_3^0\delta_3^0-\frac{1}{2}(\rho-p)g_{33}$$

$$=0-\frac{1}{2}(\rho-p)(-r^2R^2\sin^2\theta) \; [\because g_{33}=-r^2R^2\sin^2\theta]$$

$$=\frac{1}{2}(\rho-p)r^2R^2\sin^2\theta$$

Now, from Einstein field equation, we have

$$R_{\mu\upsilon}=K(T_{\mu\upsilon}-\frac{1}{2}Tg_{\mu\upsilon})$$

$$R_{00}=K(T_{00}-\frac{1}{2}Tg_{00})$$

$$=\frac{1}{2}K(\rho+3p)$$

But from Robertson- Walker line element, we obtain

$$R_{00}=\frac{3R''}{R}$$

Therefore, $\frac{3R''}{R}=\frac{1}{2}K(\rho+3p)$

$$\Rightarrow R''=\frac{1}{6}KR(\rho+3p)$$

$$\Rightarrow R''R=\frac{1}{6}KR^2(\rho+3p) \quad \text{...} \quad (3)$$

Again $R_{22}=K(T_{22}-\frac{1}{2}Tg_{22})$

$$\Rightarrow -(RR''+2R'^2+2k)r^2=\frac{1}{2}K(\rho-p)r^2R^2$$

$$\Rightarrow RR''+2R'^2+2k=-\frac{1}{2}K(\rho-p)R^2 \quad \text{.............................} \quad (4)$$

$$\Rightarrow \frac{1}{6}KR^2(\rho+3p)+2R'^2+2k=-\frac{1}{2}K(\rho-p)R^2 \qquad \text{[Using equation (3)]}$$

$$\Rightarrow 2R'^2+2k=-\frac{1}{2}K(\rho-p)R^2-\frac{1}{6}KR^2(\rho+3p)$$

$$\Rightarrow R'^2+k=\frac{1}{2}\{-\frac{1}{2}K(\rho-p)R^2-\frac{1}{6}KR^2(\rho+3p)\}$$

$$\Rightarrow R'^2+k=\frac{1}{2}(-\frac{1}{2}K\rho R^2+\frac{1}{2}KpR^2-\frac{1}{6}K\rho R^2-\frac{1}{2}KpR^2)$$

$$\Rightarrow R'^2+k=\frac{1}{2}(-\frac{1}{2}K\rho R^2-\frac{1}{6}K\rho R^2)$$

$$\Rightarrow R'^2+k=\frac{1}{2}(-\frac{4}{6}K\rho R^2)$$

$$\Rightarrow R'^2+k=-\frac{1}{3}KR^2\rho \text{ ....(5)}$$

Putting K = $-8\pi g$ (a constant) in R.H.S of equation (5), we get

$$R'^2+k=\frac{8\pi g}{3}R^2\rho \text{ .... (6)}$$

This is general form of Robertson-Walker model. Here K is scalar and $8\pi g$ is gravitational constant.

We shall refer to this equation as Friedmann equation. It is noted that the pressure has completely cancelled out of this equation. We know that $T^{\mu\upsilon}, \mu=0$ yield the continuity equation and the equation of motion of the fluid particles with C = 1 those becomes

$(\rho u^{\mu}), \mu + pu^{\mu}, \mu = 0$ .... (7)

$(\rho+p)\, u^{\upsilon},\ \mu^{u^{\mu}} = (g^{\mu\upsilon}-u^{\mu}u^{\upsilon})\, p, \mu$ ....(8)

From equation (7), we have

$$\rho, \mu^{u^{\mu}} + (\rho+p)(u^{\mu}, \upsilon + \Gamma^{\mu}_{\upsilon\mu}u^{\upsilon}) = 0$$

And with $u^{\mu}=\delta_0^{\mu}$ this reduces to

$$\rho' + (\rho+p)\frac{3R'}{R} = 0 \text{ .... (9)}$$

It contains the pressure. As for the standard and Fridmann model, we put p= 0 in the above equation and it becomes

$$\rho' + \frac{3R'}{R}\rho = 0$$

$$\Rightarrow \rho' = -\frac{3R'}{R}\rho$$

$$\Rightarrow \frac{\rho'}{\rho} = \frac{-3R'}{R}$$

Integrating the above equation, we have

$$\log\rho = -3\log R + \log C$$

$$\Rightarrow \log\rho + 3\log R = \log C$$

$$\Rightarrow \log\rho + \log R^3 = \log C$$

$$\Rightarrow \log\rho R^3 = \log C$$

$\therefore \rho R^3 = C$ (constant) .... (10)

This is known as Friedmann model of the universe where ρ = density, R = magnification. This leads to three possible models, each of which has *R(t)* =0 at some point in time and it is natural to take this point as the origin of t, so that *R (t)* = 0 and t is then taken the age of the universe.

Let t_0 be the present age of the universe and $R_0 = R(t_0)$ and $\rho_0 = \rho(t_0)$ are the present day values of R and ρ .

We may write equation (10) as

$$\rho R^3 = \rho_0 R_0^{\ 3}$$

Therefore equation (6) becomes

$$R'^2 + k = \frac{8\pi g}{3R} R_0^{\ 3}\rho_0 = \frac{A^2}{R}$$

$R'^2 + k = \dfrac{A^2}{R}$ .... (11)

where $A^2 = \dfrac{8\pi g}{3} R_0^{\ 3}\rho_0$

The three Friedmann models arise from integrating equation (11) for the three possibilities of k

k=0,1,-1

(i) k = 0 (Flat Model)

If k = 0, then equation (11) becomes

$$R'^2 = \frac{A^2}{R}$$

$$\therefore R' = \frac{A}{\sqrt{R}}$$

$$\Rightarrow \sqrt{R}R' = A$$

$$\Rightarrow \sqrt{R}\frac{dR}{dt} = A$$

$$\Rightarrow \int_0^R \sqrt{R}dR = \int_0^t Adt$$

$$\Rightarrow \frac{R^{3/2}}{3/2} = At$$

$$\Rightarrow 2R^{3/2} = 3At$$

$$\Rightarrow R^{3/2} = \frac{3}{2}At$$

$$R(t) = (\frac{3}{2}A)^{2/3} t^{2/3}$$

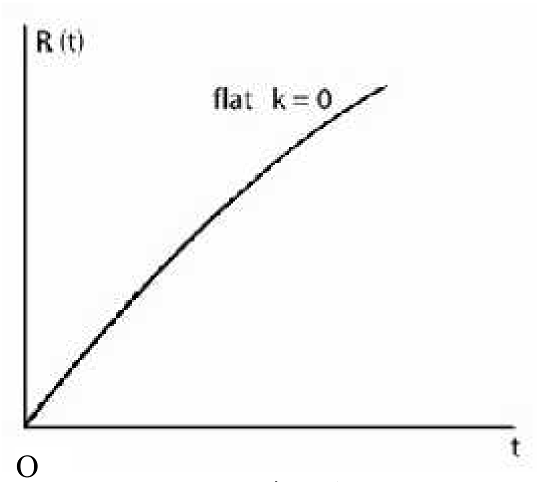


Fig.7.1

This model is known as Einstein de-Sitter model.

Here $R' \to 0 \;\; as \;\; t \to \infty$

where $a = (\frac{3}{2}A)^{2/3}$

$$\therefore R(t) = at^{2/3}$$

(ii) k = 1 (Closed Model)

If k = 1, then from equation (11), we get

$$R'^2 + 1 = \frac{A^2}{R}$$

$$\Rightarrow R'^2 = \frac{A^2}{R} - 1$$

$$\Rightarrow R'^2 = \frac{A^2 - R}{R}$$

$$\Rightarrow R' = \sqrt{\frac{A^2 - R}{R}}$$

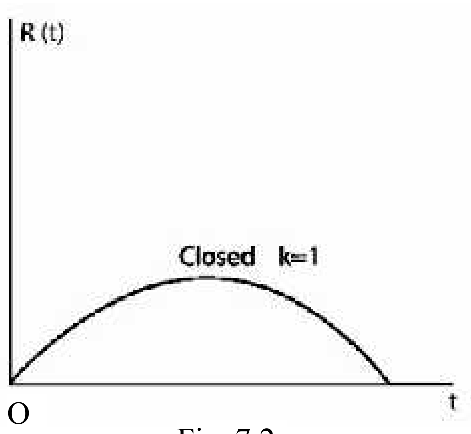


Fig. 7.2

$$\frac{dR}{dt}=\sqrt{\frac{A^2-R}{R}}$$

$$\Rightarrow\int_0^t dt=\int_0^R\sqrt{\frac{R}{A^2-R}}dR$$

Putting $R=A^2\sin^2\theta/2\ \therefore\frac{dR}{d\theta}=A^2.2(\sin\theta/2)(\cos\theta/2).\frac{1}{2}$

$$=A^2(\sin\theta/2)(\cos\theta/2)$$

$$\therefore\int_0^t dt=\int_0^\theta\left(\frac{A^2\sin^2\theta/2}{A^2-A^2\sin^2\theta/2}\right)^{\frac{1}{2}}.A^2\sin\theta/2\cos\theta/2d\theta$$

$$\therefore t=\int_0^\theta\left\{\frac{A^2\sin^2\theta/2}{A^2(1-\sin^2\theta/2)}\right\}^{\frac{1}{2}}.A^2\sin\theta/2\cos\theta/2d\theta$$

$$=\int_0^\theta\left(\frac{\sin^2\theta/2}{\cos^2\theta/2}\right)^{\frac{1}{2}}.A^2\sin\theta/2\cos\theta/2d\theta$$

$$=\int_0^\theta\frac{\sin\theta/2}{\cos\theta/2}.A^2\sin\theta/2\cos\theta/2d\theta$$

$$=\frac{A^2}{2}\int_0^\theta 2\sin^2\theta/2d\theta$$

$$[t]_0^t=\frac{A^2}{2}\int_0^\theta(1-\cos\theta)d\theta$$

$$\therefore\ t=\frac{A^2}{2}\int_0^\theta d\theta-\frac{A^2}{2}\int_0^\theta\cos\theta d\theta$$

$$=\frac{A^2}{2}[\theta]_0^\theta-\frac{A^2}{2}\int_0^\theta[\sin\theta]_0^\theta$$

$$=\frac{A^2}{2}[\theta-0]-\frac{A^2}{2}[\sin\theta-\sin 0]$$

$$=\frac{A^2\theta}{2}-\frac{A^2\sin\theta}{2}$$

$$=\frac{A^2}{2}(\theta-\sin\theta)$$

$$\therefore\frac{dt}{d\theta}=\frac{A^2}{2}(1-\cos\theta)$$

$$\Rightarrow R(t) = \frac{A^2}{2}(1-\cos\theta)$$

The graph R(t) is a cycloid.

(iii) k = -1 (Open Model)

If K = -1, then from equation (11), we get

$$R'^2 - 1 = \frac{A^2}{R}$$

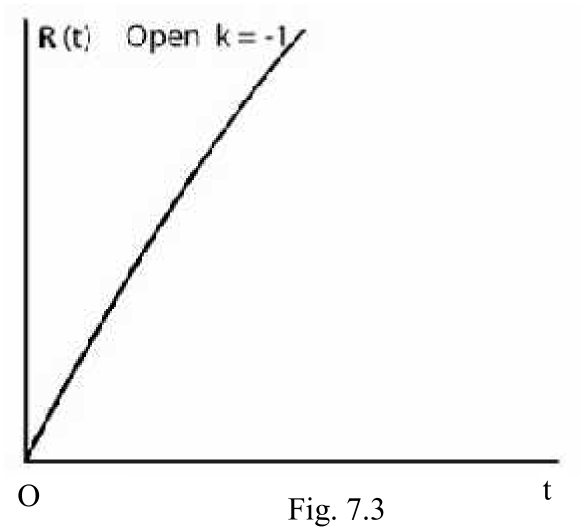


Fig. 7.3

$$\Rightarrow R'^2 = \frac{A^2}{R} + 1$$

$$\Rightarrow R'^2 = \frac{A^2 + R}{R}$$

$$\Rightarrow R' = \sqrt{\frac{A^2 + R}{R}}$$

$$\Rightarrow \frac{dR}{dt} = \sqrt{\frac{A^2 + R}{R}}$$

$$\Rightarrow \int_0^t dt = \int_0^R \sqrt{\frac{R}{A^2 + R}}\, dR$$

$\Rightarrow$ Now, Putting $R = A^2 \sinh^2 \theta/2$

$$\Rightarrow \frac{dR}{d\theta} = A^2 2\sinh\theta/2 . \cosh\theta/2 \frac{1}{2}$$

$$\Rightarrow dR = A^2 \sinh\theta/2 \cosh\theta/2\, d\theta$$

$$\therefore \int_0^t dt = \int_0^\theta \left(\frac{A^2 \sinh^2 \theta/2}{A^2 + A^2 \sinh^2 \theta/2}\right)^{1/2} A^2 \sinh\theta/2 \cosh\theta/2\, d\theta$$

$$= \int_0^\theta \left(\frac{\sinh^2 \theta/2}{\cosh^2 \theta/2}\right)^{1/2} A^2 \sin\frac{h\theta}{2} \cosh\frac{h\theta}{2} d\theta$$

$$= \frac{A^2}{2} \int_0^\theta 2\sinh^2 \theta/2\, d\theta$$

$$= \frac{A^2}{2} \int_0^\theta (\cosh\theta - 1) d\theta$$

$$\Rightarrow [t]_0^t = \frac{A^2}{2} [\sinh\theta - \theta]_0^\theta$$

$$\therefore t = \frac{A^2}{2}\left[\sinh\theta - \theta\right]$$

Therefore, $\dfrac{dt}{d\theta} = \dfrac{A^2}{2}(\cosh\theta - 1)$

$$\therefore R(t) = \frac{A^2}{2}(\cosh\theta - 1)$$

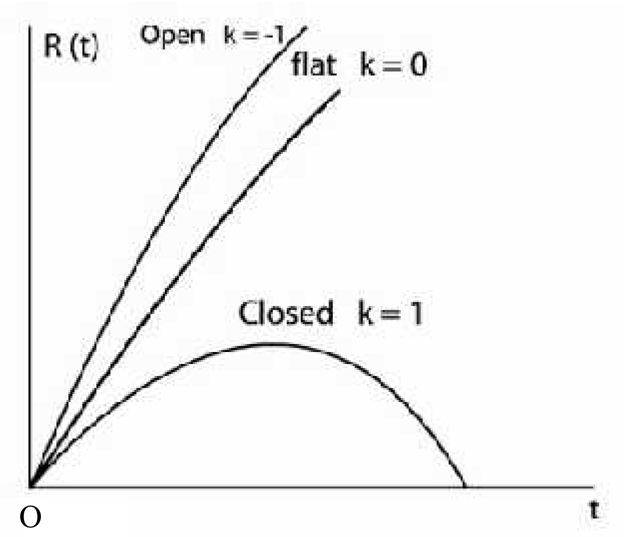


Fig. 7.4

Comments :

We see that k = 0 and k = -1 gives model which continually expand. While k = 1 gives a model which expand to maximum value of R and then contracts, so the latter is not only spatial but also temporarily closed.

7.5 Einstein's Line Element[2]:

For a static homogeneous universe, we will have the following conditions:

(i) The proper pressure p_0 as measured by a local observer shall be the same everywhere.

(ii) The proper density ρ_0 shall also be the same everywhere.

(iii) For a small values of r, the line element shall reduce to special relativity form for flat space time.

The line element satisfying the condition of spherical symmetry is given by

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) + e^{\upsilon} dt^2 \quad \text{...} (1)$$

where λ and υ are the function of r only

For the universe containing the perfect fluid,

we will have the following relations

$$8\pi p_0 = e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \quad \text{...} (2)$$

$$8\pi \rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \quad \text{...} (3)$$

$$\frac{dp_0}{dr} + (\rho_0 + p_0)\frac{\upsilon'}{2} = 0 \quad \text{..} (4)$$

According to assumption from equation (1) $\frac{dp_0}{dr} = 0$ .... (5)

Equation (4) becomes, $(\rho_0 + p_0)\frac{\upsilon'}{2} = 0$

Or $(\rho_0 + p_0)\upsilon' = 0$

either $\upsilon' = 0$ or $\rho_0 + p_0 = 0$

both $\rho_0 + p_0 = 0 = \upsilon'$

Hence $\upsilon' = 0$

Integrating, $\upsilon = c_1$ .... (6)

Applying condition (iii) i.e., $\lambda = 0 = \upsilon$ at $r = 0$ .... (7)

$\therefore c_1 = 0$

So that $\upsilon' = 0 = \upsilon$

From equation (3) $8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda$

$$\Rightarrow 8\pi\rho_0 + \Lambda = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\Rightarrow 8\pi\rho_0 + \Lambda = e^{-\lambda}\left(\frac{\lambda' r - 1}{r^2}\right) + \frac{1}{r^2}$$

$$\Rightarrow (8\pi\rho_0 + \Lambda)r^2 = e^{-\lambda}\left(\lambda' r - 1\right) + 1$$

$$\Rightarrow -e^{-\lambda}\left(\lambda' r - 1\right) = 1 - (8\pi\rho_0 + \Lambda)r^2$$

$$\Rightarrow -e^{-\lambda}.\lambda' r + e^{-\lambda} = 1 - (8\pi\rho_0 + \Lambda)r^2$$

$$\Rightarrow \frac{d}{dr}(re^{-\lambda}) = 1 - (8\pi\rho_0 + \Lambda)r^2$$

Integrating the above equation, we get

$$re^{-\lambda} = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda) + c_2$$

Applying condition (7), we get

$0 = 0 - 0 + c_2$

$$\therefore c_2 = 0$$

$$re^{-\lambda} = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda) + 0$$

$$\Rightarrow re^{-\lambda} = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda)$$

$$\Rightarrow e^{-\lambda} = 1 - \frac{r^2}{3}(8\pi\rho_0 + \Lambda)$$

Taking $\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3}$

$$\therefore e^{-\lambda} = 1 - \frac{r^2}{R^2}$$

Now, equation (1) becomes

$$ds^2 = -(1 - \frac{r^2}{R^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + dt^2$$

This is known as Einstein line element for static homogeneous universe.

7.6 Properties of Einstein Universe[2]:

Step I. Geometry of Einstein Universe:

By the transformation of co-ordinates, the Einstein line element

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\ d\varphi^2) + dt^2 \text{ ....(1)}$$

can be written into several forms.

Take the transformation

$$r = \rho \Big/ \left(1+\frac{\rho^2}{4R^2}\right)$$

Then $\quad \mathrm{r}\left(1+\frac{\rho^2}{4R^2}\right) = \rho$

Taking differential of both sides

$$dr\left(1+\frac{\rho^2}{4R^2}\right) + \frac{2\rho r}{4R^2} d\rho = d\rho$$

This gives, $dr = \dfrac{1-\dfrac{\rho r}{2R^2}}{1+\dfrac{\rho^2}{4R^2}} d\rho$

and $\dfrac{dr^2}{1-\dfrac{r^2}{R^2}} = \dfrac{1}{1-\dfrac{r^2}{R^2}}\left[\dfrac{1-\dfrac{\rho r}{2R^2}}{1+\dfrac{\rho^2}{4R^2}} d\rho\right]^2$

Simplyfying this, we get

$$\frac{dr^2}{1-\dfrac{r^2}{R^2}} = \frac{dp^2}{\left(1+\dfrac{\rho^2}{4R^2}\right)^2}$$

Now, equation (1) becomes

$$ds^2 = -\frac{1}{\left(1+\dfrac{\rho^2}{4R^2}\right)^2}[\ d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\ d\varphi^2)] + dt^2$$

This can also be transformed into

$$ds^2 = -\frac{1}{\left(1+\dfrac{\rho^2}{4R^2}\right)^2}(dx^2+dy^2+dz^2)+dt^2$$

Consider a second transformation.

$$z_1 = R - \sqrt{(1-\frac{r^2}{R^2})}$$

$$z_2 = r\sin\theta\cos\varphi$$

$$z_3 = r\sin\theta\sin\varphi$$

$$z_4 = r\cos\theta$$

We find that (1) takes the form

$$ds^2 = -(dz_1{}^2+dz_2{}^2+dz_3{}^2+dz_4{}^2)+dt^2$$

with $z_1{}^2+z_2{}^2+z_3{}^2+z_4{}^2 = R^2$

This show that the physical space of Einstein universe may be embedded in a Euclidean space of higher dimensions. This also suggests that the geometry of the Einstein universe is one which holds on the surface of a sphere embedded in a Euclidean space (z_1, z_2, z_3, z_4) of four dimensions.

By the transformation

$r = R\sin\beta,$

(1) takes the form

$$ds^2 = -R^2[d\beta^2+\sin^2\beta(d\theta^2+\sin^2\theta\, d\varphi^2)]+dt^2 \text{..................(2)}$$

We already have $0\le\theta\le\pi, 0\le\varphi\le 2\pi.$

We find that this line element (2) remains unchanged for $\beta=0$ and $\beta=\pi$, other variables θ and φ being arbitrary. That means that to an event at $\beta=0$, there is a similar event at $\beta=\pi$. That is to say that corresponding to an event at $\beta=0$, there exists a mirror image at $\beta=\pi$. In this sense the Einstein universe is taken to be spherical. The proper volume V_0 of the spherical universe is

$$\int_{\beta=0}^{\beta=\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}(Rd\beta)(R\sin\beta d\theta)(R\sin\theta\sin\beta d\varphi)$$

$$=4\pi R^3\int_0^{\pi}\frac{1}{2}(1-\cos 2\beta)d\beta = 2\pi R^3\left[\beta-\frac{1}{2}\sin 2\beta\right]_0^{\pi}$$

$$=2\pi^2 R^3$$

Thus the proper volume of the so-called spherical universe is $2\pi^2 R^3$.

Alternatively, we can take the two events at $\beta = 0$ and $\beta = \pi$, one and the same. In this sense Einstein universe is taken to be elliptical. The proper volume of this universe is

$$\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{dr.rd\theta.r\sin\theta d\varphi}{\sqrt{(1-r^2/R^2)}}$$

$$=4\pi\int_0^R\frac{r^2.dr}{\sqrt{(1-r^2/R^2)}}$$

$$=4\pi\int_0^{\pi/2}\frac{R^2\sin^2\eta(R\cos\eta)d\eta}{\cos\eta},\ \text{where}\ \frac{r}{R}=\sin\eta$$

$$=2\pi R^3\int_0^{\pi/2}(1-\cos 2\eta)d\eta = 2\pi R^3\left[\eta-\frac{1}{2}\sin 2\eta\right]_0^{\pi/2}$$

$$=\pi^2 R^3$$

Thus we see that proper volume of the so called spherical universe is just double of elliptical universe.

The proper distance around the spherical universe is

$$l_0 = 2\int_0^{\pi} Rd\beta = 2\pi R$$

In case of elliptical universe, $l_0 = 2\int_0^R \frac{dr}{\sqrt{(1-r^2/R^2}}$

$$=2\int_0^{\pi/2}\frac{R\cos\eta d\eta}{\cos\eta},\ \text{where}\ \frac{r}{R}=\sin\eta$$

$$=\pi R$$

Step II. Density and Pressure of the Matter in Einstein Universe:

We have $8\pi p_0 = e^{-\lambda}\left(\frac{\upsilon'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}+\Lambda$........................(3)

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda\text{........................(4)}$$

where $\upsilon' = 0 = \upsilon, e^{-\lambda} = 1-\frac{r^2}{R^2}$

and $d(e^{-\lambda}) = d\left(1 - \frac{r^2}{R^2}\right)$

$$\Rightarrow -e^{-\lambda}.\lambda' = 0 - \frac{2r}{R^2}$$

$$\therefore e^{-\lambda}.\frac{\lambda'}{r} = \frac{2}{R^2}$$

With this values equation (3) becomes

$$8\pi p_0 = 0 + e^{-\lambda}.\frac{1}{r^2} - \frac{1}{r^2} + \Lambda$$

$$\Rightarrow 8\pi p_0 = \left(1 - \frac{r^2}{R^2}\right)\frac{1}{r^2} - \frac{1}{r^2} + \Lambda$$

$$\Rightarrow 8\pi p_0 = \frac{1}{r^2} - \frac{1}{R^2} - \frac{1}{r^2} + \Lambda$$

$$\therefore 8\pi p_0 = -\frac{1}{R^2} + \Lambda \text{ ....(5)}$$

From equation (4)

$$8\pi \rho_0 = \frac{2}{R^2} - \left(1 - \frac{r^2}{R^2}\right)\frac{1}{r^2} + \frac{1}{r^2} - \Lambda$$

$$\Rightarrow 8\pi \rho_0 = \frac{2}{R^2} - \frac{1}{r^2} + \frac{1}{R^2} + \frac{1}{r^2} - \Lambda$$

$$\Rightarrow 8\pi \rho_0 = \frac{3}{R^2} - \Lambda \text{................(6)}$$

Adding equation (5) and (6), we have

$$8\pi(p_0 + \rho_0) = \frac{2}{R^2}$$

Or, $p_0 + \rho_0 = \frac{2}{8\pi R^2}$

$$\therefore p_0 + \rho_0 = \frac{1}{4\pi R^2} \text{ ....(7)}$$

The equation (5) and (6) are the required expressions for pressure and density.

Incoherent Matter in the Universe:

Suppose the universe is filled with fluid consisting of incoherent matter exerting no pressure. For example free particles (stars).

Then $p_0 = 0.$ Now, equation (7) becomes $\rho_0 = \frac{1}{4\pi R^2}$

Mass of the spherical universe $= V_0 \rho_0$

$$= 2\pi^2 R^3 . \frac{1}{4\pi R^2} = \frac{\pi R}{2}$$

Mass of the spherical universe $= \frac{\pi R}{2}$

Mass of the elliptical universe $= V_0 \rho_0 = \pi^2 R^3 \frac{1}{4\pi R^2} = \frac{\pi R}{4}$

Radiation in the Universe:

When the universe is completely filled with radiation for which $\rho_0 = 3p_0$

Then (7) becomes $p_0 = \frac{1}{16\pi R^2} = \frac{\rho_0}{3}$. $[\because 3p_0 = \rho_0]$

Mass of the spherical universe $= V_0 \rho_0 = 2\pi^2 R^3 . \frac{3}{16\pi R^2} = \frac{3}{8}\pi R$

Or, Mass of the spherical universe $= \frac{3}{8}\pi R$

Empty Universe:

When the universe is completely empty.

Then $p_0 = 0 = \rho_0$

Now, equation (5) and (6) becomes $\Lambda = \frac{1}{R^2}, \Lambda = \frac{3}{R^2}$

This $\Rightarrow \Lambda = \frac{1}{R^2} = 0$

This proves that the Einstein element would degenerate into a line element of special relativity form for flat space time.

Step III. Motion of a Test Particle in the Einstein Universe:

In Einstein universe the motion of a test particle has zero acceleration. It means that in Einstein universe a particle at rest remains at rest.

Step IV. Shift in Spectral Lines:

From Einstein universe, we have $\frac{\delta t_1^0}{\delta t_2^0} = 1,$ meaning thereby $\exists$ no red shift. It means that no Doppler effect is observed. In other words, the nebulaes do not seen to sun away. It contradicts the matter is receding in actual universe and the universe is expanding at every moment.

7.7 de-Sitter's Line Element[2]:

For a static homogeneous universe, we will have the following condition:

(i) The proper pressure P_0 as measured by a local observer shall be the same everywhere.

(ii) The proper density ρ_0 shall be the same everywhere.

(iii) For small values of *r*, the line element shall reduce to special relativity form for flat space time.

The line element satisfying the condition of spherical symmetry is given by

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + e^{\upsilon} dt^2 \quad \text{.........................(1)}$$

where λ and υ are the function of *r* only.

For the universe containing perfect fluids, we will have the following relations

$$8\pi p_0 = e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \text{ .... (2)}$$

$$8\pi \rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \text{ .... (3)}$$

$$\frac{dp_0}{dr} + (\rho_0 + p_0)\frac{\upsilon'}{2} = 0 \text{ .... (4)}$$

According to assumption (1)

$$\frac{dp_0}{dr} = 0$$

Equation (4) becomes

$$(\rho_0 + p_0)\frac{\upsilon'}{2} = 0$$

Or, $(\rho_0 + p_0)\upsilon' = 0$

either $\upsilon' = 0$ Or, $\rho_0 + p_0 = 0$

$$\Rightarrow \rho_0 + p_0 = 0 = \upsilon'$$

de-Sitter line element arise from the possibility that

$$\rho_0 + p_0 = 0$$

Adding equation (2) and (3), we get

$$8\pi(\rho_0 + p_0) = e^{-\lambda}\left(\frac{\lambda' + \upsilon'}{r}\right).$$

$$\Rightarrow 8\pi.0 = e^{-\lambda}\left(\frac{\lambda' + \upsilon'}{r}\right)$$

$$\Rightarrow 0 = e^{-\lambda}\,\frac{(\lambda' + \upsilon')}{r}$$

$$\lambda' + \upsilon' = 0$$

Integrating the above equation, we have

$$\lambda + \upsilon = c_1$$

Subject this to the condition (iii) $\lambda = 0 = \upsilon$ at $r = 0$ .... (5)

We obtain $c_1 = 0$

$$\therefore \lambda + \upsilon = 0$$

Or, $\upsilon = -\lambda$ .... (6)

From equation (3), we have

$$8\pi\rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda$$

$$\Rightarrow 8\pi\rho_0 + \Lambda = e^{-\lambda}\left(\frac{\lambda' r - 1}{r^2}\right) + \frac{1}{r^2}$$

$$\Rightarrow (8\pi\rho_0 + \Lambda)r^2 = e^{-\lambda}(\lambda' r - 1) + 1$$

$$\Rightarrow -e^{-\lambda}(\lambda' r - 1) = 1 - (8\pi\rho_0 + \Lambda)r^2$$

$$\Rightarrow \frac{d}{dr}(e^{-\lambda} r) = 1 - (8\pi\rho_0 + \Lambda)r^2$$

$$\Rightarrow e^{-\lambda} r = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda) + c_2$$

Subjecting this to equation (5), we get $c_2 = 0$

$$e^{-\lambda} r = r - \frac{r^3}{3}(8\pi\rho_0 + \Lambda) + 0$$

$$\Rightarrow e^{-\lambda} = 1 - \frac{r^2}{3}(8\pi\rho_0 + \Lambda)$$

Taking $\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3}$

$$\therefore e^{-\lambda} = 1 - \frac{r^2}{R^2}$$

$e^{\upsilon} = e^{-\lambda}$ [Using equation (6)]

So that $e^{\upsilon} = 1 - \frac{r^2}{R^2}$

Now, equation (1) becomes

$$ds^2 = -(1 - \frac{r^2}{R^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + (1 - \frac{r^2}{R^2})dt^2$$

This is known as de-Sitter line element for static homogenous universe.

7.8 Properties of de-Sitter Universe[2]:

Step I: Geometry of de-Sitter Universe.

By the transformation of co-ordinates, the de-Sitter line element

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \left(1 - \frac{r^2}{R^2}\right)dt^2 \text{..........(1)}$$

can be written into several forms. We make the transformation

$$\frac{r}{R} = \sin\beta$$

As a result of which (1) becomes

$$ds^2 = -R^2[d\beta^2 + \sin^2\beta(d\theta^2 + \sin^2\theta d\varphi^2)] + \cos^2\beta dt^2 \text{..........(2)}$$

On applying the transformation

$$\alpha = r\sin\theta\cos\varphi, \delta - e = \mathrm{Re}^{-t/R} / \sqrt{\left(1 - \frac{r^2}{R^2}\right)}$$

$$\beta = r\sin\theta\sin\varphi$$

$$\gamma = r\cos\theta, \delta + e = \mathrm{Re}^{t/R} / \sqrt{\left(1 - \frac{r^2}{R^2}\right)}$$

We find that equation (1) is reduced to

$ds^2 = -[d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2] + d\in^2$

Further taking $\alpha = iz_1, \beta = iz_2, \gamma = iz_3, \delta = iz_4, \in = iz_5$

We obtain $ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2$ ....(3)

with $z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = (iR)^2$.

The equation (3) suggests that the physical space in de-Sitter universe can be embedded in a Euclidean space of higher dimensions. It also shows that the geometry of this universe is one which holds on the surface of sphere embedded in a Euclidean space of five dimensions.

Lemaitre Roberstson transformation

$$r^i = -\frac{re^{t/R}}{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}, t^i = t + R\log\left\{\left(1-\frac{r^2}{R^2}\right)\right\}^{1/2}$$

with the help of this transformation (1) takes the form

$$ds^2 = -e^{2t'/R} / R\left[dr'^2 + r'^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] + dt'^2$$

Dropping dashes, we get

$$ds^2 = -e^{2t/R}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] + dt^2$$

Taking $k = 1/R$, we get

$$ds^2 = -e^{2kt}\ [dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] + dt^2$$

Its cartesian equivalent is

$$ds^2 = -e^{2kt}\ [dx^2 + dy^2 + dz^2] + dt^2$$

Thus we see that with the help of this transformation, it is possible to convert a static line element into a non-static one.

Step II. Density and Pressure of Matter in de-Sitter Universe:

The de-Sitter line element is based on the assumption

$\rho_0 + p_0 = 0$ .... (4)

Since $\rho_0 \geq 0$ and therefore we have

$\rho_0 = 0 = p_0$ .... (5)

This is the unique solution of (4). The equation (5) implies that the de-Sitter universe is completely empty. It contains neither matter nor radiation.

Step III: Motion of a Test Particle in de-Sitter Universe:

In de-Sitter universe the motion of a test particle has zero acceleration. It means that in de-Sitter universe a particle at rest at origin with $h=0$ remains at rest.

Step IV: Shift in Spectral Lines:

Form de-Sitter universe we have $\frac{\delta\lambda_0}{\lambda_0} = \frac{r}{R}$, this shows that red shift is proportional to the distance measured from the origin if we take c =1. It also supports Weyl's theory according to which nebulaes are receding with velocity proportional to the distances from us. Thus we see that de-Sitter is completely empty yet predicts the recession of nebulaes.

7.9 Similarity and Difference between Einstein and de-Sitter's Line Element:

Topics	Einstein line element	de-Sitter line element
(1) Geometry	(1)The physical space of Einstein universe may be embedded in a Euclidean space of higher dimensions. This also suggests that the geometry of the Einstein universe is one which holds on the surface of a sphere embedded in a Euclidean space (z_1, z_2, z_3, z_4) of four dimensions.	(1) The physical space in de-Sitter universe can be embedded in a Euclidean space of higher dimensions. It also shows that the geometry of this universe is one which holds on the surface of sphere embedded in a Euclidean space of five dimensions.
(2)Density and Pressure	(2) Density and pressure of the matter in Einstein universe is as follows $8\pi p_0 = -\frac{1}{R^2}+\Lambda$.................(a) $8\pi \rho_0 = \frac{3}{R^2}-\Lambda$ .... (b) From equation (a) and (b), we have $\rho_0 + p_0 = \frac{1}{4\pi R^2}$	(2) The de-Sitter line element is based on the assumption $p_0+\rho_0 = 0$.........................(a′) Since $\rho_0 \geq o$ and therefore, we have $\rho_0 = 0 = P_0$.................. (b′) This is the unique solution of (a′). The equation (b′) implies that the de-Sitter universe is completely empty. It contains neither matter nor radiation.
(3) Motion of a test particle	(3) In Einstein universe the motion of a test particle has zero acceleration. It means that in Einstein universe a particle at rest remains at rest.	(3) In de-Sitter universe the motion of a test particle has zero acceleration. It means that in de-Sitter universe a particle at rest at origin with $h=0$ remains at rest.
(4) Shift in spectral lines	(4) From Einstein universe, we have $\frac{\delta t_1^0}{\delta t_2^0}=1$ meaning there by $\exists$ no red shift. It means that no Doppler Effect is observed. In other words, the nebulaes do not seen to sun away. It contradicts the matter is receding in actual universe and the universe is expanding at every moment.	(4) From de-sitter universe, we have $\frac{\delta \lambda_0}{\lambda_0}=\frac{r}{R}$ this shown that red shift is proportional to the distance measured from the origin if we take c =1. It also supports Weyl's theory according to which nebulas are receding with velocity proportional to the distances from us.

Chapter Eight

Phenomenology and Expanding Universe

8.1 Huge Viscous Bianchi Type-1 Cosmological Model for Barotropic Fluid and Decaying Λ with Time

Introduction:

The Friedmann-Robertson-Walker models are unstable near the singularity (Patridge and Wilkinson [15]) and fail to describe the early universe. The homogenous and anisotropic Bianchi Type-1 space time is undertaken to study the universe in its early stages on evolution of the universe. Land and Magueijo [16] have described the existence of anisotropic universe that approaches the isotropic phase. The large scale distribution of galaxies of the universe shows that the matter distribution is satisfactorily described by perfect fluid. Misner [17,18] has studied the effect of viscosity on evolution of the universe. Several authors such as Roy and Prakash [19], Santos et al. [20], Coley and Tupper [21], Goenner and Kowalewski [22], Gron [23] and Ram and Singh [24,25] have studied the effect of huge viscosity on the evolution of universe at large time. In modern cosmological theories, the cosmological parameter (Λ) remains a focal point of interest. Dreitlein [26] and Linde [27] have studied its significance. Therefore, the cosmological parameter (Λ) leaving the form of Einstein field equation unchanged and preserving the conservation of energy-momentum tensor of matter content, have been investigated from time to time. Some of the authors such as Abdus-Sattar and Vishwakarma [28], Bertolami [29] and Berman [30] have investigated that the cosmological parameter Λ decreases with large time in some cosmological models.

In this section 8.1, we also have observed that the Hubble's parameter H, the pressure p, the deceleration parameter q, the matter energy density ρ and the cosmological parameter Λ on evolution of the universe at large time.

The Methodology:

The Metric and Field Equations:

We consider Bianchi Type-1 metric in the form

$ds^2 = -dt^2+A^2(t)dx^2+B^2(t)dy^2+C^2(t)dz^2$ .... (1)

The matter content is taken as huge viscous fluid given by energy-mementum tensor $T_i^j = (\rho + \overline{p})v_i v^j + \overline{p} g_i^j$ ....(2)

where $\overline{p} = p - \zeta v^i_{;i}$

Or, $\overline{p} = p - \zeta\theta$ ....(3) [$\therefore\ v^i_{;i} = \theta$]

and $v_i v^i = -1$ .... (4)

The Einstein's field equations (in geometrized unit $8\pi G = c = 1$) with time varying cosmological constant Λ (t) are given by

$R_i^j - \frac{1}{2} R g_i^j = -T_i^j + \Lambda(t) g_i^j$ ....(5)

where $v_i = (0, 0, 0, -1)$, P is the isotropic pressure and ρ is the energy density.

From (1), we have

$$g_{ij} = \begin{bmatrix} A^2(t) & 0 & 0 & 0 \\ 0 & B^2(t) & 0 & 0 \\ 0 & 0 & C^2(t) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

So that $g_{11} = A^2(t), g_{22} = B^2(t), g_{33} = C^2(t), g_{44} = -1$

and the other terms are zero.

The inverse of g_{ij} is

$$g^{ij}=\begin{bmatrix}\frac{1}{A^2(t)} & 0 & 0 & 0\\ 0 & \frac{1}{B^2(t)} & 0 & 0\\ 0 & 0 & \frac{1}{C^2(t)} & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}g^{11} & g^{12} & g^{13} & g^{14}\\ g^{21} & g^{22} & g^{23} & g^{24}\\ g^{31} & g^{32} & g^{33} & g^{34}\\ g^{41} & g^{42} & g^{43} & g^{44}\end{bmatrix}$$

So that $g^{11}=\frac{1}{A^2(t)}, g^{22}=\frac{1}{B^2(t)}, g^{33}=\frac{1}{C^2(t)}, g^{44}=-1$

and the other terms are zero.

We know the Christoffel symbol of second kind is

$$\Gamma^{\lambda}_{ij}=\frac{1}{2}g^{\lambda\sigma}\left(g_{\sigma i,j}+g_{\sigma j,i}-g_{ij,\sigma}\right)$$

The non-zero component of Christoffel symbols are

$$\Gamma^{1}_{14}=\Gamma^{1}_{41}=\frac{\dot{A}}{A},\ \Gamma^{2}_{24}=\Gamma^{2}_{42}=\frac{\dot{B}}{B},\ \Gamma^{3}_{34}=\Gamma^{3}_{43}=\frac{\dot{C}}{C},\ \Gamma^{4}_{11}=A\dot{A},\ \Gamma^{4}_{22}=B\dot{B},\ \Gamma^{4}_{33}=C\dot{C}$$

Now, $R_{\mu\upsilon}=\Gamma^{\sigma}_{\mu\sigma,\upsilon}-\Gamma^{\sigma}_{\mu\upsilon,\sigma}+\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\upsilon}-\Gamma^{\rho}_{\mu\upsilon}\Gamma^{\sigma}_{\rho\sigma}$

$$R_{11}=\Gamma^{\sigma}_{1\sigma,1}-\Gamma^{\sigma}_{11,\sigma}+\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1}-\Gamma^{\sigma}_{11}\Gamma^{\sigma}_{\rho\sigma}$$

$$\Gamma^{\sigma}_{1\sigma,1}=\Gamma^{1}_{11,1}+\Gamma^{2}_{12,1}+\Gamma^{3}_{13,1}+\Gamma^{4}_{14,1}$$

$$=0+0+0+0$$

$$=0$$

$$\Gamma^{\sigma}_{11,\sigma}=\Gamma^{1}_{11,1}+\Gamma^{2}_{11,2}+\Gamma^{3}_{11,3}+\Gamma^{4}_{11,4}$$

$$=0+0+0+\frac{\partial}{\partial t}(A\dot{A})$$

$$=A\ddot{A}+\dot{A}\dot{A}$$

$$=A\ddot{A}+\dot{A}^2$$

$$\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1}=\Gamma^{1}_{1\sigma}\Gamma^{\sigma}_{11}+\Gamma^{2}_{1\sigma}\Gamma^{\sigma}_{21}+\Gamma^{3}_{1\sigma}\Gamma^{\sigma}_{31}+\Gamma^{4}_{1\sigma}\Gamma^{\sigma}_{41}$$

$$= \Gamma^{1}_{14}\Gamma^{4}_{11} + 0 + 0 + \Gamma^{4}_{11}\Gamma^{1}_{41}$$

$$= \frac{\dot{A}}{A}.A\dot{A} + A\dot{A}.\frac{\dot{A}}{A}$$

$$= \dot{A}^2 + \dot{A}^2$$

$$= 2\dot{A}^2$$

$$\Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{11}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{11}\Gamma^{\sigma}_{3\sigma} + \Gamma^{4}_{11}\Gamma^{\sigma}_{4\sigma}$$

$$= 0 + 0 + 0 + \Gamma^{4}_{11}(\Gamma^{1}_{41} + \Gamma^{2}_{42} + \Gamma^{3}_{43} + \Gamma^{4}_{44})$$

$$= A\dot{A}(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + 0)$$

$$= \dot{A}^2 + \frac{A\dot{A}\dot{B}}{B} + \frac{A\dot{A}\dot{C}}{C}$$

$$\therefore R_{11} = 0 - A.\ddot{A} - \dot{A}^2 + 2\dot{A}^2 - \dot{A}^2 - \frac{A\dot{A}\dot{B}}{B} - \frac{A\dot{A}\dot{C}}{C}$$

$$= -A(\frac{\dot{A}\dot{B}}{B} + \frac{\dot{A}\dot{C}}{C} + \ddot{A})$$

Similarly,

$$R_{22} = -B(\frac{\dot{A}\dot{B}}{A} + \frac{\dot{B}\dot{C}}{C} + \ddot{B})$$

$$R_{33} = -C(\frac{\dot{A}\dot{C}}{A} + \frac{\dot{B}\dot{C}}{B} + \ddot{C})$$

$$R_{44} = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}$$

Now, $R^1_1 = g^{11}R_{11}$

$$= -\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} - \frac{\ddot{A}}{A}$$

$$R^2_2 = g^{22}R_{22}$$

$$= -\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{B}}{B}$$

$$R^3_3 = g^{33}R_{33}$$

$$= -\frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{C}}{C}$$

$R_4^4 = g^{44} R_{44}$

$$= -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}$$

So that the scalar curvature is

$R = R_1^1 + R_2^2 + R_3^3 + R_4^4$

$$= -2\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)$$

Now, we have

$$R_1^1 - \frac{1}{2} g_1^1 R = \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} \text{..........................(6)}$$

$$R_2^2 - \frac{1}{2} g_2^2 R = \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} \text{..........................(7)}$$

$$R_3^3 - \frac{1}{2} g_3^3 R = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} \text{..............................(8)}$$

$$R_4^4 - \frac{1}{2} g_4^4 R = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \text{........................(9)}$$

Now, the Einstein's field equation (5) can be written as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\overline{p} + \Lambda(t) \text{..........................(10)} \quad \text{[Using equation (2) and (6)]}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -\overline{p} + \Lambda(t) \text{..........................(11)} \quad \text{[Using equation (2) and (7)]}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\overline{p} + \Lambda(t) \text{..........................(12)} \quad \text{[Using equation (2) and (8)]}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = \rho + \Lambda(t) \text{..........................(13)} \quad \text{[Using equation (2) and (9)]}$$

Solution of the Field Equations:

The divergence of equation (5) leads to

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \dot{\Lambda} = 0 \text{ ....(14)}$$

For getting the model of the universe, we have assumed that the eigenvalue (σ_1^1) of shear tensor (σ_i^j) is proportional to the expansion (θ).

Here, we take $A = (BC)^n$ .... (15)

We consider the non-vacuum component of matter obeys the equation of state

p= $\omega\rho$; $0 \le \omega \le 1$ .... (16)

From equation (10) and (11), we have

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}$$

$$\Rightarrow \frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}=\frac{\dot{C}}{C}(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}) \qquad (17)$$

From equation (11) and (12), we have

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}$$

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{\dot{A}}{A}(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}) \qquad (18)$$

Differentiating equation (15) with respect to t, we have

$$\dot{A}=n(BC)^{n-1}(\dot{B}C+B\dot{C}) \qquad (19)$$

$$\ddot{A}=n(n-1)(BC)^{n-2}(\dot{B}C+B\dot{C})(\dot{B}C+B\dot{C})+n(BC)^{n-1}(\ddot{B}C+\dot{B}\dot{C}+\dot{B}\dot{C}+B\ddot{C})$$

$$\Rightarrow \ddot{A}=n(n-1)(BC)^{n-2}(\dot{B}C+B\dot{C})^2+n(BC)^{n-1}(\ddot{B}C+2\dot{B}\dot{C}+B\ddot{C}) \qquad (20)$$

From equation (17), we have

$$\frac{\ddot{B}}{B}-n(n-1)\left(\frac{\dot{B}C+B\dot{C}}{BC}\right)^2-\frac{n}{BC}(\ddot{B}C+2\dot{B}\dot{C}+B\ddot{C})=\frac{\dot{C}}{C}\left\{\frac{n}{BC}(\dot{B}C+B\dot{C})-\frac{\dot{B}}{B}\right\}$$

[Using equation (19) and (20)]

$$\Rightarrow \frac{\ddot{B}}{B}-n(n-1)\left(\frac{\dot{B}^2}{B^2}+\frac{2\dot{B}\dot{C}}{BC}+\frac{\dot{C}^2}{C^2}\right)-n\left(\frac{\ddot{B}}{B}+\frac{2\dot{B}\dot{C}}{BC}+\frac{\ddot{C}}{C}\right)= n\frac{\dot{C}^2}{C^2}+n\frac{\dot{B}\dot{C}}{BC}-\frac{\dot{B}\dot{C}}{BC}$$

$$\Rightarrow (1-n)\frac{\ddot{B}}{B}+n(n-1)\frac{\dot{B}^2}{B^2}-n^2\frac{\dot{C}^2}{C^2}-n\frac{\ddot{C}}{C}-(2n^2+n-1)\frac{\dot{B}\dot{C}}{BC}=0 \qquad (21)$$

From equation (18), we have

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{n}{BC}(\dot{B}C+B\dot{C})(\frac{\dot{B}}{B}-\frac{\dot{C}}{C})$$

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{n}{BC}(\dot{B}C+B\dot{C})(\frac{\dot{B}C-B\dot{C}}{BC})$$

$$\Rightarrow \frac{\ddot{C}B-\ddot{B}C}{BC}=\frac{n}{(BC)^2}(\dot{B}C+B\dot{C})(\dot{B}C-B\dot{C})$$

$$\Rightarrow \frac{\ddot{C}B - \ddot{B}C}{\dot{B}C - B\dot{C}} = \frac{n}{BC}(\dot{B}C + B\dot{C})$$

$$\Rightarrow \frac{-\{-(\ddot{C}B - \ddot{B}C)\}}{\dot{B}C - B\dot{C}} = \frac{n}{BC}(\dot{B}C + B\dot{C})$$

$$\Rightarrow \frac{-(\ddot{C}B - \ddot{B}C)}{\dot{B}C - B\dot{C}} = -n\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)$$

Integrating the above equation, we have

$$Log_e(\dot{B}C - B\dot{C}) = -n(\log_e B + \log_e C) + \log_e K$$

$$\Rightarrow \log_e(\dot{B}C - B\dot{C}) = \log_e(BC)^{-n} + \log_e k$$

$$\Rightarrow \log_e(\dot{B}C - B\dot{C}) = \log_e k(BC)^{-n}$$

$$\Rightarrow \dot{B}C - B\dot{C} = k(BC)^{-n}$$

$$\Rightarrow C^2 \frac{\partial}{\partial t}\left(\frac{B}{C}\right) = k(BC)^{-n} \quad \text{........................(22)}$$

where k is constant of integration.

Now, we consider

μ =BC.. (23)

and $\upsilon=\frac{B}{C}$ ....(24)

From equation (22), we have

$C^2\dot{\upsilon}$ =k μ $^{-n}$ [Using equation (23) and (24)]

$$\Rightarrow \dot{\upsilon} = \frac{k\mu^{-n}}{C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k\mu^{-n}}{\upsilon C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k\mu^{-n}}{\frac{B}{C}.C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k\mu^{-n}}{BC}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k\mu^{-n}}{\mu} \quad [\because BC = \mu]$$

$$\therefore \frac{\dot{\upsilon}}{\upsilon} = k\mu^{-(n+1)} \text{...}(25)$$

Differentiating the above equation with respect to t, we have

$$\frac{1}{\upsilon^2}(\upsilon\ddot{\upsilon} - \dot{\upsilon}\dot{\upsilon}) = -k(n+1)\mu^{-(n+2)}\dot{\mu}$$

$$\Rightarrow \frac{\ddot{\upsilon}}{\upsilon} = -k(n+1)\mu^{-(n+2)}\dot{\mu} + \left(\frac{\dot{\upsilon}}{\upsilon}\right)^2$$

$$\Rightarrow \frac{\ddot{\upsilon}}{\upsilon} = -k(n+1)\mu^{-(n+2)}\dot{\mu} + k^2\mu^{-2(n+1)} \text{.................}(26) \qquad \text{[Using equation (25)]}$$

From equation (21), we have

$$(1-2n)\frac{\ddot{\mu}}{\mu} + n(1-2n)\frac{\dot{\mu}^2}{\mu^2} + \frac{\ddot{\upsilon}}{\upsilon} - \frac{\dot{\upsilon}^2}{\upsilon^2} + (1+n)\frac{\dot{\mu}\dot{\upsilon}}{\mu\upsilon} = 0 \text{ .... }(27)$$

From equation (27), we have

$$(1-2n)\frac{\ddot{\mu}}{\mu} + n(1-2n)\frac{\dot{\mu}^2}{\mu^2} - \frac{k(n+1)\dot{\mu}}{\mu^{n+2}} + \frac{k^2}{\mu^{2(n+1)}} - \frac{k^2}{\mu^{2(n+1)}} + \frac{k(n+1)}{\mu^{n+2}}\dot{\mu} = 0$$

[Using equation (25) and (26)]

$$\Rightarrow (1-2n)\frac{\ddot{\mu}}{\mu} + n(1-2n)\frac{\dot{\mu}^2}{\mu^2} = 0$$

$$\Rightarrow (1-2n)\left(\frac{\ddot{\mu}}{\mu} + n\frac{\dot{\mu}^2}{\mu^2}\right) = 0$$

$$\Rightarrow \frac{1}{\mu}\left(\ddot{\mu} + n\frac{\dot{\mu}^2}{\mu}\right) = 0$$

$$\Rightarrow \ddot{\mu} + n\frac{\dot{\mu}^2}{\mu} = 0$$

$$\Rightarrow \frac{\ddot{\mu}}{\dot{\mu}} + n\frac{\dot{\mu}^2}{\dot{\mu}\mu} = 0$$

$$\Rightarrow \frac{\ddot{\mu}}{\dot{\mu}} + n\frac{\dot{\mu}}{\mu} = 0$$

Integrating the above equation with respect to t, we have

$$\log_e \dot{\mu} + n\log_e \mu = \log_e l$$

$$\Rightarrow \log_e(\mu^n\dot{\mu}) = \log_e l$$

$$\Rightarrow \mu^n \dot{\mu} = l$$

$$\Rightarrow l = \mu^n \frac{d\mu}{dt}$$

$$\Rightarrow \mu^n d\mu = l dt$$

Again integrating the above equation, we have

$$\frac{\mu^{n+1}}{n+1} = lt + b_1$$

$$\Rightarrow \mu^{n+1} = l(n+1)t + b_1(n+1)$$

$\Rightarrow \mu^{n+1} = at + b$ where $a = l$ (n+ 1), b = b_1(n+1) and a , b are constants.

$$\Rightarrow \left(\mu^{n+1}\right)^{\frac{1}{n+1}} = \left(at + b\right)^{\frac{1}{n+1}}$$

$$\Rightarrow \mu = (at + b)^{\frac{l}{a}} \quad \left[\because a = l(n+1)\right]$$

From equation (25), we have

$$\frac{\dot{\upsilon}}{\upsilon} = \frac{k}{\mu^{n+1}}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k}{at + b} \quad \left[\because \mu^{n+1} = at + b\right]$$

Integrating the above equation, we have

$$\log_e \upsilon = \frac{k}{a} \log_e (at + b) + \log_e L$$

$$\Rightarrow \log_e \upsilon = \log_e (at + b)^{\frac{k}{a}} + \log_e L$$

$$\Rightarrow \log_e \upsilon = \log_e L(at + b)^{\frac{k}{a}}$$

$$\therefore \upsilon = L(at + b)^{\frac{k}{a}}$$

where L is constant of integration.

From equation (23) and (24), we have

$$\mu\upsilon = BC.\frac{B}{C}$$

$$\Rightarrow B^2 = \mu\upsilon$$

$$\Rightarrow B^2 = (at + b)^{\frac{l}{a}}.L(at + b)^{\frac{k}{a}}$$

$$\Rightarrow B^2 = L(at+b)^{\frac{l}{a}+\frac{k}{a}}$$

$$\therefore B = \sqrt{L}(at+b)\frac{l+k}{2a}$$

$$= \sqrt{L}(at+b)^F$$

where $F = \dfrac{l+k}{2a}$ is constant .

Again from equation (23) and (24), we have

$$\frac{\mu}{\upsilon} = \frac{BC}{\frac{B}{C}}$$

$$\Rightarrow C^2 = \frac{\mu}{\upsilon}$$

$$\Rightarrow C^2 = \frac{(at+b)^{\frac{l}{a}}}{L(at+b)^{\frac{k}{a}}}$$

$$= \frac{1}{L}(at+b)^{\frac{l}{a}-\frac{k}{a}}$$

$$= \frac{1}{L}(at+b)^{\frac{l-k}{a}}$$

$$\Rightarrow C = \frac{1}{\sqrt{L}}(at+b)\frac{l-k}{2a}$$

$\therefore C = \dfrac{1}{\sqrt{L}}(at+b)^N$ where N=$\dfrac{l-k}{2a}$ is constant.

From equation (15), we have

A=(BC)^n

$$= \left\{\sqrt{L}(at+b)^F . \frac{1}{\sqrt{L}}(at+b)^N\right\}^n$$

$$= (at+b)^{(F+N)n}$$

$$= (at+b)^{\frac{nl}{a}} \qquad \left[\because F+N = \frac{l+k}{2a} + \frac{l-k}{2a} = \frac{2l}{2a} = \frac{l}{a}\right]$$

= (at+b)^G where G=$\dfrac{nl}{a}$ is constant.

Now, putting the value of A,B,C in equation (1),

we have

$$ds^2 = -dt^2 + (at+b)^{2G} dx^2 + L(at+b)^{2F} dy^2 + \frac{1}{L}(at+b)^{2N} dz^2 \quad ....(28)$$

where $G = \frac{nl}{a},\ F = \frac{l+k}{2a},\ N = \frac{l-k}{2a}$

Some Physical and Geometrical Properties:

The volume expansion (θ), scale factor (R), Hubble parameter (H), deceleration parameter (q) are given by

$$\theta = v^{i}_{;i} = 3\frac{\dot{R}}{R} = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \quad .... (29)$$

Hence $\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}$

$$\Rightarrow \quad \theta = \frac{Ga}{at+b} + \frac{Fa}{at+b} + \frac{Na}{at+b}$$

$$\Rightarrow \quad \theta = \left(\frac{a}{at+b}\right)(G+F+N)$$

where G + F + N = $\frac{a-l}{a} + \frac{l+k}{2a} + \frac{l-k}{2a}$

$$= \frac{2a-2l+l+k+l-k}{2a}$$

$$= \frac{2a}{2a}$$

$$= 1$$

$$\therefore \quad \theta = \frac{a}{at+b} \quad ....(30)$$

We define scale factor,

$$R = (ABC)^{1/3}$$

$$\Rightarrow R = \left\{(at+b)^{G}\sqrt{L}(at+b)^{F}.\frac{1}{\sqrt{L}}(at+b)^{N}\right\}^{1/3}$$

$$\Rightarrow R = \left\{(at+b)^{(G+F+N)}\right\}^{1/3}$$

$\therefore R = (at+b)^{1/3}$ ....(31) [$\therefore$ G+F+N=1]

From equation (29), we have

$$H = \frac{\dot{R}}{R}$$

$$\Rightarrow H = \frac{\frac{1}{3}(at+b)^{\frac{1}{3}-1}.a}{R}$$

$$\Rightarrow H = \frac{a(at+b)^{\frac{1}{3}}.(at+b)^{-1}}{3R}$$

$$\Rightarrow H = \frac{aR}{3R(at+b)}$$

$$\therefore H = \frac{a}{3(at+b)} \text{ ....(32)}$$

$$q = -\frac{\ddot{R}}{RH^2}$$

$$\Rightarrow q = \frac{\frac{2a^2}{9}(at+b)^{-\frac{5}{3}}}{(at+b)^{1/3}.\frac{a^2}{9(at+b)^2}}$$

$$\Rightarrow q = \frac{\frac{2a^2}{9}(at+b)^{-\frac{5}{3}}}{\frac{a^2}{9}(at+b)^{-\frac{5}{3}}}$$

$\therefore$ q = 2 .... (33)

From equation (10), we have

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -p + \zeta\theta + \Lambda(t) \qquad \text{[Using equation (3)]}$$

$$\Rightarrow \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\rho\omega + \zeta\theta + \Lambda(t) ----(34) \text{ [Using equation (16)]}$$

Equation (13) − (34) $\Rightarrow$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{C}\dot{A}}{CA} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = \rho(1+\omega) - \zeta\theta$$

$$\Rightarrow \rho(1+\omega) = \left(\frac{Ga}{at+b}\right)\left(\frac{Fa}{at+b}\right) + \left(\frac{Na}{at+b}\right)\left(\frac{Ga}{at+b}\right) - \frac{F(F-1)a^2}{(at+b)^2}$$

$$-\frac{N(N-1)a^2}{(at+b)^2} + \frac{a\zeta}{at+b}$$

$$\Rightarrow \rho(1+\omega) = \frac{a^2}{(at+b)^2}(GF+NG-F^2+F-N^2+N) + \frac{a\zeta}{at+b}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}(GF+NG+F+N-F^2-N^2) + \frac{a\zeta}{(at+b)(1+\omega)}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}\left\{G(F+N)+1(F+N)-F^2-N^2\right\} + \frac{a\zeta}{(at+b)(1+\omega)}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}\left\{(1+G)(F+N)-F^2-N^2\right\} + \frac{a\zeta}{(at+b)(1+\omega)}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}\left\{(1+G)(1-G)-F^2-N^2\right\} + \frac{a\zeta}{(at+b)(1+\omega)}$$

[$\therefore$ G + F + N = 1]

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}(1-G^2-F^2-N^2) + \frac{a\zeta}{(at+b)(1+\omega)}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2(1+\omega)}\{(1-(G^2+F^2+N^2)\} + \frac{a\zeta}{(at+b)(1+\omega)}$$

$$\therefore \rho = \frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right) + \frac{a\zeta}{(at+b)(1+\omega)} \qquad \text{..........................(35)}$$

where $G^2+F^2+N^2 = \frac{n^2l^2}{a^2} + \frac{(l+k)^2}{4a^2} + \frac{(l-k)^2}{4a^2}$

$$= \frac{4n^2l^2+l^2+2lk+k^2+l^2-2lk+k^2}{4a^2}$$

$$= \frac{4n^2l^2+2l^2+2k^2}{4a^2}$$

$$= \frac{2n^2l^2+l^2+k^2}{2a^2}$$

From equation (16), we have

$$\therefore p = \frac{a^2\omega}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)+\frac{a\zeta\omega}{(at+b)(1+\omega)} \qquad \text{[Using equation (35)]}$$

$$\Rightarrow p = \frac{\omega}{1+\omega}\left\{\frac{a^2}{(at+b)^2}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)+\frac{a\zeta}{at+b}\right\} \text{ .... (36)}$$

From equation (13), we have

$$\frac{GFa^2}{(at+b)^2}+\frac{FNa^2}{(at+b)^2}+\frac{NGa^2}{(at+b)^2}=\frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)+\frac{a\zeta}{(at+b)(1+\omega)}+\Lambda$$

[Using equation (35)]

$$\frac{a^2}{(at+b)^2}(GF+FN+NG)=\frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)+\frac{a\zeta}{(at+b)(1+\omega)}+\Lambda$$

$$\frac{a^2}{(at+b)^2}\{G(F+N)+FN\}=\frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)+\frac{a\zeta}{(at+b)(1+\omega)}+\Lambda$$

$$\Rightarrow \Lambda = \frac{a^2}{(at+b)^2}\left\{\frac{nl}{a}\left(\frac{l+k}{2a}+\frac{l-k}{2a}\right)+\left(\frac{l+k}{2a}\right)\left(\frac{l-k}{2a}\right)\right\}$$

$$-\frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)-\frac{a\zeta}{(at+b)(1+\omega)}$$

$$\because \Lambda = \frac{a^2}{(at+b)^2}\left(\frac{nl^2}{a^2}+\frac{l^2-k^2}{4a^2}\right)-\frac{a^2}{(at+b)^2(1+\omega)}\left(1-\frac{2n^2l^2+l^2+k^2}{2a^2}\right)$$

$$-\frac{a\zeta}{(at+b)(1+\omega)} \text{ ....(37)}$$

The special volume V is given by

$$V = (ABC)^{1/3}$$

$$\Rightarrow \quad V = R \quad [\because R = (ABC)^{1/3}]$$

$$\therefore \quad V = (at+b)^{1/3} \qquad \text{[Using equation (31)]}$$

The shear tensor σ_i^j is given by

$$\sigma^2 = \frac{1}{2}\{(\sigma_1^1)^2+(\sigma_2^2)^2+(\sigma_3^3)^2\} \text{..(38)}$$

$$\sigma_1^1 = \frac{1}{3}\left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) = \frac{\dot{A}}{A} - \frac{\dot{R}}{R} \quad \left[\because \frac{\dot{R}}{R} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right]$$

$$= \frac{a}{3(at+b)}(2G - F - N)$$

$$= \frac{a}{3(at+b)}(2G + G - G - F - N)$$

$$= \frac{a}{3(at+b)}\{(3G - (G + F + N)\}$$

$$= \frac{a}{3(at+b)}(3G - 1)$$

$$= \frac{a}{3(at+b)}(\frac{3nl}{a} - 1)$$

and $\sigma_2^2 = \frac{1}{3}\left(\frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right) = \frac{\dot{B}}{B} - \frac{\dot{R}}{R}$

$$= \frac{a}{3(at+b)}(2F - G - N)$$

$$= \frac{a}{3(at+b)}(2F + F - F - G - N)$$

$$= \frac{a}{3(at+b)}\{(3F - (G + F + N)\}$$

$$= \frac{a}{3(at+b)}(3F - 1)$$

$$= \frac{a}{3(at+b)} \; \{3(\frac{l+k}{2a}) - 1\}$$

and $\sigma_3^3 = \frac{1}{3}\left(\frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = \frac{\dot{C}}{C} - \frac{\dot{R}}{R}$

$$= \frac{a}{3(at+b)}(2N - G - F)$$

$$= \frac{a}{3(at+b)}\{2N + N - (G + F + N)\}$$

$$= \frac{a}{3(at+b)}(3N - 1)$$

$$= \frac{a}{3(at+b)}\ \{\frac{3}{2a}(l-k)-1\}$$

and $\sigma_4^4 = 0$

From equation (38), we have

$$\sigma^2 = \frac{1}{2}\left\{(\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2\right\}$$

$$= \frac{1}{2}.\frac{a^2}{9(at+b)^2}\left\{(2G-F-N)^2 + (2F-G-N)^2 + (2N-G-F)^2\right\}$$

$$= \frac{a^2}{18(at+b)^2}\left\{(3G-1)^2 + (3F-1)^2 + (3N-1)^2\right\}\ [\because G+F+N=1]$$

$$= \frac{a^2}{18(at+b)^2}(9G^2-6G+1+9F^2-6F+1+9N^2-6N+1)$$

$$= \frac{a^2}{18(at+b)^2}\{9(G^2+F^2+N^2)-6(G+F+N)+3\}$$

$$= \frac{a^2}{18(at+b)^2}\left\{\frac{9(2n^2l^2+l^2+k^2)}{2a^2}-6+3\right\}$$

$$= \frac{a^2}{18(at+b)^2}\left\{\frac{9(2n^2l^2+l^2+k^2)}{2a^2}-3\right\}$$

$$= \frac{9a^2}{18(at+b)^2}\left(\frac{2n^2l^2+l^2+k^2}{2a^2}-\frac{3}{9}\right)$$

$$\therefore \sigma^2 = \frac{a^2}{2(at+b)^2}\left(\frac{2n^2l^2+l^2+k^2}{2a^2}-\frac{1}{3}\right)$$

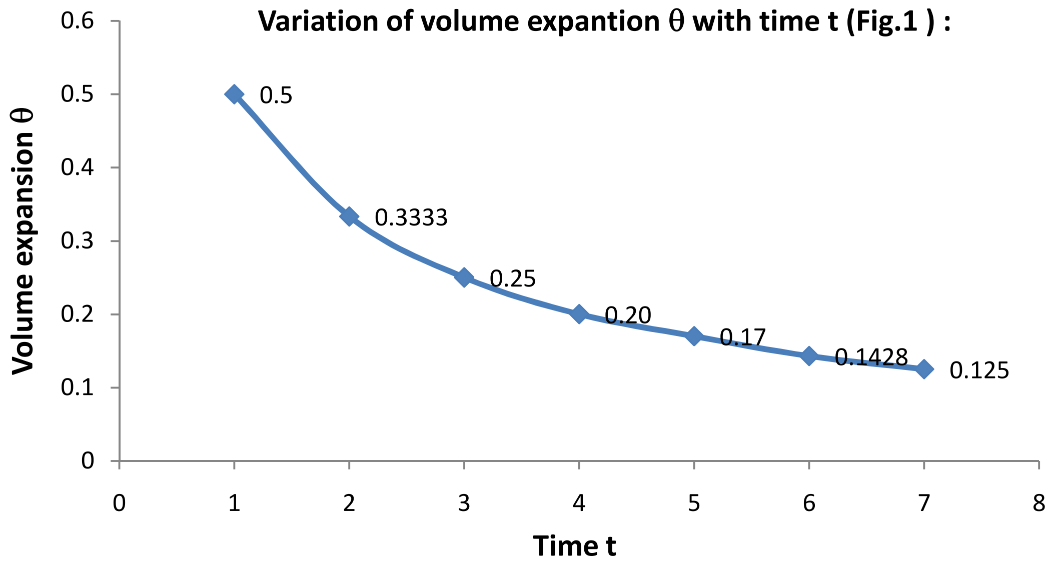


Fig. 1

Variation of Hubble's parameter H with time t (Fig. 2):

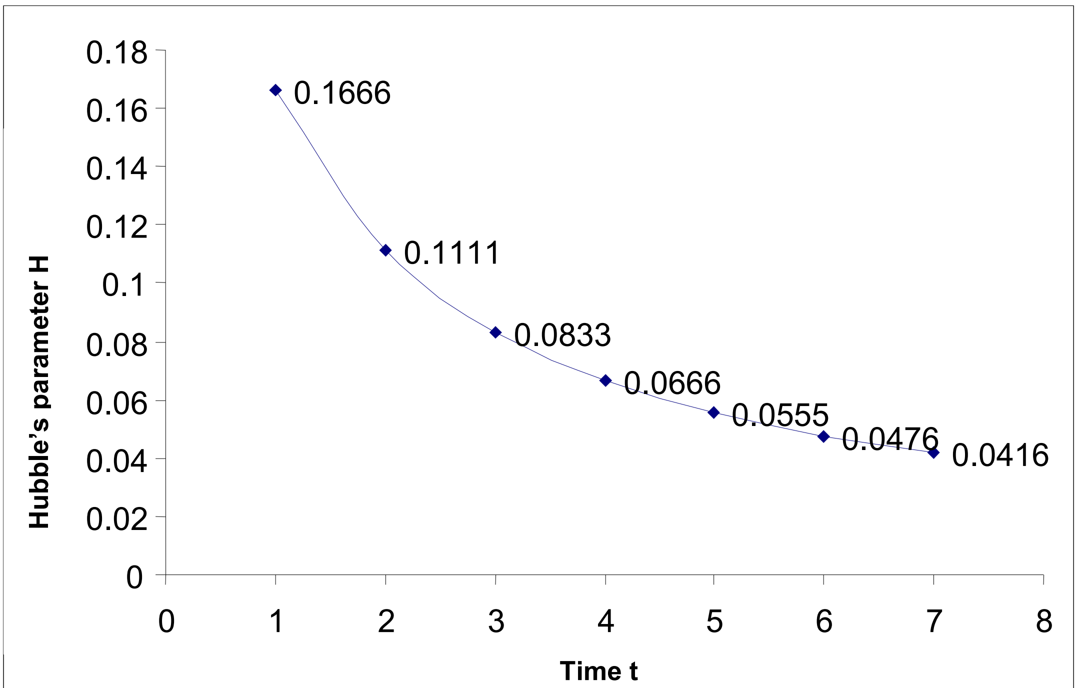


Fig. 2

Variation of pressure p with time t (Fig. 3) :

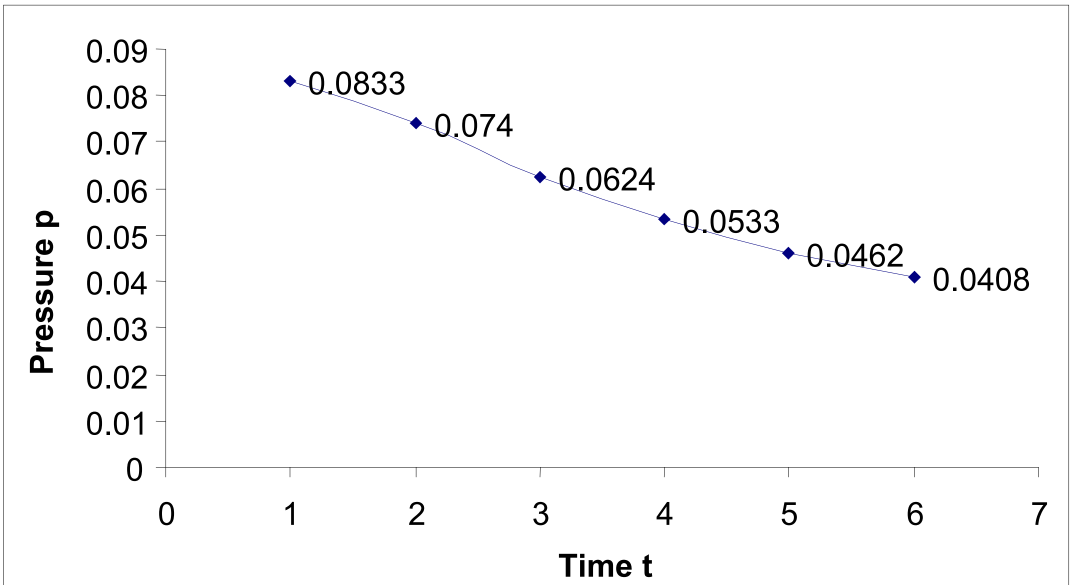


Fig.3

Variation of matter energy density ρ with time t (Fig. 4) :

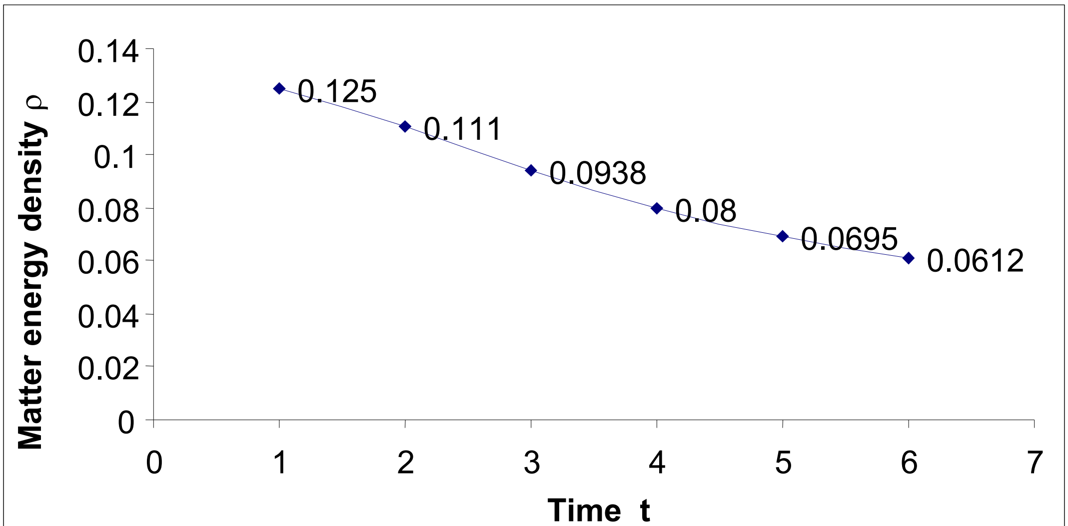


Fig. 4

Variation of cosmological parameter Λ with time t (Fig. 5) :

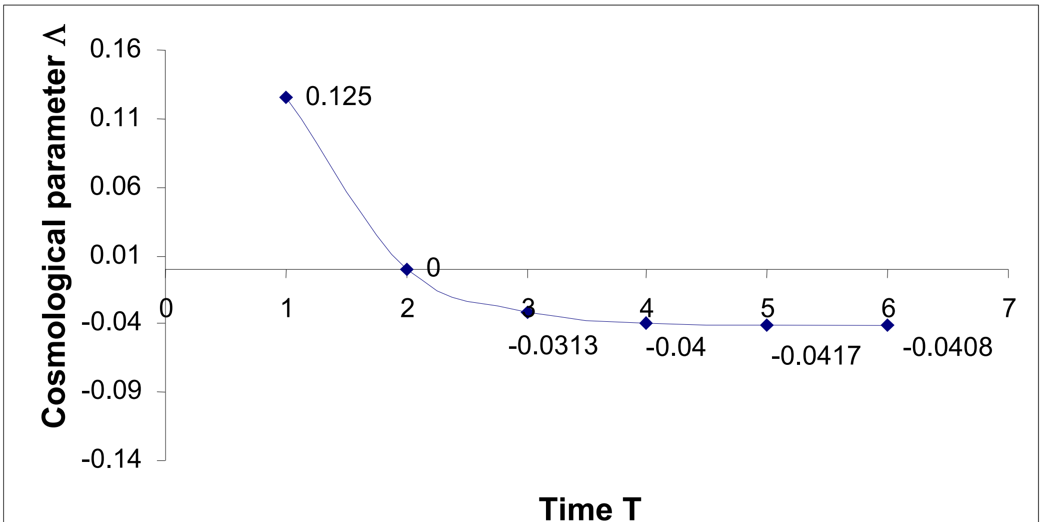


Fig. 5

Conclusion:

In this section 8.1, we have observed that the volume expansion θ, the Hubble's parameter H, the pressure p, the matter energy density ρ and the cosmological parameter Λ are decreasing with increasing of time which have shown in the figures 1, 2, 3, 4, 5 respectively in presence of huge viscosity of the universe. Here, we also have observed that the deceleration parameter q=2 and the square of the shear tensor σ^2 is decreasing with increasing of time having condition $2n^2l^2 + l^2 + k^2 > \frac{2a^2}{3}$ on evolution of the universe.

8.2 Bianchi Type-1 Cosmological Model for Fluid Distribution and Expanding Universe

Introduction:

The Friedmann-Robertson-Walker models are unstable near the singularity (Patridge and Wilkinson [15]) and fail to describe the early universe. The homogenous and anisotropic Bianchi Type-1 space time is undertaken to study the universe in its early stages on evolution of the universe. Land and Magueijo [16] have described the existence of anisotropic universe that approaches the isotropic phase. The large scale distribution of galaxies of the universe shows that the matter distribution is satisfactorily described by perfect fluid.

In this section 8.2, we have observed that the volume expansion θ, the Hubble's parameter H, the pressure p, the deceleration parameter q and the matter energy density ρ on evolution of the universe at large time.

The Methodology:

The Metric and Field Equations:

We consider Bianchi Type-1 metric in the form

$ds^2 = -dt^2+A^2(t)dx^2+B^2(t)dy^2+C^2(t)dz^2$ .... (1)

The matter content is taken as huge fluid given by energy-mementum tensor

$T_i^{\,j} = (\rho + p)v_i v^j + p g_i^{\,j}$ ....(2)

and $v_i v^i = -1$ .... (3)

The Einstein's field equations (in geometrized unit $8\pi G = c = 1$) with time are given by

$R_i^{\,j} - \frac{1}{2} R g_i^{\,j} = - T_i^j$ ....(4)

where $v_i = (0, 0, 0, -1)$, P is the isotropic pressure and ρ is the energy density.

From equation (1), we have

$$g_{ij}=\begin{bmatrix} A^2(t) & 0 & 0 & 0 \\ 0 & B^2(t) & 0 & 0 \\ 0 & 0 & C^2(t) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

So that $g_{11}=A^2(t), g_{22}=B^2(t), g_{33}=C^2(t), g_{44}=-1$

and the other terms are zero.

The inverse of g_{ij} is

$$g^{ij}=\begin{bmatrix} \frac{1}{A^2(t)} & 0 & 0 & 0 \\ 0 & \frac{1}{B^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{C^2(t)} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} g^{11} & g^{12} & g^{13} & g^{14} \\ g^{21} & g^{22} & g^{23} & g^{24} \\ g^{31} & g^{32} & g^{33} & g^{34} \\ g^{41} & g^{42} & g^{43} & g^{44} \end{bmatrix}$$

So that $g^{11}=\frac{1}{A^2(t)}, g^{22}=\frac{1}{B^2(t)}, g^{33}=\frac{1}{C^2(t)}, g^{44}=-1$

and the other terms are zero.

We know the Christoffel symbol of second kind is

$$\Gamma^{\lambda}_{ij}=\frac{1}{2}g^{\lambda\sigma}\left(g_{\sigma i,j}+g_{\sigma j,i}-g_{ij,\sigma}\right)$$

The non-zero component of Christoffel symbol are

$$\Gamma^{1}_{14}=\Gamma^{1}_{41}=\frac{\dot{A}}{A},\ \Gamma^{2}_{24}=\Gamma^{2}_{42}=\frac{\dot{B}}{B},\ \Gamma^{3}_{34}=\Gamma^{3}_{43}=\frac{\dot{C}}{C},\ \Gamma^{4}_{11}=A\dot{A},\ \Gamma^{4}_{22}=B\dot{B},\ \Gamma^{4}_{33}=C\dot{C}$$

Now, $R_{\mu\upsilon} = \Gamma^{\sigma}_{\mu\sigma,\upsilon} - \Gamma^{\sigma}_{\mu\upsilon,\sigma} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\upsilon} - \Gamma^{\rho}_{\mu\upsilon}\Gamma^{\sigma}_{\rho\sigma}$

$$R_{11} = \Gamma^{\sigma}_{1\sigma,1} - \Gamma^{\sigma}_{11,\sigma} + \Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1} - \Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma}$$

$$\Gamma^{\sigma}_{1\sigma,1} = \Gamma^{1}_{11,1} + \Gamma^{2}_{12,1} + \Gamma^{3}_{13,1} + \Gamma^{4}_{14,1}$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

$$\Gamma^{\sigma}_{11,\sigma} = \Gamma^{1}_{11,1} + \Gamma^{2}_{11,2} + \Gamma^{3}_{11,3} + \Gamma^{4}_{11,4}$$

$$= 0 + 0 + 0 + \frac{\partial}{\partial t}(A\dot{A})$$

$$= A\ddot{A} + \dot{A}\dot{A}$$

$$= A\ddot{A} + \dot{A}^2$$

$$\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1} = \Gamma^{1}_{1\rho}\Gamma^{\sigma}_{11} + \Gamma^{2}_{1\sigma}\Gamma^{\sigma}_{21} + \Gamma^{3}_{1\sigma}\Gamma^{\sigma}_{31} + \Gamma^{4}_{1\sigma}\Gamma^{\sigma}_{41}$$

$$= \Gamma^{1}_{14}\Gamma^{4}_{11} + 0 + 0 + \Gamma^{4}_{11}\Gamma^{1}_{41}$$

$$= \frac{\dot{A}}{A}.A\dot{A} + A\dot{A}.\frac{\dot{A}}{A}$$

$$= \dot{A}^2 + \dot{A}^2$$

$$= 2\dot{A}^2$$

$$\Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma} = \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma} + \Gamma^{2}_{11}\Gamma^{\sigma}_{2\sigma} + \Gamma^{3}_{11}\Gamma^{\sigma}_{3\sigma} + \Gamma^{4}_{11}\Gamma^{\sigma}_{4\sigma}$$

$$= 0 + 0 + 0 + \Gamma^{4}_{11}(\Gamma^{1}_{41} + \Gamma^{2}_{42} + \Gamma^{3}_{43} + \Gamma^{4}_{44})$$

$$= A\dot{A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + 0\right)$$

$$= \dot{A}^2 + \frac{A\dot{A}\dot{B}}{B} + \frac{A\dot{A}\dot{C}}{C}$$

$$\therefore R_{11} = 0 - A.\ddot{A} - \dot{A}^2 + 2\dot{A}^2 - \dot{A}^2 - \frac{A\dot{A}\dot{B}}{B} - \frac{A\dot{A}\dot{C}}{C}$$

$$= -A\left(\frac{\dot{A}\dot{B}}{B} + \frac{\dot{A}\dot{C}}{C} + \ddot{A}\right)$$

Similarly,

$$R_{22} = -B\left(\frac{\dot{A}\dot{B}}{A} + \frac{\dot{B}\dot{C}}{C} + \ddot{B}\right)$$

$$R_{33} = -C(\frac{\dot{A}\dot{C}}{A} + \frac{\dot{B}\dot{C}}{B} + \ddot{C})$$

$$R_{44} = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}$$

Now, $R_1^1 = g^{11} R_{11}$

$$= -\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} - \frac{\ddot{A}}{A}$$

$$R_2^2 = g^{22} R_{22}$$

$$= -\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{B}}{B}$$

$$R_3^3 = g^{33} R_{33}$$

$$= -\frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{C}}{C}$$

$$R_4^4 = g^{44} R_{44}$$

$$= -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}$$

So that the scalar curvature is

$$R = R_1^1 + R_2^2 + R_3^3 + R_4^4$$

$$= -2(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C})$$

Now, we have

$$R_1^1 - \frac{1}{2} g_1^1 R = \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} \quad \text{..........................(5)}$$

$$R_2^2 - \frac{1}{2} g_2^2 R = \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} \quad \text{..........................(6)}$$

$$R_3^3 - \frac{1}{2} g_3^3 R = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} \quad \text{..............................(7)}$$

$$R_4^4 - \frac{1}{2} g_4^4 R = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \quad \text{........................(8)}$$

Now, the Einstein's field equation (4) can be written as

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=-p \text{..........................(9)} \quad \text{[Using equation (2) and (5)]}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=-p \text{..........................(10)} \quad \text{[Using equation (2) and (6)]}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}=-p \text{..........................(11)} \quad \text{[Using equation (2) and (7)]}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}=\rho \text{..........................(12)} \quad \text{[Using equation (2) and (8)]}$$

Solution of the Field Equations:

The divergence of equation (4) leads to

$$\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\dot{\Lambda}=0 \text{..................(13)}$$

For getting the model of the universe, we have assumed that the eigen value (σ_1^1) of shear tensor (σ_i^j) is proportional to the expansion (θ).

Here, we take A= $(BC)^n$ .... (14)

From equation (9) and (10), we have

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}$$

$$\Rightarrow \frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}=\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right) \text{..(15)}$$

From equation (10) and (11), we have

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}$$

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right) \text{..(16)}$$

Differentiating equation (14) with respect to t, we have

$$\dot{A}=n(BC)^{n-1}(\dot{B}C+B\dot{C}) \text{ .... (17)}$$

$$\ddot{A}=n(n-1)(BC)^{n-2}(\dot{B}C+B\dot{C})(\dot{B}C+B\dot{C})+n(BC)^{n-1}(\ddot{B}C+\dot{B}\dot{C}+\dot{B}\dot{C}+B\ddot{C})$$

$$\Rightarrow \ddot{A} = n(n-1)(BC)^{n-2}(\dot{B}C + B\dot{C})^2 + n(BC)^{n-1}(\ddot{B}C + 2\dot{B}\dot{C} + B\ddot{C}) \text{ .... } (18)$$

From equation (15), we have

$$\frac{\ddot{B}}{B}-n(n-1)\left(\frac{\dot{B}C+B\dot{C}}{BC}\right)^2-\frac{n}{BC}(\ddot{B}C+2\dot{B}\dot{C}+B\ddot{C})=\frac{\dot{C}}{C}\left\{\frac{n}{BC}(\dot{B}C+B\dot{C})-\frac{\dot{B}}{B}\right\}$$

[Using equation (17) and (18)]

$$\frac{\ddot{B}}{B}-n(n-1)\left(\frac{\dot{B}^2}{B^2}+\frac{2\dot{B}\dot{C}}{BC}+\frac{\dot{C}^2}{C^2}\right)^2-n\left(\frac{\ddot{B}}{B}+\frac{2\dot{B}\dot{C}}{BC}+\frac{\ddot{C}}{C}\right)=n\frac{\dot{C}^2}{C^2}+n\frac{\dot{B}\dot{C}}{BC}-\frac{\dot{B}\dot{C}}{BC}$$

$$\Rightarrow (1-n)\frac{\ddot{B}}{B}+n(n-1)\frac{\dot{B}^2}{B^2}-n^2\frac{\dot{C}^2}{C^2}-n\frac{\ddot{C}}{C}-(2n^2+n-1)\frac{\dot{B}\dot{C}}{BC}=0 \quad\text{.................. (19)}$$

From equation (16), we have

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{n}{BC}(\dot{B}C+B\dot{C})(\frac{\dot{B}}{B}-\frac{\dot{C}}{C})$$

[Using equation (17)]

$$\Rightarrow \frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}=\frac{n}{BC}(\dot{B}C+B\dot{C})(\frac{\dot{B}C-B\dot{C}}{BC})$$

$$\Rightarrow \frac{\ddot{C}B-\ddot{B}C}{BC}=\frac{n}{(BC)^2}(\dot{B}C+B\dot{C})(\dot{B}C-B\dot{C})$$

$$\Rightarrow \frac{\ddot{C}B-\ddot{B}C}{\dot{B}C-B\dot{C}}=\frac{n}{BC}(\dot{B}C+B\dot{C})$$

$$\Rightarrow \frac{-\{-(\ddot{C}B-\ddot{B}C)\}}{\dot{B}C-B\dot{C}}=\frac{n}{BC}(\dot{B}C+B\dot{C})$$

$$\Rightarrow \frac{-(\ddot{C}B-\ddot{B}C)}{\dot{B}C-B\dot{C}}=-n\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)$$

Integrating the above equation, we have

$$Log_e(\dot{B}C-B\dot{C})=-n(\log_e B+\log_e C)+\log_e K$$

$$\Rightarrow \log_e(\dot{B}C-B\dot{C})=\log_e(BC)^{-n}+\log_e k$$

$$\Rightarrow \log_e(\dot{B}C-B\dot{C})=\log_e k(BC)^{-n}$$

$$\Rightarrow \dot{B}C-B\dot{C}=k(BC)^{-n}$$

$$\Rightarrow C^2\frac{\partial}{\partial t}\left(\frac{B}{C}\right)=k(BC)^{-n} \quad\text{..(20)}$$

where k is constant of integration.

Now, we consider

μ =BC.. (21)

and $\upsilon=\frac{B}{C}$ ....(22)

From equation (20), we have

$C^2\dot{\upsilon}$ =k μ $^{-n}$ [Using equation (21) and (22)]

$$\Rightarrow \dot{\upsilon}=\frac{k\mu^{-n}}{C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon}=\frac{k\mu^{-n}}{\upsilon C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon}=\frac{k\mu^{-n}}{\frac{B}{C}.C^2}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon}=\frac{k\mu^{-n}}{BC}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon}=\frac{k\mu^{-n}}{\mu}[\because BC=\mu]$$

$$\therefore \frac{\dot{\upsilon}}{\upsilon}=k\mu^{-(n+1)} \quad (23)$$

Differentiating the above equation with respect to t, we have

$$\frac{1}{\upsilon^2}(\upsilon\ddot{\upsilon}-\dot{\upsilon}\dot{\upsilon})=-k(n+1)\mu^{-(n+2)}\dot{\mu}$$

$$\Rightarrow \frac{\ddot{\upsilon}}{\upsilon}=-k(n+1)\mu^{-(n+2)}\dot{\mu}+\left(\frac{\dot{\upsilon}}{\upsilon}\right)^2$$

$$\Rightarrow \frac{\ddot{\upsilon}}{\upsilon}=-k(n+1)\mu^{-(n+2)}\dot{\mu}+k^2\mu^{-2(n+1)} \quad (24)$$ [Using equation (23)]

From equation (17), we have

$$\dot{A}=n(BC)^n(BC)^{-1}(\dot{B}C+B\dot{C})$$

$$\Rightarrow \dot{A}=nA\left(\frac{\dot{B}C+B\dot{C}}{BC}\right)$$

$$\therefore \frac{\dot{A}}{A}=n\frac{\dot{\mu}}{\mu}$$

From equation (19), we have

$$(1-2n)\frac{\ddot{\mu}}{\mu}+n(1-2n)\frac{\dot{\mu}^2}{\mu^2}+\frac{\ddot{\upsilon}}{\upsilon}-\frac{\dot{\upsilon}^2}{\upsilon^2}+(1+n)\frac{\dot{\mu}\dot{\upsilon}}{\mu\upsilon}=0 \quad \text{.............................(25)}$$

From equation (25), we have

$$(1-2n)\frac{\ddot{\mu}}{\mu}+n(1-2n)\frac{\dot{\mu}^2}{\mu^2}-\frac{k(n+1)\dot{\mu}}{\mu^{n+2}}+\frac{k^2}{\mu^{2(n+1)}}-\frac{k^2}{\mu^{2(n+1)}}+\frac{k(n+1)}{\mu^{n+2}}\dot{\mu}=0$$

[Using equation (23) and (24)]

$$\Rightarrow (1-2n)\frac{\ddot{\mu}}{\mu}+n(1-2n)\frac{\dot{\mu}^2}{\mu^2}=0$$

$$\Rightarrow (1-2n)\left(\frac{\ddot{\mu}}{\mu}+n\frac{\dot{\mu}^2}{\mu^2}\right)=0$$

$$\Rightarrow \frac{1}{\mu}\left(\ddot{\mu}+n\frac{\dot{\mu}^2}{\mu}\right)=0$$

$$\Rightarrow \ddot{\mu}+n\frac{\dot{\mu}^2}{\mu}=0$$

$$\Rightarrow \frac{\ddot{\mu}}{\dot{\mu}}+n\frac{\dot{\mu}^2}{\dot{\mu}\mu}=0$$

$$\Rightarrow \frac{\ddot{\mu}}{\dot{\mu}}+n\frac{\dot{\mu}}{\mu}=0$$

Integrating the above equation, we have

$$\log_e \dot{\mu}+n\log_e \mu=\log_e l$$

$$\Rightarrow \log_e(\mu^n\dot{\mu})=\log_e l$$

$$\Rightarrow \mu^n\dot{\mu}=l$$

$$\Rightarrow l=\mu^n\frac{d\mu}{dt}$$

$$\Rightarrow \mu^n d\mu=l dt$$

Again integrating the above equation, we have

$$\frac{\mu^{n+1}}{n+1}=lt+b_1$$

$$\Rightarrow \mu^{n+1}=l(n+1)t+b_1(n+1)$$

$\Rightarrow \mu^{n+1}=at+b$ where $a=l\,(\text{n}+1)$, b= b_1(n+1) and a, b are constants.

$$\Rightarrow \left(\mu^{n+1}\right)^{\frac{1}{n+1}} = \left(at+b\right)^{\frac{1}{n+1}}$$

$$\Rightarrow \mu = (at+b)^{\frac{l}{a}} \quad [\because a = l(n+1)]$$

From equation (23), we have

$$\frac{\dot{\upsilon}}{\upsilon} = \frac{k}{\mu^{n+1}}$$

$$\Rightarrow \frac{\dot{\upsilon}}{\upsilon} = \frac{k}{at+b}\left[\because \mu^{n+1} = at+b\right]$$

Integrating the above equation, we have

$$\log_e \upsilon = \frac{k}{a}\log_e(at+b) + \log_e L$$

$$\Rightarrow \log_e \upsilon = \log_e (at+b)^{\frac{k}{a}} + \log_e L$$

$$\Rightarrow \log_e \upsilon = \log_e L(at+b)^{\frac{k}{a}}$$

$$\therefore \upsilon = L(at+b)^{\frac{k}{a}}$$

where L is constant of integration.

From equation (21) and (22), we have

$$\mu\upsilon = BC.\frac{B}{C}$$

$$\Rightarrow B^2 = \mu\upsilon$$

$$\Rightarrow B^2 = (at+b)^{\frac{l}{a}}\, L(at+b)^{\frac{k}{a}}$$

$$\Rightarrow B^2 = L(at+b)^{\frac{l}{a}+\frac{k}{a}}$$

$$\Rightarrow B^2 = L(at+b)^{\frac{l+k}{a}}$$

$$\Rightarrow B = \sqrt{L}(at+b)^{\frac{l+k}{2a}}$$

$$\therefore B = \sqrt{L}(at+b)^{F}$$

where $F = \dfrac{l+k}{2a}$ is constant .

Again from equation (21) and (22), we have

$$\frac{\mu}{\upsilon}=\frac{BC}{\frac{B}{C}}$$

$$\Rightarrow C^2=\frac{\mu}{\upsilon}$$

$$\Rightarrow C^2=\frac{(at+b)^{\frac{l}{a}}}{L(at+b)^{\frac{k}{a}}}$$

$$\Rightarrow C^2=\frac{1}{L}(at+b)^{\frac{l}{a}-\frac{k}{a}}$$

$$\Rightarrow C^2=\frac{1}{L}(at+b)^{\frac{l-k}{a}}$$

$$\Rightarrow C=\frac{1}{\sqrt{L}}(at+b)^{\frac{l-k}{2a}}$$

$\therefore C=\frac{1}{\sqrt{L}}(at+b)^N$ where $N=\frac{l-k}{2a}$ is constant.

From equation (15), we have

$A=(BC)^n$

$$=\left\{\sqrt{L}(at+b)^F.\frac{1}{\sqrt{L}}(at+b)^N\right\}^n$$

$$=(at+b)^{(F+N)n}$$

$$=(at+b)^{\frac{nl}{a}} \qquad \left[\because F+N=\frac{l+k}{2a}+\frac{l-k}{2a}=\frac{2l}{2a}=\frac{l}{a}\right]$$

$=(at+b)^G$ where $G=\frac{nl}{a}$ is constant.

Now, putting the value of A,B,C in equation (1),

we have

$$ds^2=-dt^2+(at+b)^{2G}dx^2+L(at+b)^{2F}dy^2+\frac{1}{L}(at+b)^{2N}dz^2 \text{ ------------- } (26)$$

where $G=\frac{nl}{a},\ F=\frac{l+k}{2a},\ N=\frac{l-k}{2a}$

Some Physical and Geometrical Properties:

The volume expansion (θ), scale factor (R), Hubble parameter (H), deceleration parameter (q) are given by

$$\theta = v^i_{;i} = 3\frac{\dot{R}}{R} = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \quad \text{.................................} \quad (27)$$

Hence $\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}$

$$\Rightarrow \quad \theta = \frac{Ga}{at+b} + \frac{Fa}{at+b} + \frac{Na}{at+b}$$

$$\Rightarrow \quad \theta = \left(\frac{a}{at+b}\right)(G+F+N)$$

where $G + F + N = \frac{a-l}{a} + \frac{l+k}{2a} + \frac{l-k}{2a}$

$$= \frac{2a - 2l + l + k + l - k}{2a}$$

$$= \frac{2a}{2a}$$

$$= 1$$

$$\therefore \theta = \frac{a}{at+b} \quad \text{..} (28)$$

We define scale factor,

$$R = (ABC)^{1/3}$$

$$\Rightarrow R = \left\{(at+b)^G \sqrt{L}(at+b)^F . \frac{1}{\sqrt{L}}(at+b)^N\right\}^{1/3}$$

$$\Rightarrow R = \left\{(at+b)^{(G+F+N)}\right\}^{1/3}$$

$$\therefore R = (at+b)^{1/3} \quad \text{...} (29) \quad [\therefore G+F+N=1]$$

From equation (27), we have

$$H = \frac{\dot{R}}{R}$$

$$\Rightarrow H = \frac{\frac{1}{3}(at+b)^{\frac{1}{3}-1}.a}{R}$$

$$\Rightarrow H = \frac{a(at+b)^{\frac{1}{3}}.(at+b)^{-1}}{3R}$$

$$\Rightarrow H = \frac{aR}{3R(at+b)}$$

$$\therefore H = \frac{a}{3(at+b)} \text{ ....(30)}$$

$$q = -\frac{\ddot{R}}{RH^2}$$

$$\Rightarrow q = \frac{\frac{2a^2}{9}(at+b)^{-\frac{5}{3}}}{(at+b)^{1/3} \cdot \frac{a^2}{9(at+b)^2}}$$

$$\Rightarrow q = \frac{\frac{2a^2}{9}(at+b)^{-\frac{5}{3}}}{\frac{a^2}{9}(at+b)^{-\frac{5}{3}}}$$

$$\therefore q = 2 \text{ .... (31)}$$

From equation (9), we have

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -p$$

$$\frac{F(F-1)a^2}{(at+b)^2} + \frac{N(N-1)a^2}{(at+b)^2} + (\frac{Fa}{at+b})(\frac{Na}{at+b}) = -p$$

$$\Rightarrow -p = \frac{F(F-1)a^2}{(at+b)^2} + \frac{N(N-1)a^2}{(at+b)^2} + (\frac{Fa}{at+b})(\frac{Na}{at+b})$$

$$\Rightarrow -p = \frac{a^2}{(at+b)^2}\left(FN + F^2 - F + N^2 - N\right)$$

$$\Rightarrow -p = \frac{a^2}{(at+b)^2}\left\{\left(\frac{l+k}{2a}\right)\left(\frac{l-k}{2a}\right) + \left(\frac{l+k}{2a}\right)^2 - \frac{l+k}{2a} + \left(\frac{l-k}{2a}\right)^2 - \left(\frac{l-k}{2a}\right)\right\}$$

$$\Rightarrow -p = \frac{a^2}{(at+b)^2}\left\{\frac{l^2-k^2}{4a^2} + \frac{(l+k)^2}{4a^2} + \frac{(l-k)^2}{4a^2} - \frac{l+k}{2a} - \left(\frac{l-k}{2a}\right)\right\}$$

$$\Rightarrow -p = \frac{a^2}{(at+b)^2}\left(\frac{l^2-k^2+l^2+2lk+k^2+l^2-2lk+k^2}{4a^2} - \frac{2l}{2a}\right)$$

$$\Rightarrow -p = \frac{a^2}{(at+b)^2}\left(\frac{3l^2+k^2}{4a^2} - \frac{l}{a}\right)$$

$$\Rightarrow -p = -\frac{a^2}{(at+b)^2}\left(\frac{l}{a} - \frac{3l^2+k^2}{4a^2}\right)$$

$$\therefore p = \frac{a^2}{(at+b)^2}\left(\frac{l}{a} - \frac{3l^2+k^2}{4a^2}\right)$$

From equation (12), we have

$$\rho = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA}$$

$$\Rightarrow \rho = \left(\frac{Ga}{at+b}\right)\left(\frac{Fa}{at+b}\right) + \left(\frac{Fa}{at+b}\right)\left(\frac{Na}{at+b}\right) + \left(\frac{Na}{at+b}\right)\left(\frac{Ga}{at+b}\right)$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2}(GF + FN + NG)$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2}\left\{\frac{nl}{a}\left(\frac{l+k}{2a}\right) + \left(\frac{l+k}{2a}\right)\left(\frac{l-k}{2a}\right) + \left(\frac{l-k}{2a}\right)\frac{nl}{a}\right\}$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2}\left(\frac{2nl^2 + 2nkl + l^2 - k^2 + 2nl^2 - 2nkl}{4a^2}\right)$$

$$\Rightarrow \rho = \frac{a^2}{(at+b)^2}\left(\frac{4nl^2 + l^2 - k^2}{4a^2}\right)$$

$$\because \rho = \frac{1}{4(at+b)^2}\left(4nl^2 + l^2 - k^2\right)$$

The special volume V is given by

$$V = (ABC)^{1/3}$$

$$\Rightarrow V = R \quad [\because R = (ABC)^{1/3}]$$

$$\therefore V = (at+b)^{1/3} \quad \text{[Using equation (29)]}$$

The shear tensor σ_i^j is given by $\sigma^2 = \frac{1}{2}\{(\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2\}$...............(38)

$$\sigma_1^1 = \frac{1}{3}\left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) = \frac{\dot{A}}{A} - \frac{\dot{R}}{R}\left[\therefore \frac{\dot{R}}{R} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right]$$

$$= \frac{a}{3(at+b)}(2G - F - N)$$

$$=\frac{a}{3(at+b)}(2G+G-G-F-N)$$

$$=\frac{a}{3(at+b)}\{(3G-(G+F+N)\}$$

$$=\frac{a}{3(at+b)}(3G-1)$$

$$=\frac{a}{3(at+b)}(\frac{3nl}{a}-1)$$

and $\sigma_2^2=\frac{1}{3}\left(\frac{2\dot{B}}{B}-\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)=\frac{\dot{B}}{B}-\frac{\dot{R}}{R}$

$$=\frac{a}{3(at+b)}(2F-G-N)$$

$$=\frac{a}{3(at+b)}(2F+F-F-G-N)$$

$$=\frac{a}{3(at+b)}\{(3F-(G+F+N)\}$$

$$=\frac{a}{3(at+b)}(3F-1)$$

$$=\frac{a}{3(at+b)}\ \{3(\frac{l+k}{2a})-1\}$$

and $\sigma_3^3=\frac{1}{3}\left(\frac{2\dot{C}}{C}-\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=\frac{\dot{C}}{C}-\frac{\dot{R}}{R}$

$$=\frac{a}{3(at+b)}\ (2N-G-F)$$

$$=\frac{a}{3(at+b)}\{2N+N-(G+F+N)\}$$

$$=\frac{a}{3(at+b)}(3N-1)$$

$$=\frac{a}{3(at+b)}\ \{\frac{3}{2a}(l-k)-1\}$$

and $\sigma_4^4=0$

From equation (38), we have

$$\sigma^2 = \frac{1}{2}\left\{\left(\sigma_1^1\right)^2 + \left(\sigma_2^2\right)^2 + \left(\sigma_3^3\right)^2\right\}$$

$$= \frac{1}{2}.\frac{a^2}{9(at+b)^2}\left\{(2G-F-N)^2 + (2F-G-N)^2 + (2N-G-F)^2\right\}$$

$$= \frac{a^2}{18(at+b)^2}\left\{(3G-1)^2 + (3F-1)^2 + (3N-1)^2\right\} \; [\because G+F+N=1]$$

$$= \frac{a^2}{18(at+b)^2}(9G^2-6G+1+9F^2-6F+1+9N^2-6N+1)$$

$$= \frac{a^2}{18(at+b^2)}\{9(G^2+F^2+N^2)-6(G+F+N)+3\}$$

$$= \frac{a^2}{18(at+b)^2}\left\{\frac{9(2n^2l^2+l^2+k^2)}{2a^2} - 6 + 3\right\}$$

$$\text{where } G^2+F^2+N^2 = \left(\frac{nl}{a}\right)^2 + \left(\frac{l+k}{2a}\right)^2 + \left(\frac{l-k}{2a}\right)^2$$

$$= \frac{4n^2l^2+l^2+2lk+k^2+l^2-2lk+k^2}{4a^2}$$

$$= \frac{2n^2l^2+l^2+k^2}{2a^2}$$

$$= \frac{a^2}{18(at+b)^2}\left\{\frac{9(2n^2l^2+l^2+k^2)}{2a^2} - 3\right\}$$

$$= \frac{9a^2}{18(at+b)^2}\left(\frac{2n^2l^2+l^2+k^2}{2a^2} - \frac{3}{9}\right)$$

$$\therefore \sigma^2 = \frac{a^2}{2(at+b)^2}\left(\frac{2n^2l^2+l^2+k^2}{2a^2} - \frac{1}{3}\right)$$

Variation of volume expantion θ with time t (Fig.1) :

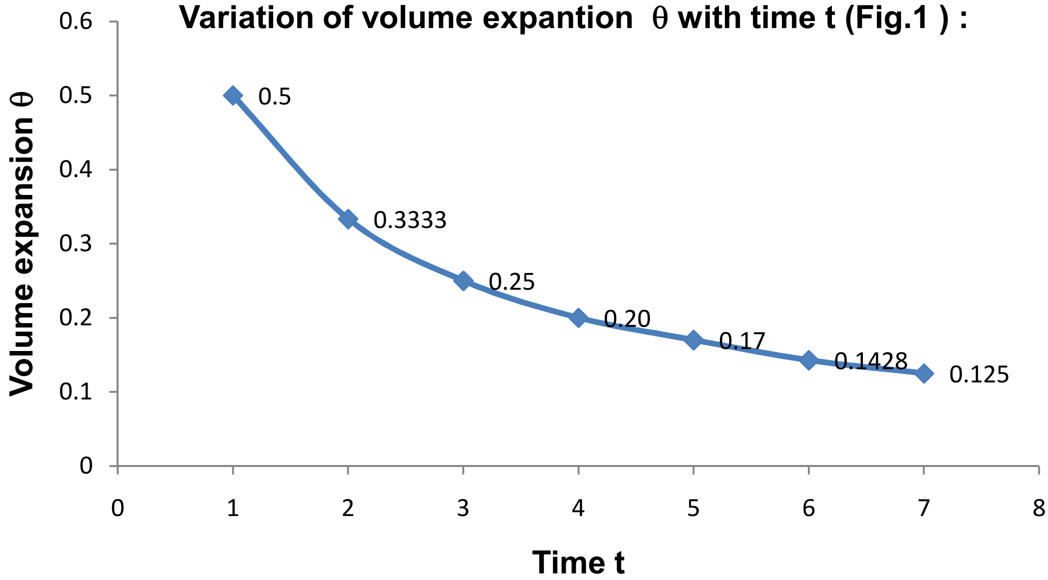


Fig. 1

Variation of Hubble's parameter H with time t (Fig. 2) :

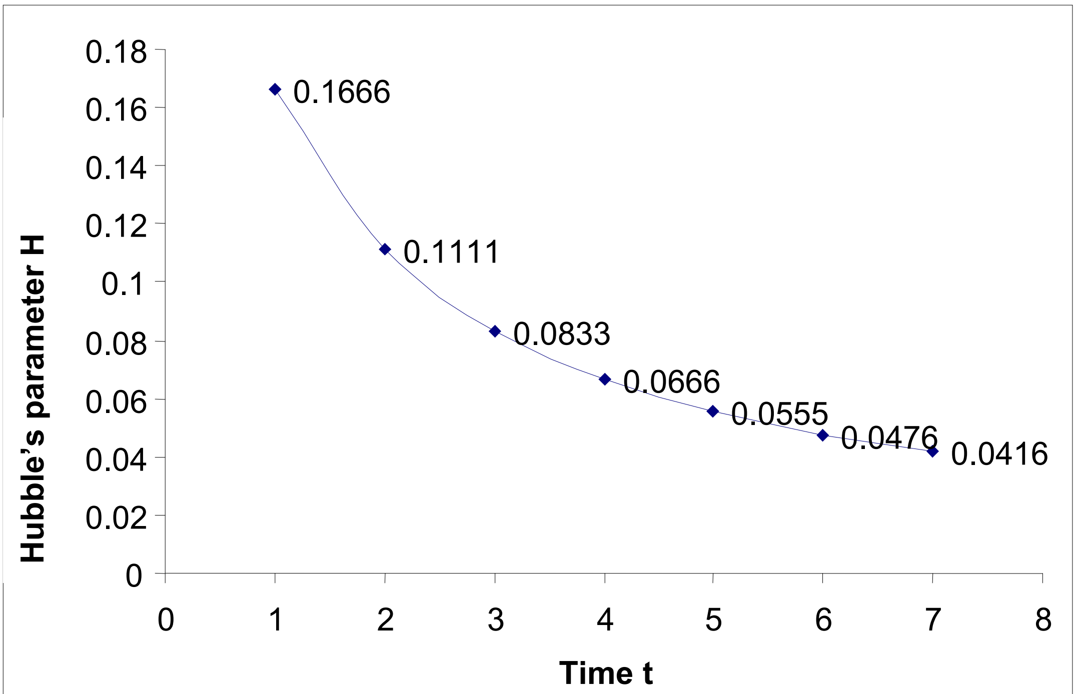


Fig. 2

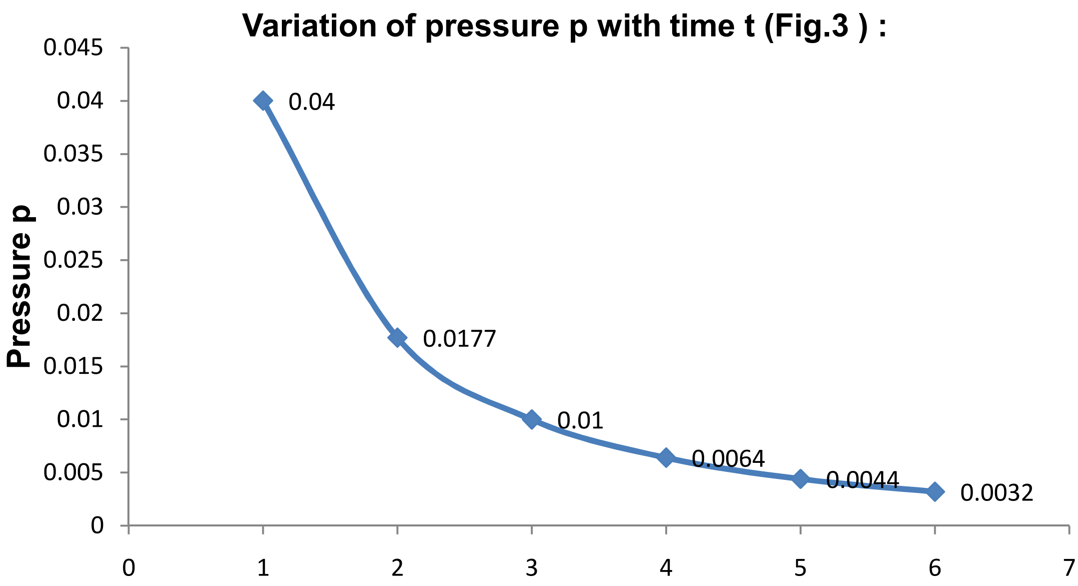


Fig.3

Variation of matter energy density ρ with time t (Fig.4):

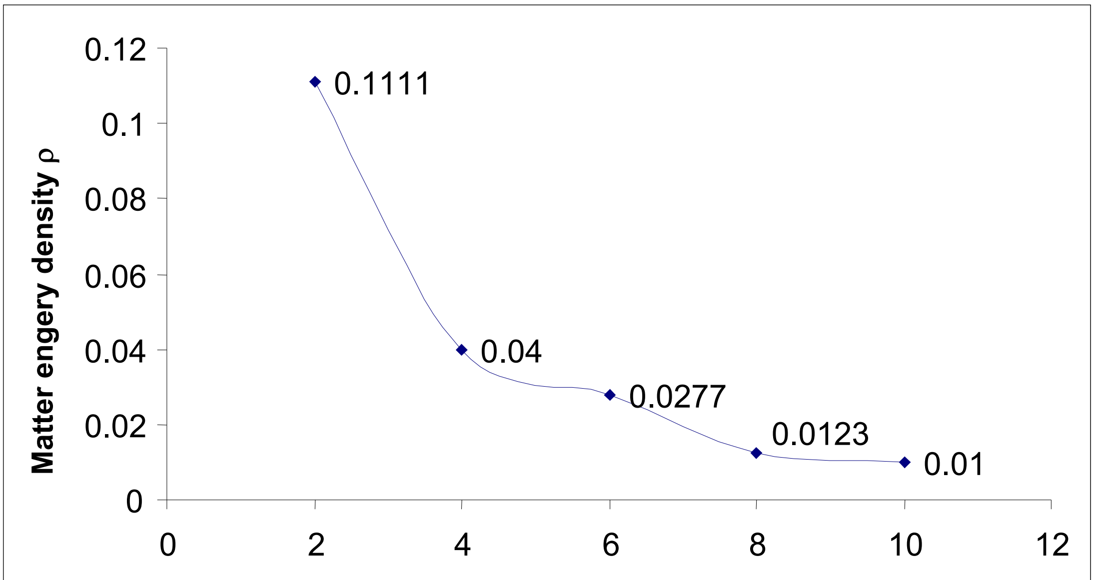


Fig. 4

Conclusion:

In this section 8.2, we have observed that the volume expansion θ, the Hubble's parameter H, the pressure p and the matter energy density ρ are decreasing with increasing of time which have shown in the figures 1, 2, 3, 4 respectively. Here, we also have observed that the deceleration parameter q=2 and the square of the shear tensor σ^2 is decreasing with increasing of time having condition $2n^2l^2 + l^2 + k^2 > \frac{2a^2}{3}$ on evolution of the universe.

8.3Phenomenology and Accelerating Universe with Time Variable Λ

Introduction:

According to the assumption of modern cosmology, the universe is primarily made of dark matter and dark energy. Various observational evidences, including SN 1a [31-34] data, support the idea of accelerating universe and it is supposed that dark energy is responsible for this effect of speeding up. At present it is accepted that about two third of the total energy density of the universe is dark energy and the other one third consists of visible matter and dark matter [35].

Although, the dark matter had a significant role during structure formation in the early universe, its composition is still unknown. It is predicted that the dark matter should be non-baryonic. Moreover, time-varying forms of dark matter [36,37] in Unstable Dark Matter (UDM) scenarios [38,39] are still not fully explored and deserve interest, giving simultaneously clustered and unclustered dark matter components.

The standard cold dark matter (SCDM) model introduced in 1980's which assume Ω_{CDM}=1 is out of favor today [40]. After the emergence of the idea of accelerating universe, the SCDM model is replaced by Λ-CDM (or LCDM) model. This model includes dark energy as a part of the total energy density of the universe and is in nice agreement with various sets of observations [41]. In this connection it is noted here that according to Λ-CDM model, acceleration of the universe should be a recent phenomenon. Some recent works [42] favor the idea that the present accelerating universe was preceded by a decelerating one and observational evidence [43] also support this.

Now, in most of the recent cosmological research, the equation of parameter ω has been taken as a constant. However, its seems that for better result ω should be taken as time-dependent [44]. A kind of this Λ-model was previously studied by Reuter and Wetterich[45] for finding out an explanation of the presently observed small value of Λ.

Therefore, motivated by the time variation of Λ and using the phenomenological model of the kind $\dot{\Lambda} \sim H^3$, the expression for the time-dependent equation of parameter ω and various physical parameters are derived in the present investigation.

In this section 8.3, we have observed that the parameter ω, the decelerating parameter q, the pressure p, the matter energy density ρ and the cosmological parameter Λ on the phenomenological evolution of the universe at large time.

The Methodology:

We consider Einstein's field equations

$$R_{\mu\upsilon} - \frac{1}{2} R\, g_{\mu\upsilon} = -8\pi G\, T_{\mu\upsilon} + \Lambda g_{\mu\upsilon} \qquad (1)$$

We consider (FRW model) the line element is

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right\}$$

In matrix form

$$g_{\mu\upsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{-a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

So that $g_{00} = 1$, $g_{11} = \dfrac{-a^2(t)}{1-kr^2}$, $g_{22} = -a^2(t)r^2$, $g_{33} = -a^2(t)r^2\sin^2\theta$

and the other terms are zero.

Now, the inverse of $g_{\mu\upsilon}$ is

$$g^{\mu\upsilon}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{kr^2-1}{a^2(t)} & 0 & 0\\ 0 & 0 & -\frac{1}{a^2(t)r^2} & 0\\ 0 & 0 & 0 & -\frac{1}{a^2(t)r^2\sin^2\theta}\end{bmatrix}$$

$$=\begin{bmatrix}g^{00} & g^{01} & g^{02} & g^{03}\\ g^{10} & g^{11} & g^{12} & g^{13}\\ g^{20} & g^{21} & g^{22} & g^{23}\\ g^{30} & g^{31} & g^{32} & g^{33}\end{bmatrix}$$

So that $g^{00}=1$, $g^{11}=\frac{kr^2-1}{a^2(t)}$, $g^{22}=-\frac{1}{a^2(t)r^2}$,

$$g^{33}=-\frac{1}{a^2(t)r^2\sin^2\theta}$$

and the other terms are zero.

$$\Gamma^0_{11}=\frac{a\dot{a}}{1-kr^2},\ \Gamma^0_{22}=a\dot{a}r^2,\ \Gamma^0_{33}=a\dot{a}r^2\sin^2\theta,\ \Gamma^0_{01}=\frac{\dot{a}}{a}=\Gamma^0_{10},$$

$$\Gamma^1_{11}=\frac{kr}{1-kr^2},\ \Gamma^1_{22}=-(1-kr^2)r,\ \Gamma^1_{33}=-(1-kr^2)r\sin^2\theta,$$

$$\Gamma^0_{02}=\frac{\dot{a}}{a}=\Gamma^0_{20},\ \Gamma^1_{12}=\frac{1}{r}=\Gamma^1_{21},\ \Gamma^2_{33}=-\sin\theta\cos\theta,$$

$$\Gamma^3_{03}=\frac{\dot{a}}{a}=\Gamma^3_{30},\ \Gamma^3_{13}=\frac{1}{r}=\Gamma^3_{31},\ \Gamma^3_{23}=\cot\theta=\Gamma^3_{32}$$

We know that

$$R_{\mu\upsilon}=\Gamma^{\sigma}_{\mu\sigma,\upsilon}-\Gamma^{\sigma}_{\mu\upsilon,\sigma}+\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\upsilon}-\Gamma^{\rho}_{\mu\upsilon}\Gamma^{\sigma}_{\rho\sigma}$$

which gives

$$R_{00}=\frac{3\dot{a}}{a}$$

$$R_{11}=-\left(\frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}\right)$$

$$\Gamma^{\sigma}_{11,\sigma} = \Gamma^{0}_{11,0} + \Gamma^{1}_{11,1} + \Gamma^{2}_{11,2} + \Gamma^{3}_{11,3}$$

$$R_{33} = -\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)r^2 \sin^2\theta$$

We have $R_{\mu\upsilon} = 0$, $\mu \neq \upsilon$, with $c = 1$

$$u^{\mu} u_{\mu} = 1,\ g^{\mu\upsilon} g_{\mu\upsilon} = 4$$

We know the energy momentum tensor for the perfect fluid is

$$T_{\mu\upsilon} = (\rho + p)u_{\mu}u_{\upsilon} - pg_{\mu\upsilon}$$

$$\Rightarrow \quad T_{\mu\upsilon}g^{\mu\upsilon} = (\rho + p)u_{\mu}u_{\upsilon}g^{\mu\upsilon} - pg_{\mu\upsilon}g^{\mu\upsilon}$$

$$\Rightarrow \quad T = (\rho + p) - 4p$$

$$= \rho - 3p$$

In our moving co-ordinate system

$$u^{\mu} = \delta^{\mu}_{0}$$

So that $u^{\mu} = g_{\mu\upsilon}\delta^{\mu}_{0} = g_{\mu 0} = \delta^{0}_{\mu}$

Here $T_{\mu\upsilon} = (\rho + p)\delta^{0}_{\mu}\ \delta^{0}_{\upsilon} - pg_{\mu\upsilon}$ ....(2)

and $T_{\mu\upsilon} - \frac{1}{2}Tg_{\mu\upsilon} = (\rho + p)\delta^{0}_{\mu}\ \delta^{0}_{\upsilon} - pg_{\mu\upsilon} - \frac{1}{2}\rho g_{\mu\upsilon} + \frac{3}{2}pg_{\mu\upsilon}$

$$= (\rho + p)\delta^{0}_{\mu}\ \delta^{0}_{\upsilon} - \frac{1}{2}\rho g_{\mu\upsilon} + \frac{1}{2}pg_{\mu\upsilon}$$

$$= (\rho + p)\delta^{0}_{\mu}\ \delta^{0}_{\upsilon} - \frac{1}{2}(\rho - p)g_{\mu\upsilon}$$

Therefore, we get from the above equation

$$T_{00} - \frac{1}{2}Tg_{00} = (\rho + p)\delta^{0}_{0}\ \delta^{0}_{0} - \frac{1}{2}(\rho - p)g_{00}$$

$$= \rho + p - \frac{1}{2}(\rho - p) \quad [\because g_{00} = 1]$$

$$= \rho + p - \frac{\rho}{2} + \frac{p}{2}$$

$$= \frac{\rho}{2} + \frac{3p}{2}$$

or, $-\left(\frac{1}{2}Tg_{00} - T_{00}\right) = \frac{1}{2}(\rho + 3p)$

$\therefore \quad \frac{1}{2}Tg_{00} - T_{00} = -\frac{1}{2}(\rho + 3p)$ ....(3)

and $T_{11} - \frac{1}{2}Tg_{11} = (\rho + p)\delta_1^0\delta_1^0 - \frac{1}{2}(\rho - p)g_{11}$

$$= 0 - \frac{1}{2}(\rho - p)\left(\frac{-a^2}{1-kr^2}\right) \quad \left[\because g_{11} = \frac{-a^2}{1-kr^2}\right]$$

or $-\left(\frac{1}{2}Tg_{11} - T_{11}\right) = \frac{1}{2}(\rho - p)\left(\frac{a^2}{1-kr^2}\right)$

$\therefore \quad \frac{1}{2}Tg_{11} - T_{11} = -\frac{1}{2}(\rho - p)\left(\frac{a^2}{1-kr^2}\right)$

We get from Einstein's field equation (1)

$R_{\mu\upsilon} = \frac{1}{2}Rg_{\mu\upsilon} - 8\pi GT_{\mu\upsilon} + \Lambda g_{\mu\upsilon}$

or, $R_{\mu\upsilon} = \frac{1}{2}KTg_{\mu\upsilon} - KT_{\mu\upsilon} + \Lambda g_{\mu\upsilon}$ $[\because R = KT \text{ and } 8\pi G = K]$

or, $R_{\mu\upsilon} = K\left(\frac{1}{2}Tg_{\mu\upsilon} - T_{\mu\upsilon}\right) + \Lambda g_{\mu\upsilon}$

or, $R_{00} = K\left(\frac{1}{2}Tg_{00} - T_{00}\right) + \Lambda g_{00}$

or, $\frac{3\ddot{a}}{a} = -\frac{1}{2}K(\rho + 3p) + \Lambda$ [Using equation (3)]

or, $\ddot{a} = -\frac{1}{6}Ka(\rho + 3p) + \frac{a\Lambda}{3}$ .... (4)

and $R_{11} = K\left(\frac{1}{2}g_{11}T - T_{11}\right) + \Lambda g_{11}$

or, $-\left(\frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}\right)=-\frac{K}{2}(\rho-p)\frac{a^2}{1-kr^2}-\frac{\Lambda a^2}{1-kr^2}$

or, $a\ddot{a}+2\dot{a}^2+2k=\frac{K}{2}(\rho-p)a^2+\Lambda a^2$

or, $-\frac{1}{6}Ka^2(\rho+3p)+\frac{a^2\Lambda}{3}+2\dot{a}^2+2k=\frac{K}{2}(\rho-p)a^2+\Lambda\, a^2$ [Using (4)]

or, $2\dot{a}^2+2k=\frac{1}{6}Ka^2(\rho+3p)+\frac{K}{2}(\rho-p)a^2+\Lambda\, a^2-\frac{a^2\Lambda}{3}$

or, $2\dot{a}^2+2k=Ka^2\left\{\frac{1}{6}(\rho+3p)+\frac{1}{2}(\rho-p)\right\}+a^2\,\Lambda\left(1-\frac{1}{3}\right)$

or, $2\dot{a}^2+2k=8\pi\, Ga^2.\frac{1}{6}(\rho+3p+3\rho-3p)+\frac{2a^2\,\Lambda}{3}$

or, $2\dot{a}^2+2k=\frac{16\pi\, Ga^2\rho}{3}+\frac{2a^2\,\Lambda}{3}$

or, $\dot{a}^2+k=\frac{8\pi\, Ga^2\rho}{3}+\frac{a^2\,\Lambda}{3}$

or, $\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}=\frac{8\pi\, G\rho}{3}+\frac{\Lambda}{3}$

or, $H^2+\frac{k}{a^2}=\frac{8\pi\, G\rho}{3}+\frac{\Lambda}{3}$ $[\because H=\frac{\dot{a}}{a}]$

or, $3H^2+\frac{3k}{a^2}=8\pi\, G\rho+\Lambda$ .... (5)

and $3H^2+3\dot{H}=-4\pi\, G(\rho+3p)+\Lambda$ ....(6)

where G, ρ and p are the gravitational constant, matter-energy density and fluid pressure respectively. Here, H denotes Hubble's parameter and *a* denotes scale factor. The generalized energy conservation law for variable G and Λ is established by Shapiro et al. [46].

Here, the time variable Λ and the constant G is a special case of the above mentioned generalized conservation law and is given by the equation

$$\dot{\rho}+3(p+\rho)H=-\frac{\dot{\Lambda}}{8\pi G} \qquad (7)$$

Cosmological Model:

We consider the relationship between fluid pressure and density of the barotropic equation is

$$p = \omega\rho \qquad (8)$$

In equation (8), ω is assumed to be a time dependent quantity so that ω=ω(t). Sometimes, it is considered as a constant quantity.

Now, using equation (8) in (7), we have

$$\Rightarrow \dot{\rho}+3\rho(1+\omega)H=-\frac{\dot{\Lambda}}{8\pi G}$$

$$\Rightarrow 8\pi G\dot{\rho}+24\pi G(1+\omega)\rho H=-\dot{\Lambda}$$

$$\because 8\pi G\dot{\rho}+\dot{\Lambda}=-24\pi G(1+\omega)\rho H \qquad (9)$$

Again differentiating equation (5) with respect to t, we get for a flat universe (k=0)

$$4\pi G\rho=-\frac{\dot{H}}{1+\omega} \qquad (10)$$

It is noted that equivalence of three phenomenological Λ-models such as

$\Lambda \sim H^2$, $\Lambda \sim \frac{\ddot{a}}{a}$ and $\Lambda \sim \rho$. We have got it studying in detail by Ray et al. [47] for constant ω.

So, it is reasonable that similar type of variable Λ may be investigated with a variable ω for a deeper understanding of both the accelerating and decelerating phase of the universe.

We use the ansatz $\dot{\Lambda} \propto H^3$,

So that $\dot{\Lambda}=A_1H^3$ (11)

where A_1 is proportional constant. This ansatz with negative A_1 can find realization in the approach of self consistent inflation [48,49], in which time variation of Λ is determined by the rate of Bose condensate evaporation [48] with $\alpha \sim (m/m_{pl})^2$ and here α is the absulate value of negative A_1 and m is the mass of scalar field and m_{pl} is the planck mass.

Using equation (8), (10) and (11), we get from equation (6)

$$\frac{2}{(1+\omega)H^3}\frac{d}{dt}\left(\frac{dH}{dt}\right)+\frac{6}{H^2}\frac{dH}{dt}=A_1 \quad \text{.................................} (12)$$

Now, we put $P=\frac{dH}{dt}$ in equation (12), we have

$$\frac{2}{(1+\omega)H^3}\frac{dP}{dt}+\frac{6P}{H^2}=A_1$$

$$\frac{2}{(1+\omega)H^3}\frac{dP}{dH}.\frac{dH}{dt}+\frac{6P}{H^2}=A_1$$

$$\frac{dP}{dH}+\frac{\frac{6P}{H^2}}{\frac{2P}{(1+\omega)H^3}}=\frac{A_1}{\frac{2P}{(1+\omega)H^3}}$$

$$\frac{dP}{dH}+\frac{6P(1+\omega)H^3}{2H^2P}=\frac{A_1(1+\omega)H^3}{2P}$$

$$\frac{dP}{dH}+3(1+\omega)H=\frac{A_1(1+\omega)H^3}{2P} \quad \text{............................} (13)$$

To arrive at any successful conclusion, we solve the equation (13) and we have

$$\omega(t)=-1+\frac{2\tau P}{H} \quad \text{...} (14)$$

Now, τ has dimension of time. The time scale τ has the physical meaning of dissipation for time varying Λ. Here, ω is the equation of parameter which depends upon time so that $\omega=\omega(t)$.

From equation (13) and (14), we have

$$\frac{dP}{dH}+6\tau P=A_1\tau H^2 \quad \text{..} (15)$$

Solution of equation (15), we have

$$H(t) = \frac{1}{6\tau}\left(1 + \tan\frac{A_2 t}{\tau}\right) \text{ .... (16)}$$

$$q = -\left\{\left(1 + \frac{6A_2}{Cos^2 \frac{A_2 t}{\tau}}\right) / \left(1 + \tan\frac{A_2 t}{\tau}\right)^2\right\} \text{ .... (17)}$$

$$\omega(t) = -1 + \frac{\frac{2A_2}{Cos^2 \frac{A_2 t}{\tau}}}{1 + \tan\frac{A_2 t}{\tau}} \text{ .... (18)}$$

$$p(t) = \frac{1}{48\pi G\tau^2}\left(1 + \tan\frac{A_2 t}{\tau} - \frac{2A_2}{Cos^2 \frac{A_2 t}{\tau}}\right) \text{ .... (19)}$$

$$\rho(t) = -\frac{1}{48\pi G\tau^2}\left(1 + \tan\frac{A_2 t}{\tau}\right) \text{ .... (20)}$$

$$\Lambda(t) = \frac{1}{6\tau^3}\left[\frac{\tau}{2}\tan^2\frac{A_2 t}{\tau} + 2\tau\log\left(\sec\frac{A_2 t}{\tau}\right) + 3\tau\tan\frac{A_2 t}{\tau} - 2A_2 t\right] \text{......(21)}$$

From equation (18), we have

$\omega > -1$ or $\omega < -1$.

According to this assumption $A_2 > 0$ or $A_2 < 0$.

Physical Nature of Parameter ω:

We assume from equation (18) as $\omega(t) = -1 + \dfrac{\dfrac{2A_2}{Cos^2 \frac{A_2 t}{\tau}}}{1 + \tan \frac{A_2 t}{\tau}}$ which unless the second term vanishes ω can not be negative as expected from the SN Ia data [50] and SN Ia data with CMB anisotropy and galaxy-cluster statistics [41] in connection to dark energy. We mention here that the physical significance of the negative density can be realized if one remembers that in the present observation the dark energy is considered not through the equation of state (8) with negative ω rather through the ansatz for Λ where Λ acts as the dark energy with acceleration universe. In that case, Λ makes a definite contribution with accelerating universe.

Physical Nature of Deceleration Parameter q:

The expression of equation (17) as $q = -\left\{ \left(1 + \dfrac{6A_2}{Cos^2 \frac{A_2 t}{\tau}} \right) \Big/ \left(1 + \tan \frac{A_2 t}{\tau} \right)^2 \right\}$ which yields the deceleration parameter q and contains tangent function. Here, in this expression q can change its sign depending on the value of the time dependent part where $A_2<0$. We can be found the decelerating-accelerating cosmic evolution from the present H^3 phenomenological Λ-CDM model.

Physical Nature of Pressure p :

It is noted that the equation (19) as $p(t) = \dfrac{1}{48\pi G \tau^2} \left(1 + \tan \frac{A_2 t}{\tau} - \dfrac{2A_2}{Cos^2 \frac{A_2 t}{\tau}} \right)$ which yields the pressure p and contains tangent function. Observing from the equation (19) and (20), it is clear that the negative energy density comes out with a positive pressure.

Physical Nature of Matter Energy Density ρ :

It is mentioned that the equation (20) as $\rho(t) = -\frac{1}{48\pi G\tau^2}\left(1 + \tan\frac{A_2 t}{\tau}\right)$ which yields the matter energy density ρ and contains tangent function. But here from the equation (19) and (20), we can clearly assume that the accelerating of the universe due to negative density with a positive pressure.

Physical Nature of Cosmological Parameter Λ:

It is noted that the equation (21) as

$\Lambda(t) = \frac{1}{6\tau^3}\left[\frac{\tau}{2}\tan^2\frac{A_2 t}{\tau} + 2\tau\log\left(\sec\frac{A_2 t}{\tau}\right) + 3\tau\tan\frac{A_2 t}{\tau} - 2A_2 t\right]$ which contains the cosmological parameter Λ and tangent function. We observe the effect of a time dependent equation of parameter on a dynamic Λ model which is selected for dark energy investigation. Assuming $\dot{\Lambda}$ ~H, expression for time dependent equation of parameter and matter density have been derived. This conception can find physical justification through the model of Bose-Einstein condensate evaporation [48,49]. The cosmological parameter Λ-dark energy acts as the role for the repulsive pressure which is responsible for the accelerated expansion. But its rule could be understood in a model with dark matter in presence of Λ where dark matter will be associated with repulsive nature due to negative density [51].

Variation of parameter ω with time t (Fig. 1) :

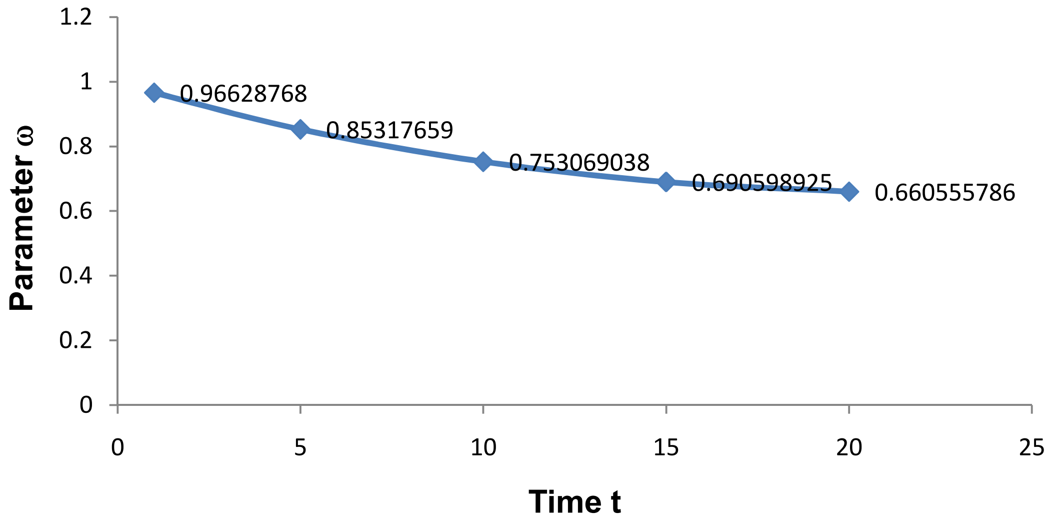


Fig. 1

Variation of deceleration parameter q with time t (Fig. 2) :

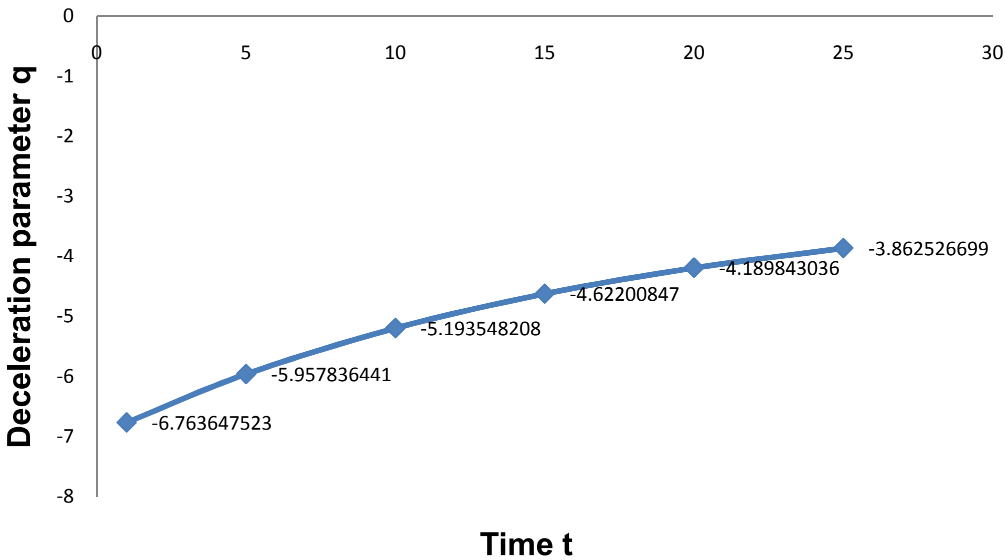


Fig.2

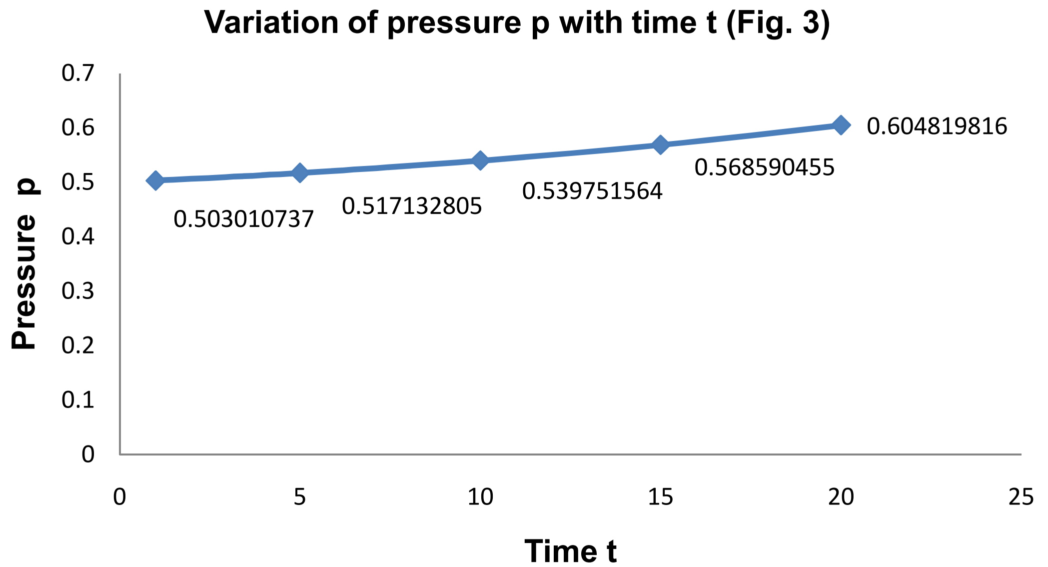

Variation of pressure p with time t (Fig. 3)
Pressure p
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
0.503010737
0.517132805
0.539751564
0.568590455
0.604819816
Time t

Fig. 3

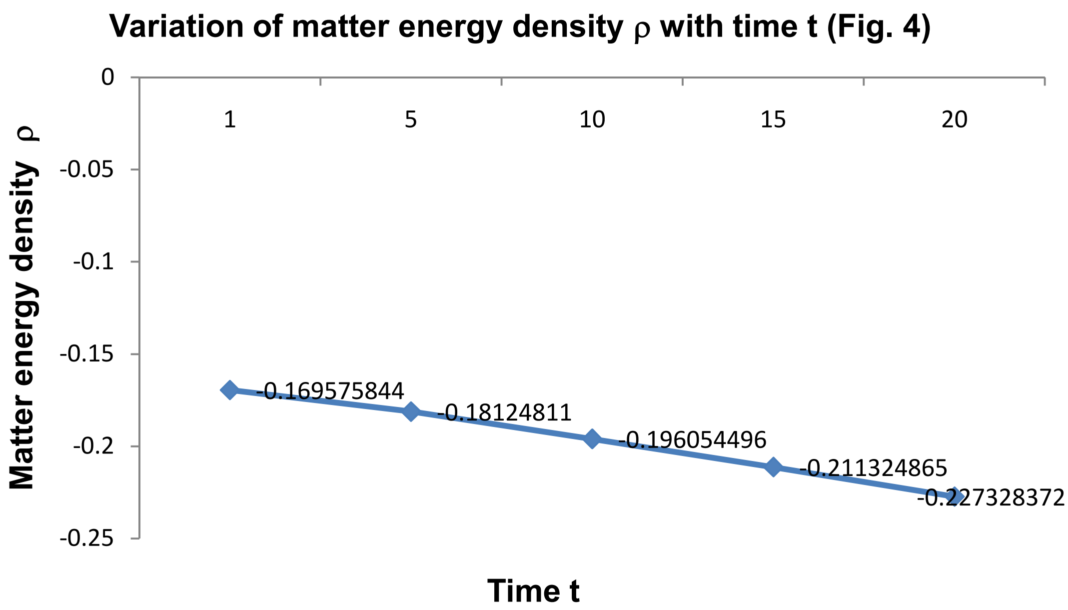

Variation of matter energy density ρ with time t (Fig. 4)
Matter energy density ρ
0
-0.05
-0.1
-0.15
-0.2
-0.25
1
5
10
15
20
-0.169575844
-0.18124811
-0.196054496
-0.211324865
-0.227328372
Time t

Fig. 4

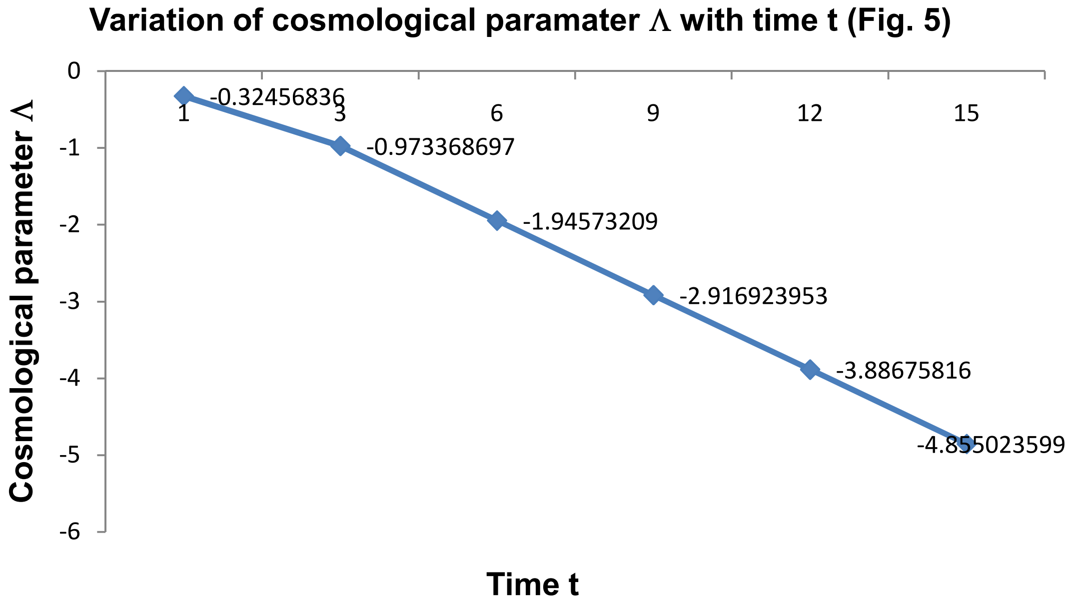

Variation of cosmological paramater Λ with time t (Fig. 5)
Cosmological parameter Λ
0
-1
-2
-3
-4
-5
-6
1
3
6
9
12
15
-0.32456836
-0.973368697
-1.94573209
-2.916923953
-3.88675816
-4.855023599
Time t

Fig. 5

Conclusion:

In this section 8.3, we have observed that the parameter ω, the decelerating parameter q, the pressure p, the matter energy density ρ and the cosmological parameter Λ on the phenomenological evolution of the universe at large time and their properties are given below:

(i) Parameter ω:

It is decreasing slowly with increasing of time on evolution of the universe which has shown in the figure 1.

(ii) Decelerating Parameter q:

It is increasing slowly with increasing of time on evolution of the universe which has shown in the figure 2.

(iii) Pressure p:

It is increasing slowly with increasing of time on evolution of the universe which has shown in the figure 3.

(iv) Matter Energy Density ρ:

It is decreasing slowly with increasing of time on evolution of the universe which has shown in the figure 4.

(v) Cosmological Parameter Λ:

It decreases with increasing of time on the phenomenological evolution of the universe which has shown in the figure 5.

Chapter Nine
Cosmology of Hubble

9.1 Introduction:

The beginning of modern cosmology, Hubble's law was published in 1929 which was based on observations of the red shift of spectral lines from remote galaxies. This was subsequently interpreted as evidence for the expansion of the univers. The concept of an expanding universe, as we are familiar with nowadays, was invented independently by the Russian scientist Alexander Friendmann and by the Belgian cosmologist Georges Lemaitre, with their solutions of Einstein's theory of general relativity applied to the cosmic fluid. Hubble's law was predicted by both Friedmann's and Lemaitre's models.

About 1929 the American astronomer Hubble demonstrated the existence of a strange correlation between distance and speed of the nebulae, they all move outwards away from us and with a velocity which increases proportional to the distance or in other words, the system of the spiral nebulae is expanding, just as the primitive comparison of this system with a gas had suggested to earlier thinkers. Now, if one regards the expansion having the same in the past as it is today, one is led to the idea that the whole system must have had a beginning when all matter was condensed in a small super nucleus and one can calculate the time interval since this beginning of the world and the present instant. The result obtained from Hubble's data was 2000 to 3000 millions of years.

Though the relativistic cosmology initiated by Einstein and de-Sitter began to ripen in the hands of Friedmann, Lamaitre, Tolmam and others. A Series of new possible models of the world were discovered between the extreme cases found by Einstein and de-Sitter and the question arose which of them fitted the empirical facts best, in particular those facts established by Hubble. Today there are many ramifications and refinements of the theory and there has been so enormous an increase of observational material that it is difficult to judge the actual situation.

Earlier ideas which seemed to be most fertile have turned out to be too narrow or even wrong.

It should be pointed out that Hubble himself was not convinced that red shift was exclusively due to Doppler effect. Up to the time of his death he maintained that velocities inferred from red shift measurements should be referred to as apparent velocities.

9.2 The Conception of Albert Einstein about Hubble's Cosmology[14]:

Interpretation of the galactic line shift discovered by Hubble as an expansion leads to an origin of this expansion which lies only about a billion years ago, while physical astronomy makes it appear likely that the development of individual stars and systems of stars take considerably longer. It is in no way known how this incongruity is to be overcome.

9.3 The Conception of Stephen Hawking about Hubble's Cosmology[14]:

For the proof of the existence of many galaxies, Hubble spent his time cataloguing their distances and observing their spectra. At that time most of the people expected the galaxies to be moving around quite randomly and so expected to find as many blue shifted spectra as red shifted ones. It was quite a surprise, therefore to find that most galaxies appeared red shifted, nearly all were moving away from us. More surprisingly still was the finding that Hubble published in 1929, even the size of a galaxy's red shift is not random but it is directly proportional to the galaxy's distance from us. Or, in other words, the farther a galaxy is the faster it is moving away and that meant that the universe could not be static, as everyone previously thought but is in fact expanding, the distance between the different galaxies is growing all the time.

In 1929, Edwin Powell Hubble made the landmark observation that wherever you look, distant galaxies are moving rapidly away from us. In other words, the universe is expanding. This means at earlier time objects would have been closer togather.

Hubble's observations suggested that there was a time, called the Big Bang, when the universe was infinitesimally small and infinitely dense.

9.4 Hubble's Law[5]:

In the 1920s Hubble measured the spectra of 18 spiral galaxies with a reasonably well-known distance. For each galaxy he could identify a known pattern of atomic spectral lines (from their relative intensities and spacing) which all exhibited a common reward frequency shift by a factor 1+z. Using the relation, $\upsilon_A(r,t)=f(t)r$, following from the assumption of homogeneity alone,

$$\upsilon = cz \quad (1)$$

He could then obtain their velocities with reasonable precision.

The Expanding Universe:

The expectation for a stationary universe was that galaxies would be found to be moving about randomly. However, some observations had already shown that most galaxies were red shifted, thus receding, although some of the nearby ones exhibited blue shift. For instance, the nearby Andromeda nebula M31 is approaching us, as its blue shift testifies. Hubble's fundamental discovery was that the velocities of the distant galaxies and he had studied increased linearly with distance,

$$\upsilon = H_0 r \quad (2)$$

This is called Hubble's law and H_o is called the parameter of Hubble. For the relatively nearby spiral galaxies he studied, he could only determine the linear, first order approximation to this function. Although the linearity of this law has been verified since then by the observations of hundreds of galaxies. The message of Hubble's law is that the universe is expanding and this general expansion is called the Hubble's flow.

9.5 Hubble's Time and Radius[5]:

We know that,

$$\upsilon = cz \quad (1)$$

$$\upsilon = H_0 r \quad (2)$$

From equations (1) and (2) one sees that the Hubble's parameter has the dimension of inverse time. Thus a characteristic time scale for the expansion of the universe is the Hubble's time,

$\tau_H = H_0^{-1} = 9.78h^{-1} \times 10^9$yr .... (3)

Here h is the commonly used dimensionless quantity

$$h = H_0/(100\ \text{kms}^{-1}\ \text{Mpc}^{-1})$$

The Hubble parameter also determines the size scale of the observable universe. In time τ_H radiation travelling with the speed of light ***c*** has reached the Hubble's radius,

$$r_H = \tau_H c = 3000h^{-1}\text{Mpc} \text{.........................(4)}$$

Or, to put it a different way, according to Hubble's nonrelativistic law, objects at this distance would be expected to attain the speed of light which is an absolute limit in the theory of special relativity. Combining equation (1) and equation (2), we obtain

$$z = H_0 \frac{r}{c} \text{..(5)}$$

9.6 Hubble's Constant[5]:

The value of this constant initially found by Hubble was H_0 = 550kms^{-1} Mpc^{-1}; an order of magnitude too large because his distance measurements were badly wrong. To establish the linear law and to determine the global value of H_0 one needs to be able to measure distances and expansion velocities well and far out. Distances are precisely measured only to nearby stars which participate in the general rotation of the galaxy and which therefore do not tell us anything about cosmological expansion. Even at distances of several Mpc, the expansion-independent, transversal peculiar velocities of galaxies are of the same magnitude as the Hubble's flow. The measured expansion at the Virgo super cluster, 18Mpc away, is about 1100 kms^{-1} whereas the peculiar velocities attain 600 km s^{-1}. At much larger distances where the peculiar velocities do not contribute appreciably to the total velocity, for instance at the coma cluster 100Mpc away, the expansion velocity is 6900 km s^{-1} and the Hubble's flow can be measured quite reliably but the imprecision in distance measurements becomes the problem. Every procedure is sensitive to small, subtle corrections and to systematic biases unless great care is taken in the reduction and analysis of data.

9.7 The Changing Views of Hubble about Cosmology:

Edwin Powell Hubble (1889-1953) established in 1924 that many nebulae are stellar systems outside the Milky Way when he discovered Cepheid variables in the Andromeda nebula using the 100 inches telescope on Mount Wilson.

In 1929 he established the famous distance- velocity relation which is also called now-a-days the law of red shift or Hubble's law. The title of his paper reads, "A relation between distance and radial velocity among extra galactic nebulae."

As a matter of fact he and his collaborator Milton L. Humason (1891-1982) never measured velocities directly. What they measured were the red shifts of these extra galactic nebulae. But in this crucial paper Hubble considered these red shifts as representing real radial velocities of these nebulae. The main conclusion of the paper, "The date in the table indicate a linear correlation between distances and velocities whether the latter are used directly or corrected for solar motion, according to the older solutions."

The outstanding feature however is the possibility the velocity-distance relation may represent the de-Sitter effect and hence that numerical data may be introduced into discussions of the general curvature of space. In the de-Sitter's cosmology, displacements of the spectra arise from two sources, an apparent slowing down of atomic vibrations and a general tendency of material particles to scatter. The latter involves an acceleration and hence introduces the element of time. The relative importance of these two effects should determine the form of the relation between distances and observed velocities and in this connection it may be emphasized that the linear relation found in the present discussion is a first approximation representing a restricted range in distance.

Willem de-Sitter (1882-1934) was a Dutch mathematician, physicist and astronomer. In 1916-1918, he had found a solution to Einstein field equations of general relativity describing the expansion of the universe. Hubble had met him in 1928 in Leiden where de-Sitter was professor of astronomy at the University of Leiden.

However, it must be pointed out that, as early as in 1935. Hubble was much more cautious when referring to velocities of recession. In a paper with R. Tolman (Hubble & Tolman,1935) already in the introductory section, the authors made a plain statement of their worries about the popper nomenclature.

Until further evidence is available, both the present writers wish to express an open mind with respect to the ultimately most satisfactory explanation of the nebular red shift and in the presentation of purely observational findings to continue to use the phrase apparent velocity of recession. They both incline to the opinion that if the red shift is not due to recessional motion, its explanation will probably involve some quite new physical principles.

Nebular spectra are peculiar in that the lines are not in the usual positions found in nearby light sources. They are displaced toward the red end of their normal position, as indicated by suitable comparison spectra. The displacements, called red shifts, increase on the average with the apparent faintness of the nebula that is observed. Since apparent faintness measures distance, it follows that red shifts increase with distance and detailed investigation shows that the relation is linear.

Small microscopic shifts, either to the red or to the violet, have long been known in the spectra of astronomical bodies other than nebulae. The same interpretation is frequently applied to the red shifts in nebular spectra and has led to the term velocity-distance relation for the observed relation between red shifts and apparent faintness. On this assumption, the nebulae are supposed to be rushing away from our region of space, with velocities that increase directly with distance.

Although no other plausible explanation of red shifts has been found, the interpretation as velocity shifts may be considered as a theory still to be tested by actual observations. Critical tests can probably be made with existing instruments. Rapidly receding light sources should appear fainter than stationary sources at the same distances and near the limits of telescopes the apparent velocities are great and that the effects should be appreciable.

Interpretations of the red shifts themselves do not inspire such complete confidence. Red shifts may be expressed as fractions, $d\lambda/\lambda$ where $d\lambda$ is the displacement of a spectral line whose normal wave length is λ. The displacements, $d\lambda$, vary systematically through any particular spectrum but the variation is such that the fraction, $d\lambda/\lambda$, remains constant. Thus $d\lambda/\lambda$ specifies the shift for any nebula and it is the fraction which increases linearly with distances of the nebulae. According to Hubble's conception that the apparent radial velocity of a nebula is to a first approximation , the velocity of light (1,86,000 miles/sec.) multiplied by the fraction, $d\lambda/\lambda$. From this point, the term red shift will be employed for the fraction, $d\lambda/\lambda$.

Moreover, the displacements, $d\lambda$, are always positive and so the wave length of a displaced line, $\lambda+d\lambda$, is always greater than the normal wave length, λ . Wave lengths are increased by the factor, $(\lambda+d\lambda)/\lambda$, or the equivalent, $1+d\lambda/\lambda$. Now, there is a fundamental relation, multiplied by the wave length of the quantum is constant.

Thus energy × wave length = constant.

Obviously, since the product remains constant, red shifts by increasing the wave lengths. Any plausible interpretation of red shifts must account for the loss of energy. The loss must occur either in the nebulae themselves or in the immensely long paths over which the light travels on its journey to the observer.

Thorough investigation of the problem has led to the following conclusions. Several ways are known in which red shifts might be produced. Of them all, only one will produce large shifts without introducing other effects which should be conspicuous but which are not observed. This explanation interprets red shifts as Doppler effects, that is to say, as velocity shifts, indicating actual motion of recession. It may be stated with some confidence that red shifts are velocity shifts or else they represent some unrecognized principle in physics.

The interpretation as velocity shifts is generally adopted by theoretical investigators and the velocity-distance relation is considered as the observational basis for theories of an expanding universe. They represent solutions of the cosmological equation which follow from the assumption of a non-static universe. They supersede the earlier solutions made upon the assumption of a static universe which are now regarded as special cases in the general theory.

Chapter Ten

The Early Universe

Introduction[4]:

The ‘cosmic background radiation’ discovered originally by Penzias and Wilson in 1965 provides evidence that the universe must have gone through a hot dense phase. We have also seen that the Friedmann models, if they are regarded as physically valid, predict that the density of mass-energy must have been very high in the early epochs of the universe. In fact, the Friedmann models imply that the mass- energy density goes to infinity as the time t approaches the initial moment or the initial singularity at t=0. This is what is referred to as the Big Bang, meaning an explosion at every point of the universe in which matter was thrown as under violently, from an infinite or near infinite density. However, the precise nature of the physical situation at t=0, or the situation before t=0; these sorts of questions are entirely unclear. Here, we simply assume that there was a catastrophic event at t=0 and try to describe the state of the universe from about t=0.01 s until about t= one million years. This will be our definition of the early universe which specifically excludes the first hundredth of a second or so, during which as speculations go, events occurred which are of a very different nature from those occurring in the early universe according to the definition given here.

We shall also describe qualitatively the state of the early universe providing a more quantitative account of this state. The description is derived largely from that given in Weinberg’s book. As indicated the spectrum of the cosmic background radiation peaks at slightly under 0.1 cm. Penzias and Wilson made their original observation at 7.35 cm. Since that time there have been many observations, both ground-based and above the atmosphere which confirm the black-body nature of the radiation, with a temperature of about 2.8 k. Below about 0.3 cm, the atmosphere becomes increasingly opaque, so such observations have to be carried out above the atmosphere. Although at times there have been slight doubts, it is now generally agreed that the cosmic background radiation is indeed the remnant of the radiation from the early universe which has been red shifted, that is, reduced in temperature to 2.8 K. The temperature of the cosmic background radiation provides us with an important datum about the universe, that there are about 1000 million photons in the universe for every nuclear particle and by the latter we mean protons and neutrons or baryons.

(i) First Frame[4]:

This is at t=0.01s when the temperature is around 10^{11} K which is well above the threshold for electron-positron pair production. The main constituents of the universe are photons, neutrinos and antineutrinos and electron-positron pairs. There is also a small contamination of neutrons, protons and electrons. The energy density of the electron-positron pairs is roughly equal to that of the neutrinos and antineutrinos, both being $\frac{7}{4}$times the energy density of the photons. The total energy density is about 21×11^{44} $eV1^{-1}$ or about 3.8×10^{11} g cm^{-3}. The characteristic expansion time of the universe is 0.02s. The neutrons and protons cannot form into nuclei, as the latter are unstable. The spatial volume of the universe would be either infinite or, if it is one of the finite models, say with density twice the critical density, its circumference would be about 4 light years.

(ii) Second Frame[4]:

This is at t=0.12s when the temperature has dropped to about 3×10^{10} K. No qualitative changes have occurred since the first frame. As in the first frame, the temperature is above electron-positron pair threshold, so that these particles are relativistic and the whole mixture behaves more like radiation than matter, with the equation of state given nearly by $p=\frac{1}{3}\in$. The total density is about 3×10^{7}g cm^{-3}. The characteristic expansion time is about 0.2 s. No nuclei and be formed yet but the previous balance between the numbers of neutrons and protons which were being transformed into each other through the reaction $n+v \rightleftarrows p+e^{-}$, is beginning to be disturbed as neutrons now turn more easily into the lighter protons than vice versa. Thus the neutron-proton ratio becomes approximately 38% neutrons and 62% protons. The thermal contact between neutrinos and other forms of matter is beginning to cease.

(iii) Third Frame[4]:

This is at t =1.1s, when the temperature has fallen to about 10^{10} K. The thermal contact between the neutrinos and other particles of matter and radiation ceases. Thermal contact is here taken to mean the conversion of electron-positron pairs into neutrino-antineutrino pairs and vice versa, the conversion of neutrino-antineutrino pairs into photons and vice versa, etc. Hence, neutrinos and antineutrinos will not play an active role but only provide a contribution to the overall mass-energy density. The density is of the order of 10^5g cm^{-3} and the characteristic expansion time is a few seconds. The temperature is near the threshold temperature for electron-positron pair production, so that these pairs are beginning to annihilate more often to produce photons than their creation from photons. It is still too hot for unclei to be formed and the neutron-proton ratio has changed to approximately 24% neutrons and 76% protons.

(iv) Fourth Frame[4]:

This is approximately at t ≈ 13 s when the temperature has fallen to about 3×10^9K. This temperature is below the threshold for electron-positron production and most of these pairs have annihilated. The heat produced in this annihilation has temporarily slowed down the rate of cooling of the universe. The neutrinos are about 8% cooler than the photons, so the energy density is a little less than if it were falling simply as the fourth power of the temperature. The neutron-proton balance has shifted to about 17% neutrons and 83% protons. The temperature is low enough for helium nuclei to exist but the lighter nuclei are unstable, so the former cannot be formed yet. By helium nuclei, we mean alpha particles, He^4 which have two protons and two neutrons. The expansion rate is still very high, so only the light nuclei form in two-particle reactions, as follows:

$p + n \rightarrow D + \gamma$, $D + p \rightarrow He^3 + \gamma$, $D + n \rightarrow H^3 + \gamma$, $He^3 + n \rightarrow He^4 + \gamma$, $H^3 + p \rightarrow He^4 + \gamma$. Here D denotes deuterium which has one neutron and one proton, He^3 is helium-3, an isotope of helium with two protons and one neutron. H^3 is tritium, an isotope of hydrogen with one proton and two neutrons and γ stands for one or more photons. Although helium is stable, the lighter nuclei mentioned here are unstable at this temperature, so helium formation is not yet possible as it is necessary to go through the above intermediate steps to form helium. The energy required to pull apart the neutron and proton in a D nucleus, for example, is one-ninth that required to pull apart a nucleon (neutron or proton) from an He^4 nucleus. In other words, the binding energy of a nucleon in deuterium is one-ninth that in an He^4 nucleus.

(v) Fifth Frame[4]:

This is about 3 minutes after the first frame when the temperature is about 10^6 K which is approximately 70 times as hot as the centre of the sun. The electron-positron pairs have disappeared and the contents of the universe are mainly photons and neutrinos plus, as before a contamination of neutrons, protons and electrons which will eventually turn into the matter of the present universe. The temperature of the photons is about 35% higher than that of the neutrinos. It is cool enough for H^3, He^3 and He^4 nuclei to be stable but the deuterium 'bottleneck' is still at work so these nuclei cannot be formed yet. The beta decay of the neutron into a proton, electron and antineutrino is becoming important, for this reaction has a time scale of about 12 minutes. This causes the neutron-proton balance to become 14% neutrons and 86% protons.

A little later than the fifth frame the temperature drops enough for deuterium to become stable, so that heavier nuclei are quickly formed but as soon as He^4 nuclei are formed other bottlenecks operate, as there are no stable nuclei at that temperature with five or eight particles. The exact temperature depends on the number of photons per baryon; if this number is 10^9 as assumed before then the temperature is about 0.9×10^9K and these events take place at some time between t = 3 minutes and t = 4 minutes. Nearly all the neutrons are used up to make He^4, with very few heavier nuclei due to the other bottlenecks mentioned. The neutron-proton ratio is about 12% or 13% neutrons to 87% protons and it is frozen at this value as the neutrons have been used up. As the He^4 nuclei have equal numbers of neutrons and protons, the proportion of helium to hydrogen nuclei by weight is about 24% or 26% helium and 76% or 74% hydrogen. This process, by which heavier nuclei are formed from hydrogen, is called nucleosynthesis. If the number of photons per baryon is lower then nucleosynthesis begins a little earlier and slightly more He^4 nuclei are formed than 24% or 26% by weight.

(vi) Sixth Frame[4]:

This is approximately at $t \approx$ 35 minutes when the temperature is about 3×10^8 K. The electrons and positrons have annihilated completely, except for the small number of electrons left over to neutralize the protons. It is assumed throughout that the charge density in any significant volume of the universe is zero. The temperature of the photons is about 40% higher than the neutrino temperature. The energy density is about 10%, the density of water of which 31% or so is contributed by neutrinos and the rest by photons. The density of matter is negligible in comparison to that of photons and neutrinos. The characteristic expansion time of the universe is about an hour and a quarter. Nuclear processes have then stopped, the proportion of He^4 nuclei being anywhere between 20% and 30% depending on the baryon: photon ratio.

We see from the preceding discussion that the proportion of helium nuclei formed in the early universe was anything from 20-30% by weight, with very few heavier nuclei due to the five and eight particle bottlenecks. For the nucleosynthesis process to take place one needs temperatures of the order of a million degrees. After the temperature dropped below about a million degrees in the early universe, the only place in the later universe where similar temperatures exist would be the centre of stars. It can be shown that no significant amount of helium (compared to the 20-30% of the early universe) could have been created in the centre of stars. This follows from the fact that such a significant amount of helium formation would have released so much energy into the interstellar and intergalactic space, that it would be inconsistent with the amount of radiation actually given off since the time of star and galaxy formation, an amount of which can be calculated from the average absolute luminosity of stars and galaxies which are known and the time scale during which these have existed which is from soon after the recombination era. Thus if the above picture is reasonable, there should be approximately 20-30% helium nuclei in the present universe, most of the rest being predominantly hydrogen, with a small amount of heavier nuclei. This is indeed found to be the case.

We have seen that the time, temperature and the extent of nucleosynthesis depends on the density of nuclear particles compared to photons. The amount of deuterium that was produced by nucleosynthesis in the early universe and the amount that survives and should be observable today, depends very sensitively on the nuclear particle to photon ratio. As an illustration of this, the abundance of deuterium as worked out by Wagoner for three values of the photon: nunclear particle ratio.

Table : Abundance of deuterium and the photon: baryon ratio.

Photons : nuclear particle	Deuterium abundance (parts/10^6)
100 million	0.00008
1000 million	16
10000 million	600

We have seen that after the first few minutes the only particles left in the universe were photons, neutrinos, neutrons, protons and electrons. The latter two particles are charged ones and in their free state they could scatter photons freely. As a result the 'mean free path' of photons, that is, the average distance that a photon travels in between scatterings by two charged particles, was small compared to the distance a photon would travel during the characteristic expansion time of the universe for that period, if it were unimpeded. This is what is meant by the matter and radiation being in equilibrium, as there is free exchange of energy between the two. Thus the universe, during the period that protons and electrons were free particles was opaque to electromagnetic radiation.

Eventually the temperature of the universe was cool enough for electrons and protons to form stable hydrogen atoms in their ground state when they combined. Now, it takes about 13.6 eV to ionize a hydrogen atom completely, that is, pull apart the electron from the proton. The energy of a particle in random motion at a temperature of T K is $k\ T$, where k is Boltzmann's constant. Thus the temperature corresponding to an energy of 13.6 eV is k^{-1} times 13.6, where k^{-1} is approximately 11605 K eV^{-1}. This gives about 1.586×10^5 K as the temperature at which a hydrogen atom is completely ionized. However, even in the excited states, in which it is not ionized, a hydrogen atom can effectively scatter photons. Thus it is only in the ground state that it ceases to interact significantly with photons. The temperature at which the primeval protons and electrons combined to form the ground state hydrogen atoms was about 3000-4000 K which occurred a few hundred thousand years after the Big Bang. This era is referred to as recombination. After this period the universe became transparent to electromagnetic radiation, that is, the mean free path of a photon became much longer than the distance traversed in a characteristic expansion time of the period. This is the reason, we get light which has hardly been impeded, except for the red shift, from galaxies billions of light years away.

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